Linear Transformations of Random Variables

Introduction

If X is a random variable and if a and b are any constants, then $a + bX$ is a **linear transformation** of X. It scales X by b and shifts it by a.

A linear transformation of X is <u>another random variable</u>; we often denote it by Z

$$Z = a + bX$$

Suppose to know both the mean value $\mathrm{E}[X]$ and the variance σ_X^2 of the r.v. X. What are the corresponding mean value and variance of the r.v. Z?

Expected Value of a Linearly Transformed Random Variable

Given a random variable X such that

$$\mathrm{E}[X] = \mu_X$$

and considering the linear transformation

$$Z = a + b \cdot X \quad a\,,\, b \in \mathbb{R}$$

then

$$\mathrm{E}[Z] = \mathrm{E}[a + b \cdot X] = a + b \cdot \mathrm{E}[X] \quad \Longrightarrow \quad \mathrm{E}[Z] = a + b \cdot \mu_X$$

Thus, the expected value of a linear transformation of a r.v. X is just the linear transformation of the expected value of X.

Variance of a Linearly Transformed Random Variable

Applying the definition of variance for the r.v. Z we get

$$\mathrm{var}[Z] = \mathrm{E}\left[(z - \mu_Z)^2\right] = \mathrm{E}\left[(a + bx - a - b\mu_X)^2\right] = \mathrm{E}\left[b^2(x - \mu_X)^2\right] \quad \Longrightarrow \quad \sigma_Z^2 = b^2 \cdot \sigma_X^2$$

Remark: the variance of $a + bX$ does not depend on a. This is appropriate: the variance is a measure of spread; adding a does not change the spread, it merely shifts the distribution to the left or to the right. It is b th responsible of changing the spread.

Applying Linear Transformation to Random Variables in MATLAB

A linear transformation of a random variable is the simplest method to obtain a r.v. with desired expected value and variance, given a r.v. with zero as mean value and one as the variance.

```
clear
close all
clc
```

An Uniform Random Variable

For example, a r.v. with (-5) as the mean value, the value +10 as the variance and distributed according the **uniform distribution** can be obtained in MATLAB using the command

```
muZ = -5; % the desired mean value
sigmaZ = +10; % the desired variance
N = 10000; % how many samples?
% the linear transformation
Z = muZ+sqrt(12*sigmaZ)*(rand(N,1)-0.5); % <-- the data
```

Please, verify experimentally that the expected value and variance of the sequence of random variables are as expected.

```
% write a piece of code for estimating the average value and the sample
% variance of the sequence
```

A Gaussian Radom Variable

Suppose now to generate a **gaussian r.v**. with (-5) as the mean value, and the value +10 as the variance. The MATLAB code is

```
muY = -5; % the desired mean value
sigmaY = +10; % the desired variance
N = 10000; % how many samples?
% the linear transformation
Y = muY+sqrt(sigmaY)*randn(N,1); % <-- the data
```

Please, verify experimentally that the expected value and variance of the sequence of random variables are as expected.

```
% write a piece of code for estimating the average value and the sample
% variance of the sequence
```