Natural reserve Control

Francesco Brand

September 2021

Inspired by the original problem that led Vito Volterra to the discovery of its equations, we here propose a feedback controlled system which regulates the number of individuals in a population of two different species, one being predated by the other.

1 Introduction

To give a bit of context to the system here proposed, let's suppose to be the supervisor of a natural reserve, in which there are two different species of fish, x and y, x being predated by y. Since the natural reserve is home of some biological studies, we are keen on having a constant population of y. Particularly, we want to ensure that, no matter how the external conditions affect the system, the population of y should not change too much and must not face the risk of extinction. In order to accomplish this, we will decide (automatically) what is the correct amount of nutrients for the prey species that needs to be injected into the water at any unit of time.

To achieve this, using MatLab and Simulink, we have simulated the dynamics of the population using a basic Lotka-Volterra equation and we have used different tools learned throughout the course to achieve control of the system. Finally, we used Moonlight to implement the verification and the falsification of formal requirements expressed in Signal Temporal Logic (STL).

1.1 The Lotka-Volterra equation

The interaction in between the species x and y will be described by the following Lotka-Volterra equation

$$\frac{dx}{dt} = x(g - \alpha y)$$
$$\frac{dy}{dt} = y(-d + e\alpha x)$$

It is a non-linear system of two first order differential equations. All parameters are numbers greater than zero; their meanings are:

- g is the generative rate for individual of the x species. It tells us how the population of x reproduces
- α is the interaction term between the species. In this context, it represents how many preys are predated per predator per unit time.
- d is the spontaneous death rate of the predator fish.
- e is the efficiency for each captured prey of the predator

The equation doesn't have a closed form solution, but has been extensively studied in the context of dynamical systems. In this form, it has a unique rest point at

$$(\bar{x}, \bar{y}) = (\frac{d}{e\alpha}, \frac{g}{\alpha})$$

but all the 2-d orbits around the rest point are periodic.

In the actual system, we have introduced a controllable parameter, $\mathbf{u}$, which will stand for the amount of nutrients added to the system, two time dependent variables, a and b, which represent the effect of the environment on the prey reproductive rate and on the predators death rate respectively. Lastly we added a Gaussian noise $\varepsilon \sim \mathcal{N}(0, \Sigma^2)$ modifying the equations to

$$\frac{dx}{dt} = x(g - \alpha y + a + u) + \varepsilon_x$$
$$\frac{dy}{dt} = y(-d + e\alpha x + b) + \varepsilon_y$$

Notice that also the equilibrium point shifts to

$$(\bar{x}, \bar{y}) = (\frac{d-b}{e\alpha}, \frac{g+u+a}{\alpha})$$

Our goal is to try to stick as close as possible to $y = \frac{g}{\alpha}$, letting x vary freely, even in the case the external conditions vary. Although unrealistically, we will suppose to be able to track perfectly the change in external conditions.

2 The system

The code to follow along is the simulink file *natural_reserve.slx*. The structure of the system is the typical one of a feedback-controlled system. The plant is coded as a MATLAB Function block which takes in input the

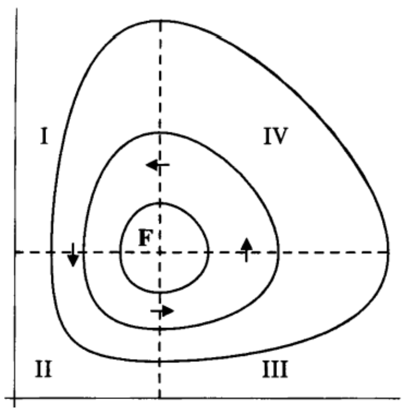

Figure 1: Lotka-Volterra equation orbits in the phase space

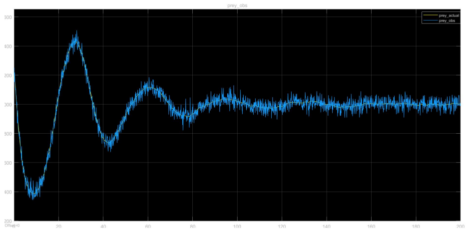

Figure 2: Preys population during a simulation

control signal and other environmental signals. Inside, it integrates the Lotka-Volterra equations given the current population. The output is given to the *sensor* subsystem that simulates the stochasticity of observations, by introducing a gaussian noise to the actual value. In our real world scenario, we assume to have installed a set of cameras/sensors, which are able to distinguish and count the number of preys and predators in a certain area of the reserve; hence allowing us to have an unbiased estimate of the population sizes effectively inhabiting the whole area.

To handle the stochasticity of both the sensor and the plant, we applied a Kalman filter, which then feeds its estimate of the population sizes to a time varying LQR controller. Since we are dealing with differential equations, and both the (Extended) Kalman Filter and the LQR controller are in practice digital devices - in our scenario, we can suppose to have a digitally controlled reservoir of prey nutrient ready to be released into the water - we need first to decide a sampling time (we have set for 0.1 units of time) which sets the amount of time that passes between two evaluations of the devices. Moreover, since we are dealing with *non-linear* equations, we need to linearize the plant around our reference trajectory everytime we make our state estimation and control phase (the reference input and control signal may vary over time).

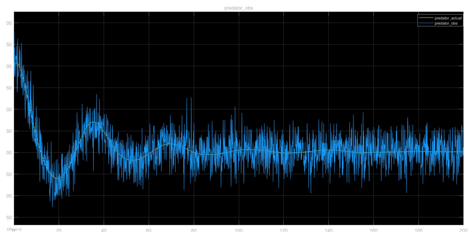

Figure 3: Predator population during a simulation

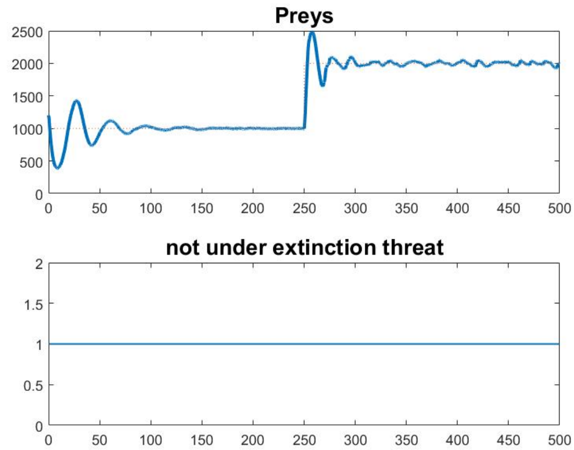

Figure 4: Prey population formal verification

3 Results

3.1 Formal Verification

parameter	value
g	0.3
α	0.001
e	0.1
d	0.1

Table 1: The base parameters used for the simulations

First of all, we have run a prototypical simulation to validate our control system. The simulation was run by considering an initial population of 1200 preys and 500 predators. We have set the parameters to their natural value and kept them fixed throughout the simulation (we haven't gathered any real world data for this experiment, so the decided natural values are based on common sense alone).

Therefore, the parameters known to the controller match precisely in every moment the actual parameters of the plant. The original equilibrium point for this system is therefore found to be at $(1000, 300)$. We have then

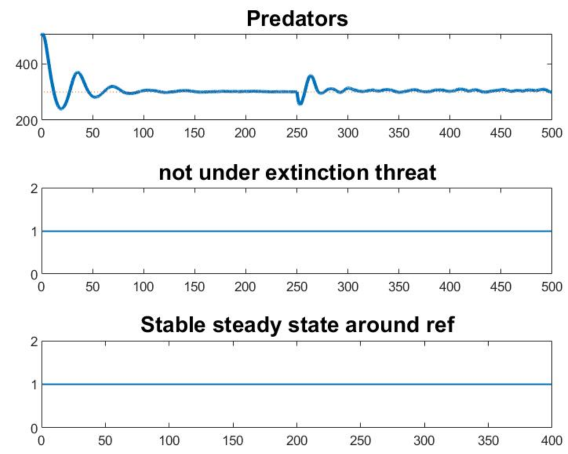

Figure 5: Predator population formal verification

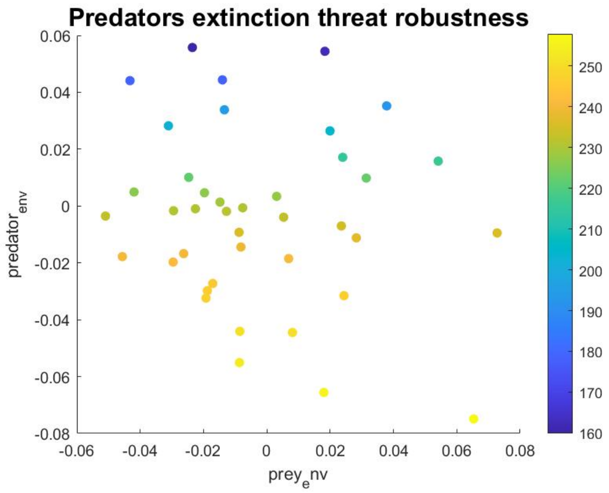

Figure 6: Falsification of extinction requirement

applied a step input signal to the a variable, which is the variation in the reproductive rate of the preys. We must ensure that neither of the population gets low enough as to risk extinction if a bad fluctuation in the reproductive rate happens.

This translates to ask that both of the populations should always stay above a certain minimum threshold. Moreover, every time there is a shift in the environment rates, we would like to have a stable predator population after a brief settling time. These requests are formalized as:

$$\text{globally x } > \varepsilon_{extinction}$$
$$\text{globally y } > \varepsilon_{extinction}$$
$$\text{globally}(step \rightarrow \text{globally}[ST](|y - y_{ref}| < t))$$

The check was carried out by assuming, $\varepsilon = 10$, $ST = 70$, $t = 50$, $y_{ref} = 300$.

As can be seen, the predator population equilibrium does not change after the introduction of the environmental bias (figure 5); in fact, the signal bumps a little bit and settles. On the other hand, in figure 4, preys population can be seen to reach a new equilibrium point, again after a brief adjustment period.

3.2 Falsification

In a similar fashion, we seek to check the validity limits of our model by varying the effects of the environment both on preys and predators in a more systematic way. To get a quantitative measure of how well does our model work under the external inputs, we measure the **robustness** function of a given formal requirement, which returns us how well the model satisfies that requirement given a certain set of input. The formal requirements will be the same already specified in section 3.1.

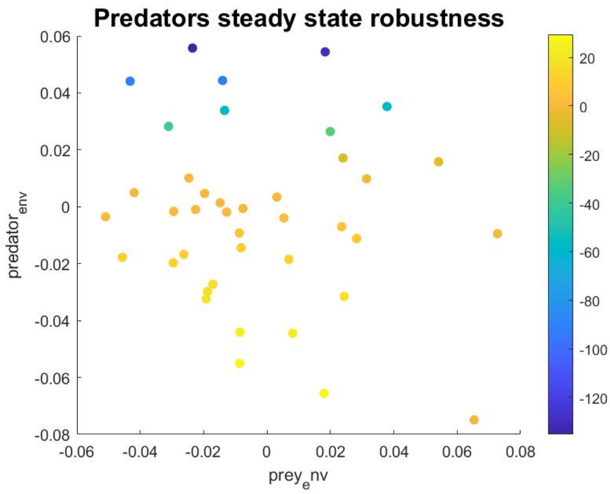

Figure 7: Falsification of steadiness requirement

The set of input has been generated by considering a test suite of different step functions. We chose both the external effect on preys reproduction (a) and on predator reproduction (b) to be of the shape

$$prey\_env = \begin{cases} 0 & t < (\tau_0 \sim \mathcal{U}[50, 300]), \\ \sim \mathcal{N}(0, 0.03) & t \geq \tau_0 \end{cases}$$
$$predator\_env = \begin{cases} 0 & t < (\tau_1 \sim \mathcal{U}[50, 300]), \\ \sim \mathcal{N}(0, 0.03) & t \geq \tau_1 \end{cases}$$

In figure 6 we see the value of the robustness (given by the colour of each dot), for a set of points in the space given by the environmental effect on the predators, on the y-axis, and on the preys on the x axis. We can clearly notice how the predators population is more likely to be under extinction threat on the top part of the scatter plot, where the environmental conditions are such that the life of predators is actually longer! This is due to the fact that having more predators with respect to our natural equilibrium triggers an external periodic orbit of the kind saw in figure 1, which means that cyclically the population will get dangerously low. This can be clearly seen in figure 7, which shows that the steadiness property is clearly falsified for a high positive impact of the environment on predators reproduction.

In order to also falsify the extinction requirement, we ran an optimal falsification routine (*optimal_falsification.m*) to sample the parameter space in a more efficient way. Allowing for the generation of random step functions, with the bottom value set to 0 and varying the step time, the property was falsified in a limited amount of iteration (8) and the argmin values for the free variables are

parameter	value
time_step_prey	96.13
prey_val	-1
time_step_predator	90.3143
predator_val	-1

Table 2: Argmin values for the extinction robustness

This means that, in order to risk the extinction of predators, environmental conditions has to be extremely difficult to survive (-1 means that at each unit of time the whole population dies, so it needs to reproduce faster than that to survive). Notice also how the optimization process allowed us to explore the parameter space further away from the region originally considered, thus exploring values that seemed to not falsify our requirements at a first glance.

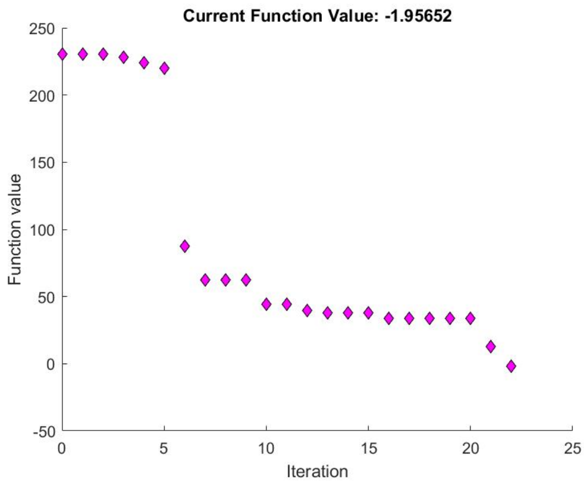

Figure 8: Falsification routine results

4 Conclusion

In this short report, we showed how to build a feedback-controlled system that allows to simulate the control of a population of 2 species, using the tools that we have learned during the lectures.