Lagrangian Duality

Alberto Santini

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Many examples in these lecture notes are adapted from popular books:

- Alexander Schrijver (1998). *Theory of linear and integer programming.* Wiley. ISBN: 0-471-98232-6.
- Vašek Chvátal (1983). *Linear Programming.* W.H. Freeman and Company. ISBN: 0-716-71195-8.
- Laurence Wolsey (2020). *Integer Programming.* 2nd Edition. Wiley. ISBN: 978-1-119-60653-6.
- Silvano Martello and Paolo Toth (1990). *Knapsack Problems: algorithms and computer implementations.* Wiley. ISBN: 978-0-471-92420-3.

Consider the formulation of an LP in standard form,

$$\max \quad \sum_{j=1}^{n} c_j x_j \tag{1}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \forall i \in \{1, \ldots, m\} \tag{2}$$

$$x_j \geq 0 \qquad \forall j \in \{1, \ldots, n\}, \tag{3}$$

and a set of non-negative numbers $\lambda_1, \ldots, \lambda_m$. We define the corresponding **Lagrangian** function as:

$$L(\vec{x}, \vec{\lambda}) = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \lambda_i \Big(b_i - \sum_{j=1}^{n} a_{ij} x_j \Big).$$

The Lagrangian function corresponds to the objective function of the LP plus one term for each constraint. Each term is the product of a λ_i with the corresponding slack of the i-th constraint. Note that if the slack is negative, the corresponding constraint is violated; otherwise, the constraint is satisfied. Numbers $\lambda_1, \ldots, \lambda_m$ are the **Lagrangian multipliers**.

We also define function $q(\vec{\lambda})$ as:

$$q(\vec{\lambda}) = \max_{\vec{x} \geq \vec{0}} L(\vec{x}, \vec{\lambda}).$$

The central observation is that q is an upper bound on the optimal value of the primal problem, no matter the choice of $\vec{\lambda} \geq \vec{0}$. In other words, if the primal admits an optimum $\vec{x}^*$, then for any $\vec{\lambda} \geq \vec{0}$ it's true that $q(\vec{\lambda}) \geq \sum_{j=1}^{n} c_j x_j^*$. This is easily shown:

$$\begin{aligned} q(\vec{\lambda}) &= \max_{\vec{x} \geq \vec{0}} L(\vec{\lambda}, \vec{x}) \overset{\text{by def. of max}}{\geq} \\ &\geq L(\vec{x}^*, \vec{\lambda}) = \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} c_j x_j^* + \sum_{i=1}^{m} \underbrace{\lambda_i}_{\geq 0} \underbrace{\left(b_i - \sum_{j=1}^{n} a_{ij} x_j\right)}_{\geq 0 \text{ by feasibility of } x^*} \geq \\
&\geq \sum_{j=1}^{n} c_j x_j^*.
\end{aligned}
$$

Because $q(\vec{\lambda})$ gives us an upper bound on the optimal solution of the primal problem, we might want to find the value of $\vec{\lambda}$ yielding the tightest possible bound. That is, we want to solve the problem:

$$
\begin{aligned}
\min \quad & q(\vec{\lambda}) \\
\text{subject to} \quad & q(\vec{\lambda}) \text{ well-defined} \\
& \vec{\lambda} \geq \vec{0}.
\end{aligned}
$$

What does it mean for $q(\vec{\lambda})$ to be well-defined? It means that it's not $+\infty$ (well, in that case, it would not be a very tight bound, right?).

And how could $q(\vec{\lambda})$ be infinite? If the inner maximisation problem over $\vec{x}$ is unbounded. Let's then write this problem in a different way:

$$
\begin{aligned}
q(\vec{\lambda}) &= \max_{\vec{x} \geq \vec{0}} \; L(\vec{x}, \vec{\lambda}) = \\
&= \max_{\vec{x} \geq \vec{0}} \; \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \lambda_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j\right) = \\
&= \max_{\vec{x} \geq \vec{0}} \; \sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} \lambda_i a_{ij}\right) x_j + \sum_{i=1}^{m} \lambda_i b_i.
\end{aligned}
$$

Because we only have non-negativity constraints on the x's, they can grow as large as they want. Therefore, as soon as the objective coefficient of any x_j is positive, the problem becomes unbounded. In other words, all terms $c_j - \sum_{i=1}^{m} \lambda_i a_{ij}$ must be non-positive. When this happens, the optimal solution to the maximisation problem over $\vec{x}$ is to set all x's equal to zero. In this case, the optimum of the problem becomes the constant coefficient and, therefore, $q(\vec{\lambda})$ takes value $\sum_{i=1}^{m} \lambda_i b_i$.

Bringing together the above observations, we can rewrite the problem of finding the tightest bound from $q(\vec{\lambda})$ as follows:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m} b_i \lambda_i \\
\text{subject to} \quad & c_j - \sum_{i=1}^{m} \lambda_i a_{ij} \leq 0 && \forall j \in \{1, \ldots, n\} \\
& \lambda_i \geq 0 && \forall i \in \{1, \ldots, m\}.
\end{aligned}
$$

It's now easy to see that we have just written the dual problem with dual variables λ_i.