

Logical Constraints

Given the optimization problem:

$$\text{Max} \quad z = f(x_1, x_2, \dots, x_n)$$

$$\text{s.t.} \quad g_1(x_1, x_2, \dots, x_n) \geq 0 \quad (1)$$

$$g_2(x_1, x_2, \dots, x_n) \geq 0 \quad (2)$$

...

Only a given “combination” of the constraints must be imposed.

Only One Constraint

Only one of the two constraints (1) and (2) must be imposed.

- Let L_1 and L_2 be the “inferior limits” of functions g_1 and g_2 , respectively (f. i., $-M$, with M very large positive number).
- * Let t_1 and t_2 be two *binary* variables such that:
 $t_k = 0$ if constraint (k) is imposed, $t_k = 1$ otherwise ($k = 1, 2$);
- * Replace constraints (1) and (2) with:
$$g_1(x_1, x_2, \dots, x_n) \geq t_1 L_1 \quad (1a)$$
$$g_2(x_1, x_2, \dots, x_n) \geq t_2 L_2 \quad (2a)$$

and impose the additional constraint:

$$t_1 + t_2 = 1, \text{ with } t_1, t_2 \text{ binary variables } (t_1, t_2 \in \{0,1\})$$

Not Both Constraints

Or only (1), or only (2), or none of the two constraints must be imposed.

* Replace constraints (1) and (2) with:

$$g_1(x_1, x_2, \dots, x_n) \geq t_1 L_1 \quad (1a)$$

$$g_2(x_1, x_2, \dots, x_n) \geq t_2 L_2 \quad (2a)$$

and impose the additional constraint:

$$t_1 + t_2 \geq 1,$$

$$\text{with } t_1, t_2 \in \{0,1\}$$

Not Only One Constraint

Or both (1) and (2), or none of the two constraints must be imposed.

* Replace constraints (1) and (2) with:

$$g_1(x_1, x_2, \dots, x_n) \geq t L_1 \quad (1b)$$

$$g_2(x_1, x_2, \dots, x_n) \geq t L_2 \quad (2b)$$

with $t \in \{0,1\}$

At Least One Constraint

Or only (1), or only (2), or both constraints must be imposed.

* Replace constraints (1) and (2) with:

$$g_1(x_1, x_2, \dots, x_n) \geq t_1 L_1 \quad (1a)$$

$$g_2(x_1, x_2, \dots, x_n) \geq t_2 L_2 \quad (2a)$$

and impose the additional constraint:

$$t_1 + t_2 \geq 1,$$

$$\text{with } t_1, t_2 \in \{0,1\}$$

* Combinations of 3 or more constraints

Minimum Production Lots

- Production of n items.
 - M_j minimum “lot” of item j ($j = 1, \dots, n$) (minimum quantity to be produced).
 - x_j = quantity of item j to be produced.
 - At least a quantity M_j of item j is produced ($x_j \geq M_j$) or item j is not produced ($x_j = 0$), $j = 1, \dots, n$.
- * Let t_j be a *binary* variable such that:
- $$t_j = 0 \text{ if } (x_j = 0), \text{ and } t_j = 1 \text{ if } (x_j \geq M_j).$$
- Impose the following constraints ($j = 1, \dots, n$) :
$$x_j \geq M_j t_j ; \quad x_j \leq G t_j ; \quad t_j \in \{0, 1\}$$
with G very large positive number.

Minimum Production Lots

- Production of n items.
- M_j minimum “lot” of item j ($j = 1, \dots, n$) (minimum quantity to be produced).
- x_j = quantity of item j to be produced.
- * Let t_j be a *binary* variable such that:

$$t_j = 0 \text{ if } (x_j = 0), \text{ and } t_j = 1 \text{ if } (x_j \geq M_j).$$

- Impose the following constraints ($j = 1, \dots, n$) :

$$x_j \geq M_j t_j ; \quad x_j \leq G t_j ; \quad t_j \in \{0,1\}$$

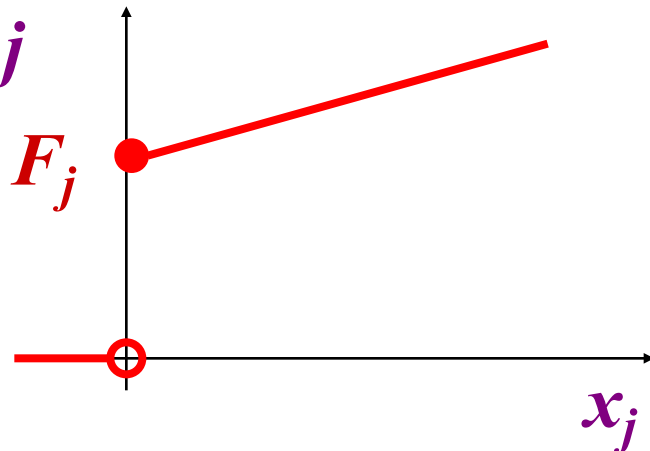
with G very large positive number.

- If $t_j = 0$: $x_j \leq 0$ and $x_j \geq 0$, hence: $x_j = 0$;
- If $t_j = 1$: $x_j \leq G$ and $x_j \geq M_j$, hence: $x_j \geq M_j$

Fixed Production Cost

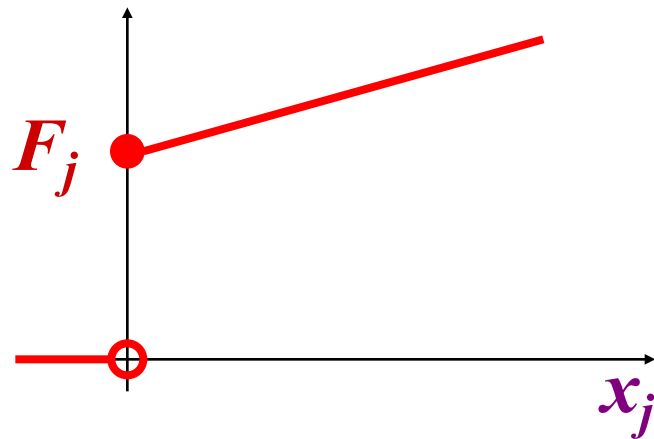
- Production Problem concerning n items.
- The cost of item j ($j = 1, \dots, n$) is given by:
 - 0 , if item j is not produced ($x_j = 0$)
 - $F_j + p_j x_j$ if item j is produced ($x_j > 0$)

Cost of
item j



- $F_j =$ fixed cost (machine)
- $p_j =$ unit production cost
- discontinuity at the origin (not algebraic function)

Fixed Production Cost (2)



$$y_j = \begin{cases} 1 & \text{if } x_j > 0 \\ 0 & \text{if } x_j = 0 \end{cases}$$

$$(j = 1, \dots, n)$$

$$\text{Min } \sum_{j=1,n} (F_j y_j + p_j x_j)$$

Logical constraints (*if ...*)

$$Ax \geq b$$

$$x \geq 0$$

$$y \in \{0,1\}$$

Variables x and y must
be connected through
linear constraints

Fixed Production Cost (3)

$$y_j = \begin{cases} 1 & \text{if } x_j > 0 \\ 0 & \text{if } x_j = 0 \end{cases} \quad \Leftrightarrow \quad x_j \leq My_j \quad (j = 1, \dots, n)$$

con $M \gg 1$ (~~x~~ + \square)

- **Satisfaction of the constraint:**

- if $x_j > 0$ y_j must be = 1 ($x_j \in M$)
- if $x_j = 0$ y_j can be = 0 or 1 ($0 \in 0$ or $0 \in M$)

or

- if $y_j = 0$ x_j must be = 0 ($x_j \in 0$) (since $x_j \geq 0$)
- if $y_j = 1$ x_j can be = 0 or > 0 ($x_j \in M$)

- **The constraint imposes only a part of the logical relation**

Fixed Production Cost (4)

$$\text{Min } \sum_{j=1,n} (F_j y_j + p_j x_j)$$

$$Ax \geq b$$

$$x_j \leq My_j$$

$$x_j \geq 0, \quad y_j \in \{0, 1\} \quad (j = 1, \dots, n)$$

MILP Model (Mixed Integer Linear Programming)

- It is not necessary to impose also the other part of the logical relation:
- a feasible solution with $x_j = 0$ and $y_j = 1$ cannot be optimal (an alternative feasible solution with a smaller cost exists: $x_j = 0, y_j = 0$)

Discrete Variables

- * Variable x must have a value among k (with $k > 1$) given different values:

$$x \in S = \{s_1, s_2, \dots, s_k\}$$

- * Let t_h ($h = 1, 2, \dots, k$) be a *binary* variable such that:

$$t_h = 1 \text{ if } (x = s_h), \text{ and } t_h = 0 \text{ if } (x \neq s_h).$$

$$x = \sum_{h=1,k} s_h t_h$$

$$\text{with } \sum_{h=1,k} t_h = 1$$

$$\text{and } t_h \in \{0, 1\} \quad (h = 1, \dots, k)$$

Discrete Variables (2)

* Example

$$x \in S = \{0.2, 0.4, \dots, 2.0\} \quad (k = 10)$$

- $t_h \in \{0, 1\}$ with $h = 1, 2, \dots, 10$

- $x = 0.2 t_1 + 0.4 t_2 + 0.6 t_3 + \dots + 2.0 t_{10}$

with

$$t_h = 1, 10 \quad t_h = 1$$

* Alternative technique:

$$x = 0.2 y \quad \text{with } y \in \{1, \dots, 10\}, y \text{ integer}$$

Transformation of an ILP model into a Binary Linear Programming (BLP) model

* x integer variable with $x \geq 0$, $x \leq k$

* Introduce $(k + 1)$ binary variables t_h , with $h = 0, \dots, k$
 ($t_h = 1$ if $x = h$, and $t_h = 0$ otherwise)

$$x = \sum_{h=0}^k h t_h$$

$$\sum_{h=0}^k t_h = 1 \quad \text{with } t_h \in \{0, 1\} \quad h = 0, \dots, k$$

Transformation of an ILP model into a Binary Linear Programming (BLP) model

- *Example:* x integer variable with $0 \leq x \leq 27$
- $z = \log_2 (27 + 1)$, $q = \lceil z \rceil = 5$
- $x = t_1 + 2 t_2 + 4 t_3 + 8 t_4 + 16 t_5$
- We must impose the constraint:

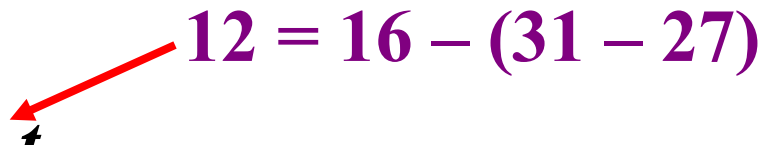
$$t_1 + 2 t_2 + 4 t_3 + 8 t_4 + 16 t_5 \leq 27 \quad (**)$$

(for $t_1 = t_2 = t_3 = t_4 = t_5 = 1$, we have: $x = 31$)

Transformation of an ILP model into a Binary Linear Programming (BLP) model

- *Example, Alternative Transformation:*

- x integer variable with $0 \leq x \leq 27$, $q = 5$

- $x = t_1 + 2 t_2 + 4 t_3 + 8 t_4 + 12 t_5$  $12 = 16 - (31 - 27)$

- We have not to impose the constraint:

$$t_1 + 2 t_2 + 4 t_3 + 8 t_4 + 12 t_5 \leq 27 \quad (**)$$

(for $t_1 = t_2 = t_3 = t_4 = t_5 = 1$, we have: $x = 27$)

Transformation of an ILP model into a Binary Linear Programming (BLP) model

- x integer variable with $x \geq b$, $x \leq k$ with $b \neq 0$

introduce q binary variables t_h , with $h = 1, \dots, q$

$$x = \sum_{h=1, q} 2^{h-1} t_h + b, \quad q = \lceil z \rceil, \quad z = \log_2 (k - b + 1)$$

$$\sum_{h=1, q} 2^{h-1} t_h \leq k - b, \quad t_h \in \{0, 1\} \quad h = 1, \dots, q \quad (**)$$

Example: $5 \leq x \leq 8$, $q = 2$, $x = 5 + t_1 + 2 t_2$

Transformation of an ILP model into a Binary Linear Programming (BLP) model

- x integer variable with $x \geq 0$, $x \leq k$

Alternative technique (binary expression of an integer):

introduce q binary variables t_h , with $h = 1, \dots, q$

$$x = \sum_{h=1, q} 2^{h-1} t_h = t_1 + 2 t_2 + 4 t_3 + \dots + 2^{q-1} t_q$$

$$\sum_{h=1, q} 2^{h-1} t_h \leq k \quad t_h \in \{0, 1\} \quad h = 1, \dots, q \quad (**)$$

$$q = \lceil \log_2 (k + 1) \rceil \quad \text{with} \quad z = \log_2 (k + 1)$$