# Complementary Slackness

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Many examples in these lecture notes are adapted from popular books:

- Alexander Schrijver (1998). *Theory of linear and integer programming.* Wiley. ISBN: 0-471-98232-6.
- Vašek Chvátal (1983). *Linear Programming.* W.H. Freeman and Company. ISBN: 0-716-71195-8.
- Laurence Wolsey (2020). *Integer Programming.* 2nd Edition. Wiley. ISBN: 978-1-119-60653-6.
- Silvano Martello and Paolo Toth (1990). *Knapsack Problems: algorithms and computer implementations.* Wiley. ISBN: 978-0-471-92420-3.

These lecture notes present a way to obtain a proof of optimality for a solution $(x_1^*, \dots, x_n^*)$ of an LP. That is, given a feasible solution, we will provide a condition called **complementary slackness** that ensures its optimality. Verifying this condition requires solving a system of linear equations.

Let us first introduce the main result of complementary slackness. Next, we will see how to use this result to develop a proof of optimality.

**Theorem 1** (Complementary slackness theorem). *Let $(x_1^*, \dots, x_n^*)$ and $(y_1^*, \dots, y_m^*)$ be a pair of primal-dual feasible solutions. These solutions are simultaneously optimal for the primal and the dual problems if and only if the following conditions hold:*

1. *For every $j \in \{1, \dots, n\}$, either $\sum_{i=1}^m a_{ij} y_i^* = c_j$ (i.e., the $j$-th dual constraint is satisfied with equality), or $x_j^* = 0$. (Eventually, both conditions can be true.)*
2. *For every $i \in \{1, \dots, m\}$, either $\sum_{j=1}^n a_{ij} x_j^* = b_i$ (i.e., the $i$-th primal constraint is satisfied with equality), or $y_i^* = 0$. (These conditions can also be both true.)*

*Proof.* Because of properties **P1**–**P3** described above, we know that the solutions are optimal for the primal and dual problems if and only if

$$\sum_{j=1}^n c_j x_j^* = \sum_{i=1}^m b_i y_i^* \tag{1}$$

is verified. Therefore, we must prove that our conditions 1. and 2. hold if and only if (1) holds.

First, we note that, because both solutions are feasible,

$$\sum_{j=1}^n a_{ij} x_j^* \le b_i \qquad \forall i \in \{1, \dots, m\} \qquad \text{and} \tag{2}$$

$$\sum_{i=1}^m a_{ij} y_i^* \ge c_j \qquad \forall j \in \{1, \dots, n\}. \tag{3}$$

Therefore, the following inequalities hold:

$$b_i y_i^* \geq \left( \sum_{j=1}^{n} a_{ij} x_j^* \right) y_i^* \qquad \forall i \in \{1, \ldots, m\} \qquad \text{and} \tag{4}$$

$$c_j x_j^* \leq \left( \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j^* \qquad \forall j \in \{1, \ldots, n\}. \tag{5}$$

Bringing all together, we have that

$$\sum_{j=1}^{n} c_j x_j^* \leq \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j^* = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} x_j^* \right) y_i^* \leq \sum_{i=1}^{m} b_i y_i^*. \tag{6}$$

Condition (1) is verified if and only if both $\leq$ conditions in (6) actually hold with equality. Let us focus on the first one.

Note that, for a given $j \in \{1, \ldots, n\}$, there are two ways in which one can have

$$c_j x_j^* = \left( \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j^*.$$

The first way is if $x_j^* = 0$, in which case both the left- and right-hand sides are equal to zero. The second way is if the coefficients of $x_j^*$ in the left- and right-hand sides are the same, i.e., if $c_j = \sum_{i=1}^{m} a_{ij} y_i^*$. This implies that the first $\leq$-inequality of (6) holds with equality if condition 1. of the theorem holds.

For the "only if" part, we have to show that there is no other way for the left- and right-hand sides to be equal other than the two ways that we have just seen. In other words, we must show that to have

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j^*$$

it must occur that each coefficient of a non-zero $x_j^*$ on the left is equal to the corresponding coefficient on the right. In general if two sums $\sum_j \alpha_j x_j^*$ and $\sum_j \beta_j x_j^*$ are equal, it does *not* follow that, for each $j$ such that $x_j^* \neq 0$, $\alpha_j$ is equal to the corresponding $\beta_j$. This is because some term $\alpha_j$ might be larger than some $\beta_j$ and some other $\alpha_{j'}$ might be smaller than $\beta_{j'}$, in a way that keeps the overall sums equal. But, in our case, this cannot happen: $(y_1^*, \ldots, y_m^*)$ is dual-feasible and, therefore, (3) holds. This means that the coefficients on the right-hand side are not smaller than those on the left-hand side. Therefore, the only way for the two sums to be equal is that *all* coefficients of non-zero $x_j^*$ on the left- and right-hand sides are pairwise equal.

Analogously, it is easy to see that the second $\leq$-inequality of (6) holds with equality if and only if condition 2. of the theorem holds. This completes the proof. □

To understand the name of the theorem (Complementary Slackness Theorem) and of conditions 1. and 2. (usually called the Complementary Slackness Conditions), recall the definition of slack variables given in the Simplex Algorithm lecture notes:

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \qquad \forall i \in \{1, \ldots, m\} \qquad \text{for the primal problem, and}$$

$$y_{m+j} = c_j - \sum_{i=1}^{m} a_{ij} y_i \qquad \forall j \in \{1, \ldots, n\} \qquad \text{for the dual problem.}$$

With these definitions in mind, we can reformulate the conditions as follows:

1. For every $j \in \{1, \ldots, n\}$ at least one between the primal decision variable $x_j^*$ and the dual slack variable $y_{m+j}^*$ is zero.
2. For every $i \in \{1, \ldots, m\}$ at least one between the dual decision variable $y_i^*$ and the primal slack variable $x_{n+i}^*$ is zero.

An intuitive interpretation of the complementary slackness conditions stems from seeing the dual variable as "shadow prices". First, recall that each dual variable is associated with a primal constraint. It is possible to show that the value of a dual variable in the optimal dual solution gives the rate of increase of the primal objective function if we could relax the corresponding primal constraint. In other words, if a constraint is binding at the primal optimum, the shadow price associated with that constraint tells us the rate at which an infinitesimal violation of that constraint would cause an infinitesimal improvement in the objective value. Note that a constraint is binding if it is satisfied with equality; therefore, the corresponding slack variable is zero.

On the other hand, if a constraint is not binding, it is "unnecessary" to violate it when it comes to (infinitesimally) improving the objective function compared to the primal optimum. In this case, its shadow price (i.e., its associated dual variable) is zero. At the same time, because the constraint is not binding, its corresponding slack variable is strictly positive.

One can then restate the complementary slackness conditions once more as:

1. In the primal problem, either the $i$-th constraint is binding ($x_{n+i}^* = 0$), or its corresponding shadow price is zero ($y_i^* = 0$, i.e., violating the $i$-th constraint "doesn't help").
2. Analogously, in the dual problem, either the $j$-th constraint is binding ($y_{m+j}^* = 0$), or its corresponding shadow price is zero ($x_j^* = 0$).

To use complementary slackness to obtain a proof of optimality, however, our next task will be to find a procedure to verify the optimality of $(x_1^*, \ldots, x_n^*)$ when no dual solution is provided. We first draw a natural corollary from Theorem 1.

**Corollary 1.1.** *A feasible solution $(x_1^*, \ldots, x_n^*)$ of the primal problem is optimal if and only if there exists $m$ numbers $(y_1^*, \ldots, y_m^*)$ such that:*

1. $\sum_{i=1}^m a_{ij} y_i^* = c_j$ *whenever* $x_j^* > 0$.
2. $y_i^* = 0$ *whenever* $\sum_{j=1}^n a_{ij} x^* j < b_i$.
3. $\sum_{i=1}^m a_{ij} y_i^* \geq c_j$ *for all* $j \in \{1, \ldots, n\}$.
4. $y_i^* \geq 0$ *for all* $i \in \{1, \ldots, m\}$.

*Proof.* The result follows immediately by noticing that the first two conditions are complementary slackness conditions, and the last two ensure the feasibility of $(y_1^*, \ldots, y_m^*)$ for the dual problem. □

Corollary 1.1 gives us immediately a way to verify the optimality of a primal-feasible solution. We solve the system of linear equalities given by the first two conditions of the corollary. If the solution is unique and satisfies the last two conditions of the corollary, then the given primal-feasible solution is optimal.

On the other hand, if the solution is unique but does not satisfy the last two conditions of the corollary, then the given primal-feasible solution is not optimal. If it was optimal, in fact, Corollary 1.1 would guarantee that there are numbers $(y_1^*, \ldots, y_m^*)$ which both satisfy all the linear equations of the system and constitute a dual-feasible solution. Because the unique numbers which solve the system do *not* constitute a dual-feasible solution, the hypothesis of the corollary must be false, i.e., $(x_1^*, \dots, x_n^*)$ is not optimal.

The above rule leaves open the case in which the system of linear equalities does *not* have a unique solution. Can this case happen? And, if so, can it happen for an optimal primal-feasible solution? The answer, unfortunately, is "yes". There is, however, positive news, too: it is extremely unlikely that a primal feasible solution found with the simplex method will produce a system with multiple solutions. The reason is the following theorem.

**Theorem 2.** *A primal-feasible solution which is basic and is not degenerate gives a system of linear inequalities (defined by the first two conditions of Corollary 1.1), which admits a unique solution.*

Therefore, we can use the optimality conditions for virtually all practical purposes.