Linear Programming Duality

Alberto Santini

Fall 2024

Many examples in these lecture notes are adapted from popular books:

- Alexander Schrijver (1998). *Theory of linear and integer programming.* Wiley. ISBN: 0-471-98232-6.
- Vašek Chvátal (1983). *Linear Programming.* W.H. Freeman and Company. ISBN: 0-716-71195-8.
- Laurence Wolsey (2020). *Integer Programming.* 2nd Edition. Wiley. ISBN: 978-1-119-60653-6.
- Silvano Martello and Paolo Toth (1990). *Knapsack Problems: algorithms and computer implementations.* Wiley. ISBN: 978-0-471-92420-3.

In this lecture notes, we introduce the concept of the dual of an LP: for each LP in maximisation form, which we call the **primal problem**, we will define an LP in *minimisation* form, which we will call the **dual problem**. Primal and dual will enjoy the following properties:

1. The objective value of any feasible solution of the dual problem is an upper bound on the optimal objective value of the primal problem.
2. Conversely, the objective value of any feasible solution of the primal problem is a lower bound on the optimal objective value of the dual problem (recall that the dual problem is a minimisation problem).
3. If any of the two problems has an optimal solution, the other one also has an optimal solution. What is more, the objective values of these solutions are the same.

Defining the dual of an LP in standard form is very easy. Let the **primal problem** be

$$\max \quad \sum_{j=1}^{n} c_j x_j \tag{1}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \forall i \in \{1, \ldots, m\} \tag{2}$$

$$x_j \geq 0 \qquad \forall j \in \{1, \ldots, n\}. \tag{3}$$

Then the **dual problem** is defined as follows:

$$\min \quad \sum_{i=1}^{m} b_i y_i \tag{4}$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad \forall j \in \{1, \ldots, n\} \tag{5}$$

$$y_i \geq 0 \qquad \forall i \in \{1, \ldots, m\}. \tag{6}$$

This definition is as easy as it is uninformative. Why did we swap n with m? Why did we replace $\vec{c}$ with $\vec{b}$ in the objective function and vice-versa in the right-hand side of the constraints? Why did we transpose the coefficient matrix (a_{ij})? (Note that in (2), each variable shares the second index of the matrix entry, but in (5), it shares the first index.) Why did a maximisation problem become a minimisation one?

To understand how problem (4)–(6) comes to life, we will first provide some motivation using an example. Before concluding this brief introduction, we remark that variables $x_1, \ldots, x_n$ are usually referred to as the **primal variables**, while $y_1, \ldots, y_m$ are the **dual variables**.

1 LP Duality: motivation

Let us start with a concrete LP in standard form:

$$\begin{aligned}
\max \quad & 4x_1 + x_2 + 5x_3 + 3x_4 && (7)\\
\text{subject to} \quad & x_1 - x_2 - x_3 + 3x_4 \leq 1 && (8)\\
& 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 && (9)\\
& -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 && (10)\\
& x_1,\ x_2,\ x_3,\ x_4 \geq 0. && (11)
\end{aligned}$$

This problem is bounded and feasible, and therefore, it admits an optimal solution (whose objective value we denote with z^*). In the next classes, we will study how we can produce such an optimal solution using an algorithm known as the *simplex algorithm*. But, for the moment, can we say something about z^* value without running the algorithm?

For example, we might want to bound z^* from below and from above, i.e., find two numbers $\underaccent{\bar}{z}$ and $\bar{z}$, such that

$$\underaccent{\bar}{z} \leq z^* \leq \bar{z}.$$

Note that any feasible solution provides a lower bound. For example, $(1, 0, 0, 0)$ is feasible, and its objective value is 4. Then $\underaccent{\bar}{z} = 4$ is a lower bound: we do not know what the optimum of the problem is, but it certainly cannot be lower than 4 because we can produce a feasible solution with objective value 4. Of course, 4 is not the best lower bound that we can produce using feasible solutions. Solution $(3, 0, 2, 0)$, for example, is also feasible, and its objective value is 22.

Lower bounds are only partially useful because no matter what a seemingly good bound we obtain, we cannot know if the corresponding solution is optimal. Lower and upper bounds work best in pair: when we derive *both* of them, we have an interval $\left[\underaccent{\bar}{z}, \bar{z}\right]$ which we know must contain the optimal objective. And, in the special case $\underaccent{\bar}{z} = \bar{z}$, we know the actual optimal objective value of the problem.

It is, thus, important to also focus on deriving a good upper bound. To get an upper bound we might consider constraint (9) and multiply both the left- and right-hand sides by $\frac{5}{3}$, to obtain

$$\frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}. \tag{12}$$

Why did we multiply by $\frac{5}{3}$? Because this is the smallest non-negative constant such that all variable coefficients in (12) are larger or equal than the corresponding coefficients in the objective value (7):

$$\frac{25}{3} \geq 4,\ \frac{5}{3} \geq 1,\ 5 \geq 5,\ \frac{40}{3} \geq 3.$$

This fact implies that the following relation holds for any non-negative value of x_1, x_2, x_3, and x_4:

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq \frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}.$$

In particular, the above relation must hold for the optimal solution. And, because the left-most term would produce z^* when we plug in the optimal solution, we have just obtained an upper bound: $z^* \leq \frac{275}{3} \simeq 91.67$.

Is this a good upper bound? Well, to be honest, we can do better: we can sum the left- and right-hand sides of (9) and (10):

$$\begin{aligned} & (5x_1 + x_2 + 3x_3 + 8x_4) + (-x_1 + 2x_2 + 3x_3 - 5x_4) \leq 55 + 3 && \Rightarrow \\ \Rightarrow \quad & 4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58. \end{aligned}$$

Because, again, each coefficient on the left-hand side is not smaller than the corresponding coefficient in the objective function, 58 is an upper bound and much tighter than the one we derived before.

At this point, it looks like we might have a strategy to search for *the tightest* possible bound: we want to find a linear combination of inequalities (8)–(10) such that the resulting variable coefficients are not smaller than the respective objective function coefficients. Furthermore, we want the resulting bound, given by the combination of the right-hand sides, to be as small as possible.

A linear combination of three inequalities is uniquely determined by choosing three coefficients that multiply them. We must be careful that these coefficients are non-negative because a negative coefficient would invert the sense of the inequality. Let us denote these three coefficients with $y_1, y_2, y_3 \geq 0$. The bound they provide is $\underline{z} = y_1 \cdot 1 + y_2 \cdot 55 + y_3 \cdot 3$. The conditions imposing that the variable coefficients are not smaller than the objective function coefficients are:

$$\begin{aligned} y_1 \cdot 1 + y_2 \cdot 5 + y_3 \cdot (-1) &\geq 4, \\ y_1 \cdot (-1) + y_2 \cdot 1 + y_3 \cdot 2 &\geq 1, \\ y_1 \cdot (-1) + y_2 \cdot 3 + y_3 \cdot 3 &\geq 5, \\ y_1 \cdot 3 + y_2 \cdot 8 + y_3 \cdot (-5) &\geq 3. \end{aligned}$$

Bringing everything together and recalling that we want to find the tightest possible $\bar{z}$, we get:

$$\begin{aligned} \min \quad & y_1 + 55y_2 + 3y_3 && (13) \\ \text{subject to} \quad & y_1 + 5y_2 - y_3 \geq 4 && (14) \\ & -y_1 + y_2 + 2y_3 \geq 1 && (15) \\ & -y_1 + 3y_2 + 3y_3 \geq 5 && (16) \\ & 3y_1 + 8y_2 - 5y_3 \geq 3 && (17) \\ & y_1,\ y_2,\ y_3,\ y_4 \geq 0. && (18) \end{aligned}$$

Problem (13)–(18) is exactly the dual of the primal (7)–(11), as defined by the primal-dual pair presented at the beginning of this chapter!

2 The duality theorem

Remark that any feasible solution of the dual is an upper bound for the primal. Or, equivalently, any feasible solution of the primal is a lower bound for the dual. This fact derives directly from the definitions: for any primal feasible solution $(x_1, \ldots, x_n)$ and any dual feasible solution $(y_1, \ldots, y_m)$,

$$\sum_{j=1}^{n} c_j x_j \overset{\text{by (5)}}{\leq} \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \overset{\text{by (2)}}{\leq} \sum_{i=1}^{m} b_i y_i. \tag{19}$$

The above relation is sometimes called *weak duality.*

However, there is a stronger relation between the primal and the dual problem. This relation is enunciated by the famous **duality theorem**, presented below. It is sometimes referred to as *strong duality.*

Theorem 1 (Duality theorem)**.** *If the primal problem has an optimal solution $(x_1^*, \ldots, x_n^*)$, then the dual also has an optimal solution, say $(y_1^*, \ldots, y_m^*)$. Moreover, the two optimal objective values coincide, i.e.,*

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*. \tag{20}$$

3 Properties of the dual

In this section, we discuss a few properties of the primal-dual pair, which will make clear *why* the dual is important for practical purposes. Theorem 1 will play a central role in deriving these properties.

If-and-only-if version of the duality theorem. The duality theorem states that if the primal has an optimal solution, so does the dual. However, a stronger if-and-only-if relation holds. That is, it is also true that if the dual has an optimal solution, then the primal has one with the same objective value. You just have to solve the following exercise to see why this is the case.

Exercise 1. Show that the dual of the dual problem (4)–(6) (once it is written in standard form), is the primal (1)–(3).

Unbounded primal implies infeasible dual. We first note that if the primal problem is unbounded, the dual must be infeasible. If the dual admitted a solution $(y_1, \ldots, y_m)$, by eq. (19) quantity $\sum_{i=1}^m b_i y_i$ would place a bound on the objective value of the primal, contradicting the hypothesis that the primal is unbounded. Because the dual of the dual is the primal, we can swap words "primal" and "dual" in the above reasoning and also conclude that if the dual is unbounded, then the primal must be infeasible.

Infeasible primal implies either unbounded or infeasible dual. While we established above the "unbounded $\Rightarrow$ infeasible" primal-dual implication, the reverse is untrue. If the primal is infeasible, then we know for sure that the dual cannot admit an optimal solution: If this were not the case, we would obtain an optimal solution for the primal using Theorem 1 and the fact that the dual of the dual is the primal. However, both options are possible for the dual: it can be either infeasible or unbounded.

Exercise 2. Consider the following LP:

$$\begin{aligned} \max \quad & 2x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 1 \end{aligned}$$

$$
\begin{aligned}
&-x_1 + x_2 \leq -2 \\
&x_1,\ x_2 \geq 0.
\end{aligned}
$$

Does this LP admit an optimal solution? If not, is it infeasible or unbounded? What about its dual?

Recap on optimality conditions. Through the duality theorem and the considerations above, we can state the following properties on the optimality of primal-dual pairs, which can serve as a quick reference on the topic:

- (**Property P1**) If the primal admits an optimal solution $(x_1^*, \ldots, x_n^*)$, then the dual also admits an optimal solution $(y_1^*, \ldots, y_m^*)$, and
$$\sum_{i=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*. \tag{21}$$
- (**Property P2**) If the dual admits an optimal solution $(y_1^*, \ldots, y_m^*)$, then the primal also admits an optimal solution $(x_1^*, \ldots, x_n^*)$, and
$$\sum_{i=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*. \tag{21}$$
- (**Property P3**) If both primal and dual have feasible solutions, respectively $(x_1^*, \ldots, x_n^*)$ and $(y_1^*, \ldots, y_m^*)$, and
$$\sum_{i=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* \tag{21}$$
holds, then these solutions are optimal for, respectively, the primal and the dual.

4 Writing down the dual: practical aspects

We have, until now, considered LPs in standard form. Although any LP can be converted into an equivalent LP in standard form, it would be convenient to write down the dual of a generic LP without explicitly performing the conversion.

We start by writing down a generic form for linear programmes. In this form, we consider two types of variables: $x_1, \ldots, x_n$ are non-negative variables, i.e., $x_j \geq 0\ \forall j \in \{1, \ldots, n\}$; $u_1, \ldots, u_l$ are free variables, i.e., $u_j \in \mathbb{R}\ \forall j \in \{1, \ldots, l\}$. We also consider two types of constraints: "$\leq$"-inequalities and equalities. With these assumptions, a generic LP can be written as

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l} p_j u_j && && (22)\\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j + \sum_{j=1}^{l} d_{ij} u_j \leq b_i && \forall i \in \{1, \ldots, m\} && (23)\\
& \sum_{j=1}^{n} e_{ij} x_j + \sum_{j=1}^{l} f_{ij} u_j = h_i && \forall i \in \{1, \ldots, q\} && (24)\\
& x_j \geq 0 && \forall j \in \{1, \ldots, n\} && (25)
\end{aligned}
$$

$$u_j \in \mathbb{R} \qquad \forall j \in \{1, \dots, l\}. \tag{26}$$

In the above formulation, the c_j's, p_j's, a_{ij}'s, d_{ij}'s, b_i's, e_{ij}'s, f_{ij}'s, and h_i's are real parameters. The advantage of formulation (22)–(26) is that it does not require any additional artificial variable and can express any LP using the x's and the u's.

One can model minimisation problems by flipping the sign in the objective coefficients. Similarly, "$\geq$"-inequalities require flipping the signs on their left- and right-hand sides. Finally, note that using only non-negative and free variables is not restrictive because all other variable bounds can be modelled as inequalities of type (23). For example, bound $x_j \geq 5$ could be modelled as $-x_j \leq -5$.

We consider two sets of dual variables to write the dual LP of (22)–(26). First, we introduce one non-negative variable y_i for each "$\leq$"-inequality, i.e., $y_i \geq 0 \ \forall i \in \{1, \dots, m\}$. Next, we introduce one free variable w_i for each equality, i.e., $w_i \in \mathbb{R} \ \forall i \in \{1, \dots, q\}$. The dual then reads as follows:

$$\min \quad \sum_{i=1}^{m} b_i y_i + \sum_{i=1}^{q} h_i w_i \tag{27}$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij} y_i + \sum_{i=1}^{q} e_{ij} w_i \geq c_j \qquad \forall j \in \{1, \dots, n\} \tag{28}$$

$$\sum_{i=1}^{m} d_{ij} y_i + \sum_{i=1}^{q} f_{ij} w_i = p_j \qquad \forall j \in \{1, \dots, l\} \tag{29}$$

$$y_i \geq 0 \qquad \forall i \in \{1, \dots, m\} \tag{30}$$

$$w_i \in \mathbb{R} \qquad \forall i \in \{1, \dots, q\}. \tag{31}$$

As we can see, each non-negative primal variable corresponds to a "$\geq$"-inequality in the dual, and each free primal variable corresponds to an equality in the dual.

Exercise 3. Write down the dual of the following linear programme:

$$\max \quad 42x_2 - 30x_3 \tag{32}$$

$$\text{subject to} \quad x_1 - x_2 + x_3 - x_4 = 0 \tag{33}$$

$$x_1 + x_3 - x_4 \leq 5 \tag{34}$$

$$5x_2 + x_3 - 5x_4 = -1 \tag{35}$$

$$x_1 \geq 0 \tag{36}$$

$$x_3 \in [0, 20] \tag{37}$$

$$x_2, x_4 \text{ free.} \tag{38}$$