A new framework for Kolyvagin systems

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Motivation: Birch and Swinnerton-Dyer conjecture

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BSD conjecture

There exists an L-function

$$L(E,s) = P_\ell(\ell^{-s})^{-1}$$

where P_ℓ are Euler factors, such that

$$r = \mathrm{ord}_{s=1} L(E,s)$$

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Introduction and motivation

Euler and Kolyvagin systems

Fitting ideals

Core rank one

Core rank 0

Ultra Kolyvagin systems

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Theorem (Kolyvagin)

If $\mathrm{ord}_{s=1} L(E,s) \leq 1$, then BSD holds true.

Idea: replace points with Galois cohomology

Fix a prime p and a natural number K.

Kummer map

There is an injection

$$E(\mathbb{Q})/p^k E(\mathbb{Q}) \hookrightarrow H^1(\mathbb{Q}, E[p^k])$$

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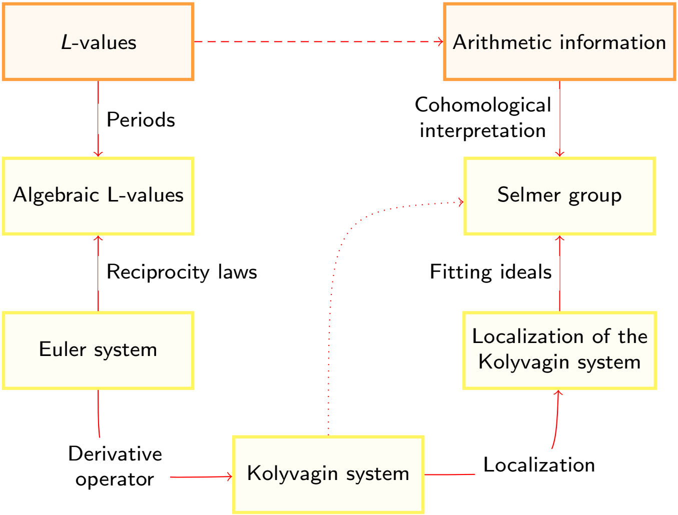
Selmer group

This is known as a *Selmer group*, and it is the object that will be studied by the Euler system machinery.

General picture



Tate module

Tate module

We will study the limit $T_pE = \varprojlim_k E[p^k]$, which is a free $\mathbb{Z}_p$-module of rank 2 endowed with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

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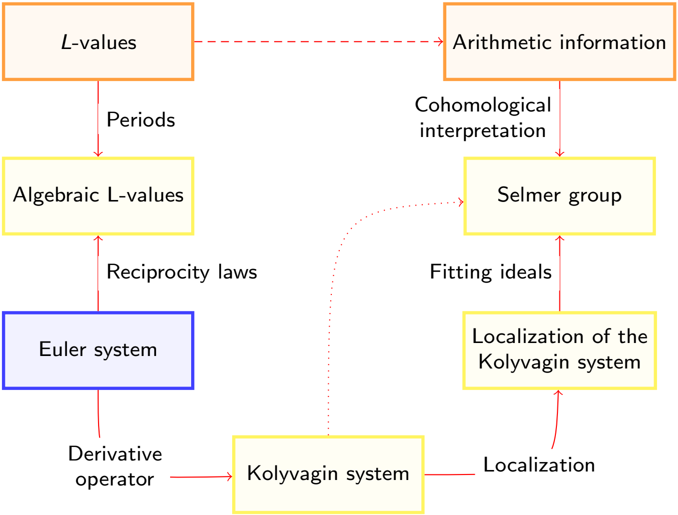
Galois representations

We only need to assume that T is a free, finitely generated $\mathbb{Z}_p$-module, endowed with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

General picture



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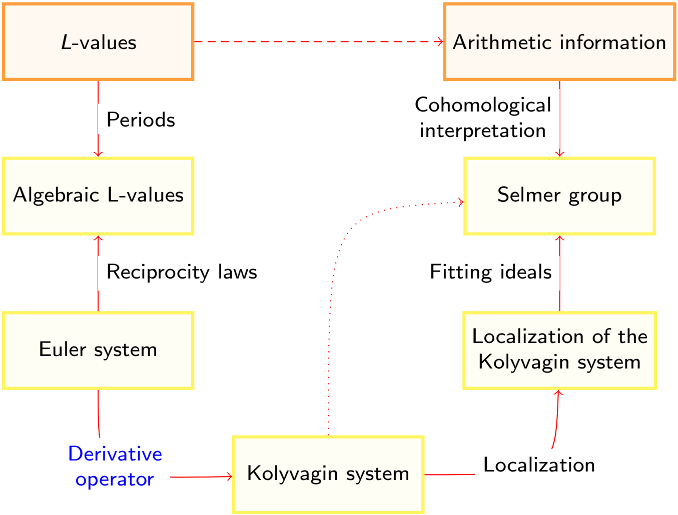
Construction

In general, it is hard to construct Euler systems and an active research area. For the particular case of an elliptic curve, an Euler system was constructed by Kato.

General picture



Kolyvagin derivative

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Fix $k \in \mathbb{N}$. Kolyvagin constructed a set of primes $\mathcal{P}$ in which

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Kolyvagin system

The collection of derivative classes $\{\kappa_n : \ n \in \mathcal{N}\}$ is a Kolyvagin system.

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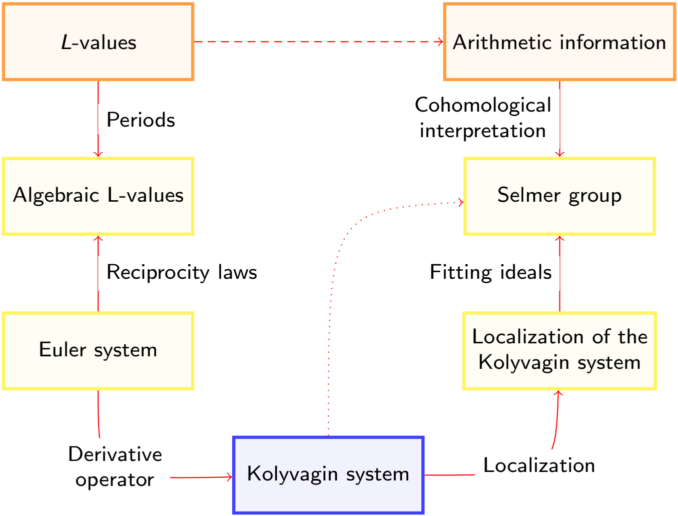

Kolyvagin systems

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A Kolyvagin system is a collection of classes $\{\kappa_n : \ n \in \mathcal{N}\}$ such that

- $\kappa_n \in H^1(\mathbb{Q}, T/p^k)$
- κ_n is unramified at primes not dividing n or p and *interestingly* ramified at primes dividing n.
- There is a relation between κ_n and $\kappa_{n\ell}$ for all $n \in \mathcal{N}$ and $\ell \in \mathcal{P}$.

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Remark

There are Kolyvagin systems for all $k \in \mathbb{N}$. However, we cannot define Kolyvagin systems on $H^1(\mathbb{Q}, T)$, since there is no prime $\ell \equiv 1 \mod p^k$ for every k.

Assumptions

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- The Selmer structure is cartesian (technical assumption).

Core rank

Dual Selmer group

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Core rank

- There exists an integer χ such that

$$\mathrm{Sel}(\mathbb{Q}, T/p^k) \cong \mathrm{Sel}(\mathbb{Q}, (T/p^k)^*) \oplus (\mathbb{Z}/p^k)^\chi$$

- χ is known as the core rank of the Selmer group.

Theorem (Mazur-Rubin, 2004)

- If $\chi = 0$, then $\mathrm{KS}(T) = 0$.

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Core rank and Kolyvagin systems

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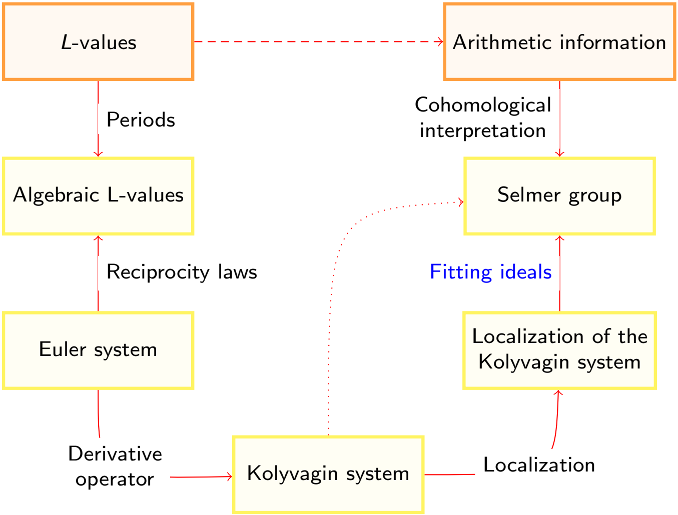
- If $\chi = 1$, then $\mathrm{KS}(T/p^k) \cong \mathbb{Z}/p^k$.

- If $\chi > 1$, then $\mathrm{KS}(T/p^k)$ is too large.

  In order to compute the Selmer group, [Mazur-Rubin, 2016] and [Burns-Sakamoto-Sano, 2025] modified the definition of Kolyvagin system in (biduals of) exterior powers of the Selmer groups.

General picture



Fitting ideals

Definition (Fitting ideal)

Let M be a finitely generated R-module. Choose a resolution

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$Fitt_0(M) = (0)$,	$\mathrm{Fitt}_1(M) = (p^5)$,
$\mathrm{Fitt}_2(M) = (p^2) + (p^3) = (p^2)$	$\mathrm{Fitt}_i(M) = (1)\ \forall i \geq 3$

Fitting ideals over Discrete Valuation Rings

Let R be a DVR (with maximal ideal $\mathfrak{m}$ and residue field κ). Then

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- For $i \geq 0$, $\alpha_i = \mathrm{Fitt}_{r+i+1}(M)\mathrm{Fitt}_{r+i}(M)^{-1}$.

Fitting ideals over the Iwasawa algebra

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Fitting ideals

Fitting ideals can recover the structure of a finitely generated Iwasawa module up to pseudo-isomorphism.

Indices and theta-ideals

Index of an element

Let M be an R-module and let $a \in M$. Denote by $M^+ = \mathrm{Hom}(M, R)$ the dual of M. There is a canonical map

$$\Phi: \ M \to M^{++}: \ x \mapsto (\varphi \mapsto \varphi(x))$$

Note that $\Phi(a) \in \mathrm{Hom}(M^+, R)$. Define the index of a as

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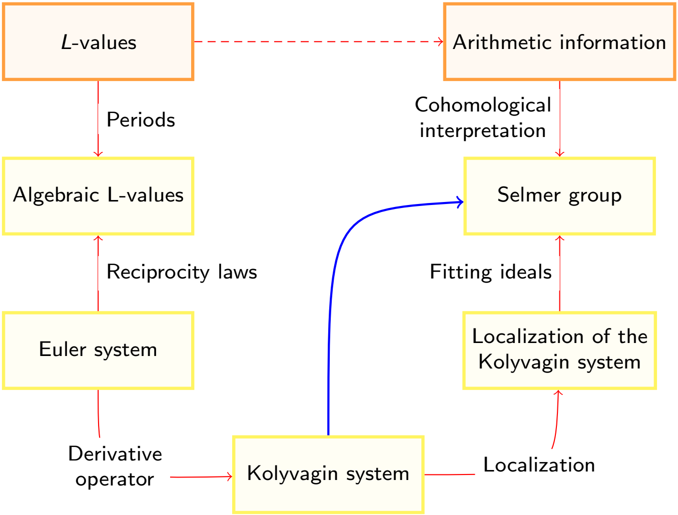
Theta ideals

We denote by $\mathcal{N}_i$ the set of square-free products of exactly i Kolyvagin primes. We define the i^{th} theta ideal of a Kolyvagin system as

$$\Theta_i(\kappa) := \sum_{n \in \mathcal{N}_i} (\kappa_n, H^1(\mathbb{Q}, T))$$

General picture



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Primitive Kolyvagin systems

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Theorem (Mazur-Rubin, 2004)

When $\chi = 1$ and κ is a primitive Kolyvagin system

$$\Theta_i(\kappa) = \mathrm{Fitt}_i\left(\mathrm{Sel}\left(\mathbb{Q}, (T/p^k)^*\right)\right) = \mathrm{Fitt}_{i+1}\left(\mathrm{Sel}(\mathbb{Q}, T/p^k)\right)$$

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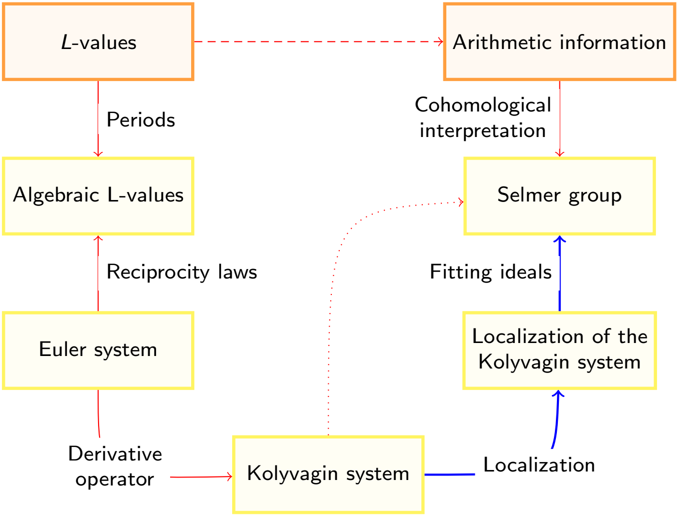

Core rank 0: non-self-dual Galois representations

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- Fix a primitive Kolyvagin system $\kappa = (\kappa_n)_{n \in \mathcal{N}}$.
- We now consider the cohomology classes

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Theorem (A., 2025)

If T is **not** a residually self-dual Galois representation, then

$$\Theta_i^{(0)}(\kappa) = \mathrm{Fitt}_i(\mathrm{Sel}(\mathbb{Q}, T/p^k))$$

Core rank 0: self-dual Galois representations

Theorem (A., 2025)

For all i, we have

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Conjecture

If T is a self-dual Galois representation, either

- $\Theta_i^{(0)}(\kappa)$ for all even i.
- $\Theta_i^{(0)}(\kappa)$ for all odd i.

Limit of Selmer groups

Note that $T_pE^* = E[p^\infty]$ and that

$$\mathrm{Sel}(\mathbb{Q}, E[p^\infty]) = \varinjlim_{k} \mathrm{Sel}(\mathbb{Q}, E[p^k])$$

Example: elliptic Curve

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Kurihara numbers

Kato's Euler system produces a Kolyvagin systems and $\mathrm{ind}(\delta_n) = \mathrm{ind}(\widetilde{\delta}_n)$, where $\widetilde{\delta}_n$ are known as the Kurihara numbers and are defined by the formula

$$\widetilde{\delta}_n := \sum_{a \in (\mathbb{Z}/n)^\times} \left(\left[\frac{a}{n}\right]^+ + \left[\frac{a}{n}\right]^- \right) \prod_{\ell | n} \log_{\eta_\ell}(a) \in \mathbb{Z}_p$$

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By the functional eqqation of the L-function, the conjecture holds true in this case.

Selmer groups with coefficients in the Iwasawa algebra

$\mathbb{Z}_p$-extensions

In Iwasawa theory, we are interested in the Selmer group over a $\mathbb{Z}_p$-extension. By Shapiro's lemma, it is equivalent to study Galois representations $\mathbf{T}$ with coefficients over the Iwasawa algebra Λ.

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If $\mathbf{T}$ is a (finitely generated, free) Λ-module, it is more difficult to study the Selmer group as a limit of Selmer groups with finite coefficients, since the representation theory over the finite quotients of Λ is more complicated.

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Idea

Use patched cohomology to overcome the lack of Kolyvagin primes and generalise the notion of Kolyvagin system to infinite coefficient rings.

Ultrafilters

Filters and ultrafilters

A filter of the natural numbers is a subset $\mathcal{U} \subset \mathbb{P}(\mathbb{N})$ such that

- $S \in \mathcal{U},\ S \subset T \Rightarrow T \in \mathcal{U}$
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Principal ultrafilters

Assume there is a finite set $S \in \mathcal{U}$. Then there is an element $a \in S$ such that

$$\mathcal{U} = \{T \subset \mathbb{N} :\ a \in T\}$$

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Fix a non-principal ultrafilter $\mathcal{U}$.

Patching

Patching (Sweeting 2021)

Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of sets/groups/rings, we define the patching via the ultrafilter $\mathcal{U}$ as

$$\mathcal{U}(M_n) = \prod_{n\in\mathcal{N}} M_n \Big/ \sim$$

where two sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are said to be equivalent if $\alpha_n = \beta_n$ for $\mathcal{U}$-many n.

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Proposition

$\mathcal{U}$ is an exact functor.

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Constant patching of finite groups

If M is a finite group and $M_n = M$ for all $n \in \mathbb{N}$, then

$$\mathcal{U}(M_n) = M$$

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Kolyvagin ultraprimes

An ultraprime $\mathfrak{u} = (\ell_i)_{i \in \mathbb{N}}$ is said to be a Kolyvagin ultraprime if, for every finite quotient of Λ, ℓ_i is a Kolyvagin prime for $\mathcal{U}$-many i.

Patched cohomology (Sweeting, 2021)

Finite coefficients

Assume T is a finite group endowed with an action a sequence of groups $G = (G_n)$. We define the patched cohomology as

$$\mathbf{H}^1(G, T) = \mathcal{U}(H^1(G, T))$$

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Assume T is profinite, the patched cohomology is defined as

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Ind-finite coefficients

If T is ind-finite, then

$$\mathbf{H}^1(G, T) = \varinjlim_{T' \hookrightarrow T} \mathbf{H}^1(G, T')$$

where T' runs through the finite submodules of T.

Local and global patched cohomology

Local patched cohomology

Let T be a Galois representation, i.e., T is endowed with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
Let $\mathfrak{u} = (\ell_n)$ be an ultraprime. Since $\mathrm{Gal}(\overline{\mathbb{Q}_{\ell_i}}/\mathbb{Q}_{\ell_i})$ is contained in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, it also acts on T.
Then the patched local cohomology

$$\mathbf{H}^1(\mathbb{Q}_{\mathfrak{u}}, T)$$

is the patching of the sequence of local Galois groups with coefficients in T. In particular, when T is finite,

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Global patched cohomology

There is also a notion of patched global cohomology unramified outside the square-free (formal) product of ultraprimes $n = \mathfrak{u}_1 \cdots \mathfrak{u}_s$, denoted by

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Selmer groups

We can extend the notion of Selmer groups to this setting and define local conditions on the local cohomology of ultraprimes. We recover the classical Selmer groups when

Ultra-Kolyvagin systems

Notation

- $\mathcal{P}$: set of Kolyvagin ultraprimes.
- $\mathcal{N}(\mathcal{P})$: set of square-free products of Kolyvagin ultraprimesprimes
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Theta ideals

$$\Theta_i(\kappa) := \sum_{n \in \mathcal{N}_i(\mathcal{P})} \mathrm{ind}(\kappa_n)$$

Selmer group over cyclotomic $\mathbb{Z}_p$-extensions

Assumptions

- T is residually irreducible.
- $\rho: \ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(T)$ is surjective.
- The Selmer structure is cartesian.
- $\frac{\mathbf{H}^1(\mathbb{Q}_\mu, T \otimes \Lambda)}{\mathbf{H}^1_{\mathcal{F}}(\mathbb{Q}_\mu, T \otimes \Lambda)}$ is Λ-torsion free

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Theorem (A., in progress)

The module of ultra-Kolyvagin systems $\mathrm{KS}(T \otimes \Lambda)$ is free of rank one over Λ. Moreover, if we choose a primitive ultra-Kolyvagin system κ, then

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Corollary

The ideals $\Theta_i(\kappa)$ determine the the structure of the Selmer group up to pseudo-isomorphism.

Thank you for your attention!