Production Planning

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In the notes on inventory management, we have studied two simple models (the EOQ and EOQ with stockouts models) that rely on a number of assumptions. They capture the main trade-off operations managers have to consider when managing inventories: placing a few large orders gives stability and low fixed costs while placing many small orders gives flexibility and low holding costs. At the same time, the models assume that the demand and the costs are constant during a long planning horizon. In these notes, we will relax these assumptions, and we will introduce Integer Programming (IP) models that can deal with variable costs and demands. Moreover, we will see how a single mathematical formulation is able to model both upstream inventories (where the goods arrive from external suppliers) and downstream ones (where the goods arrive from a production process). We will start with the latter and then show how the formulations can also solve the former.

1 The Uncapacitated Lot-Sizing Problem

A fundamental problem facing operations managers working in production processes is deciding when to produce a given product and when to free up resources for other activities. For example, a multi-product factory can have a main product, say MainProd, that they manufacture to meet the market's demand, plus other secondary products that can earn extra revenue. When there is no need to produce MainProd, the factory can produce the secondary products, perform machine maintenance, train employees, etc. The Uncapacitated Lot-Sizing Problem (ULSP) will provide an answer to the question: when should we produce MainProd, and how much should we produce in each run? Because the quantity to produce in a run is called the lot size, we call this problem a **lot sizing** problem, and hence the ULSP name. Since—for the moment—we are not placing limits on the amount of product that we can manufacture during one period, nor to the inventory we carry, we add the specifier "uncapacitated".

In the ULSP, we are given the following data.

- We work with a planning horizon discretised into periods, $T = \{1, \dots, n\}$. Depending on the nature of our company's operations, each period could represent one day, one week, one month, or some other fixed time.
- We receive a demand forecast for each period in the time horizon. Our objective is to produce and store enough products to meet the demand at each period. We denote with d_t the demand forecast for period $t \in T$.
- If we decide to produce during period $t \in T$, we pay a fixed setup cost of $s_t > 0$. This cost corresponds to organising and performing a production run during one period and only captures the fixed costs associated with the run. Therefore, this cost does not depend on the amount of product manufactured.

- To account for variable costs, we introduce parameter $p_t > 0$ (for all $t \in T$). This parameter is the cost of producing one unit during period t. Our mode is flexible and allows the production cost to vary depending on the period; for example, labour costs might be higher during some months in seasonal industries. If our unit production costs were constant, we could use a fixed value p and let $p_t = p \; \forall t \in T$.
- We can produce a larger quantity of product than there is demand during a given period. In that case, we store the extra production in an inventory, and we pay the corresponding holding costs. Again, our model is flexible enough to allow the holding costs to vary period-by-period, and we will denote with $h_t > 0$ the unit holding cost at period t. If holding costs do not vary with time, we can use a fixed value h and let $h_t = h \; \forall t \in T$.
- When we are tasked with devising a production plan, we have an initial inventory of x_0 units of product currently stored in the warehouse that we can use to satisfy future demand.

Our objective is to devise a production schedule that meets all demands d_t at the lowest possible cost, keeping in mind that demand at period t can be satisfied by both the production during period t (if any) and the inventory carried over from period $t-1$.

Because time is discretised into periods, but operations happen in real-time, we must make some assumptions about how the demand, the production and the inventory interact within the same period. For example, if we use a discretisation of one month, the factory will still be manufacturing products every day; each product unit could spend only a fraction of a month in the inventory; we could ship products to our stores on a weekly basis; etc. To reconcile daily operations with the strategic planning required by the ULSP model, we must make some simplifying assumptions about the order of events within the same period. In particular, we will assume that at each period t: (i) we first produce the entire quantity scheduled for that period, if any; (ii) then, we satisfy the entire period demand; (iii) finally, we store any extra product in the inventory, we update the inventory level, and we compute the holding costs based on this level.

To keep track of the costs incurred, we need three groups of variables. First, we will consider binary variables $y_t \in \{0, 1\}$ (for each $t \in T$) that will take the value one if and only if we produce during period t and, therefore, we pay the corresponding setup cost. Next, we need non-negative variables $w_t \geq 0$ (for each $t \in T$) that keep track of the number of units produced in period t; we will use them to account for the variable production costs. Finally, we use non-negative variables $x_t \geq 0$ (for each $t \in T$) that will hold the inventory level value at the end of period t; we will use these variables to charge holding costs. Note that we use a slight abuse of notation by employing symbol x both for a problem parameter (x_0 is the initial inventory level) and for a decision variable ($x_1, \ldots, x_n$ denote the inventory level at the end of each period). However, because the variables are indexed starting from 1, the notation is unambiguous. The reason why we use the same symbol is that (i) both the parameter and the variables refer to the same quantity, namely the inventory level, and (ii) it will simplify our mathematical formulation by not requiring that we consider the first period as a special case in the model constraints.

Table 1 summarises the problem sets, parameters and variables. An IP formulation for the ULSP is the following:

$$\min \quad \sum_{t \in T} \big(s_t y_t + p_t w_t + h_t x_t \big) \tag{1a}$$

$$\text{subject to} \quad x_t = x_{t-1} + w_t - d_t \qquad \forall t \in T \tag{1b}$$

Set	Description
T	Planning horizon.
Parameter	**Description**
d_t	Demand at period $t \in T$.
s_t	Setup (fixed) cost if producing during period $t \in T$.
p_t	Unit cost for producing during period $t \in T$.
h_t	Unit holding cost for the inventory present at the end of period $t \in T$.
x_0	Initial inventory level at the beginning of the planning horizon.
M	A very large number used for modelling purposes and not corresponding to any real-life quantity.
Variable	**Description**
x_t	Inventory level at the end of period $t \in T$.
w_t	Quantity produced during period $t \in T$.
y_t	Binary variable taking value 1 if we produce during period $t \in T$.

Table 1: Sets, parameters and variables of the Uncapacitated Lot-Sizing Problem.

$$w_t \leq M y_t \qquad \forall t \in T \tag{1c}$$
$$x_t \geq 0 \qquad \forall t \in T \tag{1d}$$
$$w_t \geq 0 \qquad \forall t \in T \tag{1e}$$
$$y_t \in \{0, 1\} \qquad \forall t \in T. \tag{1f}$$

The objective function (1a) minimises the total costs accumulated over the time horizon. For each period, the total cost is the sum of the setup cost, the variable production cost and the holding cost.

Constraint (1b) is a fundamental constraint of the model, characterising many types of lot-sizing problems. It states that the inventory at the end of period t is given by the previous inventory (at the end of time $t-1$), plus whatever we produced during period t, minus the demand at period t. Figure 1 shows this relation. Each period is represented by a circle. Incoming arrows denote the available product that comes from the inventory at the previous period or the production process. Outgoing arrows denote the product used during the period because it is sold or stored in the warehouse and made available for the next period. Constraint (1b) states that the sum of the in-flow $(x_{t-1} + w_t)$ and the out-flow $(d_t + x_t)$ must be equal, thus ensuring that each unit of product is accounted for.

Let us focus on constraint (1c). This constraint is interesting because it uses a numerical "trick" that is often employed in integer programming. The meaning of the constraint is the following: if we are manufacturing our product at time t ($w_t > 0$), then we must pay the corresponding setup cost (y_t must be one). On the other hand, if we are not paying the setup cost ($y_t = 0$), then we cannot manufacture any product (w_t must be zero). We have seen similar constraints in facility location models, but we used them to link together binary variables only. The peculiarity of constraint (1c) is that it links the binary variable y_t with the continuous variable w_t. To perform this linking, it must rely on a constant, denoted with M in the formulation, that does not correspond to any real-life quantity but is only used to ensure that the model is correct. We must think of M as a very large number that is only added to make the constraint work as intended. Indeed, we can briefly analyse that in all possible cases, the constraint is correct:

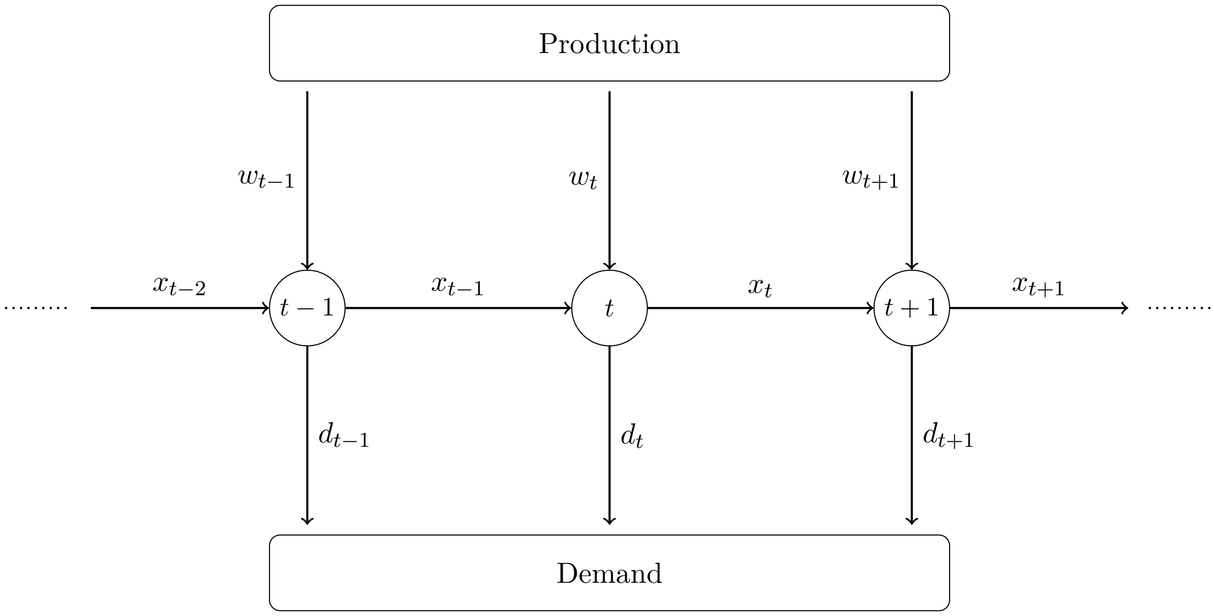

Figure 1: Relation between production, demand and inventory in the ULSP as captured by constraint (1b)

- If $y_t = 0$, we are not paying the setup costs in the objective function. This corresponds to not starting production during period t. Therefore, variable w_t must take value zero. Indeed, when $y_t = 0$, (1c) becomes $w_t \leq M \cdot 0 = 0$. Because w_t is non-negative, it must then take exactly value 0.
- If $y_t = 1$, we are paying setup costs, and we are allowed to manufacture our product during period t. In this case, the constraint becomes $w_t \leq M$ and limits the monthly production to be at most M. But, in our *uncapacitated* problem, we do not have any such limit. Hence, we must choose a very high value for M so that the constraint becomes moot and does not really enforce anything, letting w_t take any value. One way to accomplish this objective is to set $M = \sum_{t \in T} d_t$, i.e., the total demand over the entire planning horizon. Because, in the entire planning horizon, we will never produce more than we have demand for, in particular during any period t such an M is a valid upper bound on the quantity that is reasonable to produce. To summarise, using a very large value of M, we ensure that setting $y_t = 1$ allows us to manufacture as much product as we need.
- If $w_t = 0$, we are not manufacturing any product. In this case, the constraint becomes $0 \leq M y_t$, which is equivalent to $y_t \geq 0$. This latter constraint is trivially true because y_t is binary. However, note that y_t is penalised in the objective function: it appears there with a multiplier of $s_t > 0$, and we are solving a minimisation problem. Therefore, in an optimal solution of the ULSP, the solver will set $y_t = 0$ if there is no constraint preventing it from doing so. In our case, constraint (1c) (the only one where y_t appears) is imposing no restrictions, and therefore y_t will take value zero. This is correct because if we produce $w_t = 0$ units of product, then we should not pay any setup cost.
- Finally, if $w_t > 0$, then y_t must take the value one, and we must pay the setup costs. Indeed, if y_t took value zero, the constraint would read $w_t \leq 0$, which contradicts the hypothesis that $w_t > 0$. Therefore, the only value that y_t can take is one.

Model (1a)–(1f) is therefore a correct and complete model for the ULSP.

Period →	1	2	3	4	5	6	7	8	9	10	11	12
Demand	60	70	100	130	110	90	90	80	70	90	100	120
Setup cost	15	15	15	15	10	10	15	15	15	10	10	10
Production cost	1	1	1	1	2	2	1	2	1	2	2	2
Holding cost	2	2	2	2	1	1	1	1	1	1	1	1

Table 2: Input data of a ULSP instance with initial inventory $x_0 = 100$.

An interesting characteristic of the ULSP model is that it enjoys the so-called **Zero-Inventory Order** (ZIO) property. Assume that the initial inventory is empty, i.e., $x_0 = 0$. Then, in an optimal solution of the ULSP, production starts only when the inventory level at the end of the previous period reaches zero. In terms of variables, we can say that in any optimal solution, we will not find any period t such that $y_t = 1$ and $x_{t-1} > 0$.

The reason why the ZIO property holds is easy to understand. Imagine we start production at period t ($y_t = 1$), but we have a positive inventory ($x_{t-1} > 0$). Because of the empty initial inventory assumption, the x_{t-1} units in the inventory must have been produced during some earlier period. Therefore, they must have incurred some strictly positive holding costs. For example, the simple fact that we carry them in inventory from period $t-1$ to period t causes $h_{t-1}x_{t-1}$ in holding costs. Now, if instead of producing these units at earlier periods, we decided to postpone their production to period t, we would have the same total quantity of produced units but lower holding costs. At the same time, we would not be increasing the number of production runs, and thus, the setup costs would not increase. Therefore, we can modify our initial solution and find a better one by postponing the production of these units, thus proving that the initial solution with $x_{t-1} > 0$ and $y_t = 1$ cannot be optimal.

A consequence of the ZIO property is that, in any optimal solution, the produced quantity during period t is either zero or the sum of some present and future demands. In other words, if $w_t > 0$, then we must have that $w_t = d_t$ or $w_t = d_t + d_{t+1}$ or $w_t = d_t + d_{t+1} + d_{t+2}$, etc. If this property does not hold, the next time that we produce, we will have a non-empty inventory in violation of the ZIO property. And if the property does not hold and t is the last time we produce, we will have a non-empty final inventory, causing unnecessary holding costs.

Table 2 presents the data of an example ULSP instance with an initial inventory of $x_0 = 100$. After solving model (1a)–(1f), we describe an optimal solution in Figure 2. The top chart presents the inventory level at the end of each period (blue line) and the production level at each period (red line). We remark that, even if this instance does not have an empty initial inventory, after enough periods pass, the ZIO property will start holding true. For example, there is no production during periods eight and ten, because there is a positive inventory at the end of periods seven and nine. The bottom chart shows the costs incurred in each period. The yellow bars report holding costs, the purple bars are setup costs, and the pink bars are the production costs.

1.1 The ULSP for upstream inventory management

As discussed earlier, the ULSP links production and downstream inventory management. The EOQ model we have seen in previous notes, however, also applies to the case of an upstream inventory that gets its products from external suppliers. Fortunately, we can use the ULSP for this case, too. All that is required is to change the interpretation of some of the model parameters. In particular, if we are managing an upstream inventory over a discretised planning

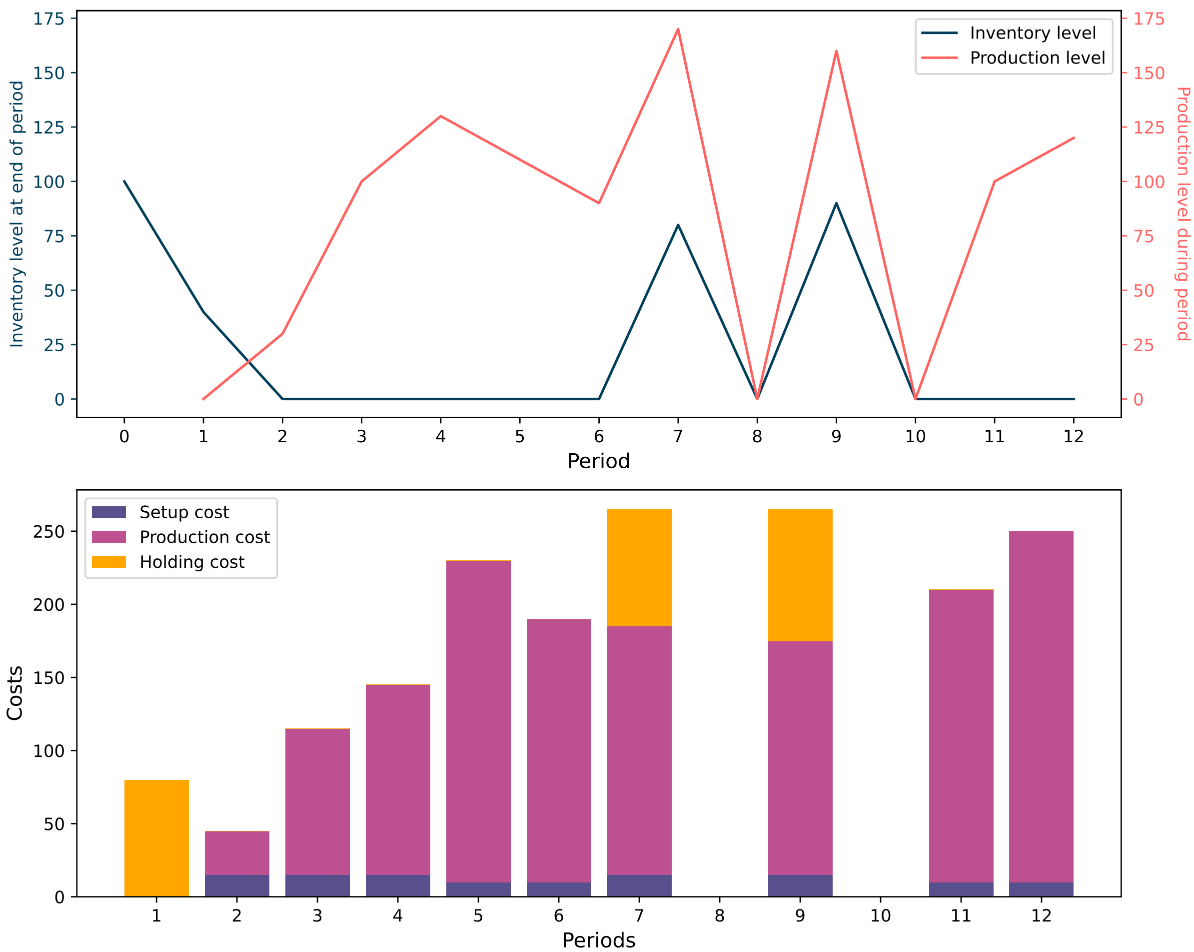

Figure 2: An optimal solution of the ULSP instance presented in Table 2. The top chart shows the inventory and production levels. The bottom chart reports the costs per period.

horizon T, we consider the following interpretation of the ULSP parameters.

- The demand forecast d_t becomes the per-period consumption of the product by subsequent productive activities. This is analogous to the EOQ's rate of consumption a. In the ULSP case, however, the consumption is discretised per period and is not continuous. On the other hand, the EOQ assumed a constant consumption, while using the ULSP, we can make the consumption vary in each period.
- The fixed setup cost s_t becomes the fixed order cost of the EOQ model, i.e., the quantity-independent cost that the firm pays each time they place an order. The ULSP model again offers additional flexibility because it allows this cost to change at each period. In contrast, the EOQ model assumes that the fixed order cost is always equal to K. On the other hand, using the ULSP, we can only place one order per period, whereas the EOQ model allows us full flexibility in the cycle length.
- The variable production cost p_t becomes the variable order cost. In this case, too, the ULSP model allows p_t to change with each t, while in the EOQ model, the variable order cost was fixed to value c.

As we can see, using the ULSP to model an upstream inventory has pros and cons. As a rule of thumb, if the consumption rate is close to constant, the EOQ model might be more suitable. On the contrary, if production is highly irregular, the ULSP will probably be more appropriate to make inventory decisions even for upstream inventories.

Exercise 1.1. We have established a link between the EOQ model and the ULSP, and we have seen that the ULSP can be used to model upstream inventories. However, as we have already seen in previous notes, the EOQ model can be extended to allow for stockouts. Can we do the same with the ULSP model? In particular, let $b_t > 0$ be the unit stockout cost in period t and assume that backorders are fulfilled immediately before fulfilling the current period demand.

For example, imagine that we have 100 units of accumulated backorders at the end of period t. In this case, we will pay $100 \cdot b_t$ in stockout costs in period t. Also, assume that we do not produce during period $t+1$ and that the demand is $d_{t+1} = 30$. Then, we will have 130 units of backorders at the end of period $t+1$, and we will pay $130 \cdot b_{t+1}$ in period $t+1$. If, in period $t+2$, we produce 200 units and have a demand of 20, we will be able to fulfil the backorders and the period's demand, and we will have $200 - 130 - 20 = 50$ units that we carry over as inventory.

Task 1. Is there a property analogous to the ZIO that involves backorders?

Task 2. Extend the ULSP model (1a)–(1f) to take into account stockouts and their relative costs. You can add new parameters, variables and constraints or change the existing ones.

Solution of Task 1. We cannot have both stockout and a positive inventory at the end of the same period. Because both carrying inventory and having backorders have associated costs, it is always convenient to use the remaining inventory to fulfil as many backorders as possible.

Solution of Task 2. Let us introduce parameter z_0, which represents the number of backorders present at the beginning of the planning horizon. Let $z_t \geq 0$ be a non-negative continuous variable defined for $t \in T$, representing the cumulative backorders at the end of period t.

Note that, by virtue of the property stated in Task 1, at most one variable between x_t and z_t will be non-zero in any optimal solution. If we recall the concept of negative inventory level we introduced for the EOQ model with stockouts, we can now express such level at the end of period t as $x_t - z_t$. If we have a positive inventory, z_t will be zero by the property defined in Task 1, and the inventory level will be $x_t - z_t = x_t > 0$. If we have backorders, x_t will be zero by

Time $t-1$	Time t	Constraint	u_t's value
Not producing $y_{t-1} = 0$	Not producing $y_t = 0$	$u_t \geq 0 - 0$	0 penalised in (3a)
Not producing $y_{t-1} = 0$	Producing $y_t = 1$	$u_t \geq 1 - 0$	1
Producing $y_{t-1} = 1$	Not producing $y_t = 0$	$u_t \geq 0 - 1$	0 penalised in (3a)
Producing $y_t = 1$	Producing $y_t = 1$	$u_t \geq 1 - 1$	0 penalised in (3a)

Table 3: The four possible pair of values taken by y_t and y_{t-1} all yield the correct value of u_t via constraint (3d).

the same property, and the inventory level will be $x_t - z_t = -z_t < 0$. Therefore, we can extend the ULSP model using this new "generalised" inventory level in place of the old x_t:

$$\begin{aligned}
\min \quad & \sum_{t \in T} \big(s_t y_t + p_t w_t + h_t x_t + b_t z_t\big) && && \text{(2a)}\\
\text{subject to} \quad & x_t - z_t = (x_{t-1} - z_{t-1}) + w_t - d_t && \forall t \in T && \text{(2b)}\\
& w_t \leq M y_t && \forall t \in T && \text{(2c)}\\
& x_t \geq 0 && \forall t \in T && \text{(2d)}\\
& w_t \geq 0 && \forall t \in T && \text{(2e)}\\
& z_t \geq 0 && \forall t \in T && \text{(2f)}\\
& y_t \in \{0, 1\} && \forall t \in T. && \text{(2g)}
\end{aligned}$$

We have added the cumulative backorder costs in the objective function (2a), and we are using the generalised inventory level in constraint (2b). □

Exercise 1.2. In the ULSP, we pay setup costs during every period when we produce. Sometimes, however, setup costs should only be paid if we turn on the machines that were previously turned off. For example, a factory that produces paint will have to pay a setup cost at period t if it was producing some other paint colour during period $t - 1$ because, in this case, they have to perform a long and expensive cleaning procedure to avoid different colours to mix. If, however, they produce blue paint during both periods $t - 1$ and t, no setup cost should be charged.

Modify model (1a)–(1f) so that the setup cost s_t is charged only if we produce during period t, but we were not producing during period $t - 1$. Assume that this cost shall also be paid if we produce during the first period ($t = 1$).

Solution. Let us introduce parameter $y_0 = 0$ to model the fact that we should pay the setup cost during the first period if we decide to produce, and consider new variables $u_t \in \{0, 1\}$ for all $t \in T$. We want u_t to take the value one if and only if the setup cost should be charged during period t. The new model then reads as follows:

$$\begin{aligned}
\min \quad & \sum_{t \in T} \big(s_t u_t + p_t w_t + h_t x_t\big) && && \text{(3a)}\\
\text{subject to} \quad & x_t = x_{t-1} + w_t - d_t && \forall t \in T && \text{(3b)}\\
& w_t \leq M y_t && \forall t \in T && \text{(3c)}
\end{aligned}$$

Set	Description
T	Planning horizon.
P	Products manufactured by the company.
Parameter	**Description**
d_{ti}	Demand for product $i \in P$ at period $t \in T$.
s_{ti}	Setup (fixed) cost if producing product $i \in P$ during period $t \in T$.
p_{ti}	Unit cost for producing product $i \in P$ during period $t \in T$.
h_{ti}	Unit holding cost for the inventory of product $i \in P$ present at the end of period $t \in T$.
x_{0i}	Initial inventory level for product $i \in P$ at the beginning of the planning horizon.
M	A very large number used for modelling purposes and not corresponding to any real-life quantity.

Table 4: Sets and parameters of the multi-product Uncapacitated Lot-Sizing Problem.

$$u_t \geq y_t - y_{t-1} \quad \forall t \in T \tag{3d}$$
$$x_t \geq 0 \quad \forall t \in T \tag{3e}$$
$$w_t \geq 0 \quad \forall t \in T \tag{3f}$$
$$u_t \in \{0,1\} \quad \forall t \in T \tag{3g}$$
$$y_t \in \{0,1\} \quad \forall t \in T. \tag{3h}$$

This model differs from (1a)–(1a) in two ways. First, in the objective function (3a), we only charge the setup cost when $u_t = 1$. Second, we added constraint (3d) to set variables u to their correct value. Let us check that these constraints work as intended by considering the four possible cases presented in Table 3. The only case when u_t takes value one is if we are not producing in period $t-1$ and producing in period t, as required. □

Exercise 1.3. A company is producing m different products $P = \{1, \dots, m\}$. They want to extend the ULSP model to optimise the manufacturing and inventory processes of all the products simultaneously. They have data about the demand, the costs and the initial inventory level on a product-by-product base. This data is described in Table 4 and extends the one presented in Table 1 by taking into account the various products.

Write an IP model that extends the ULSP model to deal with the multiple-product case. Include the additional constraint that, at each period, the company can only produce one of the products.

Solution. The data presented in Table 4 differs from the ULSP one presented in Table 1 by having two indices, one for the period and one for the product. The variables will undergo the same transformation; in particular, we will use:

- $x_{ti} \geq 0$ to describe the inventory level of product $i \in P$ at the end of period $t \in T$.
- $w_{ti} \geq 0$ to keep track of the quantity of product $i \in P$ manufactured in period $t \in T$.
- $y_{ti} \in \{0,1\}$ taking value one if and only if the company produced product $i \in P$ during period $t \in T$.

An IP formulation of the problem reads as follows:

$$\min \quad \sum_{t \in T} \sum_{i \in P} \left(s_{ti} y_{ti} + p_{ti} w_{ti} + h_{ti} x_{ti} \right) \tag{4a}$$

Set	Description
T	Planning horizon.
Parameter	**Description**
d_t	Demand at period $t \in T$.
s_t	Setup (fixed) cost if producing during period $t \in T$.
p_t	Unit cost for producing during period $t \in T$.
h_t	Unit holding cost for the inventory present at the end of period $t \in T$.
q_t	Maximum level of the inventory present at the end of period $t \in T$.
r_t	Maximum units that can be produced during period $t \in T$.
x_0	Initial inventory level at the beginning of the planning horizon.
M	A very large number used for modelling purposes and not corresponding to any real-life quantity.
Variable	**Description**
x_t	Inventory level at the end of period $t \in T$.
w_t	Quantity produced during period $t \in T$.
y_t	Binary variable taking value 1 if we produce during period $t \in T$.

Table 5: Sets, parameters and variables of the Capacitated Lot-Sizing Problem.

$$
\begin{aligned}
\text{subject to}\quad & x_{ti} = x_{t-1,i} + w_{ti} - d_{ti} && \forall t \in T,\ \forall i \in P && \text{(4b)}\\
& w_{ti} \le M y_{ti} && \forall t \in T,\ \forall i \in P && \text{(4c)}\\
& \sum_{i \in P} y_{ti} \le 1 && \forall t \in T && \text{(4d)}\\
& x_{ti} \ge 0 && \forall t \in T,\ \forall i \in P && \text{(4e)}\\
& w_{ti} \ge 0 && \forall t \in T,\ \forall i \in P && \text{(4f)}\\
& y_{ti} \in \{0, 1\} && \forall t \in T,\ \forall i \in P. && \text{(4g)}
\end{aligned}
$$

Model (4a)–(4g) straightforwardly extends (1a)–(1f). The new constraint is (4d), imposing that at most one different product is manufactured during each period. When it comes to numerically solving this formulation, one can either take a very large integer for M (for example, 10^9) or use a different M_i for each index i quantifying the constraint. In the latter case, one can take $M_i = \sum_{t \in T} d_{ti}$, analogously to what was done for the ULSP model. □

2 The Capacitated Lot-Sizing Problem

Among the assumptions of the ULSP, the ones that are most likely to clash with a company's reality are that the inventory can hold and the production process can manufacture as many units of products as desired. The Capacitated Lot-Sizing Problem (CLSP) is an extension of the ULSP that takes into account these limits. In particular, we denote with q_t the maximum inventory capacity at the end of period t and with r_t the maximum amount of units that we can manufacture during period t. As usual, we allow these quantities to change at each period for maximum flexibility, although, in real life, it is possible that they do not change during the planning horizon and take some fixed value (e.g., $q_t = q\ \forall t \in T$ and $r_t = r\ \forall t \in T$).

Table 5 extends the ULSP's Table 1 by adding these new parameters. The IP formulation for

Period →	1	2	3	4	5	6	7	8	9	10	11	12
Demand	60	70	100	130	110	90	90	80	70	90	100	120
Setup cost	15	15	15	15	10	10	15	15	15	10	10	10
Production cost	1	1	1	1	2	2	1	2	1	2	2	2
Holding cost	2	2	2	2	1	1	1	1	1	1	1	1
Production capacity	100	120	110	100	90	120	110	130	120	100	100	90
Inventory capacity	150	150	100	100	50	50	100	100	100	150	150	150

Table 6: Input data of a CLSP instance with initial inventory $x_0 = 100$.

the CLSP reads as follows:

$$\begin{aligned}
\min \quad & \sum_{t\in T}\big(s_t y_t + p_t w_t + h_t x_t\big) && && \text{(5a)}\\
\text{subject to} \quad & x_t = x_{t-1} + w_t - d_t && \forall t \in T && \text{(5b)}\\
& w_t \leq M y_t && \forall t \in T && \text{(5c)}\\
& x_t \leq q_t && \forall t \in T && \text{(5d)}\\
& w_t \leq r_t && \forall t \in T && \text{(5e)}\\
& x_t \geq 0 && \forall t \in T && \text{(5f)}\\
& w_t \geq 0 && \forall t \in T && \text{(5g)}\\
& y_t \in \{0,1\} && \forall t \in T. && \text{(5h)}
\end{aligned}$$

It differs from the ULSP formulation only because of the new constraints (5d) and (5e), which limit the inventory and production levels at each period.

It is important to remark that, differently from the ULSP, the CLSP does not enjoy the ZIO property. When production limits are imposed, it might be impossible to satisfy all demand without anticipating some production to previous periods. In this case, producing even when the inventory level is positive might be the only way to satisfy all future demand. A classic example is that of an ice cream company that has a peak in demand during the summer and must begin to ramp up production during the spring.

Table 6 extends the ULSP data presented in Table 2 by adding production and inventory capacities. After solving model (5a)–(5h), we plot the relevant characteristic of an optimal solution in Figure 3. Compared to the ULSP solution plot in Figure 2, the top chart now also reports the production and inventory capacities as dashed lines. We remark that we produce during almost all periods, while the inventory level is zero only at the end of periods five, six and eight. Therefore, as we anticipated, the CLSP does not enjoy the ZIO property.

Exercise 2.1. Is it possible for an instance of the CLSP to be infeasible? In this case, produce one such instance.

Solution. It is certainly possible for an instance to be infeasible. For example, if the demand is very high but the production capacity is very low. Consider an instance where

$$\sum_{t\in T} d_t > x_0 + \sum_{t\in T} r_t.$$

Then, no matter what the other instance parameters are (including the maximum inventory levels), it will never be possible to satisfy all the demand. The reason is that even if we used

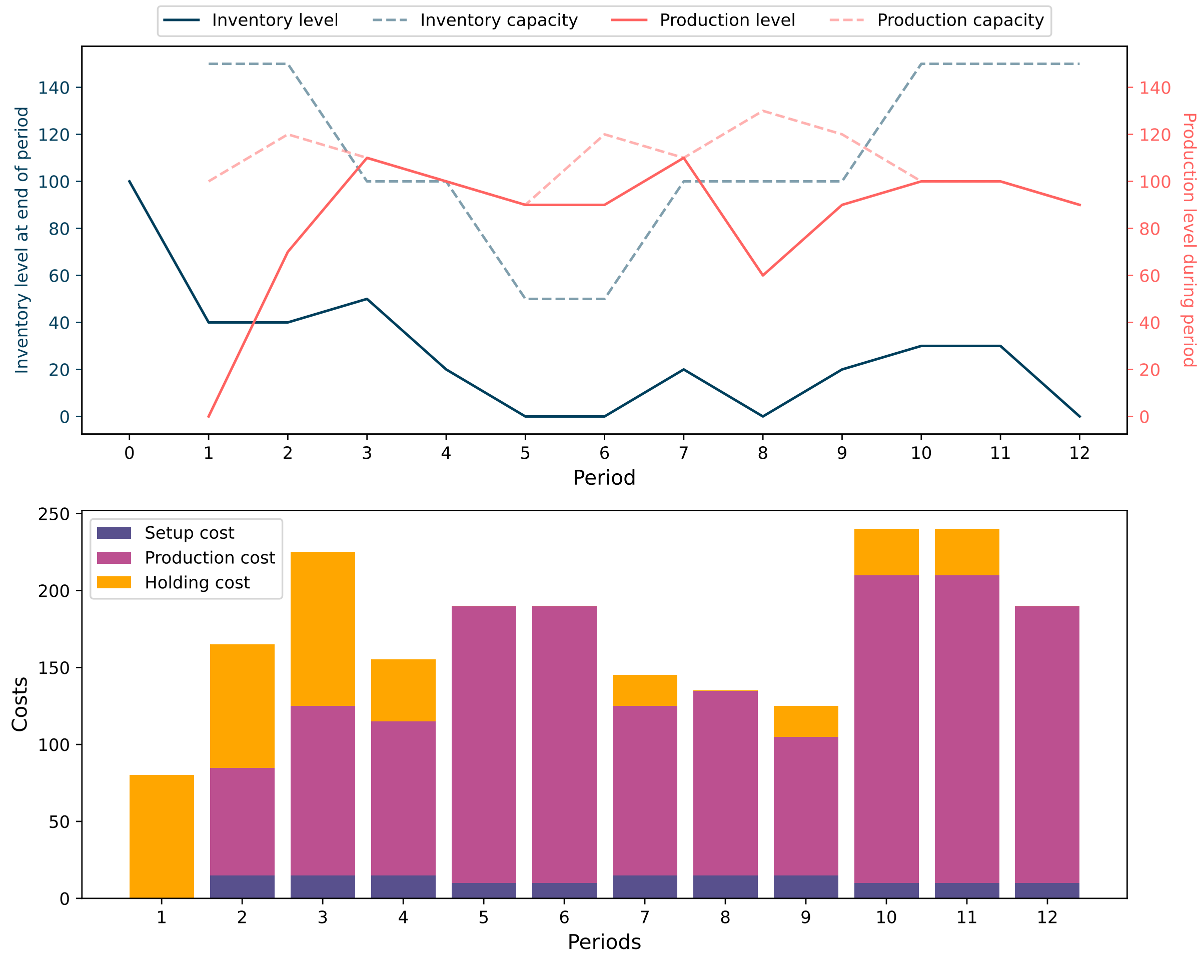

Figure 3: An optimal solution of the CLSP instance presented in Table 6. The top chart shows the inventory and production levels and capacities. The bottom chart reports the costs per period.

the entire initial inventory and produced at the maximum possible level during each period, the total amount of product available would be strictly smaller than the total demand. □

Exercise 2.2. Formulation (5a)–(5h) has four groups of constraints, excluding the constraints that define the variables' domains. These are constraints (5b), (5c), (5d), and (5e). Can you devise an alternative model for the CLSP that only uses three groups of constraints?

Solution. One way to accomplish this task is to "merge" constraints (5c) and (5e) into a single constraint

$$w_t \leq r_t y_t \quad \forall t \in T.$$

In the CLSP, we have a hard maximum r_t on the units of product we can manufacture during period t. Therefore, we can use this number as the "Big M" value of (5c). In periods t when we start production, we have $y_t = 1$, and the new constraint reads $w_t \leq r_t$, i.e., it enforces the production limit that (5e) was enforcing. On the other hand, if we do not start production in period t, we have $y_t = 0$, and the new constraint reads $w_t \leq 0$ exactly as (5c) was doing. □

Exercise 2.3. A pharmaceutical company produces a total of m products, $P = \{1, \ldots, m\}$. These products are of two types: some ($P_{\text{ref}} \subset P$) must be stored in a refrigerated warehouse, while others ($P_{\text{reg}} \subset P$) must be stored in a regular warehouse. All refrigerated products go into the same warehouse that has capacity q_{ref}. Analogously, all normal products go into another warehouse with capacity q_{reg}. These capacities do not change with time. Furthermore, each product $i \in P$ has a maximum production capacity of r_{ti} during month t, and the company can produce a maximum of K different products during each period.

All other data, presented in Table 7, are similar to those of Exercise 1.3. Note that sets P_{ref} and P_{reg} partition set P, i.e., $P_{\text{ref}} \cap P_{\text{reg}} = \emptyset$ and $P_{\text{ref}} \cup P_{\text{reg}} = P$. In other words, each product is either refrigerated or regular, but not both.

Extend the model you devised in Exercise 1.3 to deal with the case of the pharmaceutical company.

Solution. We use the same variables as in Exercise 1.3, and change model (4a)–(4g) as follows:

$$\begin{aligned}
\min \quad & \sum_{t \in T} \sum_{i \in P} \big(s_{ti} y_{ti} + p_{ti} w_{ti} + h_{ti} x_{ti} \big) && && \text{(6a)} \\
\text{subject to} \quad & x_{ti} = x_{t-1,i} + w_{ti} - d_{ti} && \forall t \in T,\ \forall i \in P && \text{(6b)} \\
& w_{ti} \leq M y_{ti} && \forall t \in T,\ \forall i \in P && \text{(6c)} \\
& \sum_{i \in P} y_{ti} \leq K && \forall t \in T && \text{(6d)} \\
& \sum_{i \in P_{\text{ref}}} x_{ti} \leq q_{\text{ref}} && \forall t \in T && \text{(6e)} \\
& \sum_{i \in P_{\text{reg}}} x_{ti} \leq q_{\text{reg}} && \forall t \in T && \text{(6f)} \\
& w_{ti} \leq r_{ti} && \forall t \in T,\ \forall i \in P && \text{(6g)} \\
& x_{ti} \geq 0 && \forall t \in T,\ \forall i \in P && \text{(6h)} \\
& w_{ti} \geq 0 && \forall t \in T,\ \forall i \in P && \text{(6i)} \\
& y_{ti} \in \{0, 1\} && \forall t \in T,\ \forall i \in P. && \text{(6j)}
\end{aligned}$$

The difference between this model and the one of Exercise 1.3 are the following. We modified constraint (4d), which allowed to manufacture at most one product per period, into (6d), which

Set	**Description**
T	Planning horizon.
P	Products manufactured by the company.
P_{ref}	Products that must be stored in a refrigerated warehouse.
P_{reg}	Products that must be stored in a regular warehouse.
Parameter	**Description**
d_{ti}	Demand for product $i \in P$ at period $t \in T$.
s_{ti}	Setup (fixed) cost if producing product $i \in P$ during period $t \in T$.
p_{ti}	Unit cost for producing product $i \in P$ during period $t \in T$.
h_{ti}	Unit holding cost for the inventory of product $i \in P$ present at the end of period $t \in T$.
q_{ref}	Maximum inventory level of the refrigerated warehouse at the end of each period.
q_{reg}	Maximum inventory level of the regular warehouse at the end of each period.
r_{ti}	Maximum amount of manufactured product of type $p \in P$ during period $t \in T$.
x_{0i}	Initial inventory level for product $i \in P$ at the beginning of the planning horizon.
K	Number of different products that can be manufactured during the same period.
M	A very large number used for modelling purposes and not corresponding to any real-life quantity.

Table 7: Sets and parameters of the multi-product Capacitated Lot-Sizing Problem faced by the pharmaceutical company.

allows a maximum of K products. Then we added constraints (6e) and (6f) imposing that, at the end of each period, the sum of the inventory levels of all refrigerated and regular products do not exceed the respective maximum levels. Finally, we added (6g) to ensure that we do not exceed the per-period production capacities for each product. □

Exercise 2.4. You are in the same setting as Exercise 2.3, but now the maximum number of different products K that can be manufactured during one period is not a hard limit, but a soft one. The pharmaceutical company can now manufacture as many different products as they want. However, each additional product they manufacture above K carries a per-period fixed extra cost of α. For example, if they manufacture $K+1$ different products during a period, they have to pay the extra cost α during that period. If they manufacture $K+2$ different products, they must pay 2α. If they manufacture 1, 2, ..., K different products, they do not pay any extra cost.

Change your solution to Exercise 2.3 to take into account the above description. You can add new variables and constraints if needed.

Solution. Let us introduce variable $z_t \in \mathbb{N}$ indicating the number of extra products manufactured during period t.[1] In other words, if the company manufactures up to K different products in period t, then z_t will be zero. Otherwise, if the company manufactures $K+\ell$ different products ($\ell > 0$ and integer), then $z_t = \ell$. The new model reads as follows:

$$
\begin{aligned}
\min \quad & \sum_{t \in T} \left(\alpha z_t + \sum_{i \in P} \left(s_{ti} y_{ti} + p_{ti} w_{ti} + h_{ti} x_{ti} \right) \right) && && \text{(7a)} \\
\text{subject to} \quad & x_{ti} = x_{t-1,i} + w_{ti} - d_{ti} && \forall t \in T,\ \forall i \in P && \text{(7b)} \\
& w_{ti} \leq M y_{ti} && \forall t \in T,\ \forall i \in P && \text{(7c)} \\
& \sum_{i \in P_{\text{ref}}} x_{ti} \leq q_{\text{ref}} && \forall t \in T && \text{(7d)} \\
& \sum_{i \in P_{\text{reg}}} x_{ti} \leq q_{\text{reg}} && \forall t \in T && \text{(7e)} \\
& w_{ti} \leq r_{ti} && \forall t \in T,\ \forall i \in P && \text{(7f)} \\
& z_t \geq \sum_{i \in P} y_{it} - K && \forall t \in T && \text{(7g)} \\
& x_{ti} \geq 0 && \forall t \in T,\ \forall i \in P && \text{(7h)} \\
& w_{ti} \geq 0 && \forall t \in T,\ \forall i \in P && \text{(7i)} \\
& y_{ti} \in \{0,1\} && \forall t \in T,\ \forall i \in P && \text{(7j)} \\
& z_t \in \mathbb{N} && \forall t \in T. && \text{(7k)}
\end{aligned}
$$

In model (7a)–(7k), we have modified the objective function to apply the required penalty. Furthermore, we removed the hard limit that was enforced by (6d) in Exercise 2.3, and we added constraint (7g). The sum on the right-hand side counts how many different products are manufactured during period t. If this number is smaller than or equal to K, then the constraint is moot because the entire right-hand side is non-positive. On the other hand, if the company produces more than K products, the right-hand side is positive and provides a lower bound for the value of z_{ti}. For example, if the company produces $K+\ell$ different products, the constraint will become $z_t \geq \ell$. At the same time, z_t is penalised in the objective function and, therefore, it will take the smallest value allowed by (7g). Therefore, in the above example, we would have $z_t = \ell$. □

[1] Recall that $\mathbb{N} = \{0, 1, \ldots\}$.