The Spectral Factorization Algorithm

Introduction

This live function introduces and explains the spectral factorisation algorithm, as well as implementing it.

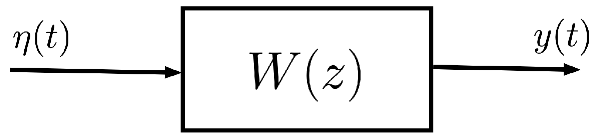
The Spectral Factorization Theorem

Given a process with rational spectrum $\Phi(z)$, there exists one and only one representation of the process as the output of an LTI system driven by a white process $\eta(\cdot)$ and with transfer function $W(z) = \frac{N(z)}{D(z)}$ if the following conditions are imposed on $W(z)$:

- $N(z)$ and $D(z)$ monic, co-prime and of the same degree;
- all roots of $N(z)$ (zeros of $W(z)$) have $|\cdot| \leq 1$;
- all roots of $D(z)$ (poles of $W(z)$) have $|\cdot| < 1$.

$$\Phi_y(z) = W(z) \cdot W\left(\frac{1}{z}\right) \cdot \Phi_\eta(z) = W(z) \cdot W\left(\frac{1}{z}\right) \cdot \lambda_\eta^2$$

A Stationary Stochastic Process from a White Noise Response of an LTI Filter



Consider a stationary stochastic process $y(t)$, generated by an LTI dynamic system with a generic transfer function

$$W(z) = \frac{b(z)}{a(z)} = \frac{b_0\, z^n + b_1\, z^{n-1} + \cdots + b_{n-1}\, z + b_n}{z^n + a_1\, z^{n-1} + \cdots + a_{n-1}\, z + a_n}$$

and white noise $\eta(t)$ as its input. Assuming that the LTI system $W(z)$ is BIBO stable. the power spectrum of the output stationary stochastic process is

$$
\begin{aligned}
\Phi_y(z) &= W(z)\cdot W(z^{-1})\cdot \lambda_\eta^2 = \frac{b(z)}{a(z)}\cdot\frac{b(z^{-1})}{a(z^{-1})}\cdot\lambda_\eta^2 \\
&= \frac{b_0\,z^n + b_1\,z^{n-1} + \cdots + b_{n-1}\,z + b_n}{z^n + a_1\,z^{n-1} + \cdots + a_{n-1}\,z + a_n}\cdot\frac{b_0\,z^{-n} + b_1\,z^{-(n-1)} + \cdots + b_{n-1}\,z^{-1} + b_n}{z^{-n} + a_1\,z^{-(n-1)} + \cdots + a_{n-1}\,z^{-1} + a_n}\cdot\lambda_\eta^2 \\
&= \frac{\beta_0 + \beta_1\,(z+z^{-1}) + \cdots + \beta_n\,(z^n + z^{-n})}{1 + \alpha_1\,(z+z^{-1}) + \cdots + \alpha_n\,(z^n+z^{-n})} = \frac{\beta(z)}{\alpha(z)}
\end{aligned}
$$

where

$$
\begin{cases}
\alpha_i = \dfrac{\sum_{k=0}^{n-1} a_k\,a_{k+i}}{\sum_{k=0}^{n} a_k^2} & , \quad i = 1, 2, \ldots, n \\[2em]
\beta_j = \dfrac{\sum_{k=0}^{n-1} b_k\,b_{k+j}}{\sum_{k=0}^{n} a_k^2}\cdot\lambda_\eta^2 & , \quad j = 1, 2, \ldots, n
\end{cases}
$$

Remark

Note the peculiar structure of the polynomials $\beta(z)$ and $\alpha(z)$ in the expression of the spectrum. They are **symmetric polynomials** in the variables z and z^{-1} . In fact, permuting the variable z with z^{-1} does not change the expression of both the polynomials.

The feature just highlighted is in fact a peculiar property of rational spectra: one can always write the two polynomials $\beta(z)$ and $\alpha(z)$ that appear in the spectrum $\Phi_y(z)$ as symmetrical polynomials in z and z^{-1}.

```
function [r,Cz, Az] = L8_spectrFactAlg(betaP, alphaP)
%  The function L8_spectrFactAlg() allows
% the spectral factorisation of a rational
% spectrum to be calculated.
% ---
% Inputs:
% betaP  <--> the coefficients of the symmetric polynomial beta(z+z^(-1)),
%             ordered from beta_n to beta_0
% alphaP <--> the coefficients of the symmetric polynomial alpha(z+z^(-1)),
%             ordered from alpha_n to alpha_0 (=1)
% ---
% Outputs:
% Az <--> the monic Hurwitz polynomial 1+a_1 z^(-1)+...a_n z^(-n_) -- the
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%          roots of Az have magnitude stricly less than 1
% Cz <--> the monic polynomial 1+c_1 z^(-1)+...c_n z^(-n_); the roots of Cz
%          have magnitude less or equal to 1
% r  <--> the variance of the white noise feeding-in the filter with
%          transfer function W(z) = Cz/Az

[r2, Cz] = poly_spectral_fact(fliplr(betaP)); % spectral factorization of teh symmetric

[r1, Az] = poly_spectral_fact(fliplr(alphaP)); % spectral factorization of teh symmetr

r = r2/r1; % the computation of the white noise variance


end
```

The Spectral Factorization Algorithm: a Summary

Given the rational spectrum

$$\Phi(z) = \frac{\beta(z)}{\alpha(z)} = \frac{\sum_{k=-m}^{m} \beta_k \, z^k}{\sum_{k=-n}^{n} \alpha_k \, z^k}$$

with

$$\begin{cases} \alpha_p = \alpha_{-p} & p = -n\,,\ -n+1\,,\ \cdots, n-1\,, n \\ \beta_q = \beta_{-q} & q = -m\,,\ -m+1\,,\ \cdots, m-1\,, m \\ \alpha_n \neq 0\,,\ \beta_m \neq 0 & \text{i.e. both polynomials are symmetric polynomials in } z \text{ and } z^{-1} \\ \Phi(e^{i\,\omega}) \geq 0\ \forall \omega & \text{i.e. } \Phi(z) \text{ is a rational spectrum} \end{cases}$$

then there is only one factorization of the spectrum $\Phi(z)$ such that

$$\Phi(z) = \frac{C(z)}{A(z)} \cdot \frac{C(z^{-1})}{A(z^{-1})} \cdot r$$

with

$$\begin{cases} \quad r & > & 0 \\ A(z) & = & 1 + a_1\, z^{-1} + a_2\, z^{-2} + \cdots + a_n\, z^{-n} \quad \text{with } z_p : A(z_p) = 0\,,\ |z_p| < 1\ \forall p \\ C(z) & = & 1 + c_1\, z^{-1} + c_2\, z^{-2} + \cdots + c_m\, z^{-m} \quad \text{with } z_q : C(z_q) = 0\,,\ |z_q| \leq 1\ \forall q \end{cases}$$

First step of the algorithm: exploiting the fact that both $\alpha(z)$ and $\beta(z)$ are symmetric polynomials, factor both the polynomials

$$\begin{cases} \alpha(z) &=& r_1\, A(z)\, A(z^{-1}) \\ \beta(z) &=& r_2\, C(z)\, C(z^{-1}) \end{cases}$$

where $r_1 > 0$, $r_2 > 0$ and the polynomials $A(z)$ and $C(z)$ are monic polynomials as defined above.

Second step of the algorithm: the spectral factorization of $\Phi(z)$ is

$$\Phi(z) = \frac{\beta(z)}{\alpha(z)} = \left(\frac{r_2}{r_1}\right) \cdot \left[\frac{C(z)}{A(z)}\right] \cdot \left[\frac{C(z^{-1})}{A(z^{-1})}\right] = r \cdot W(z) \cdot W(z^{-1})$$

An Insight on the Algorithm's Details: The Spectral Factorization of a Symmetric Polynomial

Given a symmetric polynomial

$$P(z) = p_0 + p_1\,(z + z^{-1}) + \cdots\ + p_n\,(z^n + z^{-n}) = p_0 + \sum_{k=1}^{n} p_k\,(z^k + z^{-k})\,, \quad p_n \neq 0$$

such that $p(e^{i\,\omega}) \geq 0\ \forall \omega$, then there is <u>one and only one factorization</u> of $p(z)$ as

$$p(z) = r\, q(z)\, q(z^{-1})$$

with

- $r > 0$
- $$q(z) = 1 + q_1\, z^{-1} + \cdots + q_n\, z^{-n} = 1 + \sum_{k=1}^{n} q_k\, z^{-k}$$

The roots of $q(z)$ are such that

- if $p(e^{i\,\omega}) \geq 0\ \forall \omega$, then $|z_j| \leq 1\ \forall z_j : q(z_j) = 0$
- if $p(e^{i\,\omega}) > 0\ \forall \omega$, then $|z_j| < 1\ \forall z_j : q(z_j) = 0$

Sketch of the algorithm:

- given $p(z)$ build the palindromic polynomial $\rho(z) = z^n\, p(z) = p_0\, z^n + \sum_{k=1}^{n} p_k\,(z^{n+k} + z^{n-k})$

- compute the $2n$ roots of the polynomial $\rho(z)$; $\rho(z)$ is a palindromic polynomial (by construction) so if $\tilde{z}$ is a root of $\rho(z)$ then also $\tilde{z}^{-1}$ is a root of $\rho(z)$, and with the same multiplicity;
- factor the polynomial $\rho(z)$ as

$$\rho(z) = p_n \prod_{|z_k| \leq 1} (z - z_k) \prod_{|z_k| \leq 1} \left(z - z_k^{-1}\right)$$

- consequently, the factorization of $p(z)$ assumes the expression

$$p(z) = z^{-n}\,\rho(z) = \frac{(-1)^n p_n}{\prod_{k=1}^{n} z_k} \cdot \prod_{k=1}^{n} \left(1 - z_k\, z^{-1}\right) \cdot \prod_{k=1}^{n} (1 - z_k\, z)\ , \qquad |z_k| \leq 1\ \forall k$$

- finally

$$r = \frac{(-1)^n p_n}{\prod_{k=1}^{n} z_k}\ , \qquad q(z) = \prod_{k=1}^{n} \left(1 - z_k\, z^{-1}\right)\ , \qquad |z_k| \leq 1\ \forall k$$

```
function [r, q] = poly_spectral_fact(p)
    % The function computes the spectral factorization
    %                  p(z)=rq(z)q(z^-1)
    % of the symmetric polynomial p(z) such that
    % q(z) = 1 + ... + qn z^(-n) is the unique monic spectral factor with
    % all the roots inside the circle |z|<=1

    n = length(p)-1; % the degree of p(z)
    rho = [p(n+1:-1:2) p]; % building the polynomial rho(z)

    z = roots(rho); % finding the roots of rho(z)
    z = complex(z);

    [~,i] = sort(abs(z)); % sorting the roots in ascending order, according to the magn
    z = z(i); % applying the sorting criterion

    z = z(1:n); % now the first n roots have magnitude less or equal to 1
                % (whereas the remaining roots have magnitude greater than 1)
    q = poly(z); % build the q(z) polynomial using the roots with |z|<=1

    r = ((-1)^n)*rho(1)/prod(z); % evaluating the numerical constant r
end
```