OPTIMIZATION ALGORITHMS M

Part 2 - Exercises

Exercise 1

Given n "items" and a "container", a "weight" p_j and a "cost" c_j (with p_j and c_j positive integers) are associated with each item j (j = 1, …, n).

Determine a subset M of the n items so that:

a) the sum of the weights of the items in M is not smaller than a given value a;

b) the cardinality of M is not smaller than a given value b;

c) the sum of the costs of the items in M is minimum.

1) Determine "good" Lagrangian Lower Bounds which can be computed through procedures having time complexity O(n log(n)), and describe the corresponding subgradient optimization procedures.

2) Determine a "good" Surrogate Lower Bound which can be computed through a procedure having time complexity O(n), and describe the corresponding subgradient optimization procedure.

EXERCISE 1

1.2) Possible mathematical model (BLP)

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases} \qquad j = 1, \dots, n$$

$$\min \quad z = \sum_{j=1}^{n} c_j x_j$$

s.t.

$$\sum_{j=1}^{n} p_j x_j \geq a \qquad (a)$$

$$\sum_{j=1}^{n} x_j \geq b \qquad (b)$$

$$x_j \in \{0, 1\} \qquad j = 1, \dots, n$$

1.1) Size of P: $n, a, b, (c_j), (p_j) \Rightarrow O(n)$: n

- $P \in NP$ (decision tree with n levels, 2 descendent nodes)
- KP01-min $\propto P$ (KP01-min: $\bar{n}, \bar{b}, (\bar{P}_j), (\bar{W}_j)$ size $\bar{n}$) $O(\bar{n})$

$n := \bar{n}$; $a := \bar{b}$; $c_j := \bar{P}_j$, $p_j := \bar{W}_j$ $j = 1, \dots, \bar{n}$
$\underline{b := 0}$
$\Rightarrow P \in NP\text{-Hard}$

1.3.1) $F\text{-}P \in \mathcal{P}$ (set $x_j := 1$ for $j = 1, \dots, n$; check if (a) and (b) are satisfied) $\underline{O(n)}$

1.3.2) $F\text{-}P \in \mathcal{P}$ (1. sort the n items according to non-increasing values of p_j;
2. set $x_j := 1$ for $j = 1, \dots, b$; $x_j := 0$ for $j = b+1, \dots, n$
3. check if (a) is satisfied)
$\underline{O(n \log n)}$

1.3.3) $F\text{-}P \in \mathcal{P}$: as done for 1.3.2).

1.3.4) $F\text{-}P \in NP$ (...)

$PP \propto F\text{-}P$ (... ; $b := 0$) $\Rightarrow F\text{-}P \in NP\text{-Hard}$

Exercise 3

Given a "depot" which must serve m "customers". The customers can be served by using n different "routes". In particular, each customer i ($i = 1, \ldots, m$) can be served by a subset V_i of routes (with V_i contained in the set $\{1, 2, \ldots, n\}$). Each route j ($j = 1, \ldots, n$) has a "cost" c_j and a "traveling time" t_j (with c_j e t_j non-negative). Determine a subset S of the n routes such that:

a) each customer is served by at least one route of S;

b) the sum of the traveling times of the routes of S is not smaller than a given value d;

c) the sum of the costs of the routes of S is minimum.

1) Determine "good" Lagrangian Lower Bounds which can be computed through procedures having time complexity $O(r + n)$, with $r = | V_1 | + | V_2 | + \ldots + | V_m |$, and describe the corresponding subgradient optimization procedures.

2) Determine a "good" Surrogate Lower Bound which can be computed through a procedure having time complexity $O(r + n)$, and describe the corresponding subgradient optimization procedure.

EXERCISE 3

3.1)

$$x_j = \begin{cases} 1 & \text{route } j \text{ is selected (i.e. if } j \in S) \\ 0 & \text{otherwise} \end{cases} \qquad j = 1, \dots, n$$

$$\min z = \sum_{j=1}^{n} c_j x_j \qquad (c)$$

s.t.

$$\sum_{j \in V_i} x_j \geq 1 \qquad i = 1, \dots, m \quad (a)$$

$$\sum_{j=1}^{n} t_j x_j \geq d \qquad (b)$$

$$x_j \in \{0, 1\} \qquad j = 1, \dots, n \quad (d)$$

Constraints (a) can also be written as:

$$\sum_{j=1}^{n} a_{ij} x_j \geq 1 \qquad i = 1, \dots, m \quad (a')$$

where $a_{ij} = \begin{cases} 1 & \text{if route } j \in V_i \\ 0 & \text{otherwise} \end{cases} \qquad \begin{matrix} j = 1, \dots, n \\ i = 1, \dots, m \end{matrix}$

3.2) Size of P: $m, n, (c_j), d, (t_j), (V_i) \Rightarrow O(mn)$: mn

- $P \in NP$: binary decision tree with n levels
- KP01-Min $\propto P$: size of KP01-Min: $\bar{n}, \bar{b}, (\bar{c}_j), (\bar{w}_j) \Rightarrow \bar{n}$

$O(\bar{n})\{$ $n := \bar{n}$; $d := \bar{b}$; $(c_j) := (\bar{c}_j)$; $(t_j) := (\bar{w}_j)$; $m := 1$; $V_1 := \{1, \dots, n\}$.

The optimal solution of P is also optimal for KP01-Min.

3.3) F-P $\in P$: set $x_j = 1$ for $j = 1, \dots, n$ and check (a) and (b).

3.4) • F-P2 $\in NP$: binary decision tree with n levels

$O(\bar{n})\{$ • PP $\propto$ F-P2 : ... ; $m := 1$, $V_1 := \{1, \dots, \bar{n}\}$;

PP is feasible if and only if F-P has a solution

$\Rightarrow$ F-P2 is NP-Hard

Exercise 4

Given m "items" and n "vehicles": a positive "weight" p_j is associated with each item j ($j = 1, \ , m$); a positive "capacity" a_i is associated with each vehicle i ($i = 1, \ldots, n$). Also assume: $m > n > 0$.

Determine the items to be loaded into the vehicles so that:

a) the sum of the weights of the items loaded into each vehicle i is not greater than the capacity a_i ;

b) each item j is loaded into no more than one vehicle;

c) the global number of items loaded into the vehicles is smaller than a given value k;

d) the sum of the weights of the items loaded into the vehicles is maximum.

1) Consider first the mathematical model corresponding to the surrogate relaxation of the constraints associated with point a) with surrogate multipliers all equal to 1. Then, starting from this surrogate relaxation, determine a "good" Lagrangian Upper Bound which can be computed through a procedure having time complexity $O(n + m)$, and describe the corresponding subgradient optimization procedure.

2) Determine a "good" Lagrangian Upper Bound which can be computed through a procedure having time complexity $O(m * n)$, and describe the corresponding subgradient optimization procedure. Assume $\log(m) <= n$.

Exercise 4

4.1)

$$x_{i,j} = \begin{cases} 1 & \text{if item } j \text{ loaded into vehicle } i \\ 0 & \text{otherwise} \end{cases} \qquad i=1,\dots,n;\ j=1,\dots,m$$

$$\max\ z = \sum_{i=1}^{n} \sum_{j=1}^{m} p_j x_{i,j} \qquad (d)$$

s.t.

$$\sum_{j=1}^{m} p_j x_{i,j} \le a_i \qquad i=1,\dots,n \qquad (a)$$

$$\sum_{i=1}^{n} x_{i,j} \le 1 \qquad j=1,\dots,m \qquad (b)$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} x_{i,j} < K \qquad (c)$$

$$x_{i,j} \in \{0,1\} \qquad i=1,\dots,n;\ j=1,\dots,m$$

4.2) Size: $n, m, (p_j), (a_i), K \Rightarrow O(m+n) : m$

$*\ P \in NP$: decision tree with m levels (one for each item) and $(n+1)$ descendent nodes

$*\ SSP \propto P$: size of SSP: $\bar{n}, (w_j), \bar{C} \Rightarrow \bar{n}$

$O(\bar{n}) \begin{cases} n := 1;\ m := \bar{n};\ p_j := \bar{w}_j\ (j=1,\dots,m);\ a_1 := \bar{C}; \\ K := m+1 \end{cases}$

The optimal solution of P is also optimal for SSP.

4.3.1) $F\text{-}P \in P$: no items are loaded (always feasible): $S=\emptyset$.

4.3.2) $*\ F\text{-}P2 \in NP$: decision tree with m levels and $(n+1)$ descendent nodes.

$*\ PP2 \propto F\text{-}P2$ (PP2: Partition Problem with $\bar{C} = \sum_{j=1}^{\bar{n}} \bar{p}_j / 2$.)

Size of PP2: $\bar{n}, (\bar{p}_j), (\bar{C}) \Rightarrow O(\bar{n}) : \bar{n}$

$O(\bar{n}) \{\ m := \bar{n},\ K := m,\ n = 2,\ a_1 = a_2 = \bar{C},\ p_j := \bar{p}_j\ (j=1,\dots,m)$

PP2 is feasible if and only if F-P2 has a solution.

$\Rightarrow F\text{-}P2 \in NP\text{-Hard}$

4.3.3) As for 4.3.2

Exercise 5

Given a "directed graph" $G = (V, A)$, with $|V| = n$ and $|A| = m$. A positive "cost" $c_{i,j}$ is associated with each arc (i, j) in A. Assume also that the vertex set V is partitioned into K subsets ("regions") $R_1, R_2, \ldots, R_K$, with $R_1 = \{1\}$.

Determine an "elementary circuit" of G (i.e., a circuit passing at most once through each vertex of G) visiting at least one vertex of each of the K regions, and such that the sum of the costs of the arcs of the circuit is minimum.

1) Determine "good" Lagrangian Lower Bounds which can be computed through procedures having time complexity $O(n * n)$ (some constraints could be eliminated), and describe the corresponding subgradient optimization procedures.

2) As at point 1) in the case where it is imposed that the elementary circuit visits exactly one vertex of each of the K regions.

EXERCISE 5

5.1) Size of P: $n, m, (c_{ij}), K, (R_h) \Rightarrow O(m)$; $m \ (\leq n^2)$

* $P \in NP$: decision tree with $(n-1)$ level (one for each successor vertex in the circuit) with at most $(n-1)$ descendent nodes (at the first level).

* $ATSP \propto P$

• Size of ATSP: $\bar{n}, (\bar{c}_{ij}) \Rightarrow O(\bar{n}^2)$: $\bar{n}^2$

$$O(\bar{n}^2)\begin{cases} n := \bar{n};\ c_{ij} := \bar{c}_{ij} \text{ for } i=1,\dots,\bar{n} \text{ and } j=1,\dots,\bar{n};\ m = \bar{n}^2; \\ K := n;\ R_h := \{h\} \text{ for } h=1,\dots,n. \end{cases}$$

• The optimal solution of P is also optimal for ATSP.

5.2) Transform graph G into a complete graph:

for $i=1,\dots,n$ and $j=1,\dots,n$: if $(i,j) \notin A$ then $c_{ij} := \infty$

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ in the optimal circuit} \\ 0 & \text{otherwise} \qquad i=1,\dots,n;\ j=1,\dots,n \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if vertex } i \text{ in the optimal circuit} \\ 0 & \text{otherwise} \qquad i=1,\dots,n \end{cases}$$

$$\min \ z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (a)$$

s.t.

$$\sum_{j=1}^{n} x_{ij} = y_i \qquad i=1,\dots,n \qquad (b)$$

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ji} \qquad i=1,\dots,n \qquad (c)$$

$$\sum_{i \in R_h} y_i \geq 1 \qquad h=1,\dots,K \qquad (d)$$

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S|-1 \qquad \forall S \subseteq V-\{1\},\ S \neq \emptyset$$

$$x_{ij} \in \{0,1\}\ i=1,\dots,n;\ j=1,\dots,n;\ y_i \in \{0,1\}\ i=1,\dots,n$$

5.3) Replace (d) with $\sum_{i \in R_h} y_i = 1 \quad h=1,\dots,K \qquad (d')$

Exercise 8

Given a "complete directed graph" $G = (V, A)$, with $|V| = n$: a "weight" $p_{i,j}$ and a non-negative "time" $t_{i,j}$ are associated with each arc (i, j) of A. Two disjoint subsets S and T are also given (with S and T contained in A).

Determine a "Hamiltonian circuit" H of G so that:

a) the sum of the weights of the arcs of H is maximum;

b) the sum of the times of the arcs of H is not greater than a given value d;

c) the number of arcs of H belonging to subset S is not smaller than the number of arcs of H belonging to subset T.

1) Determine a "good" Upper Bound obtained through a Lagrangian relaxation of the constraints b) and c), and which can be computed through a procedure having time complexity $O(n * n * n)$.

2) Describe the subgradient optimization procedure corresponding to the Upper Bound defined at point 1) and having time complexity $O(n * n * n)$.

3) Determine an additional "good" Lagrangian Upper Bound obtained through a Lagrangian relaxation of the constraints b) and c) (and possibly of other constraints), and which can be computed through a procedure having time complexity $O(n * n)$.

4) Describe the subgradient optimization procedure corresponding to the Upper Bound defined at point 3) and having time complexity $O(n * n * n)$.

EXERCISE 8

8.1)

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ belongs to } H \\ 0 & \text{otherwise} \end{cases} \qquad i=1,\dots,n;\ j=1,\dots,n$$

$$Z = \max \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_{ij} \qquad (1)$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i=1,\dots,n \qquad (2)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j=1,\dots,n \qquad (3)$$

$$\sum_{i\in R} \sum_{j\in R} x_{ij} \le |R|-1 \quad \forall R \subset V : |R| \ge 2 \qquad (4)$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} x_{ij} \le d \qquad (5)$$

$$\sum_{(i,j)\in S} x_{ij} \ge \sum_{(i,j)\in T} x_{ij} \qquad (6)$$

$$x_{ij} \in \{0, 1\} \qquad i=1,\dots,n;\ j=1,\dots,n \qquad (7)$$

8.2)
- Size of P: n, (p_{ij}), (t_{ij}), d, S, T: $2+4n^2 \Rightarrow n^2$
- $P \in NP$: decision tree with $(n-1)$ levels (one for each arc of the circuit) and at most $(n-1)$ descendent nodes.
- ATSP-Max $\propto P$: size of ATSP-Max: $\bar{n}$, $(\bar{p}_{ij})$: $\bar{n}^2$

$$O(\bar{n}^2) \begin{cases} n := \bar{n};\ p_{ij} := \bar{p}_{ij} \quad i=1,\dots,n;\ j=1,\dots,n \\ t_{ij} := 0 \quad i=1,\dots,n;\ j=1,\dots,n;\ d := 0; \\ S := \emptyset,\ T := \emptyset \end{cases}$$

The optimal solution of P is optimal also for ATSP-Max

Exercise 9

Given n "depots" and m "customers": each customer i ($i = 1, \ldots, m$) has a non-negative "potential profit" p_i. Each depot j ($j = 1, \ldots, n$) has a non-negative "cost" c_j and is able to "serve" a subset of the m customers. In particular, a binary matrix ($a_{i,j}$) is given, such that for each pair [depot j , customer i] (with $j = 1, \ldots, n$ and $i = 1, \ldots, m$) $a_{i,j} = 1$ if depot j is able to serve customer i, and $a_{i,j} = 0$ otherwise.

For each subset S of the n depots, the corresponding "global profit" is given by the difference: (sum of the profits of the customers which can be served by the depots of S) - (sum of the costs of the depots of S).

Determine a subset S* of the n depots so that:

a) S* contains at most d depots (with d given value greater than 0 and smaller or equal to n);
b) the global profit of S* is maximum;
c) the total cost of the depots of S* is not smaller than a given non-negative value b.

Let h denote the number of elements of the matrix ($a_{i,j}$) having value equal to 1 (with $h >= n$, $h >= m$).

Determine "good" Upper Bounds based on the following relaxations:

1) Three different Lagrangian relaxations which can be computed through procedures having time complexity O(h).

2) A Surrogate relaxation for the particular case in which all the customers must be served, and which can be computed through a procedure having time complexity O(h).

3) For at least two of the relaxations considered at point 1), describe the corresponding subgradient optimization procedures.

EXERCISE 9

9.1)

$$x_j = \begin{cases} 1 & \text{if depot } j \text{ is used (i.e. } j \in S^*) \\ 0 & \text{otherwise} \quad j = 1, \dots, n \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if customer } i \text{ is served by at least one depot in } S^* \\ 0 & \text{otherwise} \quad i = 1, \dots, m \end{cases}$$

$$z = \max \sum_{i=1}^{m} p_i y_i - \sum_{j=1}^{n} c_j x_j \qquad (1)$$

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j \geq y_i \qquad i = 1, \dots, m \qquad (2)$$

$$\sum_{j=1}^{n} x_j \leq d \qquad (3)$$

$$\sum_{j=1}^{n} c_j x_j \geq b \qquad (4)$$

$$x_j \in \{0, 1\} \qquad j = 1, \dots, n \qquad (5)$$

$$y_i \in \{0, 1\} \qquad i = 1, \dots, m \qquad (6)$$

9.3) $F\text{-}P \in P$

$O(n)$ • in constraint (2) set $y_i := 0$: (2) always satisfied;

• sort the n depots according to non increasing values

$O(n \log n)$ of the costs (c_j);

$O(n)$ • $x_j := 1$ for $j = 1, \dots, d$; $x_j := 0$ for $j = d+1, \dots, n$

$O(n)$ • If $\sum_{j=1}^{n} c_j x_j \geq b$ then a feasible solution (x_j) exists

else no feasible solution exists for F-P.

9.4.1) As for question 9.3).