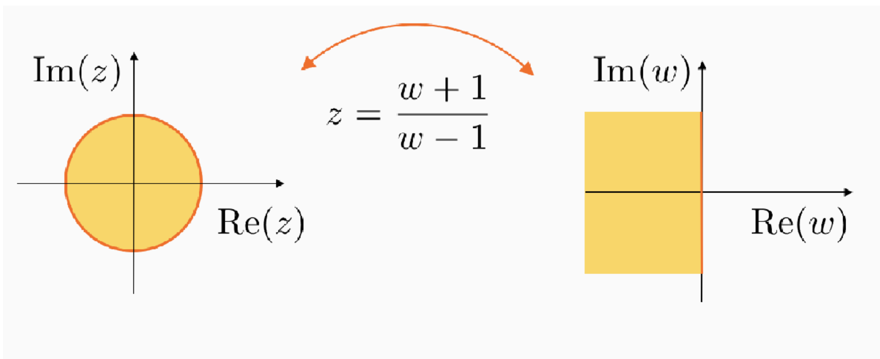
Bi-linear Transformation for the Routh-Hurwitz Criterion & MATLAB

The Bilinear Transformation: a Recap

$$z = \frac{w+1}{w-1},\ z, w \in \mathbb{C} \qquad \begin{array}{lcl} |z| < 1 & \iff & \mathrm{Re}(w) < 0 \\ |z| = 1 & \iff & \mathrm{Re}(w) = 0 \\ |z| > 1 & \iff & \mathrm{Re}(w) > 0 \end{array}$$



Substitute

$$z = \frac{w+1}{w-1},\ z, w \in \mathbb{C}$$

into $p_A(z) = \varphi_0 z^n + \varphi_1 z^{n-1} + \cdots + \varphi_{n-1} z + \varphi_n$, thus obtaining

$$q_A(w) = (w-1)^n \left[\varphi_0 \frac{(w+1)^n}{(w-1)^n} + \varphi_1 \frac{(w+1)^{n-1}}{(w-1)^{n-1}} + \cdots + \varphi_{n-1} \frac{(w+1)}{(w-1)} + \varphi_n\right]$$

and hence one gets

$$q_A(w) = q_0\, w^n + q_1\, w^{n-1} + \cdots + q_{n-1}\, w + q_n$$

with suitable coefficients $q_0, q_1, \ldots, q_n$.

Applying the Bilinear Transformation In MATLAB

Given

$$p_A(z) \ = z^3 + 2\,z^2 + z + 1$$

one gets

$$q_A(w) \;= (w-1)^3 \left[\frac{(w+1)^3}{(w-1)^3} + 2\,\frac{(w+1)^2}{(w-1)^2} + \frac{w+1}{w-1} + 1\right]$$

and after some algebra

$$q_A(w) = 5\,w^3 + w^2 + 3\,w - 1$$

Let us replicate the application of the bilinear transform using the *Symbolic Math Toolbox* in MATLAB.

```
clear
close all
clc

syms z w % let's declare a few symbolic variables
```

```
p_Az = z^3+2*z^2+z+1 % assign the characteristic polynomial
```

p_Az = $z^3 + 2\,z^2 + z + 1$

```
                        % to analyse, using the bilinear transform

bTexpr = (w+1)/(w-1) % the bilinear transform
```

bTexpr =

$$\frac{w+1}{w-1}$$

Now, using the `subs` command, let us compute the transformed polynomial

```
q_Aw = ((w-1)^3) * subs(p_Az,z,bTexpr)
```

q_Aw =

$$(w-1)^3 \left(\frac{w+1}{w-1} + \frac{2\,(w+1)^2}{(w-1)^2} + \frac{(w+1)^3}{(w-1)^3} + 1\right)$$

Finally, let us simplify the resulting expression

```
q_Aw = simplify(q_Aw)
```

q_Aw = $5\,w^3 + w^2 + 3\,w - 1$

What if the characteristic polynomial to be transformed $p_A(z)$ contains some parametric coefficients?

For example, consider the polynomial

$$p_A(z) = z^2 + az + b \quad a, b \in \mathbb{R}$$

```
clear
clc
syms z w % the symbolic variables used to describe the polynomial and the bilinear tran
syms a b % the parametric coefficients
```

Now define the polynomial

```
p_Az = z^2+a*z +b
```

p_Az = $z^2 + a\,z + b$

and the bilinear transform

```
bTexpr = (w+1)/(w-1) % the bilinear transform
```

bTexpr =

$$\frac{w+1}{w-1}$$

Apply the bilinear transform

```
q_w = ((w-1)^2)*subs(p_Az,z,bTexpr)
```

q_w =

$$(w-1)^2 \left(b + \frac{(w+1)^2}{(w-1)^2} + \frac{a\,(w+1)}{w-1}\right)$$

and simplify the result:

```
expand(q_w)
```

ans =

$$b - 2\,b\,w + \frac{1}{\sigma_1} + b\,w^2 - \frac{2\,w^2}{\sigma_1} + \frac{w^4}{\sigma_1} + \frac{a}{w-1} - \frac{a\,w}{w-1} - \frac{a\,w^2}{w-1} + \frac{a\,w^3}{w-1}$$

where

$$\sigma_1 = w^2 - 2\,w + 1$$

```
simplify(expand(q_w))
```

ans = $b - a + 2\,w - 2\,b\,w + a\,w^2 + b\,w^2 + w^2 + 1$

```
q_w = collect(simplify(expand(q_w)),w)
```

q_w = $(a+b+1)\,w^2+(2-2\,b)\,w+b-a+1$