Facility Location

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Locating facilities is a strategic decision that all organisations must take early on during their lives and re-evaluate when expanding or relocating premises.

Large **production facilities** such as factories are located according to a myriad of parameters, and only a few are directly related to the organisation's operations. For example, the availability of specialised workers, fiscal considerations, proximity with strategic partners, and strong intellectual property protection are all factors that can determine where a company will open their factories. Once the general area (say, the country) is determined, operational considerations can guide the choice of the location where the facility will open. Regarding **logistics facilities** such as depots and warehouses, their integration with the supply chain and the company's operations is fundamental when deciding their location.

Companies are sometimes also uncertain about the **number** and **size** of the facilities they want to open. These decisions are usually subject to trade-offs. Imagine, for example, a company that must decide how many warehouses to open in Spain. If they open many of them, transportation costs between the warehouse and the stores (or the customers) will be lower because average distances will be shorter. Such a choice will reduce shipping expenses and increase customer satisfaction. On the other hand, opening a warehouse is a large capital investment, and the fixed costs associated with a new warehouse must also influence the decision.

Finally, even if the number, size and location of facilities are determined, the company should also decide how to **allocate customers** to the facilities. For example, if a company opens a warehouse in Madrid and one in Barcelona, it should decide which will ship items to customers in Zaragoza.

Facility location is a complex problem with many interdependent decisions affected by multiple factors. We can summarise the decisions as follows:

- Decide how many facilities to place.
- If necessary, decide how large each facility should be.
- Decide where the facilities should be located.
- Decide which facility will serve which customers.

We can summarise the determining factors as follows:

- Reduce the fixed costs associated with opening and operating facilities.
- Reduce the variable operating costs, mainly due to shipping.
- Improve service level, e.g., by reducing the average distance between facilities and customers.

While we usually refer to logistic concepts such as warehouses and factories, in practice, Facility Location Problems (FLPs) are solved by operations managers in diverse areas. For example, to determine where to place ambulances and how to dispatch them to reduce the average wait time of patients; where to place dog parks so that most neighbours have one within a reasonable walking distance; where to place Wi-Fi access points to cover the Ciutadella Campus of Universitat Pompeu Fabra while ensuring that no access point is overloaded.

In the logistic context, we usually think about the goods in the supply chain as moving from the facilities towards the customers. For example, warehouses shipping to stores or Amazon fulfilment centres shipping to customers. However, we can use the same facility location models for reverse logistics. For example, when dealing with returns of unwanted online purchases, vehicles will pick up goods at customer's homes and bring them to the warehouses. Another example involves door-to-door trash collection, where vehicles pick up trash bags and bring them to the recycling facilities. Finally, empty container repositioning: many consumer goods are shipped on pallets, leaving the problem of collecting the empty ones to be reused.

To emphasise the diversity of FLPs, in these notes, we will cover five problems that arise in real-world situations.

1 The Uncapacitated Fixed-Charge Location Problem

We begin our exploration of FLPs from the Uncapacitated Fixed-Charge Location Problem (UFLP). Although the definition of the UFLP shows some simplifications compared to typical real-life scenarios, it is useful as a first approach to the world of mathematical modelling thanks to its simplicity.

To define this problem, consider a set of customers denoted by I. For example, if customers are stores we must deliver goods to, our set of customers could be:

$$\begin{aligned} I = \{ & \text{“Store 14. Gran Via, 14, Barcelona.”}, \\ & \text{“Store 08. Plaça de la Vila, 1, Lleida.”}, \\ & \text{“Store 21. Calle Ramon y Cajal, 14, Zaragoza.”}, \\ & \text{“Store 13. Pasaje Méndez, 3b, Catalayud.”}, \\ & \text{“Store 44. Plaza del Sol, 2, Toledo.”}, \\ & \ldots, \\ & \text{“Store 11. Avenida Buenos Aires, 155, Logroño.”} \}. \end{aligned}$$

Each customer has an associated quantity of goods that they require. For a customer $i \in I$, we denote this quantity as $h_i \in \mathbb{R}^+$. It usually refers to the goods required over a large time horizon, for example, one year. Because we are planning for the future, we likely do not know precisely how many goods each customer will order. Therefore, operations managers usually rely on estimates based on market characteristics or historical data. Suppose we are in the classical context of a large logistic operator with warehouses and stores. In that case, expressing this quantity in a unit of measure directly related to transportation costs is convenient. For example, if we move goods using trucks, we could express h_i in the number of truckloads per year. I.e., $h_i = 12$ means that customer i requires an average of one monthly truck shipment. If we pay a third-party provider to ship our goods and they charge us by the tonne, then we would rather express h_i in tonnes per year.

Set	**Description**
I	Set of customers.
J	Set of potential facility locations.
Parameter	**Description**
h_i	Yearly demand of customer $i \in I$.
c_{ij}	Unit transport cost from facility $j \in J$ to customer $i \in I$.
f_j	Yearly fixed cost if opening a facility at $j \in J$.

Table 1: Sets and parameters of the Uncapacitated Fixed-Charge Location Problem.

Operations managers or other business functions evaluate appropriate sites to open the new facilities. At the end of this search, they compile a list of potential locations that we denote with J. For example, the set of potential facility locations could be a list of addresses where to build or rent warehouses:

$$
\begin{aligned}
J = \{&\text{“Carrer de l'Industria, 2, Cerdanyola del Valles.”},\\
&\text{“Calle del Norte, 54, Riudoms.”},\\
&\text{“Calle del Progreso, 11, Leganés.”},\\
&\ldots\\
&\text{“Rúa de Arriba, 148/B, Lugo.”}\}.
\end{aligned}
$$

To determine the variable costs the company will incur if they open a facility at a given location $j \in J$, operations managers must estimate how expensive it is to transport goods from j to each customer $i \in I$. The cost of a unit shipment from j to i is denoted as $c_{ij} \in \mathbb{R}^+$ (note that, in the indices, customer i comes first and facility location j comes second). The unit of measure of c_{ij} is euros per unit shipped, where “unit shipped” is expressed in the base unit of the customer demands h_i. For example, if we measure h_i in truckloads per year, then c_{ij} will be the cost of sending a truckload from j to i. Analogously, if h_i is expressed in tonnes per year, c_{ij} will be the cost of shipping one tonne of goods.

The last input parameter of our model regards fixed costs. Each facility $j \in J$ has an associated fixed cost $f_j \in \mathbb{R}$. The company pays f_j if they open a facility at the potential location j. To make costs comparable, it is convenient that costs f_j refer to the same period as demands h_i. If demands are given annually (truckloads per year, tonnes per year, etc.), then f_j will represent the annual cost of opening and operating a facility at j. If the company is, e.g., renting a warehouse at j, then f_j could be the sum of the yearly rent plus the estimated cost of electricity, water, security, and any other cost associated with operating a warehouse. If the company builds the facility and incurs a large one-off expense, then f_j should include the yearly amortisation of the initial expense.

Table 1 resumes the parameters of the UFLP, i.e., the input data that the operations manager needs to make the relevant decisions. These decisions are where to open facilities and how to allocate customers to facilities. The operations manager uses the data to create a mathematical model, then inputs this model into a computer and uses a special software (called a solver) to make the best possible decision for the objective he wants to achieve. To this end, he needs to define **decision variables**: the “unknowns” in the problem. When the software solves the problem, it will assign concrete values (i.e., numbers) to each decision variable. As we will see, the solver will give reasonable results only if the operation manager provides a correct and

Open a facility at j	**Serve customer i from j**	**Answers valid?**	**Explanation**
Yes	Yes	✓	We open a facility at j and serve customer i from there.
Yes	No	✓	We open a facility at j, but we will serve customer i from somewhere else.
No	Yes	✗	We do not open a facility at j, but we serve customer i from location j (?!).
No	No	✓	We do not open a facility at j and, indeed, we do not serve customer i from j.

Table 2: Valid and invalid decisions for a given facility location $j \in J$ and customer $i \in I$.

complete mathematical definition of the problem. Such a definition is called a **mathematical formulation**.

Decision Variables

Let's start by considering the unknowns in our problem. In the UFLP, there are two of them. First, given a location, do we open a facility there? The answer to this question should be a yes or no. Second, given a location $j \in J$ where we open a facility, which customers do we serve from j? We can answer this question by providing a certain number of lists: for each j, we can provide a list of customers served from j (if we open a facility at j) or an empty list (if we do not). However, we can slightly reformulate the second question as asking for a yes/no. For example, we can ask: given a location j and a customer $i \in I$, will we serve i from j? The answer will be no if we do not open a facility at j. On the other hand, if we open a facility at j, the answer will be yes if i is in the list of customers served from j and no otherwise. Table 2 resumes, for a given location j and customer i, the four possible decisions that we can take. The table shows that only three possibilities make sense: we should be careful that our mathematical formulation does not allow the solver to answer as in the third row of the table (✗ sign).

We like to cast decisions as questions asking for a yes/no answer because these are easy to model mathematically using decision variables. In particular, such decisions correspond to **binary decision variables**. We call a decision variable binary if it can only take two values; mathematically, we denote them as 0 and 1. If a binary decision variable takes value 0, the answer to the corresponding question is no; if it takes value 1, the answer is yes.

With this in mind, we can model the unknowns of the UFLP using two sets of variables:

- First, for each location $j \in J$, we must decide whether to open a facility there. This is encoded into a binary decision variable $x_j \in \{0, 1\}$. In a solution of the UFLP, each variable x_j will be either 0 or 1. If $x_j = 1$, then the solver has determined that opening a facility at location j is a good idea. Otherwise, the solver has determined we should not open a facility there.
- Second, for each customer $i \in I$ and each location $j \in J$, we must decide whether i is served from j. We can represent this decision using a binary decision variable $y_{ij} \in \{0, 1\}$. In a solution of the UFLP, we shall serve i from j if $y_{ij} = 1$; otherwise, we will not use j to serve i.

We remark that these decision variables are sufficient to completely determine a solution of the UFLP. For example, assume that the solver has completed and has assigned concrete numbers (zeros and ones in our case) to all variables. If we want to know how many facilities to open, we can count how many variables x_j have value 1:

$$\text{Number of open facilities } = \sum_{j \in J} x_j.$$

If we want to know which customers we serve from location $j \in J$, we can do so in two steps:

1. Check the value of x_j. If it is 0, then we are not opening a facility at j; therefore, we are not serving any customers from there.
2. If $x_j = 1$, then check the value of all variables of type y_{ij}, where j is fixed to the location of our interest. Each index i such that $y_{ij} = 1$ corresponds to a customer that we serve from j:
$$\text{Customers served from } j = \{i \in I \text{ such that } y_{ij} = 1\}.$$

Objective function

We need a few more ingredients before our solver can give us a solution to the problem. The first such ingredient is a description of the objective that the solver should strive to reach. Note that, for each UFLP, there are many theoretically valid solutions. For example, we could open only one facility and serve all customers from there, or we could open four facilities and split the customers roughly equally among them, etc. We need a way to determine which of these solutions is preferable.

To do so, we must first decide on a business objective we want to achieve and then translate it into a mathematical formula that uses the problem parameters and the decision variables. In the UFLP, the business objective is to serve all customers at the smallest possible cost. That is, we want to **minimise** the total cost of a solution, where the total cost is the sum of the fixed costs and the variable (transport) costs. We must inform the solver that this is our objective; in this way, the solver will find us the best possible solution, i.e., the one giving the lowest cost. To this end, we must write the cost as a mathematical formula to pass to the solver. Because the total cost is the sum of fixed and variable costs, we can devise separate formulas for these costs and then sum them.

- Recall that we pay a fixed (annual) cost f_j if we open a facility at j. The total fixed cost is then the sum of the f_j's over all the locations where we open facilities. We might be tempted to write the total fixed cost as follows:
$$\text{Total fixed cost } = \sum_{\substack{j \in J \\ x_j = 1}} f_j,$$
which reads "sum the fixed costs f_j of all the locations $j \in J$ where we open a facility ($x_j = 1$)". Such a formula, however, has a problem: it relies on us already knowing the solution! We can check whether $x_j = 1$ only if we already have access to the solution, i.e., the values the solver assigns to the decision variables. However, we must inform the solver about our objective *before* it starts. Therefore, we have to find another formula for the fixed costs that works without knowing the value of the x_j's, i.e., a formula that contains the variables and works independently of the value they will take. For example, the following formula would work:
$$\text{Total fixed cost } = \sum_{j \in J} f_j x_j. \tag{1}$$

The difference might seem subtle, but it is substantial. By avoiding any explicit mention of the value of the variables, we have produced a formula that gives us the correct fixed cost, no matter which facilities we open. Each term of the sum is either f_j if the solution will have $x_j = 1$ or 0 if the solution will have $x_j = 0$. This is correct in both cases because we must pay fixed cost f_j if we open a facility ($x_j = 1$), and we do not pay anything if we do not ($x_j = 0$).

To summarise, no matter how the solver assigned values to the variables, if we compute (1), we will obtain the correct value of the fixed costs. Furthermore, (1) is written in a way that doesn't assume that we already have access to the variable values and, therefore, we can use it in the problem specification we provide to the solver before it starts solving the UFLP.

- Regarding variable costs, recall that we pay c_{ij} for each unit of goods shipped from j to i. If we assign customer i to facility j, we will send h_i units of goods from j to i and, therefore, the transport costs will be $h_i c_{ij}$. If we do not, the transport cost between j and i will be zero. Again, a naive attempt at writing the variable costs could be:

$$\text{Total variable costs } = \sum_{i \in I} \sum_{\substack{j \in J \\ y_{ij}=1}} h_i c_{ij},$$

but, as we have already seen, we are not allowed to assume that we know the variable values at this stage. Therefore, we can rewrite this formula in a way that works independently of the y_{ij}'s values:

$$\text{Total variable costs } = \sum_{i \in I} \sum_{j \in J} h_i c_{ij} y_{ij}.$$

Summing together the two formulas above, we obtain the **objective function** for our problem. All problems we consider ask to maximise or minimise an objective function, depending on the context. In the UFLP, because we want the solution with the smallest cost, we are *minimising* the objective function. Therefore, we can write:

$$\min \quad \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} h_i c_{ij} y_{ij}. \tag{2}$$

Problem constraints

The next ingredient in our problem formulation is the constraints. Remember that the solver is software: an algorithm extremely efficient at finding good solutions but, essentially, dumb. The solver lacks all context for the problem we ask it to solve; it only sees variables and parameters but does not know the meaning of these symbols. It doesn't know that c_{ij} is the unit annual transport cost between j and i; it only sees it as a parameter appearing somewhere in objective function (2). Analogously, it doesn't know that $x_j = 1$ means "open a facility at location j"; it just knows that x_j is a binary variable and that it must assign it either value 0 or 1 (whichever helps the most in minimising the objective function). Despite impressive advancements in Artificial Intelligence, we are not yet at a point where we can describe our UFLP in plain English and expect that the solver provides the optimal solution. Instead, we must provide a precise mathematical characterisation of the constraints that any solution (that is, any assignment of numeric values to the variables) must obey.

For example, the solver doesn't know we want to serve *all* our customers. From its point of view, we have never informed it that each customer must be assigned to some open facility (and, in particular, to exactly one open facility). If we asked the solver to find the best solution that minimises (2), without any constraint, the answer would be obvious: if we open facilities, we incur some cost, while if we do not open any facility, we do not incur any. Therefore, the best possible solution must be: "do not open any facility", i.e., $x_j = 0$ for all $j \in J$. This solution is absurd to us (if we do not open facilities, we cannot serve customers). Still, it is perfectly reasonable to the solver because we have never instructed it that all customers must be served. These instructions defining which solutions are acceptable from a business, real-life point of view and which ones aren't are called **problem constraints**.

In the UFLP, we have two problem constraints:

- **All customers must be served, and each customer must be served by exactly one facility.** For example, there cannot be any customer $i \in I$ such that $y_{ij} = 0$ for all locations $j \in J$ because such a customer would not be served. Analogously, there cannot be any customer i such that $y_{ij} = 1$ and $y_{ij'} = 1$ for two different facilities j and j' because such a customer would be "over-served", i.e., it would receive duplicate shipments. [1] These are business requirements, and they are part of the definition of the UFLP. As operations managers, we translate them into mathematical constraints that we can pass on to the solver. Mathematical constraints are expressions involving decision variables and the problem input data and can be of three types: "$\leq$"-inequalities, "$\geq$"-inequalities, and equalities (with a "$=$" sign). We remark here that we are not considering "$<$"- and "$>$"-inequalities on purpose: we will not allow these expressions to appear in our problem constraints.

  To model the first business requirement, it is enough to verify that, for each customer i, there is exactly one facility location j such that $y_{ij} = 1$. The most straightforward way to enforce this requirement is using a sum and an equality:

  $$\sum_{j \in J} y_{ij} = 1 \quad \forall i \in I. \tag{3}$$

  Formula (3) defines a group of constraints; in particular, there is one constraint for each customer i, as specified by the quantifier on the right. For a given customer i, the equality says that exactly one variable y_{ij} must take value one. If all variables y_{ij} were zero (i.e., customer i is not served), the sum on the left-hand side of the equality would be zero. Therefore, the equality would read "$0 = 1$" and be violated. If more than one variable y_{ij} were one (i.e., customer i is over-served), then the sum on the left-hand side of the equality would be strictly larger than one. For example, if there are two locations j, j' both serving i, then $y_{ij} = y_{ij'} = 1$ and the equality would read "$2 = 1$" and be violated. The only way for the equality to be respected is if exactly one location j serves the given customer i. Therefore, we can ask the solver to respect constraints (3), forcing it to produce solutions that respect our business requirements. It is extremely important that mathematical constraints faithfully translate business requirements because constraints are the only way we have to communicate with the solver.

- **We cannot serve a customer from a location where we do not open a facility.** This requirement excludes the invalid situation we introduced in Table 2. In terms of variables, it says that it is not possible that there are a customer $i \in I$ and a location $j \in J$ such that $x_j = 0$ but $y_{ij} = 1$. We can enforce this constraint using a "$\leq$"-inequality:

  $$y_{ij} \leq x_j \quad \forall i \in I,\ \forall j \in J. \tag{4}$$

[1] We could disregard the requirement that each customer must be served by exactly one facility in the UFLP. See Exercise 1.1.

Using formula (4), we are defining multiple constraints: one for each pair of customer $i \in I$ and location $j \in J$. For each such pair, is the constraint allowing the valid cases and preventing the invalid one, as described in Table 2? We can build another table to verify that this is the case:

Open a facility at j	**Serve customer** i **from** j	**Valid**
Yes $x_j = 1$	Yes $y_{ij} = 1$	✓ $y_{ij} \leq x_j \;\Rightarrow\; 1 \leq 1$
Yes $x_j = 1$	No $y_{ij} = 0$	✓ $y_{ij} \leq x_j \;\Rightarrow\; 0 \leq 1$
No $x_j = 0$	Yes $y_{ij} = 1$	✗ $y_{ij} \leq x_j \;\Rightarrow\; 1 \leq 0$
No $x_j = 0$	No $y_{ij} = 0$	✓ $y_{ij} \leq x_j \;\Rightarrow\; 0 \leq 0$

There are four possible cases corresponding to four possible assignments of values zero and one to two binary variables. In the three cases that are valid from a business point of view, the constraint is satisfied. In the case that is invalid, the constraint is violated. Therefore, instructing the solver to respect constraints (4) will force it to produce solutions that respect the business requirements.

Mathematical formulation

Bringing together the components that we developed in this section (decision variables, objective function, and constraints), we obtain the following mathematical formulation:

$$
\begin{aligned}
\min \quad & \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} h_i c_{ij} y_{ij} && && \text{(5a)} \\
\text{subject to} \quad & \sum_{j \in J} y_{ij} = 1 && \forall i \in I && \text{(5b)} \\
& y_{ij} \leq x_j && \forall i \in I,\ \forall j \in J && \text{(5c)} \\
& x_j \in \{0, 1\} && \forall j \in J && \text{(5d)} \\
& y_{ij} \in \{0, 1\} && \forall i \in I,\ \forall j \in J. && \text{(5e)}
\end{aligned}
$$

Formulation (5a)–(5e) is called an **Integer Linear Programme** (ILP) because the decision variables are integer numbers, and the objective function and all constraints contain functions that are linear in the decision variables. In some text, you can find the more precise name "Binary Linear Programme", which emphasises that all variables are not only linear but, more specifically, binary; i.e., they take values zero or one. The majority of the formulations that we see in our class use integer variables. Sometimes, however, we will use formulations where (some or all) variables take values in the real numbers $\mathbb{R}$; these variables are called continuous.

No matter the type of variables, absolutely *all* the formulations we study are linear. Therefore, we will never be allowed to multiply or divide variables with each other: doing so would make the formulation quadratic or rational. Analogously, we will not use non-linear functions such as logarithms, exponentials or trigonometric functions. Although, in recent years, solver support for special classes of non-linear formulations has improved, using linear ones is still the preferred

Objective function or constraint	Type	Allowed in a linear programme?
$\max\ 3x - 2y$	Linear objective function	✓
$\min\ \frac{1}{2}x + y - \frac{1}{3}z$	Linear objective function	✓
$\max\ x(x+2)$	Quadratic objective function	✗
$\min\ 4x - yz + 2$	Quadratic objective function	✗
$\max\ \frac{x}{x+y} + \frac{1}{2}z$	Rational objective function	✗
$\min\ \ln x$	Logarithmic objective function	✗
$x \leq y$	Linear constraint	✓
$x + y \geq z$	Linear constraint	✓
$\frac{3}{2}x = y$	Linear constraint	✓
$xy = 3$	Quadratic constraint	✗
$\frac{x}{y} \leq z$	Rational constraint	✗
$y + z \geq x^2 - 3x + 1$	Quadratic constraint	✗
$\sin x \leq \frac{\pi}{2}$	Trigonometric constraint	✗
$10^x \leq 300$	Exponential constraint	✗

Table 3: Examples of functions that can and cannot appear in the objective function and constraints of a linear programme. Symbols x, y and z are decision variables.

strategy in the industry. They enjoy wider solver support, and, as a general rule, such formulations are easier to solve, i.e., the solver is likely to find the optimal solution faster. Table 3 shows examples of functions that we are and are not allowed to use in linear formulations.

Before seeing an example UFLP with its solution, let us write the formulation again and highlight its various components. This is a useful exercise because mathematical formulations are communicated between operations managers and operations researchers in a "standard" way, with few variations. The first step is always to make clear what are the input data to the problem, often listing sets and parameters separately. We did this with our Table 1. Next, we present decision variables; for each group of variables, we must specify clearly and unambiguously:

1. Its indices and where they take values. For example, we can write "x_j for $j \in J$". Alternatively, we can also use "x_j for each potential facility location j". The first expression is more precise, while the second is easy to understand even if we momentarily forget what set J is.
2. Its domain, i.e., the value the variables can take. For example, we write "$x_j \in \{0, 1\}$".
3. Its meaning and the values we intend the variable to take in each situation. For example, we write "x_j takes value one if we open a facility at location j and value zero otherwise".

It is extremely important to clarify the distinction between input data and decision variables: mixing the two is the capital sin of mathematical modelling.

Finally, we are ready to write down the formulation using the following template (here, we use the UFLP formulation as an example):

$$\min \quad \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} h_i c_{ij} y_{ij} \tag{5a}$$

$$\text{subject to} \quad \sum_{j \in J} y_{ij} = 1 \qquad \forall i \in I \tag{5b}$$

$$y_{ij} \leq x_j \qquad \forall i \in I,\ \forall j \in J \tag{5c}$$
$$x_j \in \{0, 1\} \qquad \forall j \in J \tag{5d}$$
$$y_{ij} \in \{0, 1\} \qquad \forall i \in I,\ \forall j \in J\ . \tag{5e}$$

Every formulation starts by specifying if it refers to a maximisation or a minimisation problem. The top left symbol, in the blue box, informs the reader of the formulation of the "direction" that we want the objective function to take. When we translate the formulation in a digital format that the solver can read, it will guide it when looking for good solutions and, ultimately, for the best one. In our case, because the objective function contains cost terms, it is natural that we want to minimise it. If our objective function contained, say, the difference between revenue and costs, then we would have maximised it.

Next to the *min*/*max* symbols is the objective function itself (in the red box). This is a linear function containing the input data and the variables; it must not necessarily contain *all* decision variables. Underneath, the words "subject to" signal that we are stating the problem constraints. We write one group of constraints on each row (in the green boxes): we write the constraint itself on the left and the quantifiers on the right. These quantifiers specify how many constraints we have in each group and, for each constraint, to which indices it refers. Note that absolutely all indices appearing in a constraint must be accounted for, i.e., they must appear in the quantifiers or under a summation symbol. For example, the following constraint would be meaningless:

$$y_{ij} \leq x_j \quad \forall j \in J.$$

Index i is not accounted for anywhere. By contrast, any of the following two constraints would be valid:

$$y_{ij} \leq x_j \qquad \forall i \in I,\ \forall j \in J, \text{ or}$$
$$\sum_{i \in I} y_{ij} \leq x_j \qquad \forall j \in J.$$

As mentioned before, we have three types of constraints (two inequalities, $\leq$ and $\geq$, and equality $=$), and we require that both the left- and the right-hand sides only contain expressions which are linear in the decision variables. Finally, it is customary to restate the variable definition domains at the bottom of the formulation (in the yellow boxes).

In our introduction to the UFLP, we have abundantly described the meaning of the objective function and the constraints. When presenting a mathematical formulation more succinctly (as we will do in the remainder of these notes), we will instead describe them one by one after the formulation. For example, we could write:

- Objective function (5a) minimises the sum of the fixed costs to open facilities (first sum) and the variable transportation costs (second sum).
- Constraints (5b) ensure that each customer is assigned to exactly one facility location.
- Constraints (5c), which link the x and the y variables, make sure that each location with assigned customers has an open facility.

An example UFLP

We present here the instance and the solution of a UFLP. The corresponding code is in notebook `uflp.ipynb`, which also contains all input data. In this instance, we have a set of five

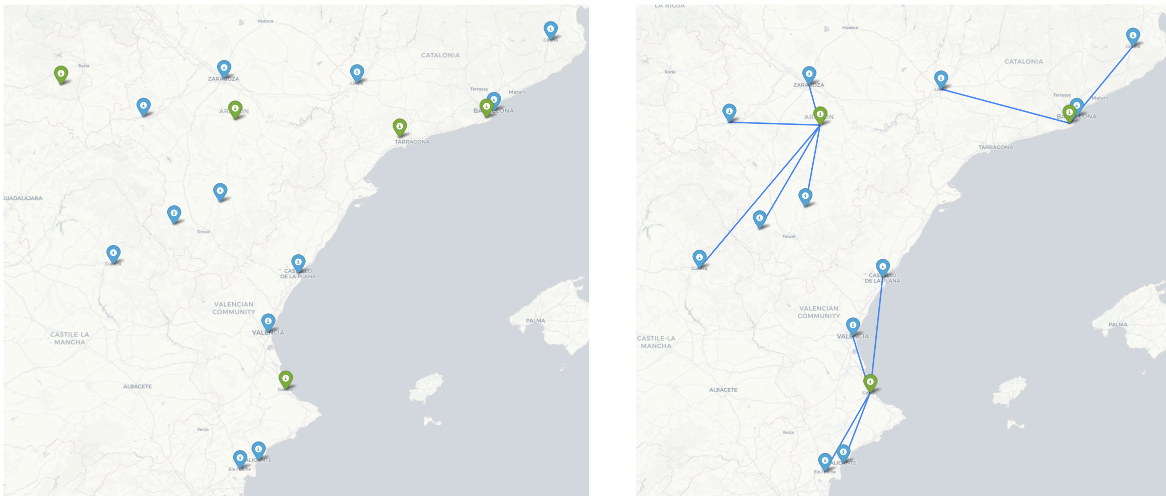

Figure 1: Left: example UFLP instance in eastern Spain. Right: optimal solution.

potential facility locations, and we must serve twelve customers. The locations (in green) and the customers (in blue) are depicted in the left part of Figure 1. After solving the instance with Gurobi, we determine that the best course of action is to open three facilities. The right part of Figure 1 shows the open facilities and the customer assignment.

Exercise 1.1. Assume that we relax the requirement that each customer is served by exactly one facility, and we allow multiple facilities to send goods to the same customer. We still send h_i units of goods to each customer $i \in I$, but we allow any open facility to send only a part of this demand. For example, there could be two distinct open facilities $j, j' \in J$, and facility j serves 70% of i's demand while j' serves the remaining 30%. In this case, the transport costs would be a proportional combination of the two costs c_{ij} and $c_{ij'}$. The part of the transport costs relative to i would be

$$(30\% \cdot h_i)c_{ij} + (70\% \cdot h_i)c_{ij'}.$$

Prove that, for any instance of the UFLP, there is always an optimal solution where all customers are served by exactly one open facility.

Solution. Assume to have an optimal solution in which customer i is served by two distinct open facilities $j, j' \in J$. For example, j sends αh_i units of goods, with $\alpha \in (0, 1)$, and j' sends $(1 - \alpha)h_i$ units.

We first consider the case in which the location-customer costs are the same for both locations j and j', i.e., $c_{ij} = c_{ij'} = C$. In this case, we can move the entire demand to any of the two facilities and obtain a solution with the same variable costs related to i:

$$\begin{aligned}
\text{current cost} &= (\alpha h_i)c_{ij} + \big((1-\alpha)h_i\big)c_{ij'} = \\
&= \alpha h_i C + (1-\alpha)h_i C = \\
&= (\alpha + 1 - \alpha)h_i C = \\
&= h_i C = \begin{cases} h_i c_{ij} & (j \text{ sends all demand}) \\ h_i c_{ij'} & (j' \text{ sends all demand}). \end{cases}
\end{aligned}$$

On the other hand, if $c_{ij} \neq c_{ij'}$, then we prove that the solution where i is served by both j and j' could not be optimal to begin with. This shows the impossibility that we find ourselves in this case. Assume, without loss of generality, that $c_{ij} < c_{ij'}$; the reasoning is analogous if

$c_{ij} > c_{ij'}$. In this case, serving all of i's demand from j is strictly more convenient compared to the current solution because

$$\begin{aligned}\text{current cost} &= (\alpha h_i)c_{ij} + \big((1-\alpha)h_i\big)c_{ij'} > \\ &> (\alpha h_i)c_{ij} + \big((1-\alpha)h_i\big)c_{ij} = \\ &= (\alpha + 1 - \alpha)h_i c_{ij} = \\ &= h_i c_{ij} = \text{cost of the solution where } j \text{ serves 100\% of } i\text{'s demand.}\end{aligned}$$

Therefore, if there is an optimal solution in which two open facilities serve a customer, we have proved that the location-customer transport costs must be equal. We can then produce an optimal solution in which i is served by only one facility. We can easily generalise this procedure to all customers and any number of facilities that serve a customer, thus concluding the proof. □

Exercise 1.2. A company is operating a set $\bar{J}$ of warehouses to ship products to its stores I. In recent years, the company opened many new stores farther away from the warehouses, and their transport costs have skyrocketed. The CEO is considering opening new warehouses to complement the operations of the existing ones. The planning department has identified a set J of new potential locations, and the CEO has a question that the Chief Operating Officer (COO) must answer: *Should we open any new facilities or keep the current ones?*

Use the following information to answer the question. The current location-customer assignment is given by input parameter $\delta_{ij} \in \{0,1\}$ for each customer $i \in I$ and each existing warehouse $j \in \bar{J}$. Parameter δ_{ij} takes value one if customer i is currently served from warehouse j and zero otherwise. The unit transport costs from (current or potential) location $j \in \bar{J} \cup J$ to customer i is c_{ij}. The unit demand of customer i in the next years is forecast to be h_i. The fixed yearly cost of opening and running a warehouse at location $j \in J$ is f_j. We also indicate with f_j $(j \in \bar{J})$ the fixed yearly cost of continuing operating an old warehouse at j. Customer assignment need not be preserved: first, if the company opens new warehouses, it can reassign customers from old to new warehouses; second, it can also reassign customers between existing warehouses. Finally, the CEO does not want to close any currently operating warehouse.

Solution. The COO asks his best operations manager to solve a variation of the UFLP to answer the CEO's question. This variation uses variables:

- $x_j \in \{0,1\}$ for $j \in \bar{J} \cup J$, taking value one if and only if a facility is open at location j.
- $y_{ij} \in \{0,1\}$ for $i \in I$ and $j \in \bar{J} \cup J$, taking value one if and only if customer i is served from location j.

The mathematical formulation is the following:

$$\begin{aligned}
\min \quad & \sum_{j \in \bar{J} \cup J} f_j x_j + \sum_{i \in I} \sum_{j \in \bar{J} \cup J} h_i c_{ij} y_{ij} && && (7a)\\
\text{subject to} \quad & \sum_{j \in \bar{J} \cup J} y_{ij} = 1 && \forall i \in I && (7b)\\
& y_{ij} \le x_j && \forall i \in I,\ \forall j \in J && (7c)\\
& x_j = 1 && \forall j \in \bar{J} && (7d)\\
& x_j \in \{0,1\} && \forall j \in \bar{J} \cup J && (7e)\\
& y_{ij} \in \{0,1\} && \forall i \in I,\ \forall j \in \bar{J} \cup J. && (7f)
\end{aligned}$$

The objective function (7a) minimises the fixed costs of opening new facilities at locations J and the variable costs of serving the customers from either old or new facilities. Constraint (7b) ensures that each customer is served from either a new or an old facility. Constraint (7c) makes sure that facilities are open at all locations serving customers. Finally, the new constraint (7d) marks the old facilities as open (recall that the CEO does not want to close existing warehouses).

After solving this model, the operations manager obtains a solution, i.e., an assignment of concrete zero/one values to variables x and y. We denote this solution as x^*_j (for $j \in \bar{J} \cup J$) and y^*_{ij} (for $i \in I$ and $j \in \bar{J} \cup J$). To answer the CEO's question, the operations manager must compare the total cost the company will incur if they don't open any new facility with the new cost associated with solution (x^*, y^*).

- The total cost of maintaining the status quo is computed using the current location-customer assignment and is:

$$C_{\text{status quo}} = \sum_{j \in \bar{J}} f_j + \sum_{i \in I} \sum_{j \in \bar{J}} h_i c_{ij} \delta_{ij}.$$

- The total cost of implementing the new solution is:

$$C_{x^*,y^*} = \sum_{j \in \bar{J} \cup J} f_j x^*_j + \sum_{i \in I} \sum_{j \in \bar{J} \cup J} h_i c_{ij} y^*_{ij}.$$

If C_{x^*,y^*} is lower than $C_{\text{status quo}}$, then the company should implement the new solution.

Note that we are not assuming that the current location-customer allocation (given by δ_{ij}) is optimal for the set of existing warehouses $\bar{J}$. Therefore, if the new solution has a lower cost than the status quo, this could be either because the customer allocation has improved without opening any new warehouse ($x^*_j = 0$ for all $j \in J$) or because we are opening some new warehouse ($x^*_j = 1$ for at least one $j \in J$). □

Exercise 1.3. Imagine being in the same setting as Exercise 1.2, but now the CEO allows the possibility of selling existing warehouses. If you decide to sell a warehouse $j \in \bar{J}$, you earn a profit p_j, stop paying fixed costs f_j, but can no longer serve any customer from j. How would you modify your model to account for this possibility?

Solution. A possible formulation reads as follows:

$$\begin{aligned}
\min \quad & \sum_{j \in \bar{J} \cup J} f_j x_j + \sum_{i \in I} \sum_{j \in \bar{J} \cup J} h_i c_{ij} y_{ij} - \sum_{j \in \bar{J}} p_j (1 - x_j) && && \text{(8a)} \\
\text{subject to} \quad & \sum_{j \in \bar{J} \cup J} y_{ij} = 1 && \forall i \in I && \text{(8b)} \\
& y_{ij} \leq x_j && \forall i \in I,\ \forall j \in \bar{J} \cup J && \text{(8c)} \\
& x_j \in \{0, 1\} && \forall j \in \bar{J} \cup J && \text{(8d)} \\
& y_{ij} \in \{0, 1\} && \forall i \in I,\ \forall j \in \bar{J} \cup J. && \text{(8e)}
\end{aligned}$$

This model differs from (7a)–(7f) in two ways:

- We have changed the objective function, adding a third term. Recall that our objective function minimises costs; therefore, any profit we obtain must be subtracted from it. The third sum in (8a) states that, for each existing warehouse $j \in \bar{J}$, we obtain a profit of p_j if we sell it. Note that we can sell a warehouse at j only if we no longer use it. We mark a

Set	Description
I	Set of customers.
J	Set of potential facility locations.
Parameter	**Description**
h_i	Yearly demand of customer $i \in I$.
c_{ij}	Unit transport cost from facility $j \in J$ to customer $i \in I$.
f_j	Yearly fixed cost if opening a facility at $j \in J$.
v_j	Yearly capacity of a facility opened at $j \in J$.

Table 4: Sets and parameters of the Capacitated Fixed-Charge Location Problem.

warehouse as unused when $x_j = 0$. Whenever an existing warehouse is unused, it always makes sense to sell it to obtain the corresponding profit. Therefore, we want to sum the profits obtained from selling each unused warehouse. Or, in mathematical terms, we want to sum profit p_j for each warehouse $j \in \bar{J}$ such that $x_j = 0$. The sum accomplishes this, as we can realise by observing that $(1 - x_j) = 1$ (and therefore, we count profit p_j in the sum) if and only if $x_j = 0$.

- The second difference is that we have removed fixing constraint (7d) because the CEO no longer requires that existing warehouses stay open. □

2 The Capacitated Fixed-Charge Location Problem

Analysing the name of the UFLP, "fixed-charge" refers to the fact that we incur fixed costs to open facilities and "uncapacitated" means that we assume that all facilities can handle an unlimited quantity of goods. Therefore, if it was economical to do so, we could open only one facility and serve the demand of all customers from there. While this assumption might hold in some cases, in the majority of business applications, facilities will have a capacity.

To address the case of capacitated facilities, we extend the UFLP and introduce the Capacitated Fixed-Charge Location Problem (CFLP). In the CFLP, each potential facility location has an associated capacity $v_j \geq 0$. If we open a facility at j, we can serve at most v_j units of demand from j during the planning horizon (typically, one year). Adding this constraint also opens an interesting scenario: it might now be convenient to serve a customer from multiple open facilities. If a facility does not have enough residual capacity to fulfil a customer's entire demand, one or more other facilities can ship goods to the customer. We will change our mathematical formulation to allow this possibility.

The CFLP uses the input data reported in Table 4 and the following sets of variables:

- $x_j \in \{0, 1\}$ for each location $j \in J$. These variables take value one if we open a facility at j or value zero otherwise.
- $y_{ij} \in [0, 1]$ for each customer $i \in I$ and location $j \in J$. These are continuous variables defined in the closed interval from zero to one. In the model, variable y_{ij} will represent the fraction of i's demand that is served from j. In particular, $y_{ij} = 0$ if we do not open a facility at j or if we open it but do not use it to serve i. On the other hand, $y_{ij} = 1$ if we serve the entire demand of i from a facility opened at location j. Intermediate values mean that there is more than one facility serving i's demand. For example, if $y_{ij} = 0.4$

and $y_{ij'} = 0.6$ for two distinct locations j and j', then we open facilities both at j and j', and the facility at j serves 40% of i's demand while the one at j' serves 60% of it.

A Mixed-Integer Programme for the CFLP reads as follows (here we use "mixed" to signal that not all variables are integers—the y's are continuous):

$$\begin{aligned}
\min \quad & \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} h_i c_{ij} y_{ij} && && \text{(9a)} \\
\text{subject to} \quad & \sum_{j \in J} y_{ij} = 1 && \forall i \in I && \text{(9b)} \\
& y_{ij} \leq x_j && \forall i \in I,\ \forall j \in J && \text{(9c)} \\
& \sum_{i \in I} h_i y_{ij} \leq v_j && \forall j \in J && \text{(9d)} \\
& x_j \in \{0, 1\} && \forall j \in J && \text{(9e)} \\
& y_{ij} \in [0, 1] && \forall i \in I,\ \forall j \in J. && \text{(9f)}
\end{aligned}$$

The objective function (9a) is identical to the UFLP's one and also aims at minimising the sum of fixed and variable costs. The same formula works now that variables y are fractional. If we serve a fraction y_{ij} of customer i's demand from location j, the quantity of goods transported from j to i is $y_{ij} h_i$. The transportation costs are given by the goods shipped times the unit transport cost, i.e., $y_{ij} h_i \cdot c_{ij}$, which is (rearranged) the term appearing in the second sum. Constraint (9b) ensures that 100% of each customer's demand is served. Constraint (9c), identical to UFLP's (5c), makes sure that we open a facility at location j if we are serving any customer from there. Finally, constraint (9d) prevents capacity violations by preventing the total quantity of goods shipped from each location j from exceeding capacity v_j.

An example CFLP

We use the same data as in the UFLP example (see Figure 1), but we now add quite restrictive capacities to the facility locations. These capacities, which you can find in notebook `cflp.ipynb`, force the operations manager to open all five facilities and to serve some customers from more than one facility. Figure 2 depicts the optimal solution for this instance. The width of the lines connecting open facilities with their assigned customers is proportional to the fraction of demand they serve.

Exercise 2.1. Show that constraints (9c) and (9d) can be replaced by a single group of constraints.

Solution. We can replace these two constraints with the following one:

$$\sum_{i \in I} h_i y_{ij} \leq v_j x_j \quad \forall j \in J.$$

Note that this constraint enjoys the following properties:

- If $x_j = 0$ (we do not open a facility at location j), then the left-hand side of the inequality must be zero. Because variables y take value in $[0, 1]$, all the variables appearing on the left-hand side must be zero, i.e., $y_{ij} = 0 \ \forall i \in I$. Therefore, the constraint is correctly imposing that no fraction of the demand of any customer is served from j.
- If $x_j = 1$ (we open a facility at location j), then the right-hand side of the constraint becomes v_j and the constraint is identical to (9d).

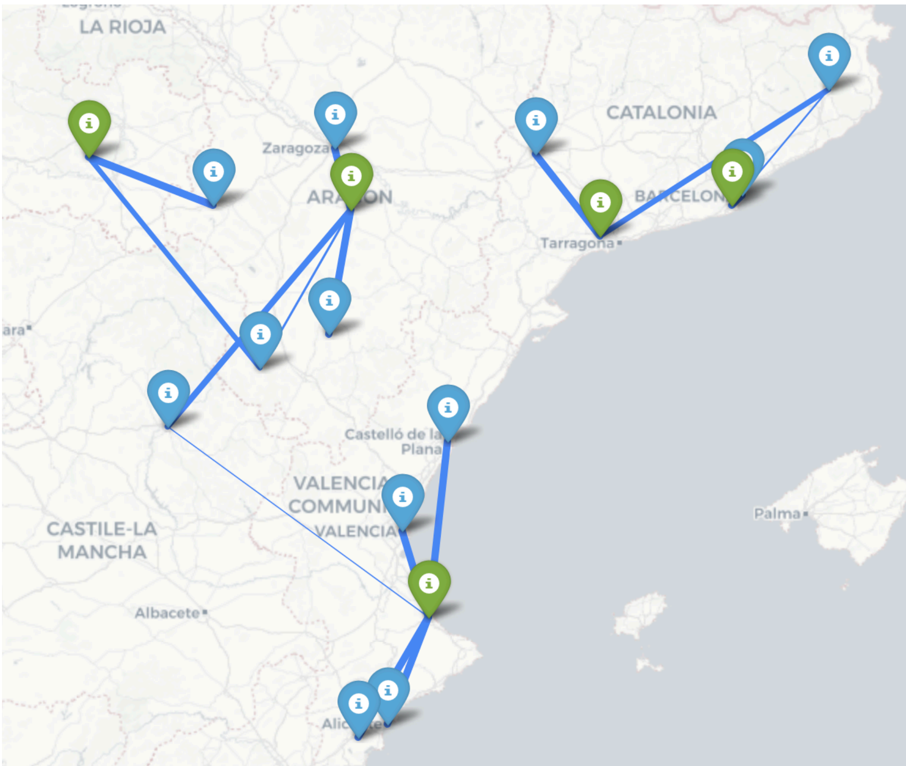

Figure 2: Optimal solution of a CFLP instance in eastern Spain.

- For a given location j, if any variable y_{ij} is strictly positive, then x_j must take value one to satisfy the constraint. Therefore, as soon as one client has some demand served from j, we must open a facility at j.

The points above are enough to justify the correctness and completeness of the constraint. □

Exercise 2.2. You are in a situation similar to that of Exercise 1.2, but now assume that all the existing warehouses are capacitated. In particular, each warehouse $j \in \bar{J}$ has capacity v_j. The capacities are enough to serve the current customer demands $\bar{h}_i$ (for each customer $i \in I$). However, the company is growing, and the sales department has forecast that future demands h_i will be higher than the current ones. Indeed, if you try to solve CFLP's formulation (9a)–(9f) using current warehouses $\bar{J}$ and future demands h_i, the solver informs you that it's impossible to satisfy all customers. Such a problem is called **infeasible**, i.e., it does not admit any solution respecting all constraints.

Therefore, as in (1.2), the CEO asks for the best way to expand the company's supply chain capabilities by building new warehouses. Using the same notation as the previous exercise, for each new potential location $j \in J$, the company has determined that they can build there a facility with capacity v_j. Inform the CEO of the best way to expand the warehouse network. The CEO again wants you to keep the existing warehouses operational.

Solution. We want to solve a variation of the CFLP with the following variables:

- $x_j \in \{0, 1\}$ for all $j \in \bar{J} \cup J$, taking value one if and only if a warehouse is open at location j.
- $y_{ij} \in [0, 1]$ for all $i \in I$ and $j \in \bar{J} \cup J$, indicating the fraction of i's demand served from j.

The modified CFLP formulation reads as follows:

$$\min \sum_{j \in \bar{J} \cup J} f_j x_j + \sum_{i \in I} \sum_{j \in \bar{J} \cup J} h_i c_{ij} y_{ij} \tag{10a}$$

$$\text{subject to} \sum_{j \in \bar{J} \cup J} y_{ij} = 1 \quad \forall i \in I \tag{10b}$$

$$y_{ij} \leq x_j \quad \forall i \in I,\ \forall j \in \bar{J} \cup J \tag{10c}$$

$$\sum_{i \in I} h_i y_{ij} \leq v_j \quad \forall j \in \bar{J} \cup J \tag{10d}$$

$$x_j = 1 \quad \forall j \in \bar{J} \tag{10e}$$

$$x_j \in \{0, 1\} \quad \forall j \in \bar{J} \cup J \tag{10f}$$

$$y_{ij} \in [0, 1] \quad \forall i \in I,\ \forall j \in \bar{J} \cup J. \tag{10g}$$

This modification of formulation (9a)–(9f) is completely analogous to the one used to transform the UFLP formulation (5a)–(5e) into (7a)–(7f) in Exercise 1.2.

Assuming the new formulation has a feasible solution (in an extreme case, there could not be enough capacity even if we open new warehouses at all locations J), its optimal solution gives an expansion plan for the CEO. □

Exercise 2.3. Imagine being in the same setting as in Exercise 2.2, but now the CEO gives you one more option to deal with the forecast demand increase. You can build new warehouses as in the previous exercise, but you can also upgrade existing ones. If you upgrade an existing warehouse $j \in \bar{J}$, you incur additional yearly costs u_j on top of the existing f_j, but you add extra capacity e_j on top of the existing v_j. Write a mathematical formulation indicating the best course of action for the CEO.

Solution. To tackle this problem, we must reason about the possible decisions that the company can make. In particular, compared to the CFLP, in our case, we have one further decision: for each existing warehouse, do we upgrade it or not? This decision cannot be encoded with existing variables, as we did for the "should we sell" decision in Exercise 1.3. Therefore, to keep track of which warehouses we upgrade, we must add new variables to our model. The complete set of variables is:

- $x_j \in \{0, 1\}$ for $j \in \bar{J} \cup J$, taking value one if and only if there is an open facility at j.
- $y_{ij} \in [0, 1]$ for $i \in I$ and $j \in \bar{J} \cup J$, denoting the fraction of i's demand served from j.
- $z_j \in \{0, 1\}$ for $j \in \bar{J}$, taking value one if and only if we upgrade the existing warehouse at location j.

Then, a formulation for the problem we must solve is:

$$\min \sum_{j \in \bar{J} \cup J} f_j x_j + \sum_{i \in I} \sum_{j \in \bar{J} \cup J} h_i c_{ij} y_{ij} + \sum_{j \in \bar{J}} u_j z_j \tag{11a}$$

$$\text{subject to} \sum_{j \in \bar{J} \cup J} y_{ij} = 1 \quad \forall i \in I \tag{11b}$$

$$y_{ij} \leq x_j \quad \forall i \in I,\ \forall j \in \bar{J} \cup J \tag{11c}$$

$$\sum_{i \in I} h_i y_{ij} \leq v_j \quad \forall j \in J \tag{11d}$$

$$\sum_{i \in I} h_i y_{ij} \leq v_j + e_j z_j \quad \forall j \in \bar{J} \tag{11e}$$

Set	Description
I	Set of customers.
J	Set of potential facility locations.
Parameter	**Description**
h_i	Yearly demand of customer $i \in I$.
c_{ij}	Unit transport cost from facility $j \in J$ to customer $i \in I$.
p	Number of facilities to open.

Table 5: Sets and parameters of the p-Median Problem.

$$x_j = 1 \qquad \forall j \in \bar{J} \tag{11f}$$
$$x_j \in \{0,1\} \qquad \forall j \in \bar{J} \cup J \tag{11g}$$
$$y_{ij} \in [0,1] \qquad \forall i \in I,\ \forall j \in \bar{J} \cup J \tag{11h}$$
$$z_j \in \{0,1\} \qquad \forall j \in \bar{J}. \tag{11i}$$

The differences between the new formulation and the one presented in Exercise 2.2 are the following:

- The objective function includes a third term computing the total extra yearly cost due to warehouse upgrades.
- Constraint (10d) is now split into two constraints, (11d) and (11e). The first ensures that capacity is respected at the new warehouses. The second has the same role but for the existing warehouses. For these latter, the capacity can be either the original value v_j if no upgrade takes place ($z_j = 0$) or the new value $v_j + e_j$ in case of upgrade ($z_j = 1$). □

3 The p-Median Problem

In the UFLP and the CFLP, there is no hard limit on the number of facilities to open. Rather, the fixed cost incurred when opening a facility implicitly limits their number by penalising opening too many facilities. In some cases, however, the exact number of facilities is determined a priori and fixed costs do not play a crucial role because they are sunk. For example, if a company has reached an agreement with a government to open three new plants in their country or if a political decision has determined that exactly ten ambulances must serve a city.

The resulting problem, which only considers variable costs, takes the name of p-Median Problem (pPM). In its base version, we must open exactly p uncapacitated facilities, and each customer must be assigned to exactly one open facility. The input data of the pMP is summarised in Table 5, and the decision variables required for its mathematical formulation are:

- $x_j \in \{0,1\}$ for each $j \in J$, taking value one if we open a facility at location j or zero otherwise.
- $y_{ij} \in \{0,1\}$ for each customer $i \in I$ and location $j \in J$, taking value one if and only if we open a facility at j and we serve customer i from j.

An ILP formulation for the pMP reads as follows:

$$\min \quad \sum_{i \in I} \sum_{j \in J} h_i c_{ij} y_{ij} \tag{12a}$$

$$
\begin{aligned}
\text{subject to} \quad & \sum_{j \in J} y_{ij} = 1 && \forall i \in I && (12b) \\
& y_{ij} \leq x_j && \forall i \in I,\ \forall j \in J && (12c) \\
& \sum_{j \in J} x_j = p && && (12d) \\
& x_j \in \{0, 1\} && \forall j \in J && (12e) \\
& y_{ij} \in \{0, 1\} && \forall i \in I,\ \forall j \in J. && (12f)
\end{aligned}
$$

The objective function (12a) minimises the variable transport costs, which are the only costs relevant to this problem. Constraints (12b) and (12c) are identical to the corresponding constraints (5b) and (5c) in the UFLP and achieve the same objectives. Constraint (12d) states that we must open exactly p facilities.

Because there is no fixed cost associated with opening more facilities, there is no harm in opening exactly p facilities even if there is an optimal solution that opens strictly fewer. Therefore, replacing "$=$" with "$\leq$" in constraint (12d) yields an equivalent formulation. However, we emphasise that if opening strictly fewer than p facility is considered preferable in real life, then perhaps using the pMP is not the correct modelling approach, and an explicit opening cost should be imposed as in the UFLP and CFLP. Ultimately, the decision of which model to use must be made on a case-by-case basis. As we have seen in the previous exercises, models are, in any case, changed and extended, and a rigid classification is only partly useful.

Exercise 3.1. An electric car company has started selling a new model, "Fabra". The new Fabra comes with many extras. Indeed, looking at the catalogue, a customer would see 30 extras they can add to the car. From the point of view of an operations manager running the assembly line, however, such a variety risks becoming a customisation nightmare. Given thirty possible extras, and considering that each customer can make an independent decision on whether to add each one of them, there are a whopping 2^{30} possible model configurations. That's more than one billion, and even the most optimistic Vice President of Sales would acknowledge that there are more possible model configurations than sold cars.

The COO then proposes this clever strategy. The factories will only assemble cars according to twenty configurations out of the 2^{30} possible ones. In this way, the COO expects to be able to keep complexity down and take advantage of good economies of scale. When a customer places a custom order, the company will deliver a car that has all the requested extras. The company, however, allows the care to have *more* extras if none of the twenty configurations matches the order exactly. As long as no extra is missing, the customer will be happy to have some free upgrades.

Denote with C the set of all possible combinations. A careful market analysis has determined which extras are chosen independently of each other and which ones are often selected together. Using this information and data from various focus groups, the marketing department has estimated the demand d_i of each combination $i \in C$. (In practice, d_i will be zero or very close to zero for many combinations, but the rest of our exercise does not rely on nor use this fact.) The COO wants to know which twenty configurations should be chosen from C to minimise the total cost of the free upgrades. If $i \in C$ is the configuration requested by a customer and $j \in C$ is the configuration delivered by the company, we denote with e_{ij} the extra cost incurred because of the free upgrades.

Write a mathematical formulation to answer the COO's request.

Solution. Perhaps surprisingly, there is a pMP hiding under the above problem description. In particular, let the customer set be $I = C$ and the facility set also be $J = C$. The problem of selecting twenty configurations then becomes the problem of opening $p = 20$ facilities in the pMP. Because the chosen configurations will be among the available ones, the customer and facility sets coincide. If the solver opens a facility at j in the pMP (i.e., if $x^*_j = 1$ in an optimal solution), then the car company should select configuration j in the subset of configurations that it will assemble. Analogously, if the solver chooses a customer-location pair (i, j) in the pMP (i.e., if $y^*_{ij} = 1$ in an optimal solution), then the car company should offer selected configuration j to customers ordering configuration i.

Customer demands h_i are simply the estimated demands d_i (i.e., $h_i = d_i$ for all $i \in I$). Finally, customer-location costs c_{ij} can be defined as follows:

$$c_{ij} = \begin{cases} e_{ij} & \text{if configuration } j \text{ has all the extras of } i \\ +\infty & \text{otherwise.} \end{cases}$$

In other words, if a configuration j is a viable replacement for i because it offers all the required extras, then the company pays extra cost e_{ij}. Otherwise, if j is not a viable replacement for i, we associate an infinitely large cost with pair (i, j). In this way, we ensure that the optimal solution of the pMP will *never* set variable $y_{ij} = 1$. Such an assignment would correspond to an infeasible solution in practice because j cannot replace i and would give an objective value of $+\infty$. The solver will disregard such solutions because of their high cost and will provide us with an optimal solution in which all chosen customer-location pairs (i.e., configuration replacements) are feasible. We also remark that we cannot input value ∞ into a computer solver: in practice, we will use an extremely large (but finite) value.

If you are uncomfortable using these (infinitely) high costs, note that this is a "trick" to obtain feasible solutions without imposing a constraint. Alternatively, you can add explicit variable fixing constraints:

$$y_{ij} = 0 \quad \forall i \in I,\ \forall j \in J \text{ such that } j \text{ cannot replace } i.$$

□

Exercise 3.2. In formulation (12a)–(12f), there are $|I| \cdot |J|$ linking constraints in group (12c). Replace these constraints with another group with only $|J|$ constraints.

Solution. A possible solution is to note that the largest number of customers assigned to a facility is $|I|$, i.e., all of them. This can happen in the extreme case of $p = 1$ or if, for a given location $j \in J$, transport costs c_{ij} are very low, making j the best assignment option for all customers and leaving the remaining $p - 1$ facilities open but unused. Although these cases are rare in practice, we cannot rule them out completely and must make our formulation account for them. At the same time, we can use this bound (at most $|I|$ customers assigned to any facility) to solve the exercise by introducing the following constraint as a replacement of (12c):

$$\sum_{i \in I} y_{ij} \leq |I| \cdot x_j \quad \forall j \in J. \tag{13}$$

To see why the constraint is correct, let us analyse the possible cases:

1. If $x_j = 0$, we do not open a facility at location j. In this case, we cannot serve any customer from there. Indeed, the constraint becomes $\sum_{i \in I} y_{ij} \leq 0$. Because all variables y are non-negative, this implies that $y_{ij} = 0$ for all $i \in I$, i.e., we are not serving customers from j.

Set	Description
I	Set of customers.
J	Set of potential facility locations.
Parameter	**Description**
a_{ij}	Indicator telling if location j covers customer i ($a_{ij} = 1$) or not ($a_{ij} = 0$).

Table 6: Sets and parameters of the Set Covering Location Problem.

2. On the other hand, if $x_y = 1$, we open a facility at j. The constraint becomes $\sum_{i \in I} y_{ij} \leq |I|$, which is trivially true because, as noted above, $|I|$ is the maximum number of customers we can assign to a facility.
3. Finally, if we serve any customer from j, then $y_{ij} > 0$ for at least one customer i. Therefore, the left-hand side of the inequality is strictly positive. This implies that x_j must take value one. Otherwise, the inequality would be violated. □

4 The Set Covering Location Problem

The models we have studied until now all share the same broad objective: minimise costs. They do not consider customer satisfaction or service quality. Consider, for example, the UFLP: for any given customer i, there is no guarantee that it will be served from a reasonably close location. If it is economical to open all facilities at locations that are far away from i, the solver will return such a solution. For customer i, this will mean higher lead times and more uncertainty on when it's receiving its goods.

While a purely economic criterion is appropriate in some settings, there are applications in which service quality must be taken into account. When locating ambulances or fire stations, public servants must guarantee that the entire population can be served within a given amount of time from calling 112. Amazon must locate their fulfilment centres in a way that allows them to perform 1-day delivery to all their Prime subscribers. A city must locate public ambulatories (CAPs here in Barcelona) in a way that no neighbourhood is too far from their assigned centre.

In this section, we introduce a facility location problem that can be used for managerial decision-making in the above situations: the Set Covering Location Problem (SCLP). In broad terms, the objective of the problem is to open the smallest number of facilities in a way that covers all customers with a specified service level. For example, we might want to know what the smallest number of ambulances that we must deploy in Barcelona is (and where we should place them) so that any call can be answered within five minutes. The SCLP only uses one set of parameters, reported in Table 6 together with the customer and location sets. Parameter $a_{ij} \in \{0, 1\}$ is defined for each pair of customer i and location j. It takes value one if a facility opened at j can cover customer i with the required service level and value zero otherwise. For example, it could take the value one if an ambulance placed at location j can reach city block i within five minutes, or if a fulfilment centre opening at j can perform one-day deliveries to town i. Figure 3 provide a visualisation of the city of Barcelona. The central red dot represents a potential ambulance location j. Blue dots are places (or "customers", in SCLP language) i reachable within five minutes from a call ($a_{ij} = 1$). Purple dots are places i not reachable within five minutes ($a_{ij} = 0$).

We only need one set of variables to formulate the SCLP as an ILP:

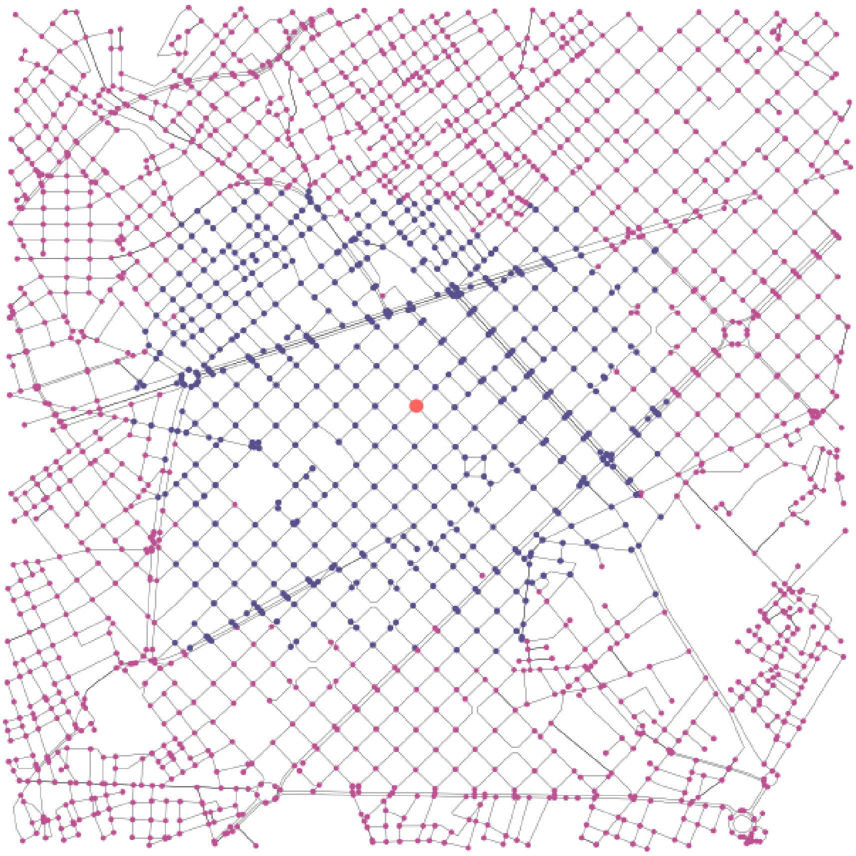

Figure 3: An example depicting parameter a_{ij} in the city of Barcelona.

- $x_j \in \{0, 1\}$ for each location $j \in J$, taking value one if and only if we open a facility at j.

The formulation reads as follows:

$$\min \quad \sum_{j \in J} x_j \tag{14a}$$

$$\text{subject to} \quad \sum_{j \in J} a_{ij} x_j \geq 1 \qquad \forall i \in I \tag{14b}$$

$$x_j \in \{0, 1\} \qquad \forall j \in J. \tag{14c}$$

The objective function (14a) minimises the number of open facilities. The only group of constraints is (14b), ensuring that each customer is covered by at least one open facility: for the left-hand side to be at least one, there must be at least one location such that $a_{ij} = 1$ and $x_j = 1$, i.e., the location covers i, and we open a facility there.

Exercise 4.1. You are the campus director of the Ciutadella facilities of Universitat Pompeu Fabra. After many student complaints, you finally decided to upgrade the Wi-Fi infrastructure to give all classrooms good coverage. Let I be the set of classrooms and J the set of potential access point (AP) locations. Because of the characteristics of the building, placing APs has different costs depending on the location. The cost of installing an AP at location j is $f_j \geq 0$. The technicians have figured out which locations they can use to cover each classroom. They provide you with parameters $a_{ij} \in \{0, 1\}$ for $i \in I$ and $j \in J$, taking value one if and only if an AP installed at j covers classroom i. However, you learned a long time ago never to trust the technicians. You want to develop an AP installation plan that ensures that at least two APs cover each classroom. Develop a mathematical programming model to determine where to install the access points. Satisfy the above requirements and devise a plan that minimises the total installation costs.

Solution. We can extend the SCLP's formulation to take into account the fixed costs and the double-coverage requirement as follows:

$$\min \quad \sum_{j \in J} f_j x_j \tag{15a}$$

$$\text{subject to} \quad \sum_{j \in J} a_{ij} x_j \geq 2 \qquad \forall i \in I \tag{15b}$$

$$x_j \in \{0, 1\} \qquad \forall j \in J. \tag{15c}$$

We have modified the objective function and, in (15a), we now minimise the total installation cost of the APs. We have also changed the right-hand side to two; therefore, (15b) now ensures that at least two APs cover each classroom.

Exercise 4.2. After you propose your solution to Exercise 4.1, you are informed that APs are not all of the same type. Indeed, the chosen supplier offers a set K of different models with different power and range. As a rule of thumb, the more expensive models have a better range and provide wider coverage. The IT department asks you to back up from your plan of double-covering the entire building and, instead, proposes a more fine-tuned approach. They will divide each classroom into small areas (comprising a new set I), and they will carefully measure signal strength in each of these areas for all locations $j \in J$ and all AP types $k \in K$. Signal strength is expressed in percentage, and they use parameter $s_{ijk} \in [0, 1]$ to represent the strength that an AP of type k installed at location j provides to area i. When $s_{ijk} = 1$, then signal strength is 100%; when $s_{ijk} = 0$, the AP is not able to cover the given area; intermediate values mean that the AP covers the area, but the connection quality is not perfect.

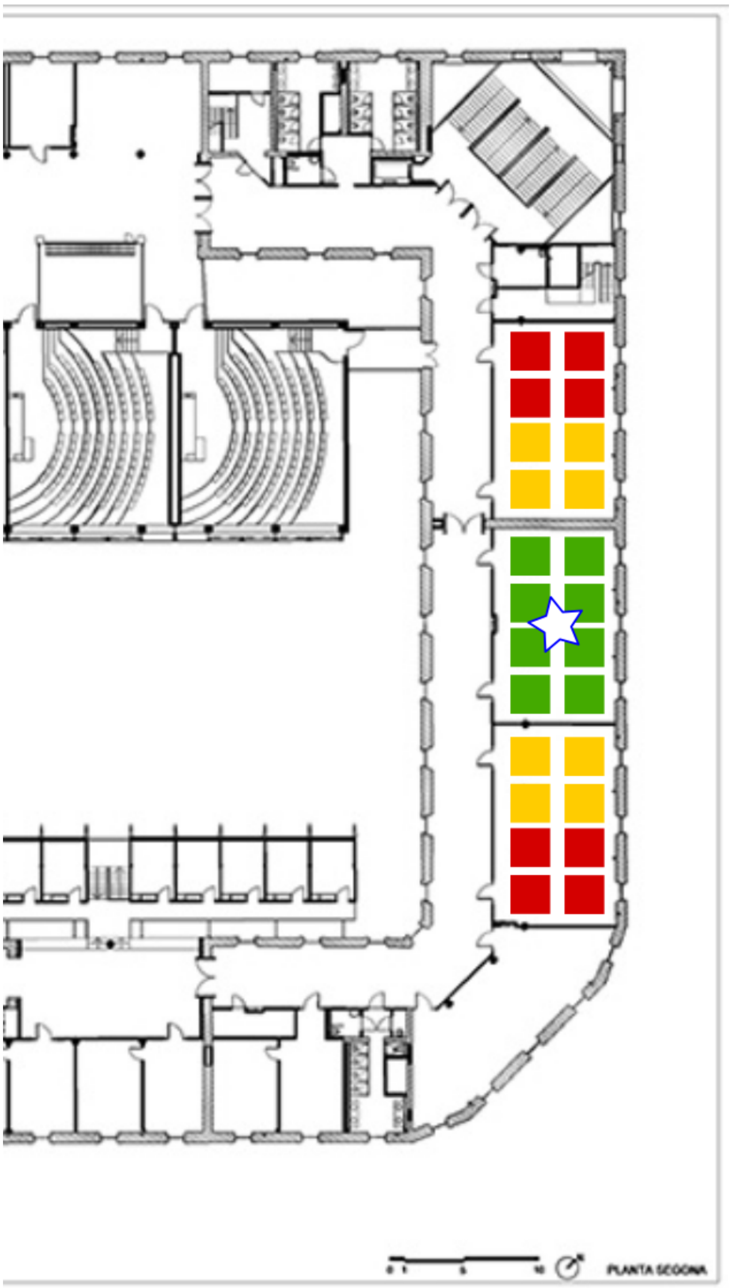

Figure 4: Signal strength of an Access Point placed in a classroom of building 40 of Universitat Pompeu Fabra.

Figure 4 shows different signal strengths for an AP installed in a classroom of building 40 of the university. A star indicates the installation point j of an AP of type k. Coloured rectangles are areas i with some signal ($s_{ijk} > 0$). If the area is coloured in green, then the connection is excellent ($s_{ijk} \geq 0.9$); yellow areas correspond to a good connection ($0.7 \leq s_{ijk} < 0.9$); red areas correspond to a bad connection ($s_{ijk} < 0.7$).

Devise a new installation plan to determine where to place APs and which type each AP should be in order to satisfy the following requirements at the minimum possible cost:

1. At least one AP should cover each area with excellent connection quality. Connection quality is excellent if the signal strength is at least 90%.
2. Each area should be covered by at least two APs with good or better connection quality. Connection quality is good if the signal strength is at least 70%.
3. You can install at most one AP in each location.

For example, an area covered by two access points with excellent (first AP) and good (second AP) connection qualities simultaneously satisfies both requirements. Analogously, an area covered by two APs such that both have excellent connection qualities would satisfy both requirements. On the other hand, an area covered by two APs such that both have good (but not excellent) connection quality would satisfy requirement 2 but violate requirement 1.

Solution. In the absence of precise indications on the cost of APs, we will take the most general approach and assume that the installation costs can depend on both location and AP type. Therefore, we denote with $f_{jk} \geq 0$ the input data parameter holding the cost of installing an AP of type $k \in K$ at location $j \in J$.

We also devise auxiliary parameter $a_{ijk} \in \{0, 1\}$ taking value one if and only if an AP of type k installed at j covers i with excellent connection quality, i.e.,

$$a_{ijk} = \begin{cases} 1 & \text{if } s_{ijk} \geq 0.9 \\ 0 & \text{otherwise.} \end{cases}$$

Analogously, let auxiliary parameter $b_{ijk} \in \{0, 1\}$ take value one if and only if an AP of type k installed at j covers i with good connection quality or better, i.e.,

$$b_{ijk} = \begin{cases} 1 & \text{if } s_{ijk} \geq 0.7 \\ 0 & \text{otherwise.} \end{cases}$$

Note that both a_{ijk} and b_{ijk} are input data and not variables; they are completely determined by input data s_{ijk}, and they are not associated with any decision.

We use decision variables $x_{jk} \in \{0, 1\}$ for $j \in J$ and $k \in K$, taking value one if and only if we install an AP of type k at location j. Then, a formulation for our problem reads as follows:

$$\begin{aligned} \min \quad & \sum_{j \in J} \sum_{k \in K} f_{jk} x_{jk} && && (16) \\ \text{subject to} \quad & \sum_{j \in J} \sum_{k \in K} a_{ijk} x_{jk} \geq 1 && \forall i \in I && (17) \\ & \sum_{j \in J} \sum_{k \in K} b_{ijk} x_{jk} \geq 2 && \forall i \in I && (18) \\ & \sum_{k \in K} x_{jk} \leq 1 && \forall j \in J && (19) \end{aligned}$$

Set	Description
I	Set of customers.
J	Set of potential facility locations.
Parameter	**Description**
h_i	Demand or importance of customer $i \in I$.
a_{ij}	Indicator telling if location j covers customer i ($a_{ij} = 1$) or not ($a_{ij} = 0$).
p	Number of facilities to open.

Table 7: Sets and parameters of the Uncapacitated Fixed-Charge Location Problem.

$$x_{jk} \in \{0, 1\} \qquad \forall j \in J,\ \forall k \in K. \tag{20}$$

The objective function (16) minimises the total installation cost of the APs. Constraints (17) and (18) enforce requirements 1 and 2, guaranteeing that each area is covered by at least one AP with excellent and two AP with good connection quality. Constraint (19) enforces requirement 3, making sure that at most one access point is installed in each location j.

5 The Maximum Coverage Location Problem

The SCLP is characterised by a strong commitment to cover all customers. The UFLP, CFLP and pMP all share this feature. Such an approach is appropriate when a contract or a regulator demands that all customers are served by at least one facility. In other cases, however, these strong constraints do not apply, and, for example, the capital budget limits the number of facilities to open. Still, the decision-maker might want to serve as many customers as possible within the given budget limit. The Maximum Coverage Location Problem (MCLP) deals with precisely this case.

Imagine that IKEA wants to enter a new country market. They have decided to budget enough to open four new stores and want to determine the best possible locations. Clearly, IKEA has no duty to serve the entire population of a country. Similarly, it must not ensure that each town has an IKEA store reachable within a 1-hour drive, etc. Still, and given the maximum of four stores they can build, it would be convenient if they could serve as many customers as possible.

Table 7 reports the input data to the MCLP. The parameters, in particular, are a "blend" of those used for the other facility location problems we have encountered before. Each customer i has an associated demand h_i, which we can also see as the importance of serving i. In the IKEA example, if I is a set of cities and towns, h_i might represent their population. (And, if the marketing department is really good, h_i will more precisely represent the part of the population that is likely to shop at IKEA.) This parameter was also present in the UFLP and the CFLP. Parameter a_{ij} indicates whether a customer i is considered covered if we open a facility at location j. For example, we might set $a_{ij} = 1$ for all cities within a 1-hour drive from j. This parameter was also used in the MCLP. Finally, parameter p, already introduced for the pMP, is the number of facilities to open.

We will solve the CFLP using a formulation with two sets of variables:

- $x_j \in \{0, 1\}$ for each potential location $j \in J$, taking value one if and only if we open a facility at j.
- $z_i \in \{0, 1\}$ for each customer $i \in I$, taking value one if and only if there is at least one

open facility serving i.

Then the formulation reads as follows:

$$\max \quad \sum_{i \in I} h_i z_i \tag{21a}$$

$$\text{subject to} \quad z_i \leq \sum_{j \in J} a_{ij} x_j \qquad \forall i \in I \tag{21b}$$

$$\sum_{j \in J} x_j = p \tag{21c}$$

$$x_j \in \{0, 1\} \qquad \forall j \in J \tag{21d}$$

$$z_i \in \{0, 1\} \qquad \forall i \in I. \tag{21e}$$

The objective function (21a) maximises the covered demand or the total importance of covered customers. Constraint (21b) links the x and z variables. If there is no open facility that can serve i, the right-hand side will be zero and, therefore, it will force the left-hand side to be zero; this is correct because, in this case, i is not served. On the other hand, if there is at least one open facility that can serve i, the right-hand side will be at least one. In this case, the constraint will be moot, i.e., it will not impose any strict limit on z_i because a condition such as $z_i \leq 1$ or $z_i \leq 2$ is uninformative due to the variable's domain definition $z_i \in \{0, 1\}$. Therefore, in this case, variable z_i will be free to take whatever value is more convenient. However, note that we are maximising the sum of the z variables. Thus, in an optimal solution, any time that z_i can take value 1, it will do so. To summarise, then, constraint (21b) forces $z_i = 0$ when no facility can serve i and implicitly makes $z_i = 1$ when i is served by at least one facility. Finally, constraint (21c) is completely analogous to constraint (12d) in the pMP and ensures that we open p facilities.

Exercise 5.1. The **U**nited **P**eople's **F**ront is at war with their enemies, the **U**niversally **Ac**knowledged **B**uffoons. You are at the helm of the United People's Front army and must ensure the security of your divisions against the enemy's aerial attacks. Given the position of your divisions, you must place p anti-aircraft missile batteries at some of the potential sites J located by your generals. The batteries have a certain range, and, if a division is within range, it is considered protected. If I is the set of positions occupied by your divisions, parameter $a_{ij} \in \{0, 1\}$ takes value one if and only if a battery placed at j protects a division at i. There is, however, a macabre twist. You have recently learnt that the chief of staff does not value all divisions equally. What is worse, they are convinced that the army will be better off without some divisions! As such, they have partitioned set I into three subsets:

- Set $I_1 \subset I$ contains all the divisions that you must protect at all costs. For example, the **D**ivision of **E**nlightened **I**ndividuals is too important to lose.
- Set $I_2 \subset I$ contains divisions that you should try to save if you can. Each division $i \in I_2$ has an associated importance h_i, and you should maximise the importance of the saved divisions. For example, the **D**ivision of **T**echonologically **I**mproved **C**omrades contains many cyborgs that might prove valuable to win the war.
- Unfortunately, the chief of staff has given you a third, final set $I_3 \subset I$. It contains divisions that are considered dangerous for the stability and the effectiveness of the army. Reading through the list, you can see the **D**ivision of **C**ertainly **P**owerless **S**oldiers among others. You are given the explicit command that these divisions should be left unprotected.

Sets I_1, I_2, and I_3 are a partition of I, i.e., their pairwise intersection is empty, and their union is the whole I.

With a heavy heart, you approach your computer and start to write the mathematical formulation that will save a thousand lives, but will mean almost certain death for many companions. What does this formulation look like? Can you guarantee that the corresponding IP is feasible?

Solution. We modify the MCLP formulation (21a)–(21e), using the same variables and the same parameters. The new formulation reads as follows:

$$\begin{aligned}
\max \quad & \sum_{i \in I_2} h_i z_i && && (22a)\\
\text{subject to} \quad & \sum_{j \in J} a_{ij} x_j \geq 1 && \forall i \in I_1 && (22b)\\
& z_i \leq \sum_{j \in J} a_{ij} x_j && \forall i \in I_2 && (22c)\\
& x_j = 0 && \forall i \in I_3,\ \forall j \in J : a_{ij} = 1 && (22d)\\
& \sum_{j \in J} x_j = p && && (22e)\\
& x_j \in \{0,1\} && \forall j \in J && (22f)\\
& z_i \in \{0,1\} && \forall i \in I. && (22g)
\end{aligned}$$

The objective function (22a) maximises the importance of the I_2 divisions protected. Constraint (22b) enforces that all I_1 divisions are protected, and is analogous to SCLP's (14b). Constraint (22c) links the x variables and the subset of z variables relative to I_2 divisions. It is analogous to the standard MCLP constraint (21b). Constraint (22d), alas, prevents you from placing any missile battery at a location that would save I_3 divisions. Finally, constraint (22e) is analogous to (21c) and imposes that you place p batteries.

To answer the second question, we note that model (22a)–(22g) must not necessarily be feasible: it depends on the input data. For example, if it is not possible to cover all I_1 divisions using only p batteries, the model will be unfeasible. Or if it is only possible to protect an I_1 division using a battery location that also covers an I_3 division, again, the model will be unfeasible.

6 Solutions of exercises from the book

Exercise 8.22 a)

This problem is tricky because it asks to *maximises* the *minimum* distance between a neighbourhood and its *closest* open facility. It sounds like a tongue-twister.

The first problem is that maximising a minimum (a so-called max-min problem) is not allowed in integer programming: we cannot put a minimum function in our objective function that we maximise. (There are more advanced models, called "bilevel models", for which this is allowed, but they are considerably more challenging to solve, and we should not resort to them.)

When we have a max-min, we shall try to circumvent this problem. In this case, one possible way of solving the exercise is to introduce a new variable $u \geq 0$ in addition to the existing x and y suggested by the book. The meaning of this variable will be:

$u =$ the minimum distance between a neighbourhood and its closest open facility.

As you can see, we have "swept the hard part under the rug", postulating that there is a variable that will hold precisely the value that we want. If we manage to make u take this value, then the objective function becomes trivial:

$$\max \quad u.$$

This is literally what the problem asks: maximise "*the minimum distance between a neighbourhood and its closest open facility*" and u has the meaning "*the minimum distance between a neighbourhood and its closest open facility*"; therefore, we should maximise u.

Recall that the x and y variables are defined in the book as $x_j \in \{0,1\}$ $\forall j \in J$, taking the value one if and only if we open a facility at location j, and $y_{ij} \in \{0,1\}$ $\forall i \in I$, $\forall j \in J$, taking the value one if and only if j is the closest open facility to i. Now the hard part becomes to ensure that u takes the correct value. To this end, we develop the following model.

$$\begin{aligned}
\max \quad & u && && \text{(23a)}\\
\text{subject to} \quad & \sum_{j \in J} x_j = p && && \text{(23b)}\\
& \sum_{j \in J} y_{ij} = 1 && \forall i \in I && \text{(23c)}\\
& y_{ij} \le x_j && \forall i \in I,\ \forall j \in J && \text{(23d)}\\
& y_{ij} \le 1 - x_k && \forall i \in I,\ \forall j \in J,\ \forall k \in J \text{ s.t. } c_{ij} > c_{ik} && \text{(23e)}\\
& u \le \sum_{j \in J} c_{ij} y_{ij} && \forall i \in I && \text{(23f)}\\
& x_j \in \{0,1\} && \forall j \in J && \text{(23g)}\\
& y_{ij} \in \{0,1\} && \forall i \in I,\ \forall j \in J && \text{(23h)}\\
& u \ge 0. && && \text{(23i)}
\end{aligned}$$

Constraints (23b)–(23d) are borrowed from the p-Median problem. The new constraints are (23e) and (23f).

Let us start from (23e). Remark how we are quantifying this constraint over three indices and we use a "filter", i.e., we consider all neighbourhoods i, all potential facility locations j and, for each pair of i and j, we only consider those potential facility locations k which are closer to i than j is. The meaning of the constraint is then the following: j cannot be *the closest* open facility for neighbourhood i if there is another open facility at k, which is closer. Indeed, we can consider the following cases:

1. If we open a facility at location k ($x_k = 1$), the constraint becomes $y_{ij} \le 1 - 1 = 0$, i.e., $y_{ij} = 0$. This is correct because if we open a facility at k, then j, which is farther away from i than k, will never be "the closest" open facility for i.

2. On the other hand, if we do not open a facility at location k ($x_k = 0$), the constraint is moot: $y_{ij} \le 1 - 0 = 1$. This is also correct because, in this case, we do not know if j will be the closest open facility for i. Indeed, we do not even know if we open a facility at j! Therefore, this constraint should not tell us anything in this case.

As we can see, this constraint prevents the following wrong situation: we open facilities at both j and k ($x_j = x_k = 1$), k is closer to i than j is, but still we "mark" j as the closest open facility to i ($y_{ij} = 1$). If we made this mistake, the constraint would read

$$y_{ij} \le 1 - x_k \;\Rightarrow\; 1 \le 1 - 1 \;\Rightarrow\; 1 \le 0 \;\Rightarrow\; \text{Error},$$

and would thus be violated.

Finally, constraint (23f) does the heavy lifting of setting the correct value for variables u. First, remark that (23c) ensures that there is only one non-zero term in the right-hand-side sum of (23f). By definition of y, this term corresponds to the distance between i and its closest open facility. Therefore, the constraints read as

$$u \leq \text{ the distance between neighbourhood } i \text{ and its closest open facility} \quad \forall i \in I.$$

This is certainly compatible with the meaning we want to give u: if it is the *minimum* distance between a neighbourhood and its closest open facility, it must undoubtedly be not larger than the distance between any neighbourhood and its closest open facility. But is the converse true? Could u be strictly smaller than all the distances between the neighbourhoods and their closest open facility but not equal to the minimum? For example, consider the case with three neighbourhoods $I = \{1, 2, 3\}$, one facility ($J = \{1\}$ and $p = 1$), and the distances between each neighbourhood and the facility are $c_{11} = 3$, $c_{21} = 4$, $c_{31} = 6$. Then we want to have $u = \min\{3, 4, 6\} = 3$. However, the constraints only state that

$$u \leq 3 \quad u \leq 4 \quad u \leq 5.$$

Therefore, e.g., setting $u = 2$ would satisfy all the constraints but would not ensure that u takes the correct value.

To realise that u takes the correct value in any optimal solution, we must notice that we are *maximising* u in the objective function. Therefore, u will take the largest value allowed by the constraints. In our small example, u will take value 3. In general, it will take the value of the minimum right-hand side in constraints (23f) which is the minimum distance between a neighbourhood and its closest open facility, thus complying with its definition. □

Exercise 8.22 b)

We can use a similar trick to the one used to solve part a). In this case, we change the meaning of variable $u \geq 0$, as we want it to take value of the minimum distance between any two open facilities. We can then replace constraint (23f) with the following one

$$u \leq c_{jk} + M(1 - x_j) + M(1 - x_k) \quad \forall j, k \in K \ (j \neq k),$$

where M is a very large number, e.g., $M = \max_{j,k \in J} c_{jk}$.

We want u to be the minimum distance between open facilities. Again considering that u is maximised in the objective function, we can just enforce that

$$u \leq \text{distance between any two open facilities.}$$

The problem with this approach is that, a priori, we do not know which facilities we are going to open. Therefore, we have to write a constraint that works for any pair j, k of potential facility locations and, if we open a facility at both locations, enforces that $u \leq c_{jk}$. Otherwise, if one or both locations do *not* host an open facility, the constraint must be moot.

Let us, then, consider the four cases:

1. If we open facilities at both j ($x_j = 1$) and k ($x_k = 1$), both terms $1 - x_j$ and $1 - x_k$ are zero and the constraint correctly reads $u \leq c_{jk}$.
2. If we open a facility at j ($x_j = 1$) but not at k ($x_k = 0$), term $1 - x_k$ is equal to one. Thus, the constraint reads $u \leq c_{jk} + M$ and, choosing M sufficiently big, this constraint is moot because the right-hand side becomes an improbably large number.

3. Analogously, if we open a facility at k ($x_k = 1$) but not at j ($x_j = 0$), the constraint becomes $u \leq c_{jk} + M$.
4. Finally, if we do not open facilities at j or k ($x_j = x_k = 0$), the constraint becomes $u \leq c_{jk} - 2M$ which, again, is moot. □

Exercise 8.33 a)

We can solve this exercise by considering the usual variables $x_j \in \{0,1\}\ \forall j \in J$ taking the value one if and only if we open a facility at location j, and new variables $z_i \in \{0,1\}\ \forall i \in I$ taking the value one if and only if town i is double-covered. Let $h_i > 0$ be the population of town $i \in I$, and $a_{ij} \in \{0,1\}$ be a parameter taking the value 1 if location $j \in J$ covers city $i \in I$ (i.e., their distance is smaller than the radius r demanded by the exercise) and the value 0 otherwise. A model for this problem reads as follows.

$$\max \quad \sum_{i \in I} h_i z_i \tag{24a}$$

$$\text{subject to} \quad \sum_{j \in J} x_j = p \tag{24b}$$

$$2z_i \leq \sum_{j \in J} a_{ij} x_j \qquad i \in I \tag{24c}$$

$$x_j \in \{0,1\} \qquad \forall j \in J \tag{24d}$$

$$z_i \in \{0,1\} \qquad \forall i \in I. \tag{24e}$$

The new constraint is (24c), ensuring that $z_i = 1$ if and only if there are two open facilities covering i. To check the constraint's validity, let us consider the following cases:

1. If no open facility covers i, the right-hand side of the constraint is zero. Therefore, the constraint reads $2z_i \leq 0$, i.e., $z_i \leq 0$ and thus $z_i = 0$, which is correct.
2. If there is exactly one open facility covering i, we must ensure that $z_i = 0$ because, in this case, i is still *not* double-covered. In this case, the right-hand side of the constraint is one and the constraint reads $2z_i \leq 1$, i.e., $z_i \leq \frac{1}{2}$. Because z_i is binary, this inequality implies that $z_i = 0$.
3. If there are two (or more) open facilities covering i, the right-hand side is at least two and the constraint becomes moot. For example, if two open facilities cover i the constraint becomes $2z_i \leq 2$, i.e., $z_i \leq 1$, which is moot.

The constraint is not *forcing* z_i to take the value one when i is double-covered. However, because we are maximising the z_i's in the objective function, these variables will take the value one every time no constraint forbids it. □

Exercise 8.34 a)

The main difference of this model compared to the ones we have already studied is in the objective function, where we have to add a paenalty term for each population centre served by a facility. We can accomplish this task without introducing any new notation or variables with the following model.

$$\min \quad \sum_{j \in J} \left(f_j x_j + w_j \sum_{i \in I} y_{ij} \right) \tag{25a}$$

$$\text{subject to} \quad \sum_{j \in J} a_{ij} y_{ij} = 1 \qquad \forall i \in I \tag{25b}$$

$$y_{ij} \leq x_j \qquad \forall i \in I,\ \forall j \in J \tag{25c}$$

$$x_j \in \{0, 1\} \qquad \forall j \in J \tag{25d}$$

$$y_{ij} \in \{0, 1\} \qquad \forall i \in I, \forall j \in J. \tag{25e}$$

For each open facility j, in the objective function, we sum its fixed cost f_j plus cost w_j times the term $\sum_{i \in I} y_{ij}$. This sum keeps track of how many population centres facility j is serving. □