Data Envelopment Analysis

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When? May 29, 2014

ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Summary

- Introduction to DEA
- DEA model
- DEA in action
- DEA: strong and weak points

Basic definitions

DEA is a management tool to evaluate the **efficiency** of a process.

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! "efficiency" and "performance" are relative terms.

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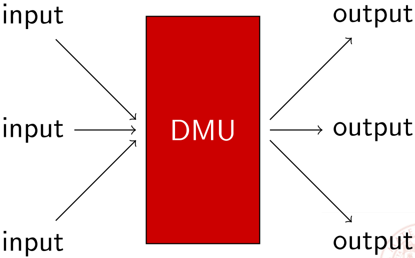

Basic definitions

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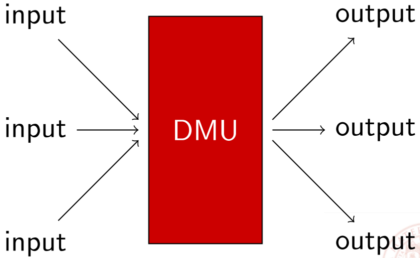

What is a problem with this definition?

DMUs

A DMU takes some **input** and produces some **output**.

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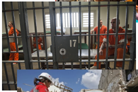
- Non-profit organisations input? output?



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- A factory's manufacturing units input? output?
- A department store's shops input? output?
- Prisons input? output?
- Non-profit organisations input? output?
- University departments input? output?

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Almost everything can be DMUs:

- A factory's manufacturing units input? output?
- A department store's shops input? output?
- Prisons input? output?
- Non-profit organisations input? output?
- University departments input? output?
- And even lecturers, such as this one

DMUs

DMU	Input	Output
Mfg unit		

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Mfg unit	Raw materials Manpower Floor space Energy	

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DMUs

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DMUs

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It can be easier for a **company** (e.g. measuring profits or added value)...

...and more difficult for a **non-profit** or for **this lecturer**.

DMUs

In its most basic form:

$$\text{Efficiency} = \frac{\text{Output}}{\text{Input}}$$

But what is Input, what is Output? Who decides what they are? Can we quantify them?

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Example: DMU = lecturer

Output desired by the **lecturer**

Output desired by his **supervisor**

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Every department has a certain number of professors and, each year, produces a certain number of graduates and is able to secure a certain amount of funding for research.

Example

University	Professors	Graduates	Funds
Unibo	6	132	9600€
Duckburg Uni	12	192	26400€
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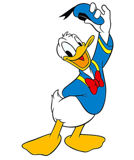

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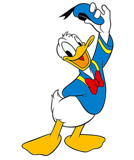

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Example

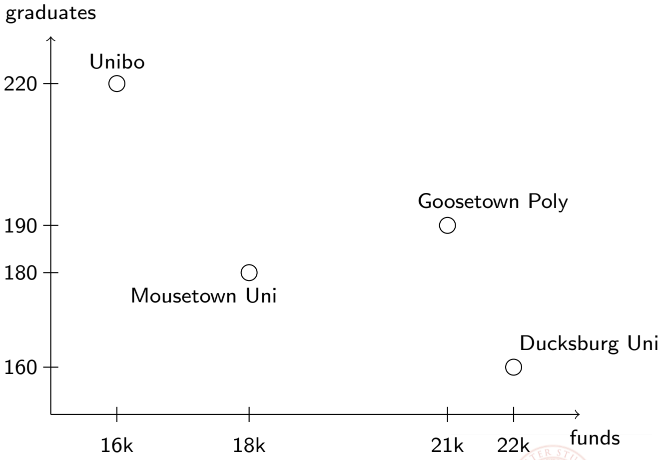
The **virtual university**

$$\text{VU} = \frac{1}{2}\ \text{Unibo} + \frac{1}{2}\ \text{Goosetown Poly}$$

Gives more output with the same input of Mousetown Uni.

Therefore, Mousetown Uni is inefficient!

Example



All inputs (# of professors) normalised to 10.

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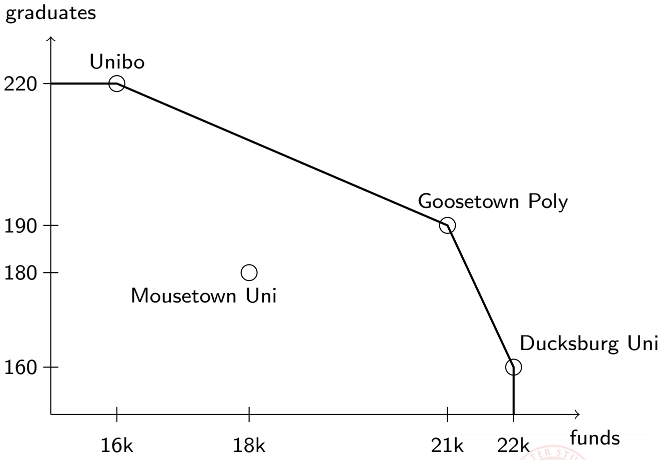

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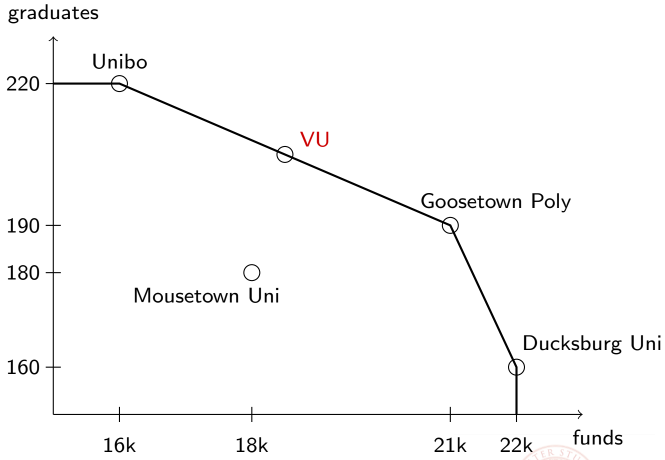

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- Outputs scale **linearly** with inputs: if 10 professors can teach to 1000 students, can 20 professors teach to 2000 students?
  - We don't consider economy of scale
  - We don't consider operational complexity
  - We assume Constant Returns to Scale (CSR)
- If an university produces Y output with X input, **every other university could be able to do the same**. But what if Mousetown Uni lectures are held in tents, because Mousetown has been hit by an earthquake?

Enter DEA

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- Score < 1: the DMU is inefficient
- Score $= 1$: the DMU is efficient

Enter DEA

Always remember that efficiency is **relative**!

I.e. a DMU is inefficient if it's possible to obtain more output with less input, by combining **other DMUs**.

DEA model

Let X_i be the vector of inputs of DMU $i \in I$.
Let Y_i be the vector of outputs of DMU $i \in I$.

We want to test the efficiency θ_j of DMU $j \in I$.

DEA model

By what we said before, the score θ_j should be 1 when the DMU is efficient and < 1 when it's inefficient.

Let's be more precise by seeing which values θ_j will attain when it is < 1.

DEA model

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Using just three quarters of the inputs

$$\sum_{i \in I} \lambda_i X_i = \frac{3}{4} X_j$$

We would expect the (in)efficiency score of j to be **at most** $\frac{3}{4}$: $\theta_j \leq \frac{3}{4}$.

DEA model

We say **at most**, because we could find another linear combination such that

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Then we would know that $\theta_j \leq \frac{1}{2}$ (which means j is very inefficient!).

DEA model

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Sounds familiar?

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These inequalities suggest a first formulation of the DEA for DMU $j \in I$ as a linear problem:

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$$\begin{aligned} \min \quad & \theta_j \\ \text{s.t.} \quad & \sum_{i \in I} \lambda_i X_i \leq \theta_j X_j \\ & \sum_{i \in I} \lambda_i Y_i \geq Y_j \\ & \theta_j \in \mathbb{R}, \quad \lambda_i \in \mathbb{R}_+ \ \forall i \in I \end{aligned}$$

Find the least θ_j such that it's possible to produce at least Y_j units of output using no more than $\theta_j X_j$ units of input, for some linear combination of all the DMUs.

DEA model

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- Under the assumptions made before, DMU j could produce the same output by using just $\sum_{i \in I} \lambda_i X_i$ input and therefore it's wasting

$$X_j - \sum_{i \in I} \lambda_i X_i \text{ units of input}$$

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- Or, alternatively, with the same input DMU j is only producing Y_j output rather than $\frac{1}{\theta_j} \sum_{i \in I} \lambda_i Y_i$ and therefore it's wasting

$$Y_j - \frac{1}{\theta_j} \sum_{i \in I} \lambda_i Y_i \text{ units of output}$$

DEA model

! The more DMUs we consider, the more difficult it is to compare them, because they become **too specialised** to be evaluated with respect to other DMUs. Therefore, more DMUs will get a score of 1.

DEA model

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Let's list our inputs $P = \{1, \ldots, N\}$ and outputs $Q = \{1, \ldots, M\}$.

Imagine we want to give unit price $u_p > 0$ to input $p \in P$ and unit prize $v_q > 0$ to output $q \in Q$.

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These are just virtual values: one unit of input p doesn't really cost u_p and one unit of output q doesn't really sells at price v_q.

DEA model

Let u and v be the vectors of the prices and the prizes and **recall** that X_j, Y_j were also vectors of inputs and outputs.

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With our vector notation, the total cost of inputs for DMU j is

$$u^{\mathsf{t}} X_j$$

and the total gain from the outputs is

$$v^{\mathsf{t}} Y_j$$

DEA model

Now **recall** that we proposed a definition of efficiency as:

$$\text{Efficiency} = \frac{\text{Output}}{\text{Input}}$$

DEA model

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This suggests that we might use, as a measure of efficiency for DMU j:

$$\text{Efficiency} = \frac{v^{\mathrm{t}} Y_j}{u^{\mathrm{t}} X_j}$$

DEA model

Since prices u and prizes v are virtual, we can "play" with them and we can ask:

- What are the prices u and prizes v that maximise the efficiency of DMU j?

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- What are the prices u and prizes v that maximise the efficiency of DMU j?

Intuitively: if the inputs used most by j cost very close to 0 and the outputs produced by j earn a big prize, then j is very efficient!

DEA model

From equation

$$\text{Efficiency} = \frac{v^{\mathsf{t}} Y_j}{u^{\mathsf{t}} X_j}$$

we see that, if we are free to "play" with u and v as we want, then there is no limit: we could take u arbitrarily small and v arbitrarily big.

DEA model

From equation

$$\text{Efficiency} = \frac{v^{\mathsf{t}} Y_j}{u^{\mathsf{t}} X_j}$$

we see that, if we are free to "play" with u and v as we want, then there is no limit: we could take u arbitrarily small and v arbitrarily big.
Therefore we impose a limit:

$$\frac{v^{\mathsf{t}} Y_j}{u^{\mathsf{t}} X_j} \leq 1$$

Notice that it's totally arbitrary: we could have said ≤ 2, ≤ 362, etc. But this choice is nice, as it gives a value between 0 and 1 for the efficiency.

DEA model

So now we can ask:

- What are the prices u and prizes v that maximise the efficiency of DMU j...
- Such that

$$\frac{v^{\mathsf{t}} Y_i}{u^{\mathsf{t}} X_i} \leq 1, \ \forall i \in I$$

Imposing Efficiency ≤ 1 for all DMUs.

DEA model

Once more this sounds familiar...

DEA model

Once more this sounds familiar…and in fact it suggests the following maximisation problem:

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\begin{aligned}
\max \quad & \frac{v^{\mathsf{t}} Y_j}{u^{\mathsf{t}} X_j} \\
\text{s.t.} \quad & \frac{v^{\mathsf{t}} Y_i}{u^{\mathsf{t}} X_i} \leq 1 \quad \forall i \in I \\
& u \in \mathbb{R}^N_+, \; v \in \mathbb{R}^M_+
\end{aligned}
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What are 2 problems with this model?

DEA model

We can transform the previous model into an (input-oriented) LP:

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\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
\max \quad & v^{\mathsf{t}} Y_j \\
\text{s.t.} \quad & u^{\mathsf{t}} X_j = 1 \\
& v^{\mathsf{t}} Y_i - u^{\mathsf{t}} X_i \leq 0 \quad \forall i \in I \\
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DEA model

Let's now have a look at the two models we presented:

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Any comment?

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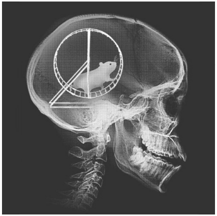
Any comment? Think hard!



DEA model

The models in action: **university departments**

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DMU	Professors	Graduates	Funds
1	6	1.32	9.6
2	12	1.92	26.4
3	10	1.9	21
4	8	1.44	14.4

DEA model

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DMU	Professors	Graduates	Funds
1	6	1.32	9.6
2	12	1.92	26.4
3	10	1.9	21
4	8	1.44	14.4

Inputs

$$X_1 = [6] \quad X_2 = [12] \quad X_3 = [10] \quad X_4 = [8]$$

DEA model

The models in action: **university departments**

DMU	Professors	Graduates	Funds
1	6	1.32	9.6
2	12	1.92	26.4
3	10	1.9	21
4	8	1.44	14.4

Inputs

$$X_1 = [6] \quad X_2 = [12] \quad X_3 = [10] \quad X_4 = [8]$$

Outputs

$$Y_1 = \begin{bmatrix} 1.32 \\ 9.6 \end{bmatrix} \quad Y_2 = \begin{bmatrix} 1.92 \\ 26.4 \end{bmatrix} \quad Y_3 = \begin{bmatrix} 1.9 \\ 21 \end{bmatrix} \quad Y_4 = \begin{bmatrix} 1.44 \\ 14.4 \end{bmatrix}$$

DEA model

First model for DMU 1 (Unibo)

$$
\begin{array}{lrrrrrrr}
\min & 1\theta_1 + & 0\lambda_1 + & 0\lambda_2 + & 0\lambda_3 + & 0\lambda_4 & & \\
\text{s.t.} & +6\theta_1 - & 6\lambda_1 - & 12\lambda_2 - & 10\lambda_3 - & 8\lambda_4 & \geq & 0 \\
 & + & 1.32\lambda_1 + & 1.92\lambda_2 + & 1.9\lambda_3 + & 1.44\lambda_4 & \geq & 1.32 \\
 & + & 9.6\lambda_1 + & 26.4\lambda_2 + & 21\lambda_3 + & 14.4\lambda_4 & \geq & 9.6
\end{array}
$$

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First model for DMU 1 (Unibo)

$$
\begin{array}{lrrrrrrl}
\min & 1\theta_1 + & 0\lambda_1 + & 0\lambda_2 + & 0\lambda_3 + & 0\lambda_4 & & \\
\text{s.t.} & +6\theta_1 - & 6\lambda_1 - & 12\lambda_2 - & 10\lambda_3 - & 8\lambda_4 & \geq & 0 \\
 & + & 1.32\lambda_1 + & 1.92\lambda_2 + & 1.9\lambda_3 + & 1.44\lambda_4 & \geq & 1.32 \\
 & + & 9.6\lambda_1 + & 26.4\lambda_2 + & 21\lambda_3 + & 14.4\lambda_4 & \geq & 9.6
\end{array}
$$

Second model for DMU 1 (Unibo)

$$
\begin{array}{llrrrl}
\max & & 0u_1 + & 1.32v_1 + & 9.6v_2 & \\
\text{s.t.} & & 6u_1 & & & = 1 \\
 & - & 6u_1 + & 1.32v_1 + & 9.6v_2 & \leq 0 \\
 & - & 12u_1 + & 1.92v_1 + & 26.4v_2 & \leq 0 \\
 & - & 10u_1 + & 1.9v_1 + & 26.4v_2 & \leq 0 \\
 & - & 8u_1 + & 1.44v_1 + & 14.4v_2 & \leq 0
\end{array}
$$

DEA model

First model for DMU 1 (Unibo)

$$
\begin{array}{llllllll}
\boxed{\min} & 1\theta_1 + & 0\lambda_1 + & 0\lambda_2 + & 0\lambda_3 + & 0\lambda_4 & & \\
\text{s.t.} & +6\theta_1 - & 6\lambda_1 - & 12\lambda_2 - & 10\lambda_3 - & 8\lambda_4 & \geq & 0 \\
 & & +1.32\lambda_1 + & 1.92\lambda_2 + & 1.9\lambda_3 + & 1.44\lambda_4 & \geq & 1.32 \\
 & & +\ 9.6\lambda_1 + & 26.4\lambda_2 + & 21\lambda_3 + & 14.4\lambda_4 & \geq & 9.6
\end{array}
$$

Second model for DMU 1 (Unibo)

$$
\begin{array}{lllll}
\boxed{\max} & 0u_1 + 1.32v_1 + 9.6v_2 & & \\
\text{s.t.} & 6u_1 & = & 1 \\
 & -\ 6u_1 + 1.32v_1 + 9.6v_2 & \leq & 0 \\
 & -\ 12u_1 + 1.92v_1 + 26.4v_2 & \leq & 0 \\
 & -\ 10u_1 + 1.9v_1 + 26.4v_2 & \leq & 0 \\
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\begin{array}{lrrrrrcl}
\boxed{\min} & \boxed{1\theta_1 + 0\lambda_1 + 0\lambda_2 + 0\lambda_3 + 0\lambda_4} & & & & & & \\
\text{s.t.} & +6\theta_1 - 6\lambda_1 - 12\lambda_2 - 10\lambda_3 - 8\lambda_4 & \geq & 0 \\
 & +1.32\lambda_1 + 1.92\lambda_2 + 1.9\lambda_3 + 1.44\lambda_4 & \geq & 1.32 \\
 & +9.6\lambda_1 + 26.4\lambda_2 + 21\lambda_3 + 14.4\lambda_4 & \geq & 9.6
\end{array}
$$

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$$
\begin{array}{lrcl}
\boxed{\max} & \boxed{0u_1 + 1.32v_1 + 9.6v_2} & & \\
\text{s.t.} & 6u_1 & = & 1 \\
 & -6u_1 + 1.32v_1 + 9.6v_2 & \leq & 0 \\
 & -12u_1 + 1.92v_1 + 26.4v_2 & \leq & 0 \\
 & -10u_1 + 1.9v_1 + 26.4v_2 & \leq & 0 \\
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\end{array}
$$

It's the dual!

DEA model

- By definition, the optimal solution gives us the efficiency of DMU j:

$$v^{\mathsf{t}} Y_j = \frac{v^{\mathsf{t}} Y_j}{1} = \frac{v^{\mathsf{t}} Y_j}{u^{\mathsf{t}} X_j} = \theta_j$$

- The "profit" $v^{\mathsf{t}} Y_j - u^{\mathsf{t}} X_j$ will be negative for inefficient DMUs and zero for efficient ones.

DEA in action

Let's see what happens when we run the model!

Summary

DEA: strength points

- No **semantics**: just deal with numbers

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  - No assumption on the process: the DMU is a black box

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- Different **units** for input and output
  - They can represent entirely different things (e.g. graduates and research funds)
  - No need to translate into a common unit when it's difficult to do so...
  - ...e.g. output can be "human lives" and "money" with no need to convert one into the other!

Summary

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  - A manual, qualitative, subjective, error-prone process
  - E.g. how important is the innovation brought by a supplier vs. their low price?

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  - E.g. if DMU 1 is made inefficient by a combination of DMU 2 and 3, maybe 1 can learn something from them!
  - Useful to identify worst-practices
  - E.g. to remove some DMU from a sample for statistical analysis

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  - What? This was a strength point!
  - It tells you how strong you are vs. your peers, but not vs. a theoretical maximum

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Summary

DEA: limitations

- A very careful **selection** of inputs and outputs is necessary
  - DEA is not tolerant of measurement noises
  - DEA doesn't perform well when there are too many inputs and outputs
  - As we mentioned before, DMUs tend to be "all optimal" with too many inputs or outputs
  - Furthermore, inputs and output need to be homogeneous and non-correlated

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Summary

DEA: overcoming *some* limitation

- Our formulation of DEA assumes Constant Returns to Scale (CRS)
- Other formulations have been presented in order to drop this assumption
  - **Advantage**: most practical applications don't have CRS
  - **Advantage**: besides telling us if a DMU is inefficient, they also tell us if it's too small or too large
  - **Disadvantage**: efficiency scores change when we use input- vs. output-oriented versions
  - Therefore, we also have to choose an orientation for the model, according to the control we have over inputs vs. outputs

Questions?