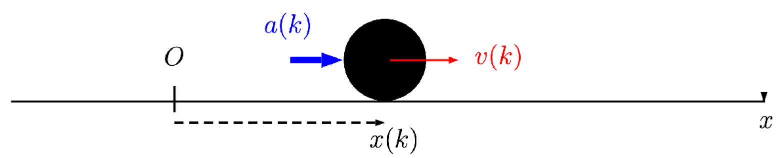
A simple Application of the Kalman Predictor

Tracking of a Moving Object

Let us study the motion of an object moving along a straight line, as schematised in the following figure



The following assumptions hold:

- the moving object has negligible dimensions, so that it can be described as a **point material**, using only kinematics equations
- position and speed of the object are observed, both affected by measurement noise

$$\begin{cases} y_1(k) &= x(k) + \eta_1(k) \\ y_2(k) &= v(k) + \eta_2(k) \end{cases} \qquad \eta_1(\cdot) \sim \mathrm{WNG}(0, 1/100)\,,\ \eta_2(\cdot) \sim \mathrm{WNG}(0, 4) \qquad (1)$$

- at each sampling instant $k\,\Delta$ [with **sampling period** $\Delta = 0.01\ \mathrm{s}$], acceleration $a(k)$ is applied to the system

It is requested to

1. describe the system with a model in state equations, with the state variables of the system described by position [$x_1(k) = x(k)$] and speed of the material point [$x_2(k) = v(k)$], taking into account the measurement noise in equations (1).
2. simulate the motion of the object (taking into account the measurement noise) in the time interval $t_k = k \cdot \Delta,\ k = 0, 1,\ \ldots 200$
3. determine the recursive **Kalman predictor** that provides the optimal prediction of the state $\widehat{x}(k+1|k)$ for the system.
4. compare the performance of the recursive Kalman predictor with the one of the **steady-state Kalman predictor**.

Let us assume that the process noise is given by $\epsilon(k) = [\epsilon_1(k)\ \epsilon_2(k)]^\top$, where $\epsilon_i(k) \sim \text{WNG}(0, 1/1000) \quad i = 1, 2$ and ϵ_1, ϵ_2 independent r.v.

Moreover, the acceleration $a(k)$ acting on the system is identically zero.

Finally, assume that the initial state of the system is $[\hat{x}_1(1),\ \hat{x}_2(1)]^\top = [0,\ -0.5]^\top$ and the corresponding uncertainty matrix is $P_1 = 10 \cdot I_2$ (I_2 is an identity matrix of order 2), whereas the effective initial state is $[x_1(1),\ x_2(1)]^\top = [-0.43\ 0.5979]^\top$

Solution

```
clear
close all
clc

Delta = 0.01; % sampling period [s]
```

The model - Kinematics of moving objects 1D

Assuming zero acceleration, the material point moves with constant speed. The state equations describing the object movement are as follows

$$\begin{cases} x(k+1) &= x(k) + \Delta \cdot v(k) \\ v(k+1) &= v(k) + \Delta \cdot a(k) = v(k) \\ y_1(k) &= x(k) \\ y_2(k) &= v(k) \end{cases} \implies \begin{cases} x_1(k+1) &= x_1(k) + \Delta \cdot x_2(k) \\ x_2(k+1) &= x_2(k) \\ y_1(k) &= x_1(k) \\ y_2(k) &= x_2(k) \end{cases}$$

The model needed by the Kalman predictor, taking into account also the process and measurement noises, becomes

$$\begin{cases} x_1(k+1) &= x_1(k) + \Delta \cdot x_2(k) + \epsilon_1(k) \\ x_2(k+1) &= x_2(k) + \epsilon_2(k) \\ y_1(k) &= x_1(k) + \eta_1(k) \\ y_2(k) &= x_2(k) + \eta_2(k) \end{cases} \implies \begin{cases} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \epsilon_1(k) \\ \epsilon_2(k) \end{bmatrix} \\ \\ \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} \end{cases}$$

with

$$V_1 = \text{var}\begin{bmatrix} \epsilon_1(k) \\ \epsilon_2(k) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad V_2 = \text{var}\begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$$

Initialization

```
proc_noiseSTD = sqrt(0.001); % the process noise standard deviation
meas_noiseSTD_1 = .1;
meas_noiseSTD_2 = 2; % the measurement noise standard deviation

V1 = [proc_noiseSTD.^2 , 0 ;...
          0, proc_noiseSTD.^2]; % the process noise variance matrix
V2 = diag([meas_noiseSTD_1.^2 ; meas_noiseSTD_2.^2]); % the measurement noise variance

hat_x1 = [0;-0.5]; % initial state
P_1 = 10*eye(2); % initial state variance

time_steps = Delta .* (0:200);

meas_noise = diag([meas_noiseSTD_1, meas_noiseSTD_2]) * randn(2,numel(time_steps));
proc_noise = diag([proc_noiseSTD, proc_noiseSTD]) * randn(2,numel(time_steps));
```

The Moving Object

```
X = zeros(2, numel(time_steps));
Y = zeros(2, numel(time_steps));

% ------------------------------
% system model equations
%
% x(k+1) = A x(k)
%   y(k) = C x(k) + v2(k)
%
%  v2(k) <--> measurement noise
%
% state vector x(k)
%
% x(k) = [x1(k) x2(k)]'
%

A = [1 Delta; 0 1];
C = eye(2);

x_init = [-0.43;.5979]; % initial state <-- NB different from the initial guess!!
X(:,1) = x_init;
Y(:,1 ) = meas_noise(:,1);
```

```
% -- simulation of the moving object ---
for k=1:numel(time_steps)-1
    Y(:,k) = C * X(:,k) + meas_noise(:,k);
    X(:, k+1) = A * X(:,k); % no process noise <-- no uncertainty in the model!!
end % for k
Y(:,numel(time_steps)) = C * X(:,numel(time_steps)) + meas_noise(:,numel(time_steps));
```

The Kalman Predictor

Given the system dynamics described by

$$\begin{cases} x(t+1) &= Fx(t) + v_1(t) \\ y(t) &= Hx(t) + v_2(t) \qquad x, v_1 \in \mathbb{R}^n,\, y, v_2 \in \mathbb{R}^p \end{cases}$$

the Kalman filter acting as one-step ahead predictor is

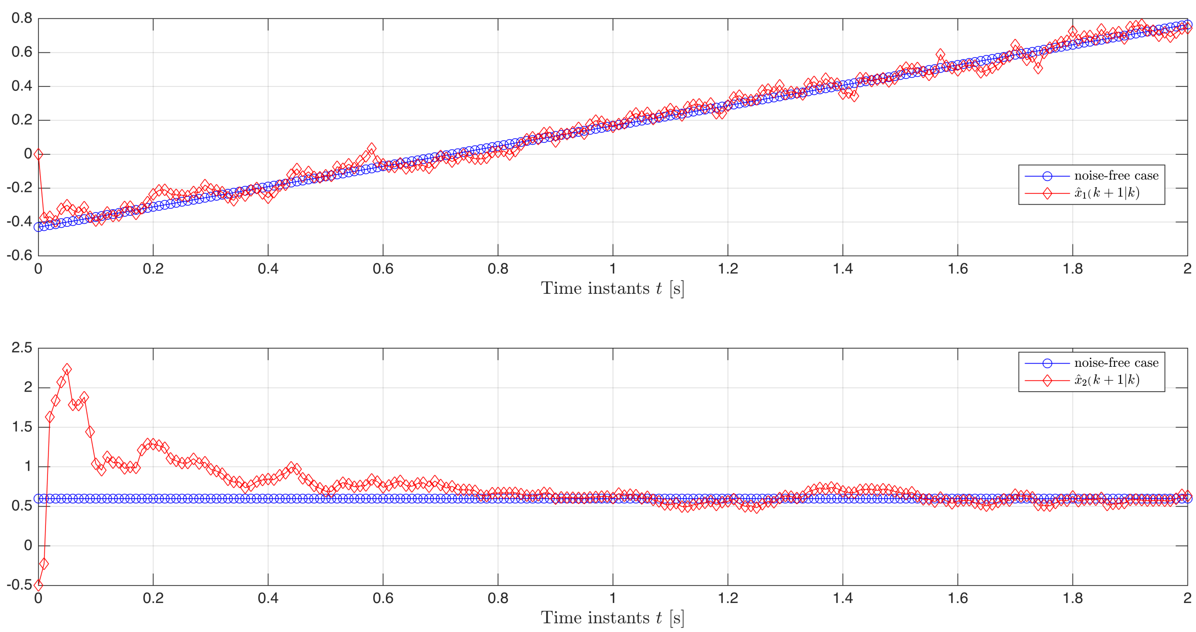
$$\begin{cases} \hat{x}(t+1\,|\,t) &= F\,\hat{x}(t\,|\,t-1) + K(t)\cdot\big(y(t) - H\,\hat{x}(t\,|\,t-1)\big) \\ K(t) &= F\cdot P(t)\cdot H^\top\big[H\cdot P(t)\cdot H^\top + V_2\big]^{-1} \\ P(t+1) &= F\,\big\{P(t) - P(t)H^\top\,\big[V_2 + HP(t)H^\top\big]^{-1}\,HP(t)\big\}\,F^\top + V_1 \end{cases}$$

```
F = A;
H = C;
Nx = size(F,1);


N = numel(time_steps);

xhat = zeros(Nx,N); % output of the Kalman 1-step ahead predictor
xhat(:,1) = hat_x1; % 1st state a priori estimation

y = Y; % the measurements
e = zeros(size(y)); % innovation

P = zeros(Nx,Nx,N);
P(:,:,1) = P_1; % 1st P matrix

% ------ evaluating the state prediction ---------
for k = 1:(N-1)

    e(:,k) = y(:,k) - H*xhat(:,k);
    cP = P(:,:,k);
    K = (F*cP*H')/(H*cP*H'+V2);
    xhat(:,k+1) = F*xhat(:,k) + K*e(:,k);
    Pnew = V1+F*(cP-(cP*H'/(V2+H*cP*H'))*H*cP)*F';
```

```
    if (k<N)
        P(:,:,k+1) = Pnew;
    end

end % for k
```

State Prediction vs State

```
figure('Units','normalized','Position',[0.1, 0.1, 0.9, 0.75]);
hp1 =subplot(2,1,1);
plot(time_steps, X(1,:),'bo-'); hold on
plot(time_steps,  xhat(1,:),'rd-')
grid on; zoom on;xlim([min(time_steps), max(time_steps)]);
xlabel('Time instants $t$ [s]', 'Interpreter','latex','FontSize',12);
legend('noise-free case', '$\hat{x}_{1}(k+1 | k)$', 'Interpreter', 'latex',...
         'Location', 'best');

hp2= subplot(2,1,2);
plot(time_steps, X(2,:),'bo-'); hold on
plot(time_steps,  xhat(2,:),'rd-');
grid on; zoom on;xlim([min(time_steps), max(time_steps)]);
xlabel('Time instants $t$ [s]', 'Interpreter','latex','FontSize',12);
legend('noise-free case', '$\hat{x}_{2}(k+1 | k)$', 'Interpreter', 'latex',...
         'Location', 'best');
```



Note the convergence of the predictions to the state variables, despite the "wrong" initial guess.

The Steady-State Kalman Predictor

What happens to the filter gain $K(t)$ and to the state covariance matrix $P(t)$ when $t \to \infty$?

```
NN = 5000; % try using N = 500 or 50000. What happens to the final matrix P?
PP = zeros(Nx,Nx,NN);
PP(:,:,1) = P_1; % 1st P matrix

% ------ evaluating the state prediction ---------
for k = 1:(NN-1)


    cP = PP(:,:,k);
    KK = (F*cP*H')/(H*cP*H'+V2);

    Pnew = V1+F*(cP-(cP*H'/(V2+H*cP*H'))*H*cP)*F';
    if (k<NN)
        PP(:,:,k+1) = Pnew;
    end

end % for k
PP(:,:,1)
```

```
ans = 2×2
    10     0
     0    10
```

```
PP(:,:,N)
```

```
ans = 2×2
    0.0038    0.0019
    0.0019    0.0550
```

```
KK
```

```
KK = 2×2
    0.2751    0.0005
    0.1363    0.0135
```

Solving the Algebraic Riccati Equation

```
[Pbar, Kbar]= idare(F, H, V1, V2)
```

```
Pbar = 2×2
    0.0037    0.0001
    0.0001    0.0638
Kbar = 2×2
    0.2702    0.0096
    0.0000    0.0157
```

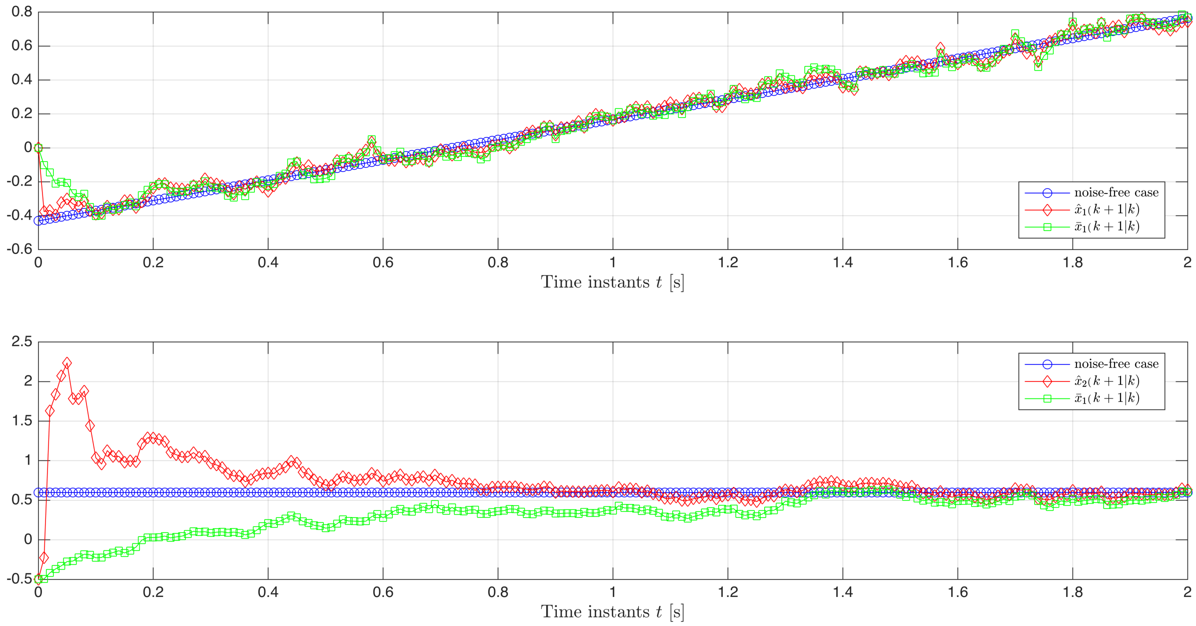
So the steady-state predictor is

```
xxhat = zeros(Nx,N); % output of the Kalman 1-step ahead predictor
xxhat(:,1) = hat_x1; % 1st state a priori estimation
```

```
y = Y; % the measurements
e = zeros(size(y)); % innovation

Pinf = Pbar;

% ------ evaluating the state prediction ---------
for k = 1:(N-1)

    e(:,k) = y(:,k) - H*xxhat(:,k);


    xxhat(:,k+1) = F*xxhat(:,k) + Kbar*e(:,k);


end % for k
```

Comparison of the Results

```
figure('Units','normalized','Position',[0.1, 0.1, 0.9, 0.75]);
hp1 =subplot(2,1,1);
plot(time_steps, X(1,:),'bo-'); hold on
plot(time_steps,  xhat(1,:),'rd-');
plot(time_steps,  xxhat(1,:),'gs-');
grid on; zoom on;xlim([min(time_steps), max(time_steps)]);
xlabel('Time instants $t$ [s]', 'Interpreter','latex','FontSize',12);
legend('noise-free case', '$\hat{x}_{1}(k+1 | k)$', '$\bar{x}_{1}(k+1 | k)$', 'Interpre
         'Location', 'best');

hp2= subplot(2,1,2);
plot(time_steps, X(2,:),'bo-'); hold on
plot(time_steps,  xhat(2,:),'rd-');
plot(time_steps,  xxhat(2,:),'gs-');
grid on; zoom on;xlim([min(time_steps), max(time_steps)]);
xlabel('Time instants $t$ [s]', 'Interpreter','latex','FontSize',12);
legend('noise-free case', '$\hat{x}_{2}(k+1 | k)$', '$\bar{x}_{1}(k+1 | k)$','Interpret
         'Location', 'best');
```


noise-free case
$\hat{x}_1(k+1|k)$
$\bar{x}_1(k+1|k)$
Time instants t [s]
noise-free case
$\hat{x}_2(k+1|k)$
$\bar{x}_1(k+1|k)$
Time instants t [s]