Hands-On: A Bayesian Estimation Experiment

Introduction

Given random variables x, w and z, **uncorrelated with each other** and such that:

$$x \sim \mathcal{G}\left(\mu_x, \lambda_x^2\right) \quad w \sim \mathcal{G}(0,1) \quad z \sim \mathcal{G}(0, \overline{\lambda}^2)$$
$$\mathrm{cov}(x,w) = 0 \quad \mathrm{cov}(x,z) = 0 \quad \mathrm{cov}(w,z) = 0$$
$$\mu_x = 2 \quad \lambda_x^2 = 1 \qquad \overline{\lambda}^2 \in \left[0.001, 1000\right]$$

we would like to know the current value of variable x, but x **is not accessible**. Then we make **two joint observations** (at the same instant) of two quantities (also random variables) d_1 and d_2, which depend linearly on x (and also on w and z respectively)

$$\begin{cases} d_1 &= 2\,x + w \\ d_2 &= -x + z \end{cases}$$

with

$$\mathrm{E}\left[d_1\right] = \mathrm{E}\left[2x + w\right] = 2\mathrm{E}\left[x\right] + \mathrm{E}\left[w\right] = 2\,\mu_x \qquad \mathrm{E}\left[d_2\right] = \mathrm{E}\left[-x + z\right] = -\mathrm{E}\left[x\right] + \mathrm{E}\left[z\right] = -\,\mu_x$$

- How can we determine the best estimate $\hat{x}$?
- How does the estimate $\hat{x}$ change if the variance of z assumes the minimum value? What if it is the largest possible?

```
clear
close all
clc
```

Setting the Random Variables

```
mu_x = 2.0;
lambda2_x = 1;

x = mu_x+sqrt(lambda2_x)*randn % the effective value of x to be estimated
```

```
x = 2.5377
```

```
w = randn; % the gaussian noise w
```

```
lambda2Z_min = 1e-3; lambda2Z_max = 1e3;
% the extremum values for the variance of the r.v. z

lambda2_z = 1; % the variance of z
z = sqrt(lambda2_z)*randn; % the gaussian noise z
```

Acquiring the Data

Let's simulate the acquisition of the measurements d_1 and d_2

```
d1 = 2*x + w % the first observation
```

```
d1 = 6.9092
```

```
d2 = -x + z  % the second observation
```

```
d2 = -4.7965
```

Bayesian Estimate $\widehat{x}$

According to the Bayes estimation theory, the a-posteriori estimate $\widehat{\vartheta}$ is

$$\begin{cases} \widehat{\vartheta} = \vartheta_m + \Lambda_{\vartheta d}\,{\Lambda_{dd}}^{-1}\,(d - d_m) \\ \text{var}\,(\vartheta - \widehat{\vartheta}) = \Lambda_{\vartheta\vartheta} - \Lambda_{\vartheta d}\,{\Lambda_{dd}}^{-1}\Lambda_{d\vartheta} \end{cases}$$

where

$$\Lambda_{dd} = \begin{bmatrix} \text{var}[d_1] & \text{cov}[d_1 \cdot d_2] \\ \text{cov}[d_2 \cdot d_1] & \text{var}[d_2] \end{bmatrix}$$

$$\text{var}[d_i] = \text{var}[\vartheta] + \text{var}[\eta_i] = \sigma_\vartheta^2 + \lambda_i^2 \qquad i = 1, 2$$

$$\text{cov}[d_1 \cdot d_2] = \text{E}\Big[(d_1 - d_m) \cdot (d_2 - d_m)\Big] = \text{E}\Big[(\vartheta + \eta_1 - \mu_\vartheta) \cdot (\vartheta + \eta_2 - \mu_\vartheta)\Big] = \text{var}[\vartheta] + \text{E}[\eta_1 \cdot \eta_2] = \sigma_\vartheta^2$$

Moreover

$$\Lambda_{\vartheta d} = \Big[\text{cov}[\vartheta \cdot d_1] \quad \text{cov}[\vartheta \cdot d_2]\Big] \qquad \Lambda_{d\vartheta} = \Big[\Lambda_{\vartheta d}\Big]^{\text{T}}$$

$$\text{cov}[\vartheta \cdot d_i] = \text{E}\Big[(\vartheta - \mu_\vartheta) \cdot (d_i - d_m)\Big] = \text{E}\Big[(\vartheta - \mu_\vartheta) \cdot (\vartheta + \eta_i - \mu_\vartheta)\Big] = \text{E}\Big[(\vartheta - \mu_\vartheta) \cdot (\vartheta - \mu_\vartheta)\Big] = \sigma_\vartheta^2 \quad i = 1, 2$$

```
% your code
```