

Chapter 1

Modular symbols

1.1 Néron lattice

Let

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

be a minimal Weierstrass equation for the elliptic curve. The Néron differential on E is defined as

$$\omega_E = \frac{dx}{2y + a_1x + a_3} \in \Omega_E$$

$E(\mathbb{C})$ is a Riemann surface of genus 1, so its fundamental group

$$\pi^1(E, O) \cong \mathbb{Z} \oplus \mathbb{Z}$$

Definition 1.1.0.1. The Néron lattice of E is defined as

$$\mathcal{L} := \left\{ \int_{\gamma} \omega_E \mid \gamma \in \pi^1(E, O) \right\}$$

[angurel]

1.2 Modular symbols

Throughout this section, let E be an elliptic curve defined over \mathbb{Q} and let N be its conductor. By the modularity theorem from [Wiles], there exists a Hecke eigenform $f \in S_k(\Gamma_0(N))$ with Fourier expansion $f = \sum_n a_n q^n$ such that for all primes p of good reduction,

$$a_p = (p + 1) - \#\tilde{E}(\mathbb{F}_p)$$

where $\tilde{E}(\mathbb{F}_p)$ is the number of points in the reduced curve at p . In this section, f

Definition 1.2.0.1. Let r be a rational number. The modular symbol $\left[\frac{a}{m}\right]$ is defined as

$$\lambda := \int_{\infty}^{a/m} f(z) dz$$

where the integral is taking along any path in the upper half plane from the cusp at infinity to the cusp at $\frac{a}{m}$.

Proposition 1.2.0.2. The modular symbols of $\frac{a}{m}$ and $\frac{-a}{m}$ are conjugates of each other:

$$\lambda(-r) = \overline{\lambda(r)}$$

In order to define the algebraic part of modular symbols, we need to consider first its real and imaginary part:

$$\begin{aligned}\lambda^+(r) &:= \frac{\lambda(r) + \lambda(-r)}{2} = \operatorname{Re}(\lambda(r)) \\ \lambda^-(r) &:= \frac{\lambda(r) - \lambda(-r)}{2} = \operatorname{Im}(\lambda(r))\end{aligned}$$

The algebraic modular symbols are the above quantities normalized by the Néron periods.

Definition 1.2.0.3. Let $\frac{a}{m}$ be a rational number expressed in its simplest terms. The algebraic modular symbols of $\frac{a}{m}$ are:

$$\left[\frac{a}{m}\right]^+ := \frac{\lambda^+\left(\frac{a}{m}\right)}{\Omega_E^+}; \quad \left[\frac{a}{m}\right]^- := \frac{\lambda^-\left(\frac{a}{m}\right)}{\Omega_E^-}$$