DMO — Midterm

Student ID: ……………………………………………

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1 Graphical LP (15%)

Solve the following LP using the graphical method. If multiple solutions exist, characterise all of them.

$$
\begin{aligned}
\max \quad & 5x_1 + \frac{14}{3}x_2 \\
\text{subject to} \quad & -x_1 + 3x_2 \leq 9 \\
& 5x_1 + 3x_2 \leq 27 \\
& 8x_1 - 3x_2 \leq 25 \\
& 2x_1 - 3x_2 \leq 4 \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

2 Duality theorem implications (20%)

For each of the following primal-dual pair, either write a simple primal LP that provides an example of the requested condition, or state that such a condition is impossible.

1. Unbounded primal with infeasible dual.
2. Unbounded primal with unbounded dual.
3. Infeasible primal with unbounded dual.

3 Production planning revisited (65%)

A widget factory must plan its production for the next twelve months. An operations manager is given the following data about the factory's operations:

- The current workforce at "month zero" is $w_0 = 300$ employees.
- For each month $t \in \{1, \dots, 12\}$, the operations manager receives an estimated demand $d_t \geq 0$. It represents the number of monthly workers necessary to produce enough widgets to satisfy market demand. That is, to simplify the task of the operations manager, demand is not given in the number of widgets but in the number of workers needed to produce the widgets.
- Each employee producing widgets during a month allows the factory to earn revenue $r \geq 0$. If there are more employees than those required to meet market demand, then demand limits the revenue that can be earned. For example, if during the first month, the factory employs 300 people, but only 200 are necessary to satisfy the demand ($d_1 = 200$), then the revenue is $200 \cdot r$; the remaining 100 employees are inactive and do not "produce" any revenue.
- On the other hand, each "missing" employee causes a monthly loss of $m \geq 0$. For example, if the first month's demand requires 400 employees but only 300 are working, the factory loses $100 \cdot m$. This cost could represent contract penalties for not fulfilling buyers' orders.

- The monthly employee salary is $\ell \geq 0$.
- At the beginning of each month, the factory can hire workers. When workers are hired at the beginning of month t, they spend the entire month training and can only perform their job at the beginning of month $t+1$. What's more, an existing employee must train the new hire. Therefore, for each worker hired at the beginning of month t, an old employee will not be productive this month because he will be training the new hire. Given our starting workforce of 300, if we hire 100 people during the first month ($t = 1$) and we don't hire during the second month ($t = 2$), then there will be 200 workers in the production line when $t = 1$ (because 100 out of 300 are busy training the new hires). On the other hand, 400 workers will be producing widgets when $t = 2$, the 300 initial workers plus the 100 hired at $t = 1$.
- Thanks to strong unions, workers cannot be fired.
- Widgets are perishable and must be sold during the month of production. As such, there is no inventory and extra production is never carried over to the next month.

Write a Linear Programming formulation of a model that solves the production/workforce planning problem described above, with the objective of maximising the company's total profit. Each month, the profit is the difference between the revenue and the sum of salary costs and penalties. (This quantity can be positive or negative.) The total profit is the sum of monthly profits.