Data-driven and Learning-based Control

Linear Quadratic Regulator

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Optimal control

1 A brief recap

- Formal definition of an optimal control problem in terms of
  - performance index (e.g., cost or reward)
  - physical constraints
- Observability and reachability properties of a dynamical system
  - special case of linear dynamical systems
- Towards the solution of an optimal control problem
  - Bellman's principle of optimality
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2 Linear quadratic regulator

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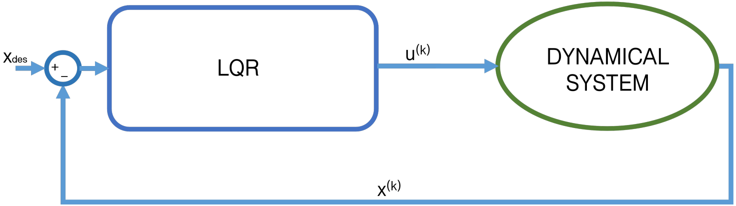
Linear Quadratic Regulator

The linear quadratic regulator (LQR) is the simplest optimal control problem we can tackle.

Due to its simplicity, it is one of the most effective and widely used methods in robotics and control system design.

It consists of identifying a control law π that minimizes a quadratic cost of a linear time-invariant system.



Infinite horizon LQR control problem

Find $\pi^*\left(x^{(k)}\right)$ solution of:

$$\arg\min_{\pi} \sum_{k=0}^{\infty} {x^{(k)}}^\top Q x^{(k)} + {u^{(k)}}^\top R u^{(k)}$$

s.t.:

$$x^{(k+1)} = Ax^{(k)} + Bu^{(k)},$$

$$x^{(0)} = x_{\text{ini}},\ x^{(\infty)} \to 0$$

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$$x^{(0)} = x_{\text{ini}},\ x^{(\infty)} \to 0$$

To use LQR correctly, several assumptions must be satisfied.

Necessary definitions

1. A matrix M is **positive definite** ($M^\top M = I,\ M > 0$) if

$$x^\top M x > 0 \ \forall x \neq 0$$

2. A matrix M is **positive semidefinite** ($M^\top M = I,\ M \geq 0$) if

$$x^\top M x \geq 0 \ \forall x$$

In practice, direct application of the definition is one of the most difficult ways to determine whether a matrix is positive definite or semidefinite

Fortunately, there are other ways to simplify this process.

Positive definite matrix

The following properties are equivalent to a matrix M being positive definite:

- M is symmetric $M^\top M = I$ and all eigenvalues of M are positive

$$\text{Solve } \det(M - \lambda I) = 0 \text{ for } \lambda \text{ and check that all the solutions } \lambda \text{ are } > 0.$$

Positive definite matrix

The following properties are equivalent to a matrix M being positive definite:

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$$\text{Solve } \det(M - \lambda I) = 0 \text{ for } \lambda \text{ and check that all the solutions } \lambda \text{ are } > 0.$$

- *Sylvester's criterion*: M is symmetric $M^\top M = I$ and all of the leading principal minors of M are positive

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & \dots & m_{1n} \\ m_{21} & m_{22} & m_{23} & \dots & \\ m_{31} & m_{32} & m_{33} & \dots & \vdots \\ \vdots & & & \ddots & \\ m_{n1} & & \dots & & m_{nn} \end{bmatrix}$$

$$\det() > 0 \quad \det() > 0 \quad \det() > 0 \quad \dots$$

Positive semidefinite matrix

The following properties are equivalent to a symmetric matrix M ($M^\top M = I$) being positive semidefinite:

- M is symmetric $M^\top M = I$ and all eigenvalues of M are non-negative.

Solve $\det(M - \lambda I) = 0$ for λ and check that all the solutions λ are ≥ 0.

Positive semidefinite matrix

The following properties are equivalent to a symmetric matrix M ($M^\top M = I$) being positive semidefinite:

- M is symmetric $M^\top M = I$ and all eigenvalues of M are non-negative.

  $$\text{Solve } \det(M - \lambda I) = 0 \text{ for } \lambda \text{ and check that all the solutions } \lambda \text{ are } \geq 0.$$

- *Sylvester's criterion*: M is symmetric $M^\top M = I$ and all of the leading principal minors of M are non-negative

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & \dots & m_{1n} \\ m_{21} & m_{22} & m_{23} & \dots & \\ m_{31} & m_{32} & m_{33} & \dots & \vdots \\ \vdots & & & \ddots & \\ m_{n1} & & \dots & & m_{nn} \end{bmatrix}$$

$\det() \geq 0 \quad \det() \geq 0 \quad \det() \geq 0 \quad \dots$

LQR assumptions

1. **The matrix Q must be positive semidefinite, i.e., $\mathbf{Q} \geq \mathbf{0}$**: this assumption ensures the state is driven to 0 because otherwise, a state error different from 0 might actually *decrease* the cost, thereby misleading the minimization

   example: Set $Q = I$:

$$x^\top Qx = x_1 * 1 * x_1 + x_2 * 1 * x_2 + x_3 * 1 * x_3 + \ldots + x_n * 1 * x_n$$

   - if we are close to $x = 0$ the cost is positive but small
   - if we are far from $x = 0$ the cost is positive but big

1. **The matrix Q must be positive semidefinite, i.e., $\mathbf{Q} \geq \mathbf{0}$**: this assumption ensures the state is driven to 0 because otherwise, a state error different from 0 might actually *decrease* the cost, thereby misleading the minimization

example: Set $Q = \begin{bmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & 1 \end{bmatrix}$:

$$x^\top Qx = x_1 * -1 * x_1 + x_2 * 1 * x_2 + x_3 * 1 * x_3 + \ldots + x_n * 1 * x_n$$

- if $x_2 * 1 * x_2 + x_3 * 1 * x_3 + \ldots + x_n * 1 * x_n = x_1 * -1 * x_1$ even if we are far from $x = 0$ we obtain a null cost

LQR assumptions

1. **The matrix $\mathbf{Q}$ must be positive semidefinite, i.e., $\mathbf{Q} \geq \mathbf{0}$**: this assumption ensures the state is driven to 0 because otherwise, a state error different from 0 might actually *decrease* the cost, thereby misleading the minimization

example: Set $Q = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & 1 \end{bmatrix}$:

$$x^\top Qx = x_1 * 0 * x_1 + x_2 * 1 * x_2 + x_3 * 1 * x_3 + \dots + x_n * 1 * x_n$$

- if x_1 is far from 0 we obtain the cost will not be affected, i.e., we are not interested in leading the x_1 component of x towards 0.

LQR assumptions

2. **The matrix $\mathbf{R}$ must be positive definite, i.e., $\mathbf{R} > \mathbf{0}$**: this ensures that there is always a penalty for applying control effort

example: Set $R = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & 1 \end{bmatrix}$:

$$u^\top R u = u_1 * 0 * u_1 + u_2 * 1 * u_2 + u_3 * 1 * u_3 + \dots + u_m * 1 * u_m$$

- if $u_1 \to \infty$ we are not penalizing the choice. Therefore we are not avoiding infinite control efforts.

LQR assumptions

3. **The state is accessible** or at least (A, C) observable
4. $(\mathbf{A}, \mathbf{B})$ **is fully reachable**

Formal definition of an infinite horizon LQR control problem

Find $\pi^* \left(x^{(k)}\right) = -Kx^{(k)}$ solution of:

$$\arg\min_{\pi} \sum_{k=0}^{\infty} {x^{(k)}}^\top Q x^{(k)} + {u^{(k)}}^\top R u^{(k)}$$

s.t.:

$$x^{(k+1)} = Ax^{(k)} + Bu^{(k)},$$

$$x^{(0)} = x_{\text{ini}},\; x^{(\infty)} \to 0$$

$$\mathbf{Q} = \mathbf{Q}^\top,\; \mathbf{Q} \geq \mathbf{0}$$

$$\mathbf{R} = \mathbf{R}^\top,\; \mathbf{R} > \mathbf{0}$$

Formal definition of an infinite horizon LQR control problem

Find $\pi^*\left(x^{(k)}\right) = -\mathbf{K}x^{(k)}$ solution of:

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s.t.:

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$$\mathbf{Q} = \mathbf{Q}^\top,\; \mathbf{Q} \geq \mathbf{0}$$

$$\mathbf{R} = \mathbf{R}^\top,\; \mathbf{R} > \mathbf{0}$$

Notice that the control policy of interest is a **constant feedback gain matrix** K

Infinite horizon LQR solution

1. Assume that the optimal value function has the form $V_\pi^* \left(x^{(k)}\right) = x^{(k)\top} P x^{(k)}$

Infinite horizon LQR solution

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2. Following the Bellman equation we know

$$V_\pi^*\left(x^{(k)}\right) = {x^{(k)}}^\top Q x^{(k)} + \pi\left(x^{(k)}\right)^\top R \pi\left(x^{(k)}\right) + V_\pi^*\left(x^{(k+1)}\right)$$

and therefore

$$ {x^{(k)}}^\top P x^{(k)} = {x^{(k)}}^\top Q x^{(k)} + \pi\left(x^{(k)}\right)^\top R \pi\left(x^{(k)}\right) + {x^{(k+1)}}^\top P x^{(k+1)}$$

1. Assume that the optimal value function has the form $V^*_\pi\left(x^{(k)}\right) = x^{(k)^\top} P x^{(k)}$
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$$V^*_\pi\left(x^{(k)}\right) = x^{(k)^\top} Q x^{(k)} + \pi\left(x^{(k)}\right)^\top R \pi\left(x^{(k)}\right) + V^*_\pi\left(x^{(k+1)}\right)$$

and therefore

$$x^{(k)^\top} P x^{(k)} = x^{(k)^\top} Q x^{(k)} + \pi\left(x^{(k)}\right)^\top R \pi\left(x^{(k)}\right) + x^{(k+1)^\top} P x^{(k+1)}$$

3_a. By substituting the dynamics in the above formula and the control policy of interest $\left(x^{(k+1)} = Ax^{(k)} + Bu^{(k)}\right.$ and $u^{(k)} = \pi^*\left(x^{(k)}\right) = -Kx^{(k)}\big)$ we obtain

$$x^{(k)^\top} P x^{(k)} = x^{(k)^\top} Q x^{(k)} + \left(Kx^{(k)}\right)^\top R K x^{(k)} + \left((A - BK)x^{(k)}\right)^\top P \left((A - BK)x^{(k)}\right)$$

Infinite horizon LQR solution

4_a. Recalling that $(AB)^\top = B^\top A^\top$ we obtain

$$x^{(k)\top} P x^{(k)} = x^{(k)\top} Q x^{(k)} + x^{(k)\top} K^\top R K x^{(k)} + x^{(k)\top} (A - BK)^\top P (A - BK) x^{(k)}$$

thus

$$x^{(k)\top} P x^{(k)} = x^{(k)\top} \left(Q + K^\top R K + (A - BK)^\top P (A - BK) \right) x^{(k)}$$

namely

$$Q + K^\top R K + (A - BK)^\top P (A - BK) - P = 0$$

Infinite horizon LQR solution

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3_b. By considering again

$$x^{(k)}{}^\top P x^{(k)} = x^{(k)}{}^\top Q x^{(k)} + u^{(k)}{}^\top R u^{(k)} + x^{(k+1)}{}^\top P x^{(k+1)}$$

and by substituting only the dynamics in the above formula we obtain

$$x^{(k)}{}^\top P x^{(k)} = x^{(k)}{}^\top Q x^{(k)} + u^{(k)}{}^\top R u^{(k)} + \left(Ax^{(k)} + Bu^{(k)}\right)^\top P \left(Ax^{(k)} + Bu^{(k)}\right)$$

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$$x^{(k)^\top} P x^{(k)} = x^{(k)^\top} Q x^{(k)} + u^{(k)^\top} R u^{(k)} + x^{(k+1)^\top} P x^{(k+1)}$$

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4_b. By differentiating the above formula with respect to $u^{(k)}$ we obtain

$$R u^{(k)} + B^\top P \left(A x^{(k)} + B u^{(k)}\right) = 0$$

namely

$$u^{(k)} = -\left(R + B^\top P B\right)^{-1} B^\top P A x^{(k)}$$

Infinite horizon LQR solution

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$$x^{(k)\top} P x^{(k)} = x^{(k)\top} Q x^{(k)} + u^{(k)\top} R u^{(k)} + x^{(k+1)\top} P x^{(k+1)}$$

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4_b. By differentiating the above formula with respect to $u^{(k)}$ we obtain

$$R u^{(k)} + B^\top P \left(A x^{(k)} + B u^{(k)}\right) = 0$$

namely

$$u^{(k)} = -\underbrace{\left(R + B^\top P B\right)^{-1} B^\top P A}_{K}\, x^{(k)}$$

Infinite horizon LQR solution

5. By substituting

$$K = \left(R + B^\top P B\right)^{-1} B^\top P A$$

into

$$Q + K^\top R K + (A - BK)^\top P (A - BK) - P = 0$$

we obtain the **Riccati equation**

$$A^\top P A - P + Q - A^\top P B \left(R + B^\top P B\right)^{-1} B^\top P A = 0$$

which is quadratic in P

Infinite horizon LQR solution

6. By solving the Riccati equation for P we will find the unique positive definite solution $P^* = {P^*}^\top > 0$ and we can substitute it into the equation of K thus obtaining the optimal gain K^*

Infinite horizon LQR solution

6. By solving the Riccati equation for P we will find the unique positive definite solution $P^* = {P^*}^\top > 0$ and we can substitute it into the equation of K thus obtaining the optimal gain K^*

   Notice that in order to achieve this result we substituted the dynamics of the system (3_a. and 3_b.).

   Therefore this procedure design:

   — is an **offline** procedure
   — needs the **full knowledge** of the system dynamics

LQR solution via Policy Iteration

Initialization. Select a guess $\pi_i = \pi_0$

Policy evaluation (PE). Determine the value of the current policy V_{π_i}

Policy improvement (PI). Determine an improved policy

$$\pi_{i+1} = \arg\min_{\pi} V_{\pi_i}$$

Terminal condition. PE and PI are repeated until

$$\pi_{i+1} = \pi_i$$

LQR solution via Policy Iteration

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Policy evaluation (PE). Determine the value of the current policy V_{π_i}

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$$\pi_{i+1} = \arg\min_{\pi} V_{\pi_i}$$

Terminal condition. PE and PI are repeated until

$$\pi_{i+1} = \pi_i$$

Initialization. Select a guess $K_i = K_0$

Policy evaluation (PE). Select a guess $P^j = P^0$

Repeat for N fixed steps

$P^{j+1} = Q + K_i^\top R K_i + (A - BK_i)^\top P^j (A - BK_i)$

P^N is the evaluation of the policy K_i

Policy improvement (PI).

$$K_{i+1} = \left(R + B^\top P^N B\right)^{-1} B^\top P^N A$$

LQR solution via Policy Iteration

Initialization. Select a guess $\pi_i = \pi_0$

Policy evaluation (PE). Determine the value of the current policy V_{π_i}

Policy improvement (PI). Determine an improved policy

$$\pi_{i+1} = \arg\min_{\pi} V_{\pi_i}$$

Terminal condition. PE and PI are repeated until

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Repeat for N fixed steps

$P^{j+1} = Q + K_i^\top R K_i + (A - BK_i)^\top P^j (A - BK_i)$

P^N is the evaluation of the policy K_i

Policy improvement (PI).

$$K_{i+1} = \left(R + B^\top P^N B\right)^{-1} B^\top P^N A$$

Notice that by setting $N = \infty$ PE is equivalent to solving

$$Q + K_i^\top R K_i + (A - BK_i)^\top P (A - BK_i) - P = 0$$

Finite horizon LQR control problem

Find $\pi^*\left(x^{(k)}\right)$ solution of:

$$\arg\min_{\pi} x^{(H)^\top} Q_H x^{(H)} + \sum_{k=0}^{H-1} x^{(k)^\top} Q x^{(k)} + u^{(k)^\top} R u^{(k)}$$

s.t.:

$$x^{(k+1)} = A x^{(k)} + B u^{(k)},$$

$$x^{(0)} = x_{\text{ini}},\ x^{(H)} = 0$$

Finite horizon LQR control problem

Find $\pi^*\left(x^{(k)}\right)$ solution of:

$$\arg\min_{\pi} \mathbf{x}^{(\mathbf{H})^\top} \mathbf{Q_H} \mathbf{x}^{(\mathbf{H})} + \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{H}-\mathbf{1}} \mathbf{x}^{(\mathbf{k})^\top} \mathbf{Q} \mathbf{x}^{(\mathbf{k})} + \mathbf{u}^{(\mathbf{k})^\top} \mathbf{R} \mathbf{u}^{(\mathbf{k})}$$

s.t.:

$$\mathbf{x}^{(\mathbf{k}+\mathbf{1})} = \mathbf{A}\mathbf{x}^{(\mathbf{k})} + \mathbf{B}\mathbf{u}^{(\mathbf{k})},$$

$$x^{(0)} = x_{\text{ini}},\ x^{(H)} = 0$$

Finite horizon LQR control problem

Find $\pi^* \left(x^{(k)}\right)$ solution of:

$$\arg\min_{\pi} \mathbf{x^{(H)}}^\top \mathbf{Q_H x^{(H)}} + \sum_{\mathbf{k=0}}^{\mathbf{H-1}} \mathbf{x^{(k)}}^\top \mathbf{Q x^{(k)}} + \mathbf{u^{(k)}}^\top \mathbf{R u^{(k)}}$$

s.t.:

$$\mathbf{x^{(k+1)}} = \mathbf{A x^{(k)}} + \mathbf{B u^{(k)}},$$

$$x^{(0)} = x_{\text{ini}},\ x^{(H)} = 0$$

To use LQR correctly, several assumptions must be satisfied.

LQR assumptions

1. **The matrices $\mathbf{Q}$ and $\mathbf{Q_H}$ must be positive semidefinite, i.e., $\mathbf{Q_H} \geq \mathbf{0}$, $\mathbf{Q} \geq \mathbf{0}$**: this assumption ensures the state is driven to 0 because otherwise, a state error different from 0 might actually *decrease* the cost, thereby misleading the minimization

2. **The matrix $\mathbf{R}$ must be positive definite, i.e., $\mathbf{R} > \mathbf{0}$**: this ensures that there is always a penalty for applying control effort

3. **The state is accessible** or at least $(A,\ C)$ observable

LQR assumptions

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3. **The state is accessible** or at least $(A,\ C)$ observable

4. **The state $\mathbf{x}^{(\mathbf{H})} = \mathbf{0}$ must be reachable in $\mathbf{H}$ steps**, in the better case $(A,\ B)$ fully reachable

Formal definition of a finite horizon LQR control problem

Find $\pi^* \left(x^{(k)}\right) = -K_k x^{(k)}$ solution of:

$$\arg\min_{\pi} x^{(H)^\top} Q_H x^{(H)} + \sum_{k=0}^{H-1} x^{(k)^\top} Q x^{(k)} + u^{(k)^\top} R u^{(k)}$$

s.t.:

$$x^{(k+1)} = A x^{(k)} + B u^{(k)},$$

$$x^{(0)} = x_{\text{ini}},\ x^{(H)} = 0$$

$$\mathbf{Q} = \mathbf{Q}^\top,\ \mathbf{Q} \geq \mathbf{0}$$

$$\mathbf{Q_H} = \mathbf{Q_H}^\top,\ \mathbf{Q_H} \geq \mathbf{0}$$

$$\mathbf{R} = \mathbf{R}^\top,\ \mathbf{R} > \mathbf{0}$$

Formal definition of a finite horizon LQR control problem

Find $\pi^*\left(x^{(k)}\right) = -\mathbf{K_k} x^{(k)}$ solution of:

$$\arg\min_{\pi} \; x^{(H)^\top} Q_H x^{(H)} + \sum_{k=0}^{H-1} x^{(k)^\top} Q x^{(k)} + u^{(k)^\top} R u^{(k)}$$

s.t.:

$$x^{(k+1)} = Ax^{(k)} + Bu^{(k)},$$

$$x^{(0)} = x_{\text{ini}}, \; x^{(H)} = 0$$

$$\mathbf{Q} = \mathbf{Q}^\top, \; \mathbf{Q} \geq \mathbf{0}$$

$$\mathbf{Q_H} = \mathbf{Q_H}^\top, \; \mathbf{Q_H} \geq \mathbf{0}$$

$$\mathbf{R} = \mathbf{R}^\top, \; \mathbf{R} > \mathbf{0}$$

Notice that, in this case, the control policy of interest is a **time-varying feedback gain matrix** K_k

Finite horizon LQR solution via DP

Starting from the optimal value of the performance index at the final state $J^*_{x(H)}$ and proceeding **backward** in time from $k = H - 1$ to $k = 0$

$$J^*_{x(H-1)} = J^*_{x(H)} + \min \, x^{(H-1)^\top} Q x^{(H-1)} + u^{(H-1)^\top} R u^{(H-1)}$$

$$J^*_{x(k)} = J^*_{x(k+1)} + \min \, x^{(k)^\top} Q x^{(k)} + u^{(k)^\top} R u^{(k)}.$$

1. Assume that the optimal value function has the form

$$V^*_\pi\left(x^{(k)}\right) = x^{(k)^\top} P_k x^{(k)}$$

2. At $k = H$ we have $P_H = Q_H$ and the corresponding value function is

$$V^*_\pi\left(x^{(H)}\right) = x^{(H)^\top} P_H x^{(H)}$$

3. Following dynamic programming we have that $\forall k = H - 1, \ldots, 0$

$$V^*_\pi\left(x^{(k)}\right) = x^{(k)^\top} P_k x^{(k)} = x^{(k+1)^\top} P_{k+1} x^{(k+1)} + \min_{u \in \mathcal{U}} \left(x^{(k)^\top} Q x^{(k)} + u^{(k)^\top} R u^{(k)}\right)$$

Finite horizon LQR solution via DP

That can also be written:

$$V_\pi^*\left(x^{(k)}\right) = {x^{(k)}}^\top P_k x^{(k)}$$

$$= \min_{u \in \mathcal{U}} \left(\left(Ax^{(k)} + Bu^{(k)}\right)^\top P_{k+1} \left(Ax^{(k)} + Bu^{(k)}\right) + {x^{(k)}}^\top Q x^{(k)} + {u^{(k)}}^\top R u^{(k)} \right)$$

4. By differentiating the above formula with respect to $u^{(k)}$ we obtain

$$u^{(k)} = -\left(R + B^\top P_{k+1} B\right)^{-1} B^\top P_{k+1} A x^{(k)}$$

5. And substituting $u^{(k)}$ in $V_\pi^*\left(x^{(k)}\right)$

$$V_\pi^*\left(x^{(k)}\right) = {x^{(k)}}^\top \left(A^\top P_{k+1} A - P_{k+1} + Q - A^\top P_{k+1} B \left(R + B^\top P_{k+1} B\right)^{-1} B^\top P_{k+1} A\right) x^{(k)}$$

Finite horizon LQR solution via DP

Thus leading to the general computation of

$$P_k = A^\top P_{k+1} A + Q - A^\top P_{k+1} B \left(R + B^\top P_{k+1} B \right)^{-1} B^\top P_{k+1} A$$

and

$$K_k = \left(R + B^\top P_{k+1} B \right)^{-1} B^\top P_{k+1} A$$

6. Iterating backward in time, we can find $P_0,\ P_1, \ldots, P_H$ and therefore the related control inputs.

$$\downarrow$$

Riccati recursion

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3 Time-Varying LQR

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The previous derivations hold even if the dynamics is the system we want to control is a discrete-time linear time-varying system:

$$x^{(k+1)} = \mathbf{A}(\mathbf{k})x^{(k)} + \mathbf{B}(\mathbf{k})u^{(k)}$$

and similarly, the cost functions are also time-varying:

$$x^{(H)^\top} Q_H x^{(H)} + \sum_{k=0}^{H-1} x^{(k)^\top} \mathbf{Q}(\mathbf{k}) x^{(k)} + u^{(k)^\top} \mathbf{R}(\mathbf{k}) u^{(k)}$$

Formal definition of time-varying LQR control problem

Find $\pi^*\left(x^{(k)}\right) = -K_k x^{(k)}$ solution of:

$$\arg\min_{\pi} \; {x^{(H)}}^\top Q_H x^{(H)} + \sum_{k=0}^{H-1} {x^{(k)}}^\top Q(k) x^{(k)} + {u^{(k)}}^\top R(k) u^{(k)}$$

s.t.:

$$x^{(k+1)} = A(k) x^{(k)} + B(k) u^{(k)},$$

$$x^{(0)} = x_{\text{ini}},\; x^{(H)} = 0$$

$$\mathbf{Q(k) = Q(k)^\top,\; Q(k) \geq 0}$$

$$\mathbf{Q_H = Q_H^\top,\; Q_H \geq 0}$$

$$\mathbf{R(k) = R(k)^\top,\; R(k) > 0}$$

Formal definition of time-varying LQR control problem

Find $\pi^* \left(x^{(k)}\right) = -\mathbf{K}(\mathbf{k})x^{(k)}$ solution of:

$$\arg\min_{\pi} x^{(H)^\top} Q_H x^{(H)} + \sum_{k=0}^{H-1} x^{(k)^\top} Q(k) x^{(k)} + u^{(k)^\top} R(k) u^{(k)}$$

s.t.:

$$x^{(k+1)} = A(k)x^{(k)} + B(k)u^{(k)},$$

$$x^{(0)} = x_{\text{ini}},\ x^{(H)} = 0$$

$$\mathbf{Q}(\mathbf{k}) = \mathbf{Q}(\mathbf{k})^\top,\ \mathbf{Q}(\mathbf{k}) \geq \mathbf{0}$$

$$\mathbf{Q_H} = \mathbf{Q_H^\top},\ \mathbf{Q_H} \geq \mathbf{0}$$

$$\mathbf{R}(\mathbf{k}) = \mathbf{R}(\mathbf{k})^\top,\ \mathbf{R}(\mathbf{k}) > \mathbf{0}$$

Notice that the control policy of interest is a **time-varying feedback gain matrix** K_k

The equivalent Riccati equation will be:

$$A(k)^\top P(k)A(k) - P(k) + Q(k) - A(k)^\top P(k)B(k)\left(R(k) + B(k)^\top P(k)B(k)\right)^{-1} B(k)^\top P(k)A(k) = 0$$

At each k-th time instant, we will obtain a unique solution $P(k) = P(k)^\top > 0$.

We will use it in order to compute

$$K(k) = \left(R(k) + B(k)^\top P(k)B(k)\right)^{-1} B(k)^\top P(k)A(k)$$

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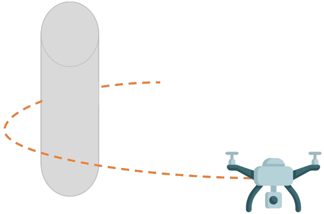
Trajectory tracking by LQR

Consider a task in which, instead of minimizing the state error with respect to the null state vector $x = 0$, we want to drive the system following a desired trajectory $\{\tilde{x}^{(0)}, \tilde{x}^{(1)}, \ldots, \tilde{x}^{(H)}\}$, still minimizing the controller effort.

To apply the LQR formulation, it is necessary to apply some transformations that allow us to restore the conditions useful for employing the previous derivations.



Formal definition of tracking LQR control problem

Find $\pi^* \left(x^{(k)}\right) = -Kx^{(k)}$ solution of:

$$\arg\min_{\pi} \sum_{k=0}^{H-1} \left(x^{(k)} - \tilde{x}^{(k)}\right)^\top Q \left(x^{(k)} - \tilde{x}^{(k)}\right) + u^{(k)^\top} R u^{(k)}$$

s.t.:

$$x^{(k+1)} = Ax^{(k)} + Bu^{(k)}$$

$$\mathbf{Q} = \mathbf{Q}^\top,\ \mathbf{Q} \geq \mathbf{0}$$

$$\mathbf{R} = \mathbf{R}^\top,\ \mathbf{R} > \mathbf{0}$$

1. By expanding the term corresponding to the state error we get

$$\left(x^{(k)} - \tilde{x}^{(k)}\right)^\top Q \left(x^{(k)} - \tilde{x}^{(k)}\right) = {x^{(k)}}^\top Q x^{(k)} + {\tilde{x}^{(k)}}^\top Q \tilde{x}^{(k)} - 2{\tilde{x}^{(k)}}^\top Q x^{(k)}$$

   Notice that at the k-th time instant ${\tilde{x}^{(k)}}^\top Q \tilde{x}^{(k)} = d^{(k)}$ is constant, and also $2{\tilde{x}^{(k)}}^\top Q = 2{q^{(k)}}^\top$.
   Therefore we can write

$$\left(x^{(k)} - \tilde{x}^{(k)}\right)^\top Q \left(x^{(k)} - \tilde{x}^{(k)}\right) = {x^{(k)}}^\top Q x^{(k)} + d^{(k)} - 2{q^{(k)}}^\top x^{(k)}$$

2. Then by assuming to apply the homogeneous transformation:

$$\hat{x}^{(k)} = \begin{bmatrix} x^{(k)} \\ 1 \end{bmatrix}$$

$$\hat{x}^{(k+1)} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \hat{x}^{(k)} + \begin{bmatrix} B \\ 0 \end{bmatrix} u^{(k)} = \hat{A}\hat{x}^{(k)} + \hat{B}u^{(k)}$$

we can state the following auxiliary problem

Tracking LQR solution

Find $\pi^* \left(\hat{x}^{(k)}\right) = -K\hat{x}^{(k)}$ solution of:

$$\arg\min_{\pi} \sum_{k=0}^{H-1} \hat{x}^{(k)^\top} \begin{bmatrix} Q & -q^{(k)} \\ -q^{(k)^\top} & d^{(k)} \end{bmatrix} \hat{x}^{(k)} + u^{(k)^\top} R u^{(k)}$$

s.t.:

$$\hat{x}^{(k+1)} = \hat{A}\hat{x}^{(k)} + \hat{B}u^{(k)}$$

where the desired trajectory is inglobed in q and p.
Then the previous derivations hold.

LQR limits

The LQR controller itself has significant limitations:

- The system must be linear
- The dynamics of the system must be known
- Constraints on states and controls cannot be represented
- Weights placed on the cost function often have no physical meaning and cause difficulties in controller tuning.

How can we handle **nonlinear systems**? How should we deal with **significant constraints on states and controls**?

Questions for you

1. What happens if the terminal state is not equal to 0? Can we use the LQR?

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1. What happens if the terminal state is not equal to 0? Can we use the LQR?

2. How do I choose Q and R? If I want to work harder to reduce the control input, what values do I assign to Q and R?

3. In the case of constraints on x and u can I benefit from appropriate settings of Q and R?

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Control nonlinear systems via LQR

Real-world systems rarely approach linear behavior, so we need a way to optimally control even non-linear dynamical systems.

What we already know about non-linear systems is that:

“their behavior within a certain operating range of an equilibrium state can often be reasonably approximated by that of a linear dynamical system”

Therefore if we would like to solve the problem of finding $\pi^* \left(x^{(k)}\right) = -Kx^{(k)}$ solution of:

$$\arg\min_{\pi} \sum_{k=0}^{\infty} x^{(k)^\top} Q x^{(k)} + u^{(k)^\top} R u^{(k)}$$

s.t.:

$$\mathbf{x^{(k+1)}} = \mathbf{f}\left(\mathbf{x^{(k)}}, \mathbf{u^{(k)}}\right)$$

where $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is the non-linear state transition function, we can work on the linearized version of the non-linear system

Control nonlinear systems via LQR

The procedure is as follows:

1. Compute an equilibrium state $\bar{x} \in \mathcal{X}$ of the DT system by setting $u^{(k)} = \bar{u} \in \mathcal{U}$ to a constant value and by computing

$$f\left(\bar{x}, \bar{u}\right) = \bar{x}.$$

Thus obtaining the equilibrium point $\left(\bar{x},\ \bar{u}\right)$ of the DT system.

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1. Compute an equilibrium state $\bar{x} \in \mathcal{X}$ of the DT system by setting $u^{(k)} = \bar{u} \in \mathcal{U}$ to a constant value and by computing
$$f\left(\bar{x}, \bar{u}\right) = \bar{x}.$$
Thus obtaining the equilibrium point $\left(\bar{x},\ \bar{u}\right)$ of the DT system.

2. Linearize the non-linear system around the equilibrium point thus obtaining
$$\delta x^{(k+1)} = A\delta x^{(k)} + B\delta u^{(k)}$$
where $A = \left[\frac{\partial f}{\partial x}\right]_{x=\bar{x},u=\bar{u}}$, $B = \left[\frac{\partial f}{\partial u}\right]_{x=\bar{x},u=\bar{u}}$, while $\delta x^{(k)} = x^{(k)} - \bar{x}$ and $\delta u^{(k)} = u^{(k)} - \bar{u}$ are the variations of $x^{(k)}$ and $u^{(k)}$ from their equilibrium values.

3. Solve the problem of finding $\pi^*\left(\delta x^{(k)}\right) = -K\delta x^{(k)}$ solution of:

$$\arg\min_{\pi} \sum_{k=0}^{\infty} \delta x^{(k)^\top} Q\delta x^{(k)} + \delta u^{(k)^\top} R\delta u^{(k)}$$

s.t.:

$$\delta\mathbf{x}^{(\mathbf{k+1})} = \mathbf{A}\delta\mathbf{x}^{(\mathbf{k})} + \mathbf{B}\delta\mathbf{u}^{(\mathbf{k})}$$

applying one of the LQR resolution procedures.

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   s.t.:

$$\delta \mathbf{x}^{(\mathbf{k+1})} = \mathbf{A}\delta\mathbf{x}^{(\mathbf{k})} + \mathbf{B}\delta\mathbf{u}^{(\mathbf{k})}$$

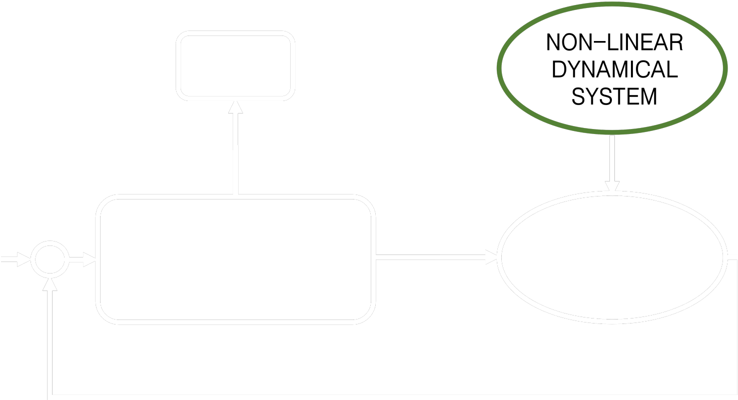
   applying one of the LQR resolution procedures.

4. Use K to compute the control inputs to be applied on the non-linear system $x^{(k+1)} = f\left(x^{(k)}, u^{(k)}\right)$ as follows:

$$u^{(k)} = -K\left(x^{(k)} - \bar{x}\right) + \bar{u}$$

Control nonlinear systems via LQR

5 Iterative LQR



Control nonlinear systems via LQR

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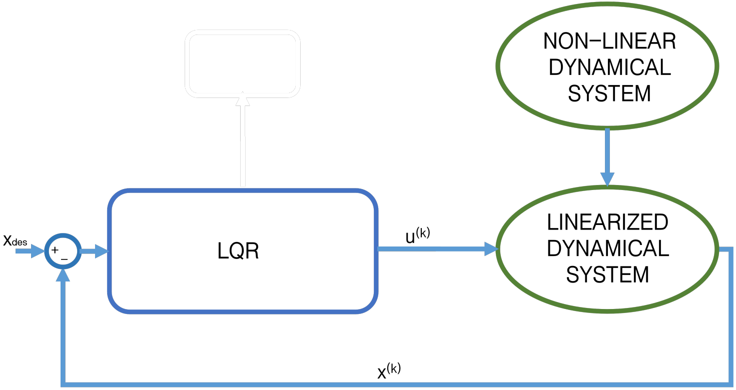

Control nonlinear systems via LQR

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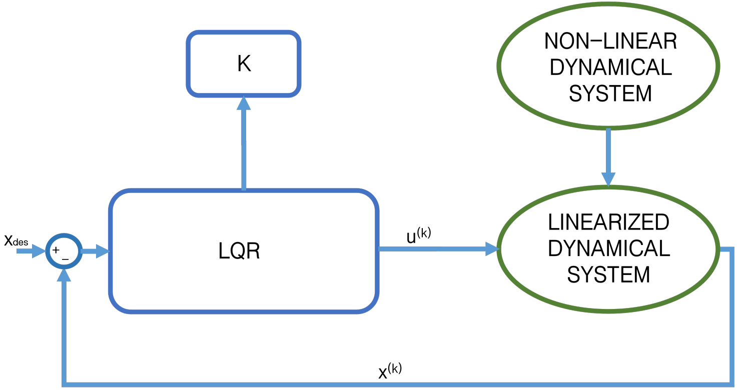
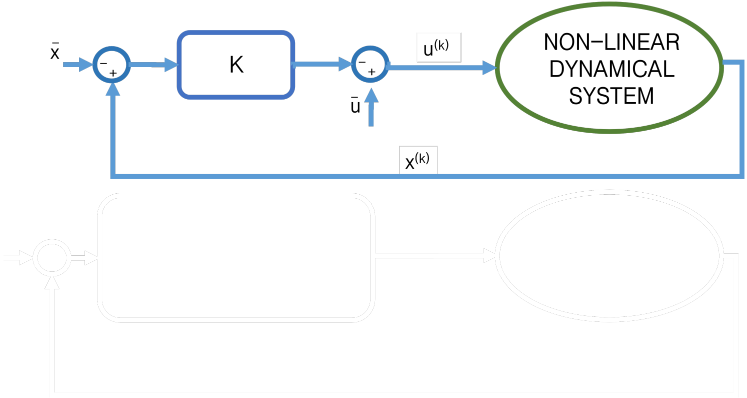


Control nonlinear systems via LQR



Control nonlinear systems via LQR

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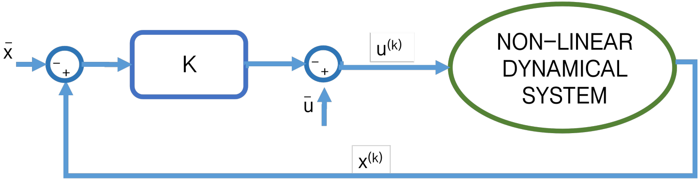

Does it work well? Does it always work?

Iterative LQR

If there is no equilibrium point or if the dynamics need to be approximated not only around an equilibrium point, the procedure defined above is not the correct choice. Another approach consists of repeating the following step until convergence

1. Use linearization of the nonlinear system around a trajectory that is considered *nominal*
2. Compute a locally optimal feedback control law
3. Use the locally optimal feedback control law to obtain a new nominal trajectory

We would like to solve the problem of finding $\pi^* \left(x^{(k)}\right) = -Kx^{(k)}$ solution of:

$$\arg\min_{\pi} \sum_{k=0}^{\infty} x^{(k)\top} Q x^{(k)} + u^{(k)\top} R u^{(k)}$$

s.t.:

$$\mathbf{x^{(k+1)}} = \mathbf{f}\left(\mathbf{x^{(k)}}, \mathbf{u^{(k)}}\right)$$

where $f : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}$ is the non-linear state transition function

Iterative LQR

The procedure is as follows:

1. Record a state-input trajectory of H steps from the non-linear system $\left\{\bar{x}^{(k)}, \bar{u}^{(k)}\right\}_{k=0}^{H-1}$

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1. Record a state-input trajectory of H steps from the non-linear system $\left\{\bar{x}^{(k)}, \bar{u}^{(k)}\right\}_{k=0}^{H-1}$
2. Linearize the non-linear system around the trajectory
$$\delta x^{(k+1)} = A(k)\delta x^{(k)} + B(k)\delta u^{(k)}$$
where $A(k) = \left[\frac{\partial f}{\partial x}\right]_{x=\bar{x}^{(k)}, u=\bar{u}^{(k)}}$, $B(k) = \left[\frac{\partial f}{\partial u}\right]_{x=\bar{x}^{(k)}, u=\bar{u}^{(k)}}$, while $\delta x^{(k)} = x^{(k)} - \bar{x}^{(k)}$ and $\delta u^{(k)} = u^{(k)} - \bar{u}^{(k)}$ are the variations of $x^{(k)}$ and $u^{(k)}$ from their nominal values.

   The cost term then becomes
$$\left(x^{(k)} - \bar{x}^{(k)}\right)^\top Q \left(x^{(k)} - \bar{x}^{(k)}\right) + \left(u^{(k)} - \bar{u}^{(k)}\right)^\top R \left(u^{(k)} - \bar{u}^{(k)}\right)$$

3. Compute the second-order Taylor series expansion of the cost around the nominal trajectory thus obtaining a quadratic approximation

$$\hat{x}^{(k)\top} \hat{Q(k)} \hat{x}^{(k)} + \hat{u}^{(k)\top} R \hat{u}^{(k)}$$

where $\hat{x}^{(k)}$ and $\hat{u}^{(k)}$ is the homogeneous transformation of the time-variant linearization of the non-linear system

4. Solve an LQR problem:
   Find $\pi^* \left(\hat{x}^{(k)}\right) = -K_k \hat{x}^{(k)}$ solution of:

$$\arg\min_{\pi} \hat{x}^{(k)^\top} \hat{Q_H(k)} \hat{x}^{(k)} \sum_{k=0}^{H} \hat{x}^{(k)^\top} \hat{Q(k)} \hat{x}^{(k)} + \hat{u}^{(k)^\top} R \hat{u}^{(k)}$$

   s.t.:

$$\hat{x}^{(k+1)} = \begin{bmatrix} A(k) & 0 \\ 0 & 1 \end{bmatrix} \hat{x}^{(k)} + \begin{bmatrix} B(k) \\ 0 \end{bmatrix} \hat{u}^{(k)}$$

   where $\hat{x}^{(k)} = \begin{bmatrix} x^{(k)} - \bar{x}^{(k)} \\ 1 \end{bmatrix}$ and $\hat{u}^{(k)} = u^{(k)} - \bar{u}^{(k)}$.

   The solution is a controller K_k which returns $u^{(k)} = -K_k \hat{x}^{(k)} + \bar{u}^{(k)}$

5. Apply the controller on the non-linear system and store a new state-input trajectory of H steps from the non-linear system $\left\{\bar{x}^{(k)}, \bar{u}^{(k)}\right\}_{k=0}^{H-1}$ and repeat form step 2.

On the convergence of iterative LQR

- The optimal policy for the LQR approximation may end up not staying close to the sequence of points around which the LQR approximation was calculated by Taylor expansion.

On the convergence of iterative LQR

- The optimal policy for the LQR approximation may end up not staying close to the sequence of points around which the LQR approximation was calculated by Taylor expansion.
  We can solve this issue by modifying the cost:

$$(1-\alpha)\left({x^{(k)}}^\top Q x^{(k)} + {u^{(k)}}^\top R u^{(k)}\right) + \alpha\left(x^{(k)} - \bar{x}^{(k)}\right)$$

  Assuming ${x^{(k)}}^\top Q x^{(k)} + {u^{(k)}}^\top R u^{(k)}$ is bounded, for α close enough to 1, the second term will dominate and ensure the linearizations approximations around the trajectory of the solution found by LQR

On the convergence of iterative LQR

- f is non-linear, hence the resulting optimization is non-convex. Can get stuck in local optima! Good initialization matters!

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On the convergence of iterative LQR

- f is non-linear, hence the resulting optimization is non-convex. Can get stuck in local optima! Good initialization matters!

- g could be non-convex, then the LQR approximation can fail to have positive-definite cost matrices
  If $Q(k)$ is not positive definite the solution could be to increase the penalty $Q(k) = Q(k) + \xi I$ for deviating from the current state until resulting in $Q(k)$ positive semidefinite.

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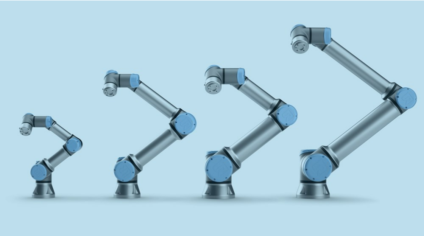
Constrained dynamical systems

What we observed so far works without constraints on state and control input.

As you will see in the **Control Theory course**, you can add linear constraints on states in the LQR formulation, but you cannot add constraints on control inputs.

However, systems have physical limitations or constraints, e.g., an optimal control problem for a robotic arm might have constraints on the joint angles and torques.



Therefore, constraints in optimal control problems serve to ensure that the resulting control policy or trajectory meets practical and task-specific requirements.

The optimization process aims to find the best control strategy while respecting these constraints.

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The optimization process aims to find the best control strategy while respecting these constraints.

Model Predictive Control (MPC) is a popular control strategy that explicitly takes into account the possibility of defining **constraints as an integral part of the control design**.

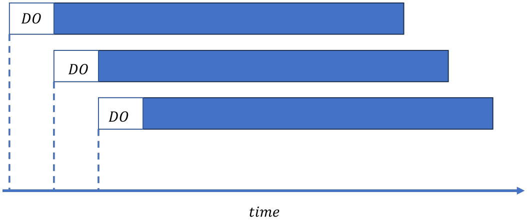
It involves a **predictive model** of the system and optimization to determine the control input that minimizes a cost function while adhering to various constraints

Model Predictive Control

Model predictive control (also known as ***receding horizon*** control) entails solving finite-time optimal control problems in a receding horizon fashion.

Specifically, given a model of the system:

- Record a state measurement
- Solve a finite-time optimal control problem for a pre-specified planning horizon
- Execute the first control action
- Repeat



Formal definition of MPC problem

At each l-th step given $x^{(l)}$, find $\pi^*(k)$ solution of:

$$\arg\min_{\pi} \; {x^{(l+H)}}^\top Q x^{(l+H)} + \sum_{k=l}^{l+H-1} {x^{(k)}}^\top Q x^{(k)} + {u^{(k)}}^\top R u^{(k)}$$

s.t.:

$$x^{(k+1)} = Ax^{(k)} + Bu^{(k)},$$
$$x^{(k)} \in \mathcal{X}, \quad u^{(k)} \in \mathcal{U},$$
$$x^{(H)} \in \mathcal{X}_H,$$
$$x^{(0)} = x^{(l)}$$

$$Q = Q^\top \geq 0$$
$$Q_H = Q_H^\top \geq 0$$
$$R = R^\top > 0$$

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s.t.:

$$\begin{aligned} x^{(k+1)} &= Ax^{(k)} + Bu^{(k)}, \\ x^{(k)} &\in \mathcal{X}, \quad u^{(k)} \in \mathcal{U}, \\ x^{(H)} &\in \mathcal{X}_H, \\ x^{(0)} &= x^{(l)} \end{aligned}$$

$$Q = Q^\top \geq 0$$

$$Q_H = Q_H^\top \geq 0$$

$$R = R^\top > 0$$

The result is a sequence of optimal control inputs $u^{(l)^*}, \ldots, u^{(l+H-1)^*}$.

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The result is a sequence of optimal control inputs ${u^{(l)}}^*, \dots, {u^{(l+H-1)}}^*$.
Apply ${u^{(l)}}^*$ measure $x^{(l+1)}$ and repeat the procedure

MPC problems

- Computationally demanding (important when embedding controller on hardware)
- May or may not be feasible
- May or may not be stable

Therefore we need to derive conditions on

- terminal function so that closed-loop stability is guaranteed
- terminal constraint set so that persistent feasibility is guaranteed

Questions' time!

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