

Equation of state for directed percolation

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Equation of state for directed percolation

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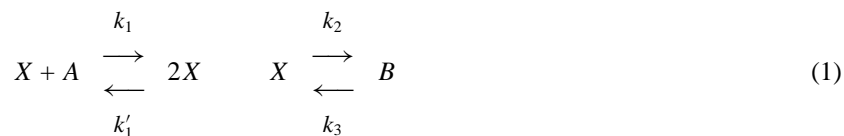
Abstract. Using field-theoretic renormalization group methods we calculate the equation of state for non-equilibrium systems belonging to the universality class of directed percolation to second order in $\epsilon = 4 - d$. By introducing a parametric representation the result can be written to this order in a very simple form. We use our result to obtain a universal amplitude ratio to second order in ϵ .

1. Introduction

Numerous physical systems that show continuous nonequilibrium phase transitions display, near the transition point, space-time structures that can be described by directed percolation clusters. Well known examples are epidemic processes without immunization [1], poisoning of catalytic surfaces [2], roughening transitions in growth processes [3], Schlögl's reaction models [4, 5], contact processes [6, 7] and certain cellular automata [8, 9]. The model known in particle physics as reggeon field theory [10–12] has also been related to directed percolation and epidemic processes [13–16]. All these systems share the property that their critical behaviour is characterized by the existence of an absorbing state. This property determines their universality class [5, 17].

The behaviour of these systems near the transition point is characterized by universal quantities such as critical exponents. The exponents for directed percolation have been calculated to second order in $\epsilon = 4 - d$ [5] and numerically for $d = 1$ and $d = 2$ (for recent results see, e.g. [18–20]). It is well known from the theory of equilibrium critical phenomena that universality not only holds for critical exponents but also for certain amplitude ratios and scaling functions [21–23]. An example for a universal relationship between physical quantities near a critical point is the equation of state. In this paper we derive an equation that describes the stationary state of epidemic processes with an absorbing state near the transition point.

Before we turn to the mesoscopic description used in the analysis presented below let us briefly discuss two examples for microscopic models that belong to the universality class of directed percolation. Schlögl's autocatalytic reaction scheme [4] is defined by the reaction equations



where the concentration of A s and B s is kept constant (e.g. by reservoirs). Here two remarks are in order: (i) if $A = B$ (i.e. the particles are of the same type) and their concentration is

not kept constant (but governed by the reaction–diffusion dynamics) the system belongs to a different universality class. In that case the total particle density is conserved and affects the long-time dynamics of the system [24, 25]. (ii) Only if the reaction $B \rightarrow X$ has zero probability the system has an absorbing state. In the case of chemical systems it is natural to take this process into account since back reactions cannot in general be avoided. Thus the B -particles represent a source for X . Below a critical value of the reaction rate k_2 the density of X -particles is nonvanishing in the stationary state for $k_3 \rightarrow 0$.

In the theory of infectious diseases [1] one considers two types of individuals: susceptibles S who can catch the disease and infectives I who have the disease and can transmit it. The dynamics of the system is characterized by the following processes.

- (i) Healthy (susceptible) individuals may catch the disease by contact with infectives. Infected individuals may recover spontaneously and thereby become susceptible again.
- (ii) Interaction of infected individuals can lead to recovery ('saturation').
- (iii) Diffusive spreading of the disease.
- (iv) Susceptibles may catch the disease spontaneously.

The process (iv) models, for instance, a continuous flow of germs into the system. Note that the system has an absorbing state if (and only if) the process (iv) has zero probability.

A mesoscopic description for the dynamics near the transition point requires a single density $n(\mathbf{r}, t)$ which represents the slow degrees of freedom. In the case of Schlögl's model (1) n corresponds to the density of X -particles, in the context of poisoning of a catalytic surface n is the density of vacant sites and in epidemic processes it represents the density of infectives. The density satisfies the Langevin equation

$$\partial_t n = \lambda \nabla^2 n + R[n]n + \lambda q + \zeta \quad (2)$$

where the reaction rate $R[n]$ can be expanded in powers of n . In order to investigate the universal properties near the transition point we only need the two leading terms,

$$R[n] = \lambda \left(\tau + \frac{g}{2} n \right) \quad (3)$$

where $g > 0$ and τ is a temperature-like critical parameter. For $q = 0$ the system has an absorbing state with $n = 0$ if the random force ζ is multiplicative:

$$\langle \zeta(\mathbf{r}, t) \zeta(\mathbf{r}', t') \rangle = \lambda \tilde{g} n(\mathbf{r}, t) \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') \quad (4)$$

where we have again neglected irrelevant higher powers in n . In the case of the reaction–diffusion system (1) the (constant) source term q vanishes for a zero reaction rate k_3 . By a simple rescaling it is possible to render the coupling constants g and \tilde{g} equal.

2. Renormalized field theory and perturbational calculation of the equation of state

Hereafter we work with the dynamic functional [5, 29–32]

$$\mathcal{J}[\tilde{s}, s] = \int dt \int d^d r \tilde{s} \left[\partial_t s + \lambda(\tau - \nabla^2)s + \frac{\lambda g}{2} s(s - \tilde{s}) - \lambda q \right] \quad (5)$$

where $s \propto n$ and \tilde{s} denotes a Martin–Siggia–Rose response field [33]. Response and correlation functions can be computed by integrating products of the fields \tilde{s} and s with the weight $\exp(-\mathcal{J})$. We are especially interested in the average $M = \langle s \rangle$ which corresponds to the mean particle density. The external field λq describes the spontaneous creation of particles and breaks the symmetry $\tilde{s}(t) \leftrightarrow -s(-t)$.

In order to obtain the equation of state—i.e. M as a function of τ and q in the stationary state—we perform the shift $s = M + \phi$ and determine M by the no-tadpole requirement $\langle \phi \rangle = 0$. Since the stationary state is homogeneous the shift leads to the functional

$$\mathcal{J}_\phi[\tilde{\phi}, \phi] = \mathcal{J}_0[\tilde{\phi}, \phi] + \mathcal{J}_G[\tilde{\phi}, \phi] + \mathcal{J}_I[\tilde{\phi}, \phi] \quad (6)$$

where $\tilde{\phi} = \tilde{s}$,

$$\mathcal{J}_0[\tilde{\phi}, \phi] = \int dt \int d^d r \lambda \tilde{\phi} \left[M \left(\tau + \frac{g}{2} M \right) - q \right] \quad (7)$$

$$\mathcal{J}_G[\tilde{\phi}, \phi] = \int dt \int d^d r \tilde{\phi} \left[\partial_t \phi + \lambda (\tau + gM - \nabla^2) \phi - \frac{\lambda g}{2} M \tilde{\phi} \right] \quad (8)$$

and

$$\mathcal{J}_I = \int dt \int d^d r \frac{\lambda g}{2} \tilde{\phi} \phi (\phi - \tilde{\phi}). \quad (9)$$

The equation of state can now be written in the form

$$\lambda q = \lambda M \left(\tau + \frac{g}{2} M \right) - \text{bubble} \quad (10)$$

where the bubble represents the sum of all one-particle irreducible Feynman diagrams of the field theory (6)–(9) with one external $\tilde{\phi}$ -leg. Details of the diagrammatic analysis are given in the appendix. After dimensional regularization of ultraviolet divergences one obtains at two-loop order

$$q = M \left[\tau + \frac{g}{2} M - \frac{G_\epsilon g^2 \bar{\tau}^{1-\epsilon/2}}{\epsilon(2-\epsilon)} - \frac{g^4 \bar{\tau}^{-\epsilon}}{24} \left(gM \frac{1-\epsilon}{2} I_1 + (2I_1 + 3I_2) \bar{\tau} \right) \right] \quad (11)$$

where $G_\epsilon = \Gamma(1 + \epsilon/2)/(4\pi)^{d/2}$ is a geometrical factor and

$$\bar{\tau} = \tau + gM. \quad (12)$$

The integrals I_1 and I_2 are defined in the appendix.

The poles in ϵ can be absorbed into renormalizations of the fields and the coupling constants. The renormalizations read

$$\begin{aligned} \tilde{\phi} &\rightarrow \mathring{\tilde{\phi}} = Z^{1/2} \tilde{\phi} & \phi &\rightarrow \mathring{\phi} = Z^{1/2} \phi & M &\rightarrow \mathring{M} = Z^{1/2} M \\ q &\rightarrow \mathring{q} = Z_\lambda^{-1} Z^{1/2} q & \lambda &\rightarrow \mathring{\lambda} = Z^{-1} Z_\lambda \lambda & \tau &\rightarrow \mathring{\tau} = Z_\lambda^{-1} Z_\tau \tau \\ g &\rightarrow \mathring{g} = Z^{-1/2} Z_\lambda^{-1} Z_g g & G_\epsilon g^2 &= u \mu^\epsilon \end{aligned} \quad (13)$$

where μ denotes an external momentum scale. The required renormalization factors have been determined to two-loop order in [5]. The Z -factors are

$$\begin{aligned} Z &= 1 + \frac{u}{4\epsilon} + \left(\frac{7}{\epsilon} - 3 + \frac{9}{2} \ln \frac{4}{3} \right) \frac{u^2}{32\epsilon} \\ Z_\lambda &= 1 + \frac{u}{8\epsilon} + \left(\frac{13}{\epsilon} - \frac{31}{4} + \frac{35}{2} \ln \frac{4}{3} \right) \frac{u^2}{128\epsilon} \\ Z_\tau &= 1 + \frac{u}{2\epsilon} + \left(\frac{1}{2\epsilon^2} - \frac{5}{32\epsilon} \right) u^2 & Z_g &= 1 + \frac{u}{\epsilon} + \left(\frac{5}{4\epsilon^2} - \frac{7}{16\epsilon} \right) u^2. \end{aligned} \quad (14)$$

3. Scaling form of the equation of state

In order to improve the perturbative result (11) by the renormalization group we exploit, as usual, the invariance of the bare-field theory with respect to a variation of μ at fixed bare parameters, i.e.,

$$\mu \frac{d}{d\mu} \Big|_0 \hat{q}(\bar{\tau}, \bar{M}, \bar{g}) = 0. \quad (15)$$

This equation—expressed by renormalized quantities—is the renormalization group equation

$$[\mu \partial_\mu + \kappa \tau \partial_\tau - \frac{1}{2} \gamma M \partial_M + \beta \partial_u + \zeta - \frac{1}{2} \gamma] q(\tau, M, u; \mu) = 0 \quad (16)$$

with the Wilson functions (to two-loop order)

$$\begin{aligned} \gamma(u) &= \frac{d \ln Z}{d \ln \mu} \Big|_0 = -\frac{u}{4} + \left(3 - \frac{9}{2} \ln \frac{4}{3}\right) \frac{u^2}{16} \\ \kappa(u) &= \frac{d \ln \tau}{d \ln \mu} \Big|_0 = \frac{3u}{8} - \left(\frac{49}{4} + \frac{35}{2} \ln \frac{4}{3}\right) \frac{u^2}{64} \\ \zeta(u) &= \frac{d \ln \lambda}{d \ln \mu} \Big|_0 = -\frac{u}{8} + \left(\frac{17}{2} - \ln \frac{4}{3}\right) \frac{u^2}{128} \\ \beta(u) &= \frac{du}{d \ln \mu} \Big|_0 = u \left[-\epsilon + \frac{3u}{2} - \left(169 + 106 \ln \frac{4}{3}\right) \frac{u^2}{128} \right]. \end{aligned} \quad (17)$$

The renormalization group equation (16) can be solved by characteristics. At the fixed point (u_* with $\beta(u_*) = 0$)

$$u_* = \frac{2\epsilon}{3} \left[1 + \left(\frac{169}{192} + \frac{53}{96} \ln \frac{4}{3} \right) \frac{2\epsilon}{3} + O(\epsilon^2) \right] \quad (18)$$

the result (combined with dimensional analysis) reads

$$q(\tau, M, u; \mu) = \mu^{2+d/2} l^{z+(d-\eta)/2} q(\mu^{-2} l^{-1/\nu} \tau, \mu^{-d/2} l^{-(d+\eta)/2} M, u_*; 1) \quad (19)$$

with the critical exponents [5]

$$\begin{aligned} \eta &= \gamma(u_*) = -\frac{\epsilon}{6} \left[1 + \left(\frac{25}{288} + \frac{161}{144} \ln \frac{4}{3} \right) \epsilon + O(\epsilon^2) \right] \\ \nu &= \frac{1}{2 - \kappa(u_*)} = \frac{1}{2} + \frac{\epsilon}{16} \left[1 + \left(\frac{107}{288} - \frac{17}{144} \ln \frac{4}{3} \right) \epsilon + O(\epsilon^2) \right] \\ z &= 2 + \zeta(u_*) = 2 - \frac{\epsilon}{12} \left[1 + \left(\frac{67}{288} + \frac{59}{144} \ln \frac{4}{3} \right) \epsilon + O(\epsilon^2) \right] \\ \beta &= \frac{\nu(d+\eta)}{2} = 1 - \frac{\epsilon}{6} \left[1 - \left(\frac{11}{288} - \frac{53}{144} \ln \frac{4}{3} \right) \epsilon + O(\epsilon^2) \right] \\ \delta &= 1 + \frac{\nu(z-\eta)}{\beta} = 2 + \frac{\epsilon}{3} \left[1 + \left(\frac{85}{288} + \frac{53}{144} \ln \frac{4}{3} \right) \epsilon + O(\epsilon^2) \right] \end{aligned} \quad (20)$$

and the scaling function

$$\begin{aligned} q(\tau, M, u_*, \mu) &= M \left[\tau + \frac{g}{2} M - \frac{\epsilon}{6} \bar{\tau} \left(A - B \ln(\bar{\tau}/\mu^2) - \frac{\epsilon}{12} (\ln(\bar{\tau}/\mu^2))^2 \right) \right. \\ &\quad \left. + \frac{\epsilon^2}{72} g M (1 + I - 2 \ln(\bar{\tau}/\mu^2) + (\ln(\bar{\tau}/\mu^2))^2) \right] \end{aligned} \quad (21)$$

where

$$A = 1 + \epsilon \left(\frac{13}{288} + \frac{53}{144} \ln \frac{4}{3} - \frac{1}{4} \ln 3 - \frac{I}{3} \right) \quad (22)$$

$$B = 1 + \epsilon \left(\frac{37}{288} + \frac{53}{144} \ln \frac{4}{3} \right). \quad (23)$$

An integral representation for $I = -2.343\,9072\dots$ is given in the appendix. In order to simplify the writing we will use, hereafter, the dimensionless quantities defined by

$$\mu^{-2}\tau \rightarrow \tau \quad \mu^{-2}gM \rightarrow M \quad \mu^{-4}gq \rightarrow q. \quad (24)$$

While equation (21) cannot be used directly to investigate the equation of state in the critical region ($\tau, M \rightarrow 0$) one may map the critical region by an appropriate choice of the flow parameter l in (19) on scales on which perturbation theory can be applied. For $l = M^{v/\beta}$ equations (21) and (19) yield the scaling form

$$aq = M^\delta F(b\tau M^{-1/\beta}) \quad (25)$$

with the scaling variables $x = b\tau M^{-1/\beta}$ and $z = aq M^{-\delta}$ and the universal scaling function

$$F(x) = x + 1 + \frac{\epsilon}{6} K [(x+2) \ln(x+2) - 2(x+1) \ln 2] \\ + \frac{\epsilon^2}{72} [(x+4)(\ln(x+2))^2 - 4(x+1)(\ln 2)^2] + O(\epsilon^3) \quad (26)$$

where

$$K = 1 + \epsilon \left(\frac{85}{288} + \frac{29}{72} \ln 2 - \frac{53}{144} \ln 3 \right) = 1 + \epsilon 0.169\,9728\dots \quad (27)$$

The coefficients a and b in equation (25) are

$$a = 2 \left[1 + \frac{\epsilon}{3} + \frac{\epsilon^2}{18} \left(\frac{85}{48} + \frac{53}{12} \ln 2 - \frac{89}{24} \ln 3 - \frac{5}{2} I \right) + O(\epsilon^3) \right] \quad (28)$$

$$b = 2 \left[1 + \frac{1 + \ln 2}{6} \epsilon + \frac{\epsilon^2}{36} \left(\frac{13}{48} - \frac{89}{24} \ln 3 - 3I + \left(\frac{99}{16} + \frac{59}{12} \ln 2 - \frac{53}{24} \ln 3 \right) \ln 2 \right) + O(\epsilon^3) \right]. \quad (29)$$

They have been introduced in order to normalize the scaling function, i.e.

$$F(0) = 1 \quad F(-1) = 0. \quad (30)$$

The equation of state (25) satisfies the correct scaling behaviour in the critical region but the scaling function (26) can only be used if x is not too large. Since q is an analytic function of M for positive τ (for equilibrium systems this was first discussed by Griffiths [34]) the approximation (26) breaks down for $M \ll \tau^\beta$. In order to study this limit one may choose the flow parameter in (19) as $l = \tau^\nu$. One arrives at

$$aq = (b\tau)^{\beta\delta} \bar{F}(M(b\tau)^{-\beta}) \quad (31)$$

with the scaling variables $y = M(b\tau)^{-\beta} = x^{-\beta}$ and $\bar{z} = aq(b\tau)^{-\beta\delta} = zy^\delta$ and the scaling function

$$\bar{F}(y) = y \left\{ 1 + y + \frac{\epsilon}{6} K [(1+2y) \ln(1+2y) - 2(1+y) \ln 2] \right. \\ \left. + \frac{\epsilon^2}{72} [(1+4y)(\ln(1+2y))^2 - 4(1+y)(\ln 2)^2] + O(\epsilon^3) \right\} \quad (32)$$

which is fully analytic in y :

$$\bar{F}(y) = y \sum_{k=0}^{\infty} c_k y^k. \quad (33)$$

Of course $y^{-\delta} \bar{F}(y)$ and $F(x)$ match in the ϵ -expansion up to second order.

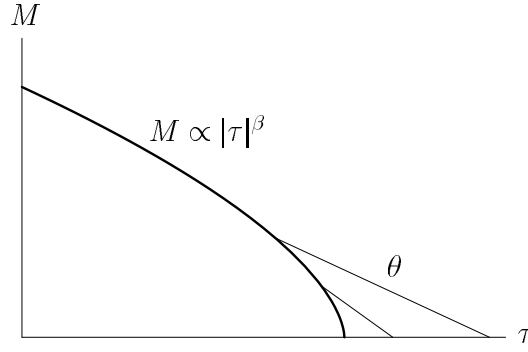


Figure 1. Sketch of the parametric representation (34). The straight lines correspond to a variation of θ at fixed R .

4. Parametric form of the equation of state

The full range of the variables τ , M (including the active phase for $\tau < 0$) can be investigated by introducing a parametric representation for the equation of state. For equilibrium systems such a representation was first suggested by Schofield *et al* [26] and Josephson [27] and later derived in an ϵ -expansion by Brézin *et al* [28]. To apply this method to directed percolation we write τ and M as

$$b\tau = R(1 - \theta) \quad M = R^\beta \theta / \theta_0. \quad (34)$$

Inserting (34) with $\theta_0 = 2$ into (25) one finds

$$aq = (R^\beta / 2)^\delta H(\theta) \quad (35)$$

with the simple scaling function

$$H(\theta) = \theta(2 - \theta) + O(\epsilon^3). \quad (36)$$

The parameter range $R \geq 0$, $0 \leq \theta < 2$ is required to describe the whole phase diagram around the critical point. The case $\tau < 0$, $q \rightarrow +0$ corresponds to the limit $\theta \rightarrow 2$ (from below). (See figure 1.)

As a simple application of the parametric representation we briefly discuss the susceptibility

$$\chi = \left. \frac{\partial M}{\partial q} \right|_\tau \quad (37)$$

which satisfies a power law for $q \rightarrow 0$,

$$\chi = \chi_\pm |\tau|^{-\gamma} \quad (38)$$

where χ_+ and χ_- correspond to the cases $\tau > 0$ and $\tau < 0$, respectively, and $\gamma = \beta(\delta - 1)$. To second order in ϵ the susceptibility can be written in terms of the parameters (34) as

$$\chi = a 2^{\delta-1} R^{-\gamma} \frac{1 - (1 - \beta)\theta}{\beta\delta\theta(2 - \theta) + 2(1 - \theta)^2}. \quad (39)$$

Therefore the universal amplitude ratio χ_-/χ_+ can be expressed to this order by the exponent β as

$$\frac{\chi_-}{\chi_+} = 2\beta - 1 + O(\epsilon^3) = 1 - \frac{\epsilon}{3} \left[1 - \left(\frac{11}{288} - \frac{53}{144} \ln \frac{4}{3} \right) \epsilon + O(\epsilon^2) \right] \quad (40)$$

$$= 1 - \frac{\epsilon}{3} (1 + \epsilon 0.067\,6885 \dots). \quad (41)$$

The parametric equation of state allows us also to express the expansion coefficient c_0 in (33) to second order in ϵ by δ . For this purpose we write

$$x = 2^{1/\beta} \theta^{-1/\beta} (1 - \theta) \quad y = \frac{\theta/2}{(1 - \theta)^\beta} \quad z = \theta^{-\delta} H(\theta) \quad \bar{z} = \frac{2^{-\delta} H(\theta)}{(1 - \theta)^{\beta\delta}} \quad (42)$$

which gives to $O(\epsilon^2)$

$$\frac{\bar{F}(y)}{y} = \frac{\bar{z}}{y} = \frac{2^{1-\delta}(2 - \theta)}{(1 - \theta)^{\beta(\delta-1)}}. \quad (43)$$

For $y \sim \theta \rightarrow 0$ this becomes

$$\begin{aligned} c_0 &= 2^{2-\delta} + O(\epsilon^3) = 1 - \frac{\epsilon}{3} \ln 2 - \frac{\epsilon^2}{864} (85 + 164 \ln 2 - 106 \ln 3) \ln 2 + O(\epsilon^3) \\ &= 1 - \epsilon 0.231\,049 - \epsilon^2 0.065\,9639 + O(\epsilon^3). \end{aligned} \quad (44)$$

5. Conclusion

We have derived a universal equation that describes the order parameter of directed percolation systems as a function of the scaling fields near the transition point to second order in $\epsilon = 4 - d$. Using a parametric representation for the thermodynamic variables our result could be written in a very simple form. This analysis was motivated by the success of a parametric equation of state (analogous to (34)–(36)) for equilibrium critical phenomena which was shown to be in good agreement with experiments [26]. Our hope is that our results will help to analyse data obtained by Monte Carlo simulations. It would be especially interesting to see how important the $O(\epsilon^3)$ corrections to equation (36) are in practice.

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Appendix

We give some details of the diagrammatic analysis required to compute the equation of state. The diagrams contributing to two-loop order are given in figure A1. The Gaussian propagator follows from the action \mathcal{J}_G in equation (8) as

$$t \text{ --- } \overbrace{\text{---}}^q \text{ --- } t' = \Theta(t - t') \exp(-\lambda(\bar{\tau} + q^2)(t - t')) \quad (A1)$$

where $\Theta(t)$ denotes the step function and q is the momentum carried by the line. The correlator reads

$$t \text{ --- } \overbrace{\text{---}}^q \text{ --- } t' = \frac{gM}{2(\bar{\tau} + q^2)} \exp(-\lambda(\bar{\tau} + q^2)|t - t'|) \quad (A2)$$

and the vertices

$$\begin{array}{c} \nearrow \\ \text{---} \\ \searrow \end{array} = - \begin{array}{c} \nwarrow \\ \text{---} \\ \swarrow \end{array} = \lambda g \quad (A3)$$

represent the interaction (9).

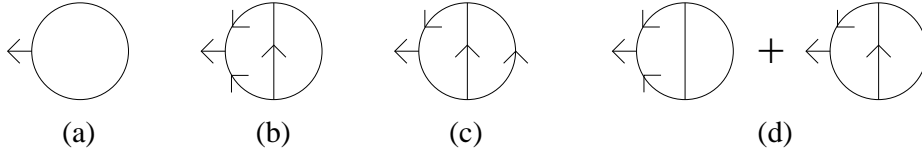


Figure A1. Contributions to the equation of state to two-loop order.

After integration over the internal time arguments the Feynman diagrams are

$$(a) = -\frac{\lambda g^2 M}{4} \int_p \frac{1}{\bar{\tau} + p^2} = \frac{G_\epsilon}{\epsilon(2-\epsilon)} \lambda g^2 M \bar{\tau}^{1-\epsilon/2} \quad (A4)$$

$$(b) = \frac{\lambda g^4 M}{12} \int_{k,p} \frac{1}{(\bar{\tau} + k^2)(\bar{\tau} + p^2)(\bar{\tau} + (\mathbf{k} + \mathbf{p})^2)} = \frac{\lambda g^4 M}{12} \bar{\tau}^{1-\epsilon} I_1 \quad (A5)$$

$$(c) = \frac{\lambda g^4 M}{8} \int_{k,p} \frac{1}{(\bar{\tau} + k^2)^2(3\bar{\tau} + k^2 + p^2 + (\mathbf{k} + \mathbf{p})^2)} = \frac{\lambda g^4 M}{8} \bar{\tau}^{1-\epsilon} I_2 \quad (A6)$$

$$(d) = -\frac{\lambda g^5 M^2}{16} \int_{k,p} \frac{1}{(\bar{\tau} + k^2)^2(\bar{\tau} + p^2)(\bar{\tau} + (\mathbf{k} + \mathbf{p})^2)} \\ = \frac{\lambda g^5 M^2}{16} \frac{1}{3} \frac{\partial}{\partial \bar{\tau}} \int_{k,p} \frac{1}{(\bar{\tau} + k^2)(\bar{\tau} + p^2)(\bar{\tau} + (\mathbf{k} + \mathbf{p})^2)} \\ = \frac{\lambda g^5 M^2}{16} \frac{1-\epsilon}{3} \bar{\tau}^{-\epsilon} I_1 \quad (A7)$$

where we have used the notation $\int_k \dots = (2\pi)^{-d} \int d^d k \dots$

The ϵ -expansions of the integrals I_1 and I_2 read

$$I_1 = \int_{k,p} \frac{1}{(1+k^2)(1+p^2)(1+(\mathbf{k}+\mathbf{p})^2)} = -\frac{6G_\epsilon^2}{\epsilon^2} \left[1 + \frac{3\epsilon}{2} + \frac{\epsilon^2}{4}(7+I) + O(\epsilon^3) \right] \quad (A8)$$

with

$$I = \int_0^1 dx \frac{\ln(x(1-x))}{1-x(1-x)} = -2.343\,9072\dots \quad (A9)$$

and

$$I_2 = \int_{k,p} \frac{1}{(1+k^2)^2(3+k^2+p^2+(\mathbf{k}+\mathbf{p})^2)} = \frac{3G_\epsilon^2}{2\epsilon} \left[1 + \frac{\epsilon}{2}(3 - \ln 3) + O(\epsilon^2) \right]. \quad (A10)$$

The finite parts of the integrals are required to calculate the coefficient K in the scaling function (26) and the normalization constants a and b (28), (29).

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