Hands-On - AR(1) Stochastic Process: Stationary vs. Non-Stationary Process

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Introduction

Consider the AR(1) process

$$v(t) = a\,v(t-1) + \eta(t) \qquad \eta(\cdot) \sim WN(0, \lambda^2)$$

Recalling that $|a| < 1$ and setting $v_0 = v(t_0)$, $t_0 = 0$. Then, in general, we can write

$$v(t) = \sum_{i=0}^{t-1} a^{t-1-i}\,\eta(i+1) + a^t\,v_0$$

Now suppose to generate N samples for the r.v. of the AR process. Compare the results obtained using the following strategies

1. Collect N samples, starting from the time instant t_0;
2. Collect $N_{skip} + N$ samples, starting from the time instant t_0, but then throw away the first N_{skip} samples, in order to avoid data containig the transient response depending on the initial condition. Use only the remaining N samples, as data of the steady-state stationary stochastic process.

```
clear variables
close all
clc
```

The AR(1) Process

Select the AR(1) parameter a, such that $|a| < 1$

```
a_AR = 0.8; % select a value in the interval ]-1 , +1[
```

Tune the variance of the white noise λ^2

```
varWG = 4;      % select the white noise variance
```

Set as initial condition v_0 a random value

```
v0 = 1e3*randn; % as initial condition a gaussian r.v., with 0 mean and unitary varian
```

Choose how many data to generate (N_{tot} samples), how many to throw away after the generation (N_{skip} samples) and how many of the generated values to keep (N samples)

```
Ndata = 3800;    % how many samples to generate and keep?

Nskip = 675;   % how many samples to generate, but then ignore?

Ntot = Nskip+Ndata; % total num. of samples to generate
```

Stationary vs. Non-Stationary Process

- Evaluate the average value, the sampled variance, the autocorrelation function and the spectrum using the N data of the "stationary" process.
- Compare the results with the corresponding values you obtain when using the whole N_{tot} data, with the transient due to the initial condition.

Sampled Estimator of the Mean Value, the Variance and the Autocorrelation Function

We have already analysed the sampled estimators of mean value and variance. We recall them for convenience.

Sample Average Estimator

Given N random variables $v(1)$, $v(2)$, …, $v(N)$ such that

$$\mathrm{E}[v(i)] = \mu_v\,, \quad i = 1\,, 2\,, \; \ldots\,, N$$

(i.e., with the same mean value) and

$$\mathrm{E}\{[v(i) - \mu_v]\,[v(j) - \mu_v]\} = 0\,, \quad \forall i \neq j$$

(i.e., the data are mutually uncorrelated), the **sampled-average estimator**

$$\widehat{\mu}_v = \frac{1}{N}\sum_{i=1}^{N} v(i)$$

is an **unbiased estimator**, i.e.

$$\mathrm{E}[\widehat{\mu}_v] = \mu_v$$

The Sample Variance

The sample variance of N observations $\{v_i\}, i = 1, \ldots, N$ of the random variable V with known mean μ_v is defined as

$$\widehat{\sigma}^2_{\mu_v,N} = \frac{1}{N}\sum_{i=1}^{N}(v_i - \mu_v)^2.$$

We have added the subindex μ to indicate that we used the exact value of the mean to calculate the variance.

In practice, **the mean value is often unknown** and replaced by the sample mean. In that case the sample variance is defined as

$$\widehat{\sigma}^2_N = \frac{1}{N-1}\sum_{i=1}^{N}(v_i - \widehat{\mu}_N)^2$$

The Sample Autocorrelation Function

Given N observations $\{v_i\}, i = 1, \ldots, N$ of the random variable V with **null mean value** $\mu_v = \mathrm{E}[v_i] = 0\ \forall i$, the autocorrelation function $\gamma_v(\tau)$ can be estimated by mean of the following expression

$$\widehat{\gamma}_v(\tau) = \begin{cases} \dfrac{1}{N-\tau}\displaystyle\sum_{n=0}^{N-\tau-1} v(n)\cdot v(n+\tau) & \tau \geq 0 \\ \widehat{\gamma}_v(-\tau) & \tau < 0 \end{cases} \qquad |\tau| \leq N-1$$

It can be proven that it is an **unbiased estimator** (refer to [1] for details). Moreover, it can be proven that the variance of the estimate converges to zero as $N \to \infty$, so the estimate $\hat{\gamma}(\tau)$ is a **consistent estimate** of $\gamma(\tau)$ (refer to [1] for details).

Generate and Collect the Samples

Remember

$$AR(1): \quad v(t) = a\, v(t-1) + \eta(t)$$

Let's generate the data

```
% the AR process
buffer = zeros(Nskip,1); % a buffer used to generate a stationary AR process, regardle
buffer(1) = v0;     % the initial condition

AR1 = zeros(Ntot-Nskip+1,1);     % the buffer used to store the data belonging to the 
ARbuffer = [buffer; AR1];        % merging the arrays

eta = sqrt(varWG)*randn(Ntot,1); % let's generate the white noise samples

% ---- the  AR(1) process ----
for ii = 2:Ntot
    ARbuffer(ii) = a_AR * ARbuffer(ii-1)+eta(ii);
end % for ii

AR1 = ARbuffer(Nskip+1:end); % skip the initial Nskip data
                             % and use the remaining samples as AR
                             % stationary process

AR0 = ARbuffer; % use the whole dataset as "candidate" AR process
```

Comparing Average Values and Variances

Let's estimate the mean value and the variance for <u>the stationary AR process</u>

```
av_valAR1 = mean(AR1)      % the average value for the stationary AR process
```

```
av_valAR1 = -0.2552
```

```
varAR1 = varWG/(1-a_AR.^2) % the theoretical variance of the stationary process
```

```
varAR1 = 11.1111
```

```
sampled_varAR1 = var(AR1)  % the sampled variance
```

```
sampled_varAR1 = 11.3466
```

Using the whole dataset, the mean and the varaince assume different values

```
% check the average value using all the dataset
av_AR0 = mean(AR0)
```

```
av_AR0 = 0.6379
```

```

% check the variance
sampled_varAR0 = var(AR0)  % the sampled variance
```

```
sampled_varAR0 = 453.2263
```

Run more than once, varying the variance of the white noise feeding in.

What About the Autocovariance Function?

```
Ncorr1 =1024; % select the max value of tau in the estimation formula
```

Check if the maximum value for the lag τ is respecting the constraint $|\tau| \leq N-1$, with N the amount of available data.

```
errMSG = 'The max lag must be less or equal to (N-1), with N the amount of available da
assert((Ncorr1 <= (Ndata-1)), errMSG)
```

Let evaluate the estimate, using the MATLAB command `xcorr()`.

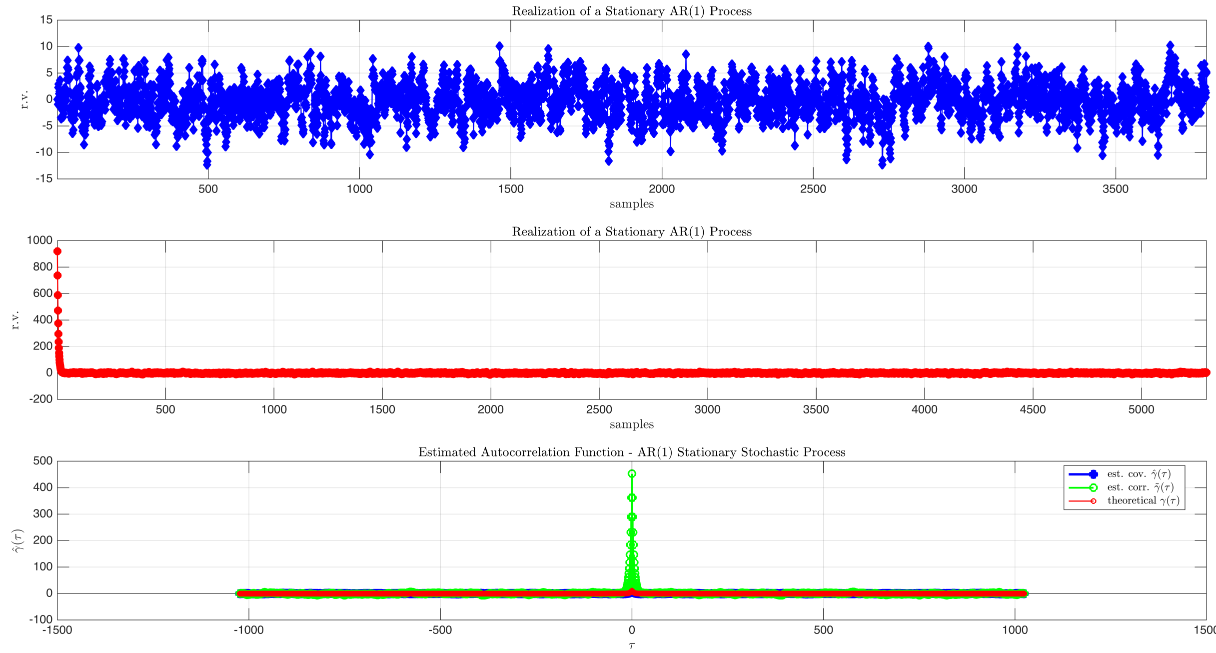
```
[gamma_v1, lags_v1] = xcorr(AR1, Ncorr1,'unbiased');
% unbiased estimation of the autocovariance function

[gamma_v0, lags_v0] = xcov(AR0, Ncorr1,'unbiased');
% unbiased estimation of the autocorr. function of tilde_v

gammaAR1 = varAR1*(a_AR.^(abs(lags_v1))); % the true autocorrelation function of the st

```

Now let's plot

- the acquired samples of the AR(1) process

- the samples of the estimated autocovaraiance and autocorrelation function toghethere with the corresponding values of the theoretical expression of the autocovariance function

```
figure('Units','normalized','Position',[0.1, 0.1, 0.85, 0.75]);

subplot(3,1,1);
plot(AR1,'db-','MarkerSize',6, 'MarkerFaceColor','b','LineWidth',1.0);grid on;
title('Realization of a Stationary AR$(1)$ Process', 'Interpreter','latex');
xlabel('samples', 'Interpreter','latex');ylabel('r.v.', 'Interpreter','latex');
xlim([1, Ndata]); % setting the extremum values on the x-axis[']]]=

subplot(3,1,2);

plot(AR0,'or-','MarkerSize',6, 'MarkerFaceColor','r','LineWidth',1.0);grid on;
title('Realization of a Stationary AR$(1)$ Process', 'Interpreter','latex');
xlabel('samples', 'Interpreter','latex');ylabel('r.v.', 'Interpreter','latex');
xlim([1, Ntot]); % setting the extremum values on the x-axis[']]]=

subplot(3,1,3);
% the covariance function of v(t)
stem(lags_v1, gamma_v1,'b','filled', 'LineWidth',2);
grid on; hold on;

% the correlation function of tilde_v
stem(lags_v0, gamma_v0,'g', 'LineWidth',1.5);

% the theoretical expression of the covaraince function
stem(lags_v1, gammaAR1, 'r', 'MarkerSize',4,'LineWidth',1.0);
xlabel('$\tau$', 'Interpreter','latex');ylabel('$\hat{\gamma}(\tau)$',...
    'Interpreter','latex');
title('Estimated Autocorrelation Function - AR$(1)$ Stationary Stochastic Process',...
    'Interpreter','latex');
legend('est. cov. $\hat{\gamma}(\tau)$', 'est. corr. $\tilde{\gamma}(\tau)$','theoreti
```



Realization of a Stationary AR(1) Process
r.v.
samples
Realization of a Stationary AR(1) Process
r.v.
samples
Estimated Autocorrelation Function - AR(1) Stationary Stochastic Process
est. cov. $\hat{\gamma}(\tau)$
est. corr. $\hat{\gamma}(\tau)$
theoretical $\gamma(\tau)$
$\hat{\gamma}(\tau)$
τ