Multifractal characterization of urban form and growth: the case of Beijing

Article *in* Environment and Planning B Planning and Design · January 2013

DOI: 10.1068/b36155

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Environment and Planning B: Planning and Design 2013, volume 40, pages 884 – 904

doi:10.1068/b36155

Multifractal characterization of urban form and growth: the case of Beijing

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Received 18 November 2009; in revised form 22 January 2013

Abstract. Urban form takes on properties similar to random growing fractals and can be described in terms of fractal geometry. However, a model of simple fractals is not effectual enough to characterize both the global and local features of urban patterns. In this paper multifractal measurements are employed to model urban form and analyze urban growth. The capacity dimension D_0, information dimension D_1, and correlation dimension D_2 of a city's pattern can be estimated utilizing the box-counting method. If $D_0 > D_1 > D_2$ significantly, the city can be treated as a system of multifractals, and two sets of fractal parameters, including global and local parameters, can be used to spatially analyze urban growth. In this case study, multifractal geometry was applied to Beijing city, China. The results based on the remote-sensing images taken in 1988, 1992, 1999, 2006, and 2009 show that the urban landscape of Beijing bears multiscaling fractal attributes. The dimension spectrum curves show several abnormal aspects, especially the upper limit of the global dimension breaks through the Euclidean dimension of embedding space and the local dimension fails to converge in a proper way. The geographical features of Beijing's spatiotemporal evolution are discussed, and the conclusions may be instructive for spatial optimization and city planning in the future.

Keywords: urban form, urban growth, monofractal, multifractals, multiscaling exponent, singularity spectrum, box-counting method, Beijing

1 Introduction

Fractal theory has been applied to urban studies for more than twenty years, and in particular fractal dimension has proved to be an effective spatial measurement for urban form and growth. Fractal geometry and related concepts have provided us with powerful tools for researching the spatial organization of urban patterns (Batty and Longley, 1994; Frankhauser, 1994). In addition, they also provide a new way of looking at cities (Batty, 1995). The patterns of urban form and the process of urban growth are in fact scaling phenomena with no characteristic scale. Therefore, urban landscapes cannot be fully described using the conventional measurements, such as length (eg, urban perimeter), area (eg, built-up area), size (eg, urban population), or density (eg, population density). Using the fractal method, the conventional measurements of cities can be converted into scaling exponents from which some kind of fractal dimension can be derived. In urban research, a city is usually treated as a simple fractal with only one scaling process. A simple fractal is also referred to as a 'monofractal' (Frankhauser, 1998). Compared with multifractal analysis, the monofractal analysis of urban morphology has at least three limitations. First, the monofractal dimension suffers from a lack of self-comparability. The use and value of a spatial measurement can be illustrated through comparison. The comparison of monofractal dimensions can be drawn in a temporal framework for a city over many years or in a cross-sectional framework for different cities at the same time. However, for a city in a specific time, the fractal dimension can hardly be brought into comparison. Second, the monofractal method cannot be used to reveal the developing details of urban patterns. The monofractal dimensions are applied mainly to global analysis, but they are not useful at the local scale. Third, monofractal models cannot provide different points of view for examining urban form and growth. For example, using the monofractal dimension, it is impossible to investigate a city at both the macro and micro levels. Due to their complexity, cities in the real world may consist of multifractals rather than monofractals.

The above shortcomings of monofractal analysis can be overcome by utilizing multifractal analysis. Actually, there is inherent compatibility between monofractals and multifractals for mathematical modeling and quantitative characterization of cities, meaning that a monofractal is a special case of multifractals. If a city can be described in multifractal terms, it can also be characterized by the monofractal dimension. Many geographers and urban scientists have attached importance to multifractal measurements (Ariza-Villaverde et al, 2013; Batty and Longley, 1994; Frankhauser, 2008; Goodchild, 1980; Lam and Quattrochi, 1992), and multifractal theory has been applied to the spatial distribution of human populations, systems of cities, and rank-size distributions of settlements (Appleby, 1996; Chen, 1995; 2012; Chen and Zhou, 2004; Haag, 1994). However, applying multifractal studies to urban morphology is a much rarer occurrence.

This paper attempts to use multifractal geometry to characterize patterns of urban form and growth. The analytical process is organized as follows. To begin with, a brief introduction is made for multifractals, two sets of multifractal parameters, and a practical approach to estimating multifractal spectrums. Then, as a case study, the multifractal method is applied to the city of Beijing, China, based on five years of remote sensing data. Further, the spatial problems of urban development are brought to light, the results of which may be helpful for our understanding of urban evolution and improving the city planning of metropolises in the future. The paper's main academic contributions to urban studies are listed below. First, this paper will attempt to illustrate a practical approach to calculating the multifractals spectrums of urban form. Second, the paper will present a framework of multifractal analysis for urban research. Third, the paper will show that actual cities may be compared with multifractals; however, the multiscaling structure can be influenced by human activities.

2 Multifractal measurements of complex systems

2.1 Multifractals and multiscaling cities

A fractal can be defined as a shape made of parts that are similar to the whole in some way (Feder, 1988). The three elements of fractals are form, chance, and dimension. Generally speaking, the Hausdorff–Besicovitch dimension of a fractal set strictly exceeds its topological dimension (Mandelbrot, 1983). The basic nature of fractals is dilation symmetry, or scaling invariance. Among various types of fractals, growing fractals attracted the attention of urban geographers because this type of fractals can be used to model or analogize urban growth. If a growing fractal develops with the same probability at different growth points and at the same rate in different directions which are orthogonal, it is a simple self-similar fractal. If the rates of development in various directions are different, but the probabilities of growth at different points are the same, the growing fractal is self-affine. If the growth probabilities of various parts are different, the growing fractal is considered to be a multiscaling process, which creates a multifractal pattern (table 1). The similarities and differences between monofractals, including a simple self-similar fractal and a simple self-affine fractal, and multifractals are illustrated in figure 1 (Jullien and Botet, 1987; Vicsek, 1989).

Table 1. Comparison between monofractals (simple fractals) and multifractals (complex fractals).

Item	Monofractal		Multifractals
	self-similar fractal	self-affine fractal	
Form/pattern	self-similarity	self-affinity	both self-similarity and self-affinity
Growth	isotropy	anisotropy	isotropy
Developing chance	equal probability at different directions and parts	inequal probability at different directions	inequal probability at different parts
Dimension	single dimension value	different dimension values at different directions	spectrums of fractal dimension values
Scaling	monoscaling process	monoscaling process	multiscaling processes

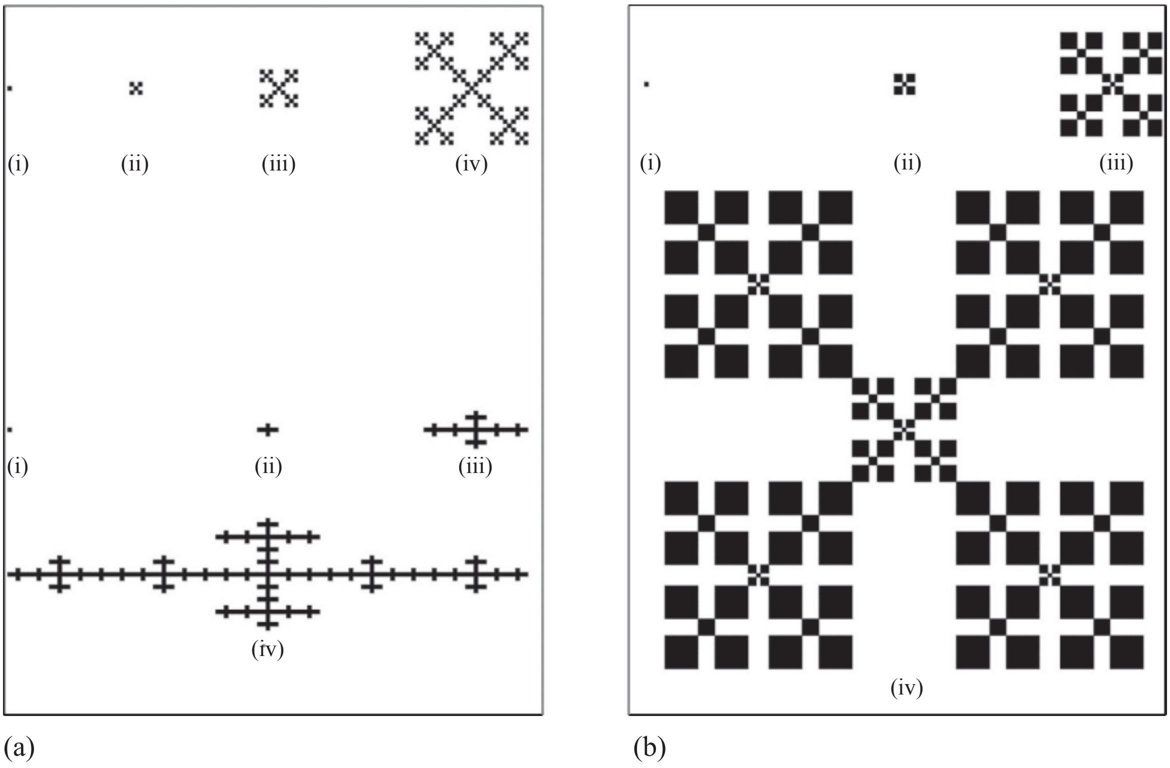


Figure 1. The first four generating steps of three types of regular growing fractals: a self-similar fractal (upper left), a self-affine fractal (lower left), and a multifractals (right).

Both monofractals and multifractals are self-similar hierarchies, yet they are different. Each hierarchy is made up of many levels, and each level consists of many fractal copies/units and corresponds to a step of fractal generation. In a monofractal hierarchy, each fractal unit at the same level has the same growth probability. However, in a multifractal hierarchy, different fractal units at the same level have different probabilities of growth (Chen, 2012). Therefore, a monofractal has one scaling process, while multifractals have two or more scaling processes. A fractal city can be reduced to a hierarchy (Batty and Longley, 1994; Frankhauser, 1994). If various places at the same level of an urban hierarchy have equal chances to develop, the city fractal will be a monofractal; otherwise, it may have a multifractal structure. It is easy to distinguish between monofractals and multifractals in regular geometric shapes (figure 1).

However, for random distributions, it is difficult to differentiate one from another. For a city, we can estimate its fractal dimension values by using the box-counting method, and then judge whether or not the urban form is a monofractal or multifractal phenomenon.

The basic properties of multifractal cities are as follows. First, a multifractal city can be treated as a monofractal, which can be measured with a kind of fractal dimension, such as a box dimension and a radial dimension. As stated above, a simple fractal is a special type of a complex fractal. The self-similar growing fractal displayed in the upper region of figure 1(a) can be likened to a special type of growing multifractals displayed in figure 1(b). Second, a multifractal city may exhibit a form and growth patterns similar to self-affine growing fractals. For the regular fractals, it is easy to see that a fractal with the self-affine growth pattern, as shown in the lower region of figure 1(a), differs from the multifractal growth pattern in figure 1(b). However, for random growth, the different rates of growth in different directions of self-affine fractals are hard to distinguish from the different probabilities of growth of different parts. An actual city is a complex system which may be composed of both self-affine patterns and multiscaling processes.

2.2 Two sets of parameters for multifractals

Two approaches can be employed to study multifractal cities: one is a theoretical approach, while the other is empirical. In the former, one begins by building regular fractal models based on one or more postulates (Chen and Zhou, 2004), while for the latter, one begins by processing remote sensing data or digital maps (Chen, 1995). No matter which method is utilized, two sets of parameters are needed to characterize a multifractal pattern. One set is the global parameters, and the other, the local parameters. The global parameters include the generalized correlation dimension and the mass exponent, while the local parameters include the Lipschitz–Hölder exponent, and the fractal dimension of the set supporting this exponent (Feder, 1988). The global and local parameters can both be calculated using the box-counting method. The generalized correlation dimension is always defined in the following form (Grassberger, 1985; Hentschel and Procaccia, 1983):

$$D_q = -\lim_{\varepsilon \to 0} \frac{I_q(\varepsilon)}{\ln \varepsilon} = \frac{1}{q-1} \lim_{\varepsilon \to 0} \frac{\ln \sum_{i=1}^{N(\varepsilon)} P_i(\varepsilon)^q}{\ln \varepsilon}, \tag{1}$$

where D_q refers to the order-q generalized correlation dimension, P to the probability of growth for the ith fractal copies/units with a linear size of ε, and N to the number of fractal copies in any given level. The parameter q is referred to as the order of moment in statistics. I denotes the Renyi's information entropy (Renyi, 1970), which is defined as:

$$I_q(\varepsilon) = \frac{1}{1-q} \ln \sum_{i=1}^{N(\varepsilon)} P_i(\varepsilon)^q . \tag{2}$$

If $q = 1$, then equation (2) transforms into the well-known formula of Shannon's information entropy. Within the set of D_q, three values are commonly used. If $q = 0$, $D_q = D_0$ refers to the *capacity dimension*; if $q = 1$, $D_q = D_2$ refers to the *information dimension*; and if $q = 2$, $D_q = D_2$ refers to the *correlation dimension*. In theory, q is a continuous variable. In other words, we have $q \in (-\infty, +\infty)$. However, in practice, q is always a discrete sequence. Generally speaking, we can take an ordered set of quantities such as $q = \ldots, -50, -49, \ldots, -2, -1, 0, 1, 2, \ldots, 49, 50, \ldots$. But sometimes, to depict the detailed changes of fractal parameters, we can utilize fractions such as $q = \ldots, -0.2, -0.1, 0, 0.1, 0.2, \ldots$ which results in a multifractal spectrum for global analysis (Grassberger, 1983; Grassberger and Procaccia, 1983; Hentschel and Procaccia, 1983).

The mass exponent can be defined by the following partition function (Halsey et al, 1986; Meakin, 1998):

$$\lim_{\varepsilon \to 0} \sum_{i=1}^{N(\varepsilon)} P_i(\varepsilon)^q \propto \varepsilon^{\tau(q)} , \tag{3}$$

where $\tau(q)$ refers to the mass exponent of order q, and the symbol '$\propto$' means 'directly proportional to'. Comparing equation (3) with equation (1) shows the relationship between $\tau(q)$ and D_q. Thus we have

$$\tau(q) = D_q(q - 1) . \tag{4}$$

Both D_q and $\tau(q)$ make up the set of global parameters of multifractal sets. If the D_q spectrum is calculated by a certain method, then the $\tau(q)$ values can be indirectly estimated through the D_q values. The relationship between D_q and q yields the multifractal dimension spectrum for global analysis of the urban form. By changing the value of q, more details of spatial development of a city can be recognized $(q \to \infty)$, or more details can be ignored while surveying all patterns $(q \to -\infty)$.

A multifractal set (the whole) has many fractal subsets (parts), so it is necessary to make local analyses of multifractal cities. In fact, various fractal subsets have different probabilities of growth. Each growing probability corresponds to a power law, such as $P_i(\varepsilon) \propto \varepsilon_i^{\alpha(q)}$, where $\alpha(q)$ denotes the Lipschitz–Hölder exponent, which is also known as the *exponent of singularity* (Feder, 1988). In a multifractal system, each fractal subsystem has its own fractal dimension. The local dimension can be defined by $N_i(\alpha, \varepsilon) \propto \varepsilon_i^{-f(\alpha)}$, where $N(\alpha, \varepsilon)$ refers to the number of the smaller fractal copies, and $f(\alpha)$ to the fractal dimension of the subset supporting the exponent $\alpha(q)$ (Feder, 1988; Frisch and Parisi, 1985). The $\alpha(q)$ and $f(\alpha)$ compose the set of local parameters of the multifractal sets. The relationship between $f(\alpha)$ and α forms the multifractal dimension spectrum for local analysis of urban form and growth. By changing the value of q, attention can be focused on locations with a high probability of growth $(q \to \infty)$, or, conversely, focus attention on locations with a low probability of growth $(q \to -\infty)$.

However, it is impractical to calculate the local parameters for each part of a random fractal. An advisable approach to revealing the local information is to make use of the moment order q. To change the values of q is to change the 'resolution' of the fractal patterns, which will cause various parts of the fractal to come into focus. Consequently, the local parameters can be estimated by the scaling process. In theory, the global and local parameters can be connected with and converted into one another by the Legendre transform (Badii and Politi, 1997; Feder, 1988):

$$\alpha(q) = \frac{\mathrm{d}\tau(q)}{\mathrm{d}q} = D_q + (q - 1)\frac{\mathrm{d}D_q}{\mathrm{d}q} , \tag{5}$$

$$f(\alpha) = q\alpha(q) - \tau(q) = q\alpha(q) - (q - 1)D_q . \tag{6}$$

Using equations (1), (4), (5), and (6), the special values of D_q, $\tau(q)$, $\alpha(q)$, and $f(\alpha)$ can be found. These values are listed in table 2, which gives a reference for the multifractal analysis of cities.

The box dimension of a fractal is greater than its topological dimension, but less than the Euclidean dimension of the embedding space in which the fractal exists (Mandelbrot, 1983; Vicsek, 1989). Since urban forms were examined through the digital maps based on remote-sensing images defined in two-dimensional space, the D_q values of a city should theoretically be between 0 and 2: ie, $d_T = 0 =< D_{+\infty} < D_q < D_{-\infty} < d_{\mathrm{E}} = 2$. Here $d_{\mathrm{T}} = 0$ refers to the topological dimension of urban form, and $d_{\mathrm{E}} = 2$ to the Euclidean dimension of

Table 2. Special values and limitations of the generalized dimension spectrum (D_q) and the related fractal parameters [τ(q), $\alpha(q)$, and $f(\alpha)$].

q	D_q	$\tau(q) = (q-1)D_q$	$\alpha(q) = \mathrm{d}\tau(q)/\mathrm{d}q$	$f(\alpha) = q\alpha(q) - \tau(q)$
$\to -\infty$	$D_{-\infty}$	$\to -qD_{-\infty}$	$\alpha_{\max} \to D_{-\infty}$	$f(\alpha_{\max}) \to f[\alpha(-\infty)]$
0	D_0	D_0	α_0	$f_{\max} = D_0$
1	D_1	0	$\alpha_1 = D_1$	$f(\alpha) = D_1$
2	D_2	D_2	$\alpha(2)$	$2\alpha(2) - D_2$
$\to +\infty$	$D_{+\infty}$	$\to qD_{+\infty}$	$\alpha_{\min} \to D_{+\infty}$	$f(\alpha_{\min}) \to f[\alpha(+\infty)]$

Note: The arrow '→' denotes 'approaching' or 'nearness' (refer to Chen and Zhou, 2004; Feder, 1988, page 84). A multifractal object in a two-dimensional embedding space differs from that in a one-dimensional embedding space.

the embedding space of urban figures. Accordingly, the $\alpha(q)$ values vary from $D_{+\infty}$ to $D_{-\infty}$ and the $f(\alpha)$ values, which are equal to or less than D_0, must fall between 0 and 2 (table 2). Therefore, if the value of D_q, $\alpha(q)$, or $f(\alpha)$ is greater than 2, it can be treated as an abnormal result. On the other hand, D_q and $\alpha(q)$ are monotonic decreasing functions, while $\tau(q)$ is a monotonic increasing function of q. That is to say, the values of D_q and $\alpha(q)$ are both monotonic decreasing quantities: $D_q > D_{q+1}$, $\alpha(q) > \alpha(q+1)$, whereas $\tau(q)$ is a monotonic increasing quantity: $\tau(q) < \tau(q+1)$. Function $f(\alpha)$ has the maximum $f[\alpha(q)]_{\max} = D_0$ when $q = 0$. In fact, if $f(\alpha)$ is a monotonic decreasing quantity, and if $q \leqslant 0$, $f(\alpha)$ is a monotonic increasing quantity. For $q \geqslant 0$, the expected result is $f[\alpha(q)] > f[\alpha(q+1)]$, while for $q \leqslant 0$, we have $f[\alpha(q)] < f[\alpha(q+1)]$. Otherwise, the values cannot be theoretically acceptable.

2.3 A practical approach to estimating multifractal dimension spectrums

In practice, multifractal dimension spectrums can be estimated using the box-counting method. Suppose that there is an urban figure consisting of n pixels on a digital map with a given resolution. If the number of pixels covered by the ith 'box' is $n_i(\varepsilon)$, then the 'probability of distribution' of the place corresponding to this box can be defined by the equation:

$$P_i(\varepsilon) = \frac{n_i(\varepsilon)}{n} = \frac{n_i(\varepsilon)}{\sum_{i=1}^{N(\varepsilon)} n_i(\varepsilon)}, \tag{7}$$

in which $N(\varepsilon)$ denotes the number of nonempty boxes with a linear size of ε. Based on the probability measurements, the values of D_q and thus $\tau(q)$ can be calculated. Using the Legendre transform, the values of $\alpha(q)$ and $f(\alpha)$ can be estimated. However, this approach is not very convenient because it involves a differential equation.

As an alternative, the μ-weight method, proposed by Chhabra and Jensen (1989), can be employed to calculate the values of $\alpha(q)$ and $f(\alpha)$. Defining a weight measurement such as

$$\mu_i(\varepsilon) = \frac{P_i(\varepsilon)^q}{\sum_i P_i(\varepsilon)^q}, \tag{8}$$

we have

$$\alpha(q) = \lim_{\varepsilon \to 0} \frac{1}{\ln \varepsilon} \sum_i \mu_i(\varepsilon) \ln P_i(\varepsilon), \tag{9}$$

and

$$f(\alpha) = \lim_{\varepsilon \to 0} \frac{1}{\ln \varepsilon} \sum_i \mu_i(\varepsilon) \ln \mu_i(\varepsilon) . \tag{10}$$

The linear regression based on the ordinary least squares (OLS) method can be adopted to estimate the values of $\alpha(q)$ and $f(\alpha)$. After evaluating $\alpha(q)$ and $f(\alpha)$ with equations (8), (9), and (10), the value of D_q can be computed using equation (6), and then $\tau(q)$ can be computed using equation (4).

3 Multifractal description of Beijing's urban development

3.1 Study area and data

In this section the multifractal measurements will be applied to the multiscaling properties of the development pattern of Beijing, the capital of China. Beijing is a typical megacity with an urban population in 2009 of approximately 14.918 million inside the metropolitan area. The area administered by Beijing municipality is about16 800 km^2, but the 'metropolitan area' of the city is almost 13 800 km^2. According to the fifth population census, there were approximately 8.5 million people within the 'metropolitan area' of Beijing in 2000. The urban development pattern will be investigated through remote sensing images for five years: 1988, 1992, 1999, 2006, and 2009. The ground resolution of these images is 30 m. In fact, a number of remote sensing images of Beijing from NASA are available for spatial analysis (appendix A). Only three Landsat TM images and two Landsat ETM+ images are more appropriate for this study because they were taken chiefly in autumn or winter so there is less obstruction from clouds and plants. The five images were rectified to a common UTM coordinate system. The development of Beijing metropolis and its neighboring rural settlements can be seen from the images (figures 2 and 3). The results comprise innumerable patches, which appears similar to the irregular and fragmented patterns with statistical self-similarity and fractional dimension.

To study the urban development and growth of Beijing using fractal geometry, we should find a viewing frame to focus on the city center. The viewing frame, or viewing window, is a rectangle on the digital maps, from which the multifractal dimensions are estimated. This frame is simply the study region. There are two ways of determining the study area. One is a fixed study area, which was illustrated by Batty and Longley (1994) and used by Shen (2002), and the other is a variable study area recommended by Benguigui et al (2000) and utilized by Feng and Chen (2010). If a fixed study area is chosen, the frame does not change because of urban growth. In contrast, if a variable study area is adopted, the frame will depend on the size of the city. For comparability of fractal dimension values, a fixed study area was defined in this study.

The method of defining a study area is as follows. First, the outline of the urban shape is drawn by extracting the city's general boundary on the digital map of 2009, creating an urban envelope (Batty and Longley, 1994; Longley et al, 1991). Next, a rectangle is determined in light of the urban envelope. The four sides of the rectangle should touch the four vertexes in the east, south, west, and north of the urban boundaries. The rectangle can be regarded as a view-finding frame, and its area should be approximately 3332 km^2 (figure 2). This study area included all the inner city districts (Dongcheng, Xicheng, Xuanwu, Chongwen), outer city districts (Haidian, Chaoyang, Fengtai, Shijingshan), and partial suburban districts (Changping, Shunyi, Mentougou, Fangshan, Daxing, Tongzhou), but did not involve exurban districts and counties (Huairou, Miyun, Pinggu, Yanqing). Finally, the same view-finding frame was applied to the urban images taken in the other years: 1988, 1992, 1999, and 2006 (figure 3).

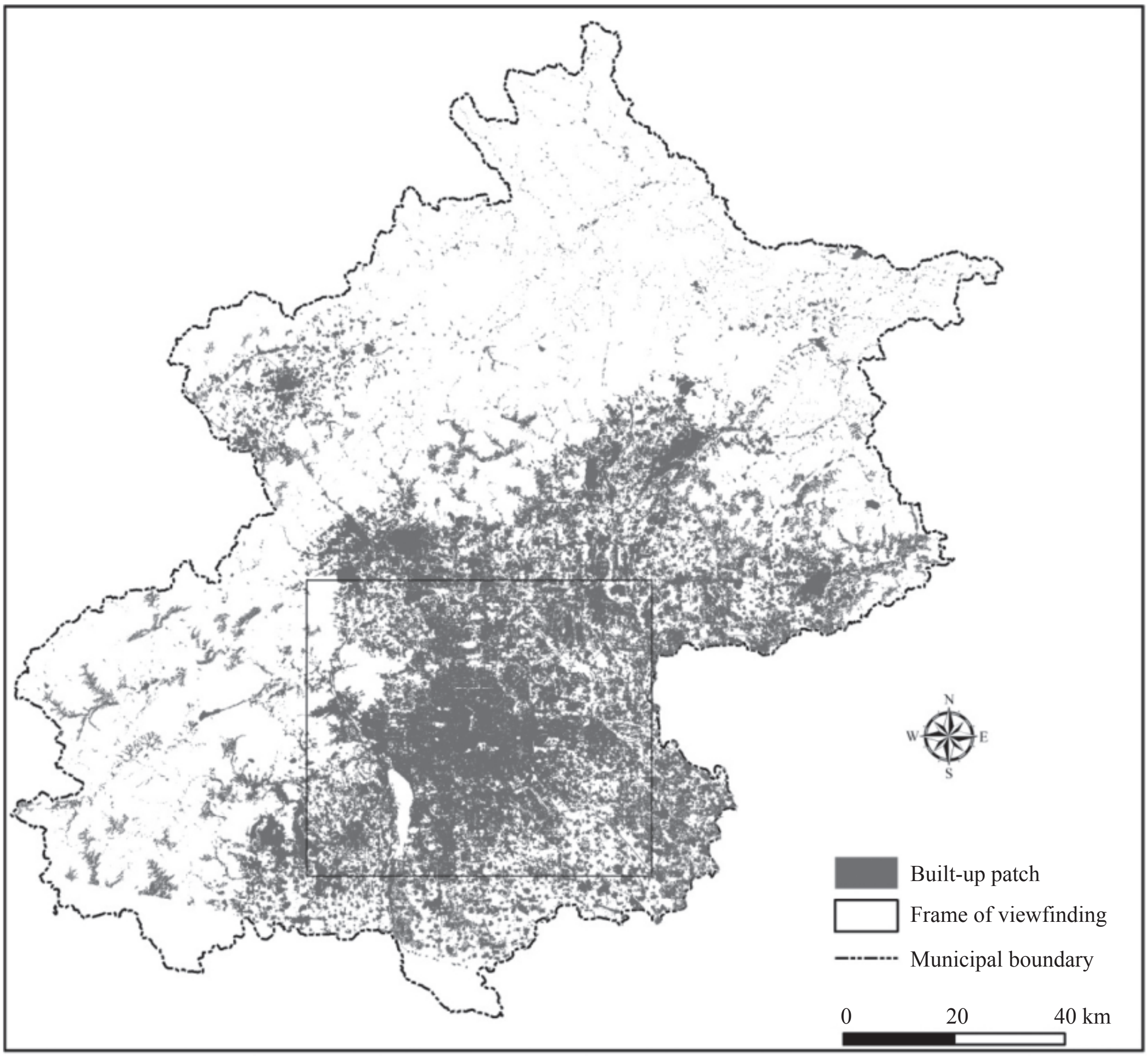
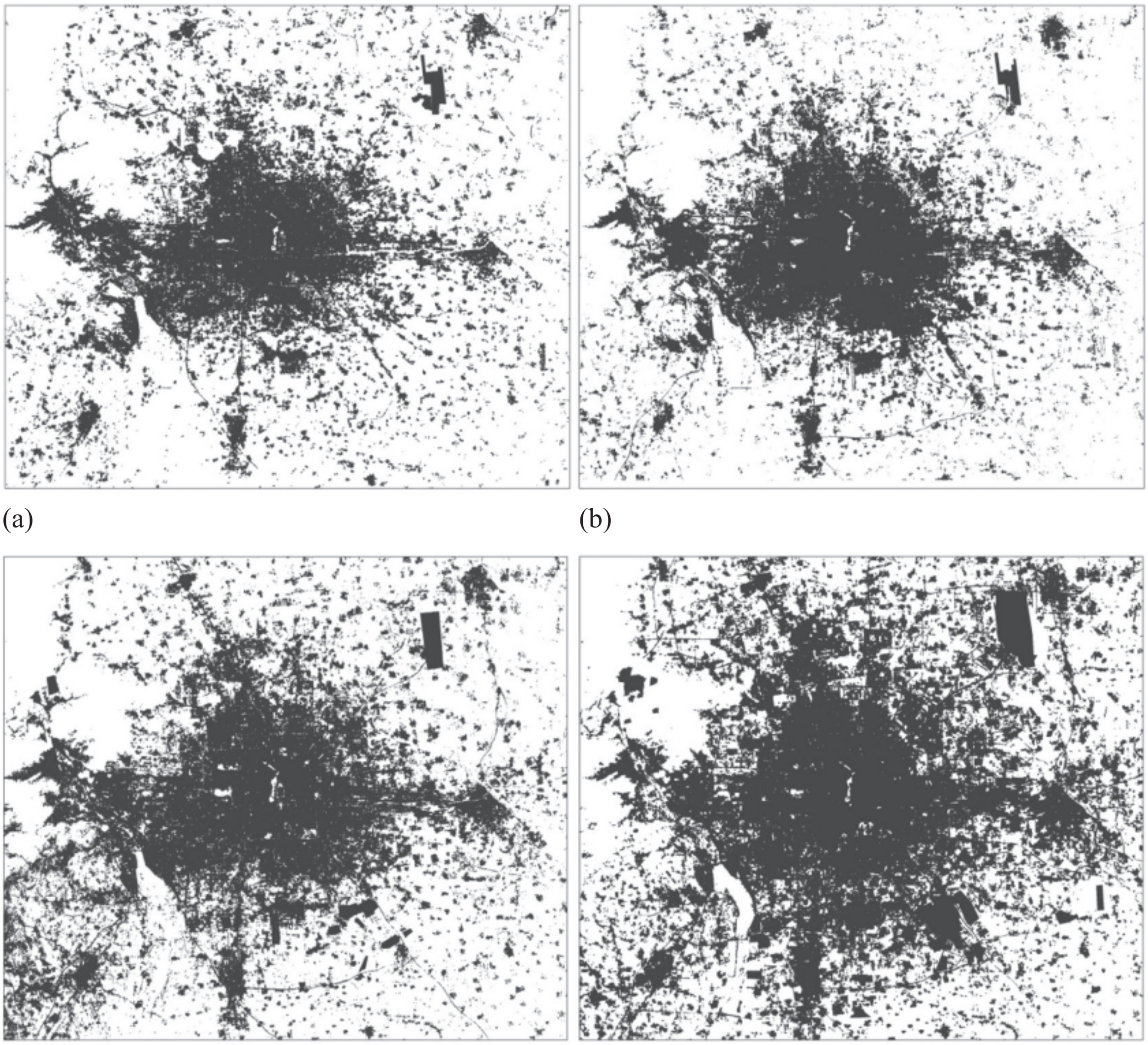


Figure 2. Urban region of Beijing and the viewing frame for fractal analysis (2009). Note: The built-up area outside the viewfinding framework mainly came from the satellite towns of Beijing.

3.2 Methods and results

The common methods of estimating fractal dimension include the area–radius scaling method (Batty and Kim, 1992; Batty and Longley, 1994; Fotheringham et al, 1989; Frankhauser, 1998; White and Engelen, 1993; 1994), the area–perimeter scaling method (Batty and Longley, 1988; Benguigui et al, 2006), the box-counting method (Benguigui et al, 2000; Shen, 2002), along with other methods (Chen, 2008; De Keersmaecker et al, 2003; Longley and Mesev, 2000; Longley et al, 1991; Thomas et al, 2007; 2008). The box-counting method and other similar methods, such as the grid method, are appropriate for measuring the multifractal dimensions of urban growth patterns. In this case study, the rectangle space subdivision (RSS) method was used to estimate fractal dimensions. RSS is a generalized box-counting method, which is similar to a type of grid method (Frankhauser, 1998). It is in fact the functional box-counting method that was originally adopted by Lovejoy et al (1987) to analyze radar rain data. In urban studies, RSS was once employed to explore fractal systems of cities and towns by Chen (1995), and was used to research fractal urban form and growth patterns by Feng and Chen (2010). The geometrical basis of the RSS method is the recursive subdivision of space (Goodchild and Mark, 1987) and the cascade structure of hierarchies (Batty and Longley, 1994). Its mathematical basis is the transformation relationship between the exponential laws based on translational symmetry and the power laws based on dilation



(a) (b) (c) (d)

Figure 3. The results of employing the viewing frame for fractal analysis of Beijing in (a) 1988, (b) 1992, (c) 1999, and (d) 2006.

symmetry (Chen, 2012; Chen and Zhou, 2006). Compared with the other methods including the dilation method and the correlation method used to obtain fractal information of urban form and growth patterns (Batty and Longley, 1994; Benguigui et al, 2000; Frankhauser, 1998), this method is more convenient and efficient and can yield a stable calculation.

The approach to estimating fractal dimension is as follows. First, draw a rectangle (find a 'view') on the digital map so that the urban figure of the most recent year is within the area. The side length of the rectangle can be regarded as one unit: ie, $\varepsilon_1 = 2^{1-1} = 1$. Accordingly, the box number is 1. Second, divide the primary rectangle into four equal smaller rectangles by using two straight lines. The side length of the subrectangles is half of one unit: ie, $\varepsilon_2 = 2^{1-2} = 1/2$. Each rectangle represents a smaller box. Thus there are four 'boxes' of secondary rectangles. Count the number of occupied 'boxes' and the pixel numbers within the occupied rectangles. Third, divide each secondary rectangle into four equal smaller rectangles. The side length of the tertiary rectangles is $\varepsilon_3 = 2^{1-3} = 1/4$, and the number of boxes is 16. The rest of the steps will follow in turn. In the mth step, the number of the rectangles is $N_m = 4^{m-1}$, and the side length of small rectangles is $\varepsilon_m = 2^{1-m} = 1/2^{m-1}$ (appendix B).

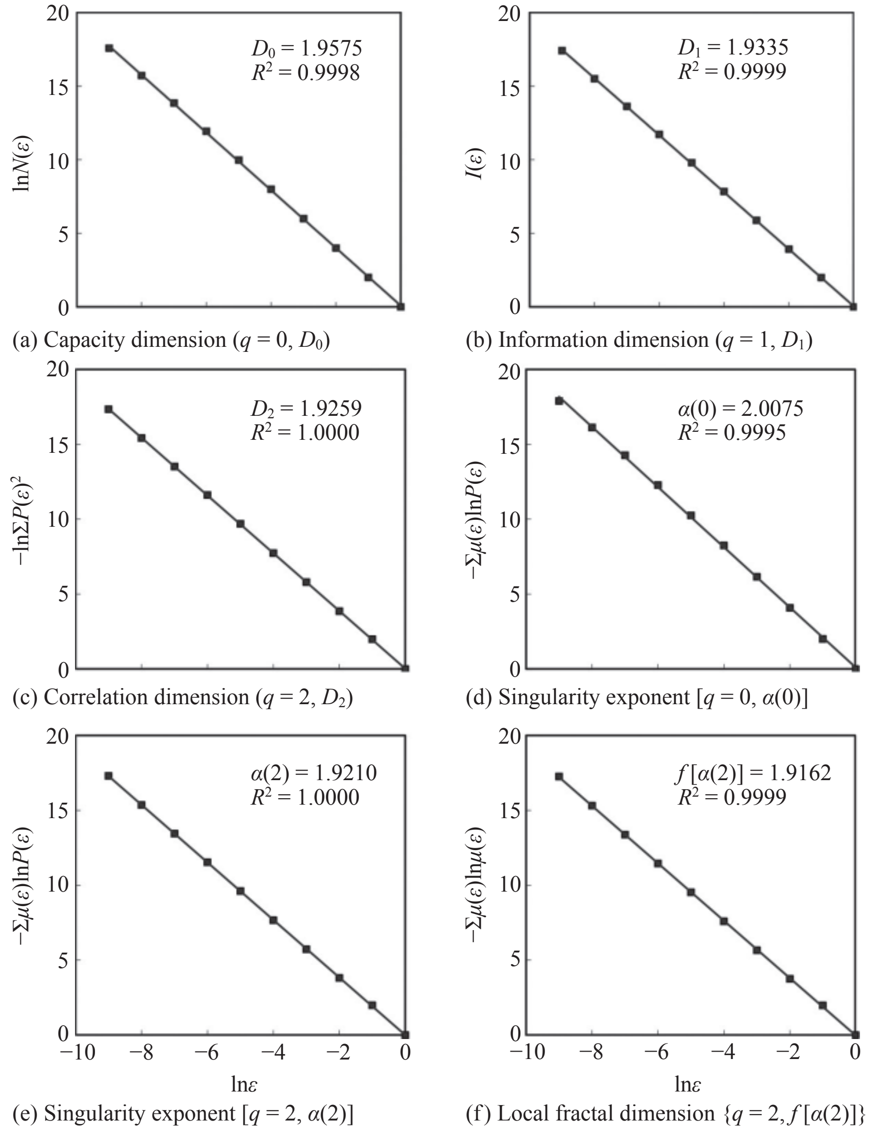


Figure 4. The plots for evaluating multifractal parameters of Beijing's urban form and growth pattern in 2009 (examples). Note: the base of the logarithm is 2. If $q = 0, f[\alpha(0)] = D_1 \approx 1.9575$; If $q = 1, f[\alpha(1)] = D_1 \approx 1.9335$.

The procedure based on the RSS method of estimating the multifractal parameters can be implemented by carefully following the steps below.

Step 1: Estimate the basic dimensions, including D_0, D_1, and D_2. First of all, compute the capacity dimension of D_0. Let $q = 0$, then equation (1) can be transformed into the inverse power function:

$$N(\varepsilon) = N_1 \varepsilon^{-D_0}, \tag{11}$$

where $N(\varepsilon)$ refers to the number of nonempty 'boxes' (rectangles), and N_1 to the proportionality coefficient. By means of this power function, we can estimate the value of D_0 using the RSS method. This is the generalized box dimension, which can be visually displayed on a log–log plot [figure 4(a)]. For the random multifractals, the box dimension is approximately equal to the value of D_0 (Chen, 1995).

Next, compute the information dimension of D_1. If $q = 1$, then the relationship between the information dimension and information quantity can be derived through equation (1) using l'Hospital's rule or the Taylor series expansion. The result is:

$$I(\varepsilon) = -\sum_{i=1}^{N(\varepsilon)} P_i(\varepsilon) \ln P_i(\varepsilon) = I_1 - D_1 \ln(\varepsilon), \tag{12}$$

where $I(\varepsilon)$ is the amount of spatial information, I_1 is a constant, and $P(\varepsilon)$ can be treated as a 'probability' defined by equation (7), in which n denotes the total number of the pixels indicating urban land-use or built-up area within the entire study region, and $n_i(\varepsilon)$ is the number of pixels or land-use area inside the ith nonempty box with a side length of ε. D_1 can then be estimated using regression analysis based on equation (12) [figure 4(b)].

Furthermore, if $q = 2$ in equation (1), the correlation dimension of urban form and growth patterns can be estimated, and we have the value of D_2 [figure 4(c)].

Step 2: Judge whether or not the urban form and growth pattern are similar to a multifractals by comparing the values of D_0, D_1, and D_2 with one another. If $D_0 \approx D_1 \approx D_2$, then the urban growth pattern can be approximately treated as a monofractal structure; if $D_0 > D_1 > D_2$ and $D_0 > D_1$ significantly, then the urban growth pattern can be seen as a multifractal structure. Specially, the inequality $D_0 < D_1 < D_2$ suggests that something went wrong during the process of fractal dimension estimation. We have $D_0 \approx 1.9575$, $D_1 \approx 1.9335$, and $D_2 \approx 1.9259$ for Beijing in 2009, and these results indicate a multifractal pattern (figure 4, table 3). Twenty years ago, in 1988, the results were $D_0 \approx 1.8507$, $D_1 \approx 1.8099$, and $D_2 \approx 1.8039$. For fractal dimension values, 1.8507 is obviously different from 1.8099. In recent years, however, the multifractal structure seems to have degenerated due to overfilling of urban space.

Step 3: Compute the spectrums of fractal dimensions and the related parameters. If the urban growth pattern is of multiscaling fractal structure, then the global parameters, D_q and $\tau(q)$, and the local parameters, $\alpha(q)$ and $f(\alpha)$, can be estimated. Based on the RSS method, the D_q values can be determined using equation (1), and the mass exponent, $\tau(q)$, can be calculated via equation (4). Thus the global parameter values of multifractal urban form can be obtained. Proceeding to the next step, the μ-weight method is used to evaluate the $\alpha(q)$ and $f(\alpha)$ values [figures 4(d)–(f)]. In fact, if the local parameters, $\alpha(q)$ and $f(\alpha)$ are calculated first, based on the RSS method, then the global parameter values can be readily calculated through the Legendre transform. According to equation (6), If $q \neq 1$, we have:

$$D_q = \frac{1}{q-1}[q\alpha(q) - f(\alpha)]. \tag{13}$$

If $q = 1$, the information dimension is simply $D_1 = \alpha(1) = f[\alpha(1)]$ (table 2).

Table 3. The multifractal dimension and related parameters of Beijing's urban form and growth pattern in 2009 (partial results).

Moment order q	Global parameter			Local parameter			
	D_q	R^2	$\tau(q)$	$\alpha(q)$	R^2	$f[\alpha(q)]$	R^2
−1	2.2268	0.9979	−4.4535	3.5060	0.9806	0.9476	0.9024
−0.5	2.0098	0.9996	−3.0147	2.3180	0.9963	1.8557	0.9995
0	1.9575	0.9998	−1.9575	2.0075	0.9995	1.9575	0.9998
1	1.9335	0.9999	0.0000	1.9335	0.9999	1.9335	0.9999
2	1.9259	1.0000	1.9259	1.9210	1.0000	1.9162	0.9999
5	1.9195	0.9999	7.6780	1.9164	0.9997	1.9039	0.9971
10	1.9182	0.9997	17.2641	1.9181	0.9994	1.9166	0.9890
15	1.9184	0.9996	26.8573	1.9191	0.9993	1.9291	0.9828
20	1.9187	0.9995	36.4544	1.9197	0.9992	1.9394	0.9782

4 Analysis of multifractal dimension spectrums

4.1 Geometric and geographical aspects of multifractal dimensions

The geometric and geographical meanings of multifractal parameter spectrums should be further explained before we look at how to analyze urban form and growth patterns using multifractal measurements. First of all, the function, use, and effect of the moment order, q, must be explained. Taking the generator (step 2) of the regular growing multifractals displayed in figure 1(b) as an example, we can see how the shapes and sizes of fractal copies change along with the values of q (figure 5). Notice that the critical value is $q = 1$, where the multifractal pattern shows no change. The probability of growth for the peripheral four parts is $P_1 = P_2 = P_3 = P_4 = 4/17$ (high probability, indicating fast growth rate), while that of the central part is $P_5 = 1/17$ (low probability, indicating slow growth rate) [figure 5(c)]. The effects of changing the value of q on the global and local parameters are different (table 4). For example, all the fractal copies are taken into account for the global parameters; however, the copies with different growth rates have different 'weight numbers'. If the value of q is increased from 1 to positive infinity (∞), the fractal copies of higher growth rates will be 'zoomed in', while the copies of slower growth rates will be relatively 'zoomed out' [figure 5(d)]. The fractal copies of different growth rates become so polarized that the parts with slower growth rates may be gradually overlooked. As a result, fewer and fewer fractal copies will be counted in fractal measurements so that the generalized correlation dimension

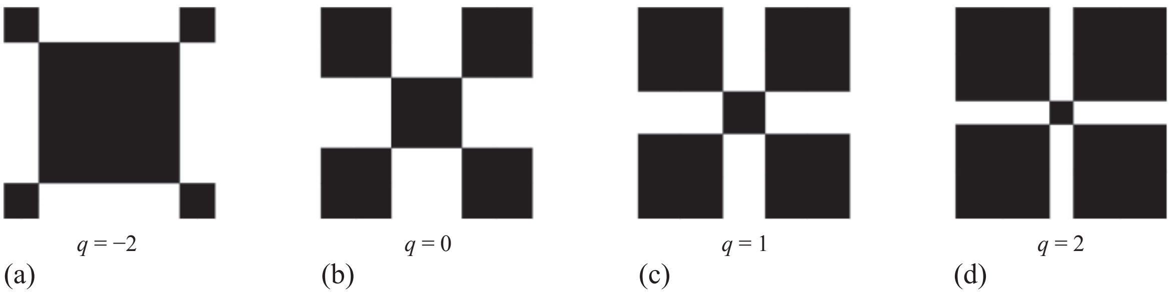


Figure 5. The changing patterns of the multifractal generator based on different values of q (four examples). Note: When $q = 1$, there will be no change in the generator and the original pattern will remain. If $q = 0$, the different fractal copies will become equal to one another. If $q = 2$, the large fractal copies will become larger, and the small ones will become smaller. If $q = -2$, the large fractal copies become smaller, while the small ones become larger. Other cases are analogous in turn.

Table 4. The geometrical and geographical effects of the multifractal parameter spectrums.

Parameter	q value	Fractal object (figure 1)	Urban form (figures 2 and 3)
D_q and τ_q	$q > 1$, $q \to \infty$	The copies with faster growth rates (higher probabilities of growth) are globally intensified	The regions of higher rate of growth are globally intensified
	$q < 1$, $q \to -\infty$	The copies with slower growth rates (lower probabilities of growth) are globally intensified	The regions of lower rate of growth are globally intensified
$f(\alpha)$ and $\alpha(q)$	$q > 1$, $q \to \infty$	The copies with faster growth rates (higher probabilities of growth) are locally put into focus	The locations of higher rate of growth are locally brought into prominence
	$q < 1$, $q \to -\infty$	The copies with slower growth rates (lower probabilities of growth) are locally put into focus	The locations of lower rate of growth are locally brought into prominence

(D_q) values become lower and lower, and finally reach the lower limit [for figure 1(b) the lower limit is $D_{+\infty} = \ln(4/17)/\ln(2/5) \approx 1.5791$]. On the other hand, if we decrease the value of q from 1 to negative infinity ($-\infty$), the fractal copies with slower growth rates will be zoomed in, while the copies with faster growth rates will be relatively zoomed out [figure 5(a)]. This is a process of reverse polarization. As a result, more and more fractal copies will be included in fractal measurements so that the generalized correlation dimension values will become higher and higher until the upper limit is reached [for figure 1(b) the upper limit is $D_{-\infty} = \ln(1/17)/\ln(1/5) \approx 1.7604$]. The degree/extent of polarization and reverse polarization can be measured with the mass exponent, $\tau(q)$.

For the local parameters, only partial fractal copies are taken into account, and other copies are ignored. The process of changing the value of q is equivalent to the process of focusing, in which the singularity exponent, $\alpha(q)$, can be used to reflect whether or not a fractal copy has been put into focus. If the probability of growth of the ith fractal copy (P_i) is close to its linear size (ε_i) to the $\alpha(q)$ power (ie, $P_i = \varepsilon_i^{\alpha(q)}$), the fractal copy will be focused, or else it will be overlooked. In this case, the local fractal dimension is $f(\alpha) = -\ln N_i/\ln(\varepsilon_i)$, where N_i refers to the number of the smaller fractal units in the ith copy. For other fractal copies, the fractal dimension cannot be properly evaluated. If we increase the value of q from 1 to plus infinity (∞), the fractal copies with faster growth rates will be focused upon individually. As soon as some kind of fractal copy is focalized, others will be neglected. If the value of q approaches plus infinity, the parts with the fastest growth rates will be brought into focus. On the other hand, if the value of q is decreased from 1 to minus infinity ($-\infty$), the fractal copies with slower growth rates will be brought into focus one by one. If the value of q approaches minus infinity, the parts with the slowest growth rates will be put into focus. The maximum value of the local dimension is $f[\alpha(0)] = D_0$, which indicates the capacity dimension [for the multifractals displayed in figure 1(b), $D_0 \approx 1.5995$].

The multifractal spectrums are based on the moment order q, which is both a parameter and a variable. Renyi (1970) employed the q to generalize the Shannon entropy. The Renyi entropy, as shown in equation (2), constitutes the principal measurement of multifractals. An entropy function is often based on a probability measure, P_i. The multifractal measurement is defined by the probability P_i to the qth power: ie, P_i^q. If $q = 1$, this will result in an information dimension of D_1 based on the Shannon entropy and the box-counting method. The information dimension mirrors the actual spatial structure of urban form and growth patterns, and it can be associated with spatial entropy, which was discussed by Batty (1974; 1976). If $q = 0$, this will result in a capacity dimension of D_0, and different probability values in different boxes will be regarded as equal. In previous studies, the capacity dimension, rather than the information dimension, was often employed to research cities (see, eg, Benguigui et al, 2000; Feng and Chen, 2010; Shen, 2002). If $q > 1$, the differences of the P_i values will be intensified and the peripheral regions (the P_i values are large) will be relatively projected. In contrast, if $q < 1$, the central regions (the P_i values are small) will be relatively highlighted. The multifractal spectrum can be assimilated to telescope and microscope in urban studies due to the value of q. If $q > 1$ and approaches ∞, a geometric ‘telescope’ will form, which will cause the spatial information in the peripheral regions of high growth rates to become stressed. If $q < 1$ and approaches $-\infty$, a geometric ‘microscope’ will be formed and the information in the central regions of low growth rates can be ‘zoomed in’ (table 4). To some extent, multifractal analysis, especially the local dimension analysis, is analogous to wavelet transform. The values of q change continuously from $-\infty$ to ∞ in theory. However, for empirical cases, multifractal dimensions can be determined only within a limited range of q. Even for regular multifractals, the values of q are also finite and fall between an upper limit q_u and a lower limit q_l ($q_u > 2$, $q_l < 0$), which can be easily identified.

If $q > q_u$ or $q < q_l$, the multifractal spectrums will diverge or break down, making the fractal parameters meaningless or incalculable.

Cities are a type of complex adaptable system (CAS), and the fractal structure of cities cannot be as normative as regular fractals. There are always various problems in urban evolution, which can be examined through mathematical modeling. The main points of the multifractal analysis of cities can be outlined as follows. First, a comparison between the empirical results and theoretical expectations of multifractal spectrums should be drawn to find the anomalies within the spectral curves. Second, the parameter values of each spectrum should be examined to determine the nature of the anomalies and where they appear. Third, the spatial information resulting from different parameters should be integrated to reveal the urban characters and reveal the urban anomalies. Compared with monofractal dimensions and other conventional measurements, multifractal dimension spectrums have several advantages. The first is self-comparability: that is, the same parameter, say D_q, can be compared with one another by changing the values of q. Another advantage is that, by considering the values of the global and local parameters, fractal dimension spectrums can provide us with multiple angles of view from which to examine a city. The most important advantage rests with the 'zooming in/out' function of the method. By changing the values of q, the high-density regions (eg, city center) can be brought into focus or the low-density regions (eg, urban fringe) can be explored more carefully.

4.2 Fractal analysis of Beijing's urban form and growth pattern

An analysis of Beijing's urban form and growth pattern can be performed in light of the geometric and geographical meanings of multifractal dimensions. Compared with the regular multifractals displayed in figure 1(b), the spatial structure of Beijing city takes on multiscaling fractal properties (table 5). First, the urban form bears clear characteristics of fractals. The scaling relationship between the number of nonempty boxes and the corresponding linear size of boxes follow the power law, and the scaling exponents indicating the fractal dimensions fall between 1 and 2 (figure 4). Second, the D_0 values are significantly different from the D_1 and D_2 values, which leads to $D_0 > D_1 > D_2$. For example, in 1988 the results were $D_0 \approx 1.8507$, $D_1 \approx 1.8099$, and $D_2 \approx 1.8039$. Obviously the value 1.8507 (D_0) differs from the values 1.8099 (D_1) and 1.8039 (D_2). Third, within a certain scale range of q,

Table 5. Comparison between Beijing's urban form and growth pattern and regular multifractal structure.

Parameter	Regular multifractals		Beijing's fractal form	
	functional property	value range	curve property	value range
D_q	monotonic decreasing	$1 \sim 2$	monotonic decreasing only when $q > q_u$	$D_q > 2$ when $q < 0$, $D_q < 2$ when $q > 0$
$\tau(q)$	monotonic increasing	$-\infty \sim \infty$	monotonic increasing	$-\infty \sim \infty$
$\alpha(q)$	monotonic decreasing	$1 \sim 2$	monotonic decreasing only when $q > q_u$	$\alpha(q) > 2$ when $q < 0$, $\alpha(q) < 2$ when $q > 0$
$f[\alpha(q)]$	monotonic increasing from 0 to D_0 when q ranges from $-\infty$ to 0, monotonic decreasing from D_0 to 0 when q ranges from 0 to ∞	$f[\alpha(q)] \leqslant D_0$	the $f[\alpha(q)]$ curve fails to converge when q ranges from 0 to ∞	$f[\alpha(q)] \leqslant D_0$, when $q > 0$; $f[\alpha(q)]$ is not always of monotonic decreasing

Note: The upper limit value q_u varied from 2 to more than 30 from 1988 to 2009. There are different upper limit values for different years.

the fractal dimension curves are very similar to the multifractal spectrums. There seems to be no clear distinction in appearance between the multifractal spectrum curves for 1988 and those for 2009 (figures 6 and 7). In fact, the multifractal spectrums of 1992, 1999, and 2006 are very similar to those of 1988 and 2009.

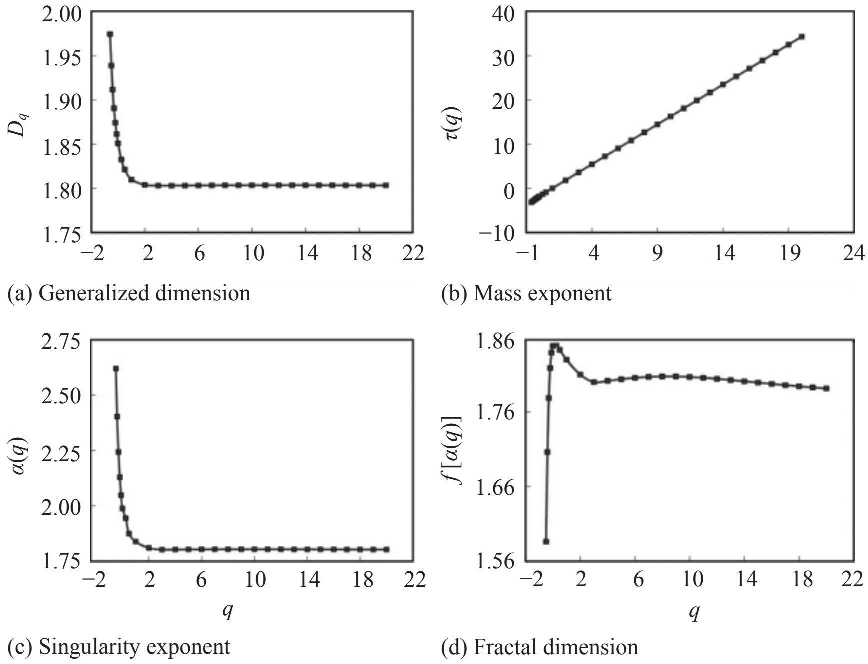


Figure 6. The spectrum of fractal dimension and the curves of related parameters of Beijing's urban form and growth in 1988. Note: The values for $q > 10$ correspond to urban fringe, the region of high growth rate; while the values for approximately $q < 0$ correspond to urban core, the region of low growth rate.

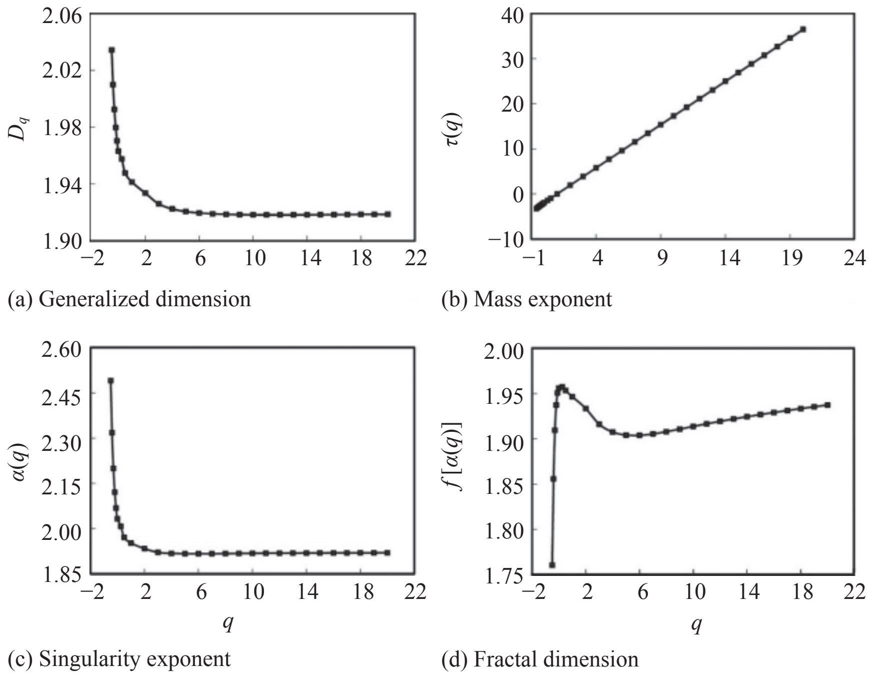


Figure 7. The spectrum of fractal dimensions and the curves of related parameters of Beijing's urban form and growth pattern in 2009.

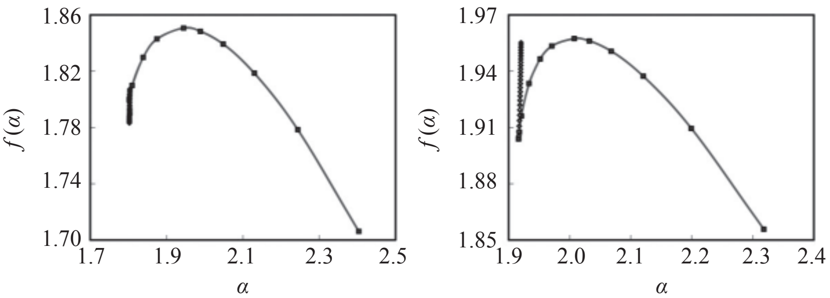
However, the actual range of Beijing's fractal parameter values is restricted. The valid scale of the multifractal parameters can be determined through the use of multifractal theory. If a multifractal parameter is theoretically determined by an increasing/decreasing function of q, and the calculated values of the parameter within some sections are consistent with the increasing/decreasing function and not greater than the Euclidean dimension $d_E = 2$, the section is regarded as a valid range. Otherwise, this section may be an invalid range. For the urban form and growth pattern of Beijing in 1988, the valid value of D_q varies from $D_{q<-0.6}$ to $D_{q>30}$. In other words, for $q \geqslant 0$, as a whole, we have $D_q > D_{q+1}$ approximately. However, If $q \leqslant -0.7$, then $D_q > d_E = 2$, and this is not normal. With regard to the $\alpha(q)$ values, the valid range of q values is relatively narrow and falls between $q = -0.2$ and $q = 30$. If $q \leqslant -0.2$, then $\alpha(q) > 2$, which is abnormal. As for the value of $f(\alpha)$, when $q < 0$, it converges too rapidly; whereas when $q > 0$, it converges slowly (figure 6). In 2009 the valid scale of fractal dimensions became narrower. The proper values of D_q varied from $D_{-0.5} \approx 2.0098$ to $D_{11} = 1.9182$. If $q < -0.5$, then $D_q > d_E = 2$, while if $q > 11$, then $D_q < D_{q+1}$. The valid scale of the $f(\alpha)$ became narrower, and if $q > 2$, then $f[\alpha(q)] < f[\alpha(q+1)]$, which suggests that the $f(\alpha)$ values diverged (figure 7).

For actual cities, a spectral curve of multifractal dimension can be divided into several sections. For example, the D_q values of Beijing's urban growth pattern in 2009 can be generally divided into three sections based on the values of q: $-\infty \sim -0.5$, $-0.5 \sim 11$, $11 \sim \infty$. The first section ($-\infty < q \leqslant -0.5$) is clearly invalid because $D_q > 2$ when $q < -0.5$. The second section ($-0.5 \leqslant q \leqslant 11$) is normal and is thus a valid range. As stated above, D_q is a decreasing function of q. The values within the interval [−0.5, 11] satisfy the relationship, $D_q > D_{q+1}$. To some extent, the third section ($-0.5 \leqslant q < \infty$) is abnormal because $D_q < D_{q+1}$ inside this range. Of course, this abnormality is not so significant where the spectrum curve is concerned [figure 7(a)]. If $q = 5$ is taken as a dividing point, the $f[\alpha(q)]$ values of Beijing in 2009 can be divided into two sections. When $0 \leqslant q < 5$, the $f[\alpha(q)]$ is a decreasing function, but for $q > 5$, the calculated values of $f[\alpha(q)]$ satisfy the following inequality: $f[\alpha(q)] < f[\alpha(q+1)]$, which is abnormal. When $q < 0$, $f[\alpha(q)]$ changes too abruptly with q, which is also a problem [figure 7(d)].

As stated above, the generalized correlation dimension and the mass exponent can be used to analyze the spatial details of urban form or to examine the entire pattern of urban growth in a comprehensive way, while the local fractal dimensions and the singularity exponent can be employed to bring the given parts into focus. Generally speaking, the core of a city is a region of higher density, which corresponds to a lower rate of growth, while the edge of the city is a zone of lower density, which corresponds to a higher rate of growth. The first important multifractal spectrums can be shown by plotting the relationships between D_q and q. Thus the global spatial information of Beijing's urban form and growth can be illustrated by the curves of multifractal dimension spectrums [figure 6(a), figure 7(a)]. More spatial components can be detected using the generalized dimension spectrum. In other words, the urban map can be redrawn by intensifying the spatial information of the peripheral regions and weakening the spatial information of the central regions to a great degree, revealing elaborate fractal structures on the transformed digital map. However, when $q < 0$, the D_q values diverge rapidly rather than converging at the limit value of $D_{-\infty} < d_E = 2$. This suggests the fractal structure of Beijing's urban growth pattern could disappear if the spatial pattern of the city were reconstructed. That is to say, if the urban map is redrawn by intensifying the spatial information of the central regions and weakening the spatial information of the peripheral regions to a great extent, we will find no fractal structure on the transformed digital map.

The local parameters can be employed to investigate different regions of Beijing city. When $q < 0$, the local fractal dimension reveals more spatial information about the high-density regions, such as the city center; when $q > 0$, the local dimension shows more spatial information about the low-density regions, such as outskirts of the city. The $f(\alpha)$ values do not converge when q approaches infinity. This suggests that the peripheral regions of highest growth rates have some anomalies to be resolved. On the other hand, the $f(\alpha)$ values converge too rapidly when $q < 0$. This suggests that the central regions of lowest growth rates contain some anomalies. If the relationships between the $\alpha(q)$ and the $f(\alpha)$ values are plotted, the graphs will provide the second most important multifractals spectrums (figure 8). An $\alpha - f(\alpha)$ spectrum is expected to be a curve with a single peak and no inflexion. When $q \to \pm\infty$, the $f(\alpha)$ curves should approach zero or a fixed value. However, the $f(\alpha)$ spectrums of Beijing became somewhat abnormal when $q > 6$ in 1992, $q > 2$ in 1999, $q > 3$ in 2006, and $q > 5$ in 2009. This phenomenon can be regarded as scaling breaking of urban fractal structure. In 1988, the abnormality is not very pronounced, but in 2009, the scaling break is significant. For $q > 1$, the abnormal segments of the $f(\alpha)$ curves mainly reflect the geographical information of the peripheral regions around the city center. The $\alpha - f(\alpha)$ spectrum suggests that the fringe area of the city of Beijing seems to fall into disorder owing to rapid urban sprawl. When $q < 0$, the $\alpha(q)$ values diverge and the $f(\alpha)$ values converge immediately; these suggest that the fractal structure of the central regions has degenerated and been reduced to a commonplace Euclidean plane because of over space-filling.

Scientific studies should proceed first by describing how a system works and then understand why (Gordon, 2005). To a degree, Beijing city is similar to the primate city defined by Jefferson (1939) despite the fact that it is not the largest city in China. Although Beijing has no absolute advantage where population size is concerned, it gains an absolute advantage over other Chinese cities in the respect of personal developing potential. Because it is the nation's capital, the city can obtain more resources for development through the central government and can provide more opportunities for local people. For these reasons, Chinese citizens are very attracted to it as an ideal place to settle. Because of the factors of history, geography, culture, economics, policy, etc, more Chinese people prefer the city proper to the outskirts. So many people try to 'swarm' into the capital to seek 'opportunities' which causes the city proper to become very crowded with workers and inhabitants. The urban population of six different censuses is 2.058 million in 1953, 4.258 million in 1964, 5.970 million in 1982, 7.945 million in 1990, 10.522 million in 2000, and 18.038 million in 2010. The population data can be fitted into an exponential curve. However, Beijing's infrastructure is limited and available urban land and resources are dwindling, especially water resources. From 1998 to 2009 Beijing's water resources were less than 300 m^3 per capita per year.



(a) (b)

Figure 8. The $f(\alpha)$ curves of Beijing's urban form and growth pattern in (a) 1988 and (b) 2009. Note: The tiny squares represent the normal data points, while tiny diamonds represent the abnormal data points indicating the declining state of urban development.

If the municipal government had not made a policy to restrict immigration, the megacity would have become a primate city in China.

Due to advantages over other cities in Mainland China, Beijing has attracted many people, who have migrated to find jobs in this city. Dense population led to overfilling of urban space, especially in the city center (inner-city districts). From 1988 to 2009 the fractal dimension values increased significantly, which suggests overcrowding in the urban areas. Since the introduction of reform policies, the 'Opening-up' at the end of 1978 and the establishment of a socialist economic system in 1992, China's national economy and cities grew rapidly. The fractal dimensions of Beijing's urban form increased gradually. The city center is still flourishing; yet, it is also experiencing many urban problems, such as traffic congestion, air pollution, and land shortages. However, urban spread is confined by limited land and water resources and the effect of national policies on land usage. The city center is overcrowded, while the settlement of people into the periphery is out of order. As a result, the multifractal structure of the city is in a state of decline because multifractal dimension spectrums are not consistent with the theoretical curves. To lessen the population pressure, the city government has planned to construct eleven new satellite cities around the main city.

5 Conclusions

Urban systems are complex and very different from the classical physical systems. Urban growth patterns are associated with a city's evolutionary process. A city's form may be fractal or nonfractal, and a city's structure may be monofractal or multifractals. Models of a city reflect its characteristics and evolution. Fractal theory is a powerful tool to explore spatial complexity. The idea of the spatial analysis of cities has been changing gradually because of the introduction of fractal geometry into urban studies. The traditional concept of scale may be replaced by the concept of scaling, and the traditional distance-based space notion could be substituted by the dimension-based space notion from fractal geometry. The theory of multifractals is necessary for studying the growth of complex structures such as megacities. This paper demonstrated that urban growth patterns take on multifractal properties; therefore, multifractal geometry can be employed to diagnose problems associated with urban growth. Normal urban structure with multiscaling properties should yield proper multifractal spectrums. If the urban space is badly filled in the whole or parts of the region, the corresponding dimension spectrum curves will be abnormal. The marks of urban decline include at least three aspects: (1) the upper limit of the D_q or $\alpha(q)$ values break through the Euclidean dimension of the embedding space, (2) the multifractal spectrum of D_q or $f(\alpha)$ diverges or fails to converge in the proper way, and (3) the spectrum curves take on some type of scaling break.

The case studies of Beijing show that the complex structure of actual cities can be modeled by using multifractal measurements. To some extent, the forms of the multifractal spectrums show the spatiotemporal information of city development. First, the fractal structure of high-density regions, such as the city center, degenerated owing to overfilling of space. Second, the scaling properties of the low-density regions, such as the urban periphery, become disordered due to the spatial constraint of urban growth. Third, the change extents of fractal parameters become narrower with the passage of time as the city is filled to capacity. Thus, Beijing's spatial structure can be loosely divided into three layers from the center to its periphery. The first layer comprises the city center ($q < 0$), where fractal structure has degenerated. The third layer is the urban fringe ($q \to \infty$), consisting of built-up areas and suburbs, where the spatial structure has also become disordered. The second layer between the city center and urban fringe is a median zone with multifractal characteristics. Urban problems occur mainly in the first and third layers, resulting from improper administration, urban land deficiency, water scarcity, population explosion, etc. New methods of spatial optimization and city planning using the concepts from fractals and self-organization will be likely to help to solve these problems.

Acknowledgements. This research was sponsored by the National Natural Science Foundation of China (Grant No. 41171129) and the Natural Science Foundation of Beijing (Grant No. 8093033). The supports are gratefully acknowledged. Many thanks to the anonymous reviewers whose interesting comments were helpful in improving the quality of this paper.

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Appendices

Appendix A. Data source and related explanation

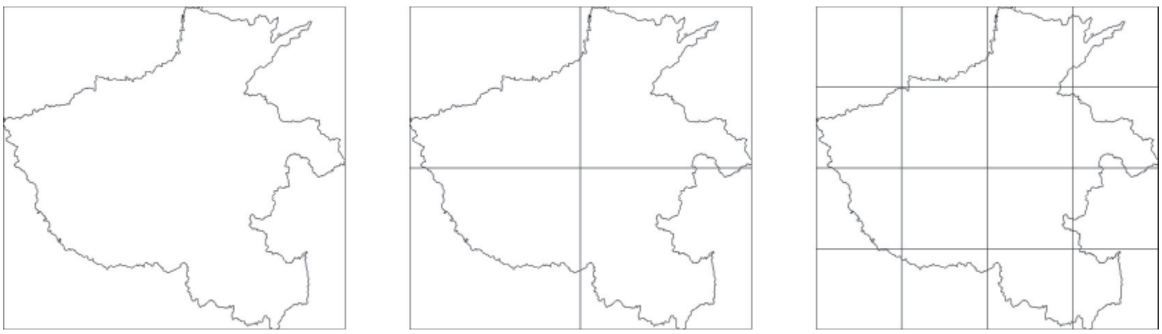
The remote sensing images used in the paper came from National Aeronautics and Space Administration. The basic information of these images is tabulated as follows (table A1). The data are available from: http://glcfapp.umiacs.umd.edu:8080/esdi/index.jsp.

Table A1. The basic information of the remote sensing images used in the paper and the area of urban figure within the rectangle framework of viewfinding.

Year	Type	Resolution (m)	Wave band	Data quality	Date	Urban area within rectangle
1988	Landsat-4 TM	30	7	a little cloud	25 December	837 158 000
1992	Landsat-5 TM	30	7	cloudless	7 September	851 491 000
1999	Landsat-7 ETM+	30	8	cloudless	1 July	1 140 490 000
2006	Landsat-7 ETM+	30	8	cloudless	6 September	1 536 300 000
2009	Landsat-5 TM	30	7	cloudless	22 September	1 907 170 000

Appendix B. A sketch map for RSS

The method of RSS was applied to fractal cities by Chen (1995, pages 81–88), who studied the fractal systems of cities and towns in the Central Plains of China. The first three steps of RSS are illustrated as follows (figure B1). The mathematical principle of this method is that a fractal is a hierarchy with a cascade structure, which can be quantified with a geometric sequence (Chen and Zhou, 2006).



(a) $\varepsilon_1 = 1$, $N_1 = 1$ (b) $\varepsilon_2 = 1/2$, $N_2 = 4$ (c) $\varepsilon_3 = 1/4$, $N_3 = 16$

Figure B1. A sketch map of the rectangle space subdivision method for fractal dimension estimation.