

and Fitting ideals o Selmer

Introduct

Selmer structure

Fittin

Dualit

Kabasa

systems

of Selmo

Euler

Elliptio

D...--6.

Kolyvagin systems and Fitting ideals of Selmer group of rank 0

Alberto Angurel Andrés

University of Nottingham

30/09/2025



General picture

Introducti

Selmer structur

Fittin

Dualit

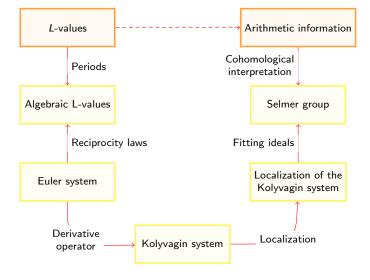
Kolyva system

Structur of Selme groups

Euler

Elliptic Curves

Proofs





■ We are going to focus on

Localization Kolyvagin system Group structure of $\kappa_n \in H^1(\mathbb{Q}, T)$ $\mathrm{loc}_p(\kappa_n) \in H^1(\mathbb{Q}_p, T)$ the Selmer group

■ We are going to focus on

Kolyvagin system $\kappa_n \in H^1(\mathbb{Q},T) \xrightarrow{\text{Localization}} \text{Group structure of the Selmer group}$

- We cannot apply the theory of Kolyvagin systems directly, because
 - The classical Selmer group is self-dual, so its core rank is zero.
 - There are no non-zero Kolyvagin systems for this Selmer group.

...c. oddetio

structu

Fittin

ideals

pairing

Kolyva system

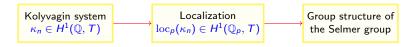
Structu of Selm

groups

Ellipti

Proof

■ We are going to focus on



- We cannot apply the theory of Kolyvagin systems directly, because
 - The classical Selmer group is self-dual, so its core rank is zero.
 - There are no non-zero Kolyvagin systems for this Selmer group.
- The general theory of Kolyvagin systems only describes the structure of the Selmer group *restricted at p.*

~ .

Fitting

ideals

pairing

systems

of Selm groups

system Elliptic Curves

SLIDE

and
Fitting
deals of
Selmer
group of
rank 0

ma odden

Fitting

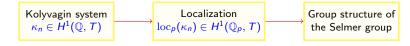
Duality

Kolyvag systems

Structure of Selmer

Euler system

Elliptic Curves Proofs We are going to focus on



- We cannot apply the theory of Kolyvagin systems directly, because
 - The classical Selmer group is self-dual, so its core rank is zero.
 - There are no non-zero Kolyvagin systems for this Selmer group.
- The general theory of Kolyvagin systems only describes the structure of the Selmer group *restricted at p*.
- We extend this theory to Selmer groups of rank zero by considering Kolyvagin systems over an auxiliary Selmer structure.



Setting and assumptions

Introducti

Selmer structui

Fittir

idea:

pairin

Kolyva system

Structu of Selm groups

groups Euler

Ellipti

)raaf

■ (H0) Let $p \ge 5$ and let **T** be free \mathbb{Z}_p -module of finite rank endowed with a continuous action of $G_{\mathbb{Q}}$, ramifying only at a finite amount of primes.

• (H1) $\rho: G_{\mathbb{Q}} \to \operatorname{Aut}(\mathbf{T})$ is surjective.

■ (H2) (will appear later)



Selmer groups are formed by the elements of the global cohomology groups $H^1(\mathbb{Q}, \mathbf{T})$ that satisfy *local conditions*.



■ Selmer groups are formed by the elements of the global cohomology groups $H^1(\mathbb{Q}, \mathbf{T})$ that satisfy *local conditions*.

lacktriangle What is a local condition? A local condition for a prime ℓ is a choice of a subgroup

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathsf{T})\subset H^1(\mathbb{Q}_\ell,\mathsf{T})$$

Selmer

Fittii

Duali

Kolyva system

of Selm groups

Euler syster

Elliptio

Proof



- Selmer groups are formed by the elements of the global cohomology groups $H^1(\mathbb{Q}, \mathbf{T})$ that satisfy local conditions.
- What is a local condition? A local condition for a prime ℓ is a choice of a subgroup

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathsf{T})\subset H^1(\mathbb{Q}_\ell,\mathsf{T})$$

Definition (Selmer pre-structure)

A Selmer pre-structure \mathcal{F} is a choice of a local condition for every prime (including the archimedean one).



■ Selmer groups are formed by the elements of the global cohomology groups $H^1(\mathbb{Q}, \mathbf{T})$ that satisfy *local conditions*.

■ What is a local condition? A local condition for a prime \(\ell \) is a choice of a subgroup

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathsf{T})\subset H^1(\mathbb{Q}_\ell,\mathsf{T})$$

Definition (Selmer pre-structure)

A Selmer pre-structure \mathcal{F} is a choice of a local condition for every prime (including the archimedean one).

Definition (Selmer group)

The Selmer group for \mathcal{F} is defined as

$$\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q},\mathbf{T}):=\ker\left(H^1(\mathbb{Q},\mathbf{T})\to\bigoplus_{\ell}\frac{H^1(\mathbb{Q}_{\ell},\mathbf{T})}{H^1_{\mathcal{F}}(\mathbb{Q}_{\ell},\mathbf{T})}\right)$$

Selmer structur

Duality pairings

> Kolyvag systems

groups

Euler
system

Elliptic Curves



Definition (finite cohomology)

$$H^1_f(\mathbb{Q}_\ell,\mathsf{T}):=\mathsf{ker}\left(H^1(\mathbb{Q},\mathsf{T})\to H^1(I_\ell,\mathsf{T}\otimes\mathbb{Q}_p)\right)$$



Definition (finite cohomology)

$$H^1_f(\mathbb{Q}_\ell,\mathsf{T}):=\mathsf{ker}\left(H^1(\mathbb{Q},\mathsf{T}) o H^1(I_\ell,\mathsf{T}\otimes\mathbb{Q}_p)
ight)$$

Definition (Selmer structure)

A Selmer structure is a Selmer pre-structure such that there is a finite set of primes Σ such that

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathsf{T})=H^1_f(\mathbb{Q}_\ell,\mathsf{T})\;orall \ell
otin\Sigma$$

Selmer

Fittin

ideals

Dualit pairin

Kolyva svstem

Structu of Selm

Euler

Ellipti Curve

Droof



Definition (finite cohomology)

$$H^1_f(\mathbb{Q}_\ell,\mathsf{T}):= \mathsf{ker}\left(H^1(\mathbb{Q},\mathsf{T}) o H^1(I_\ell,\mathsf{T}\otimes\mathbb{Q}_p)
ight)$$

Definition (Selmer structure)

A Selmer structure is a Selmer pre-structure such that there is a finite set of primes Σ such that

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell, T) = H^1_f(\mathbb{Q}_\ell, T) \; \forall \ell \notin \Sigma$$

Proposition (Selmer groups)

If \mathbb{Q}_{Σ} denotes the maximal extension of \mathbb{Q} unramified outside Σ , we have that

$$\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},\mathbf{T}) = \ker \left(H^1(\mathbb{Q}_{\Sigma}/\mathbb{Q},\mathbf{T}) \to \prod_{\ell \in \Sigma} \frac{H^1(\mathbb{Q}_{\ell},\mathbf{T})}{H^1_{\mathcal{F}}(\mathbb{Q}_{\ell},\mathbf{T})} \right)$$

Selmer structur

Fitting ideals

Duality pairings Kolyvagi

Structur of Selm

groups Euler system

Elliptic Curves



Definition (finite cohomology)

$$H^1_f(\mathbb{Q}_\ell,\mathsf{T}):= \mathsf{ker}\left(H^1(\mathbb{Q},\mathsf{T}) o H^1(I_\ell,\mathsf{T}\otimes\mathbb{Q}_p)\right)$$

Definition (Selmer structure)

A Selmer structure is a Selmer pre-structure such that there is a finite set of primes Σ such that

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathsf{T})=H^1_f(\mathbb{Q}_\ell,\mathsf{T})\;orall \ell
otin\Sigma$$

Proposition (Selmer groups)

If \mathbb{Q}_{Σ} denotes the maximal extension of \mathbb{Q} unramified outside Σ , we have that

$$\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},\mathbf{T}) = \ker \left(H^1(\mathbb{Q}_{\Sigma}/\mathbb{Q},\mathbf{T}) \to \prod_{\ell \in \Sigma} \frac{H^1(\mathbb{Q}_{\ell},\mathbf{T})}{H^1_{\mathcal{F}}(\mathbb{Q}_{\ell},\mathbf{T})} \right)$$

Corollary

 $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q},\mathsf{T})\subset H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathsf{T})$ is a finitely generated \mathbb{Z}_p -module.

Fitting ideals

Duality pairings

Colyvagir ystems

of Seln groups Euler

System Elliptic Curves



12246

Definition (Fitting ideal)

Fitting ideals

Let M be a finitely generated R-module. Choose a resolution

$$R^n \xrightarrow{A} R^m \longrightarrow M \longrightarrow 0$$

 $\operatorname{Fitt}_{i}^{R}(M)$ is the ideal generated by the minors of size (m-i) of A.

Selmer structui

ideals

Dualit

Kolyva system

Structu of Selm groups

Euler syste

Ellipti Curve

Droof



S And

Definition (Fitting ideal)

Fitting ideals

Let M be a finitely generated R-module. Choose a resolution

$$R^n \xrightarrow{A} R^m \longrightarrow M \longrightarrow 0$$

 $\operatorname{Fitt}_{i}^{R}(M)$ is the ideal generated by the minors of size (m-i) of A.

Fact: Fitting ideals are well defined.

Selmer structur

Fittin

Dualit

pairin

Kolyva system

Structu of Selm groups

Euler syster

Elliptic Curves

Proof



CLIDEC

Definition (Fitting ideal)

Fitting ideals

Let M be a finitely generated R-module. Choose a resolution

$$R^n \xrightarrow{A} R^m \longrightarrow M \longrightarrow 0$$

 $\operatorname{Fitt}_{i}^{R}(M)$ is the ideal generated by the minors of size (m-i) of A.

Fact: Fitting ideals are well defined.

Example

Consider $R = \mathbb{Z}_p$ and $M = \mathbb{Z}_p \times \mathbb{Z}_p/p^3 \times \mathbb{Z}_p/p^2$.

Fitting

ideals

Kolyva

Kolyvag

of Selr groups

Euler syste

Elliptic Curves

Droof



Fitting ideals

Definition (Fitting ideal)

Let ${\it M}$ be a finitely generated ${\it R}\text{-module}.$ Choose a resolution

$$R^n \xrightarrow{A} R^m \longrightarrow M \longrightarrow 0$$

 $\operatorname{Fitt}_{i}^{R}(M)$ is the ideal generated by the minors of size (m-i) of A.

Fact: Fitting ideals are well defined.

Example

Consider $R = \mathbb{Z}_p$ and $M = \mathbb{Z}_p \times \mathbb{Z}_p/p^3 \times \mathbb{Z}_p/p^2$. A resolution is given by

$$(\mathbb{Z}_p)^3 \xrightarrow{\mu} (\mathbb{Z}_p)^3 \xrightarrow{\varepsilon} M \longrightarrow 0$$

Here ε is the natural map and μ is given by the matrix $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p^3 & 0 \\ 0 & 0 & p^2 \end{pmatrix}$

structu

ideals

pairings Kolyvag

> Structure of Selmer

groups Euler system

Elliptic Curves



Fitting ideals

Definition (Fitting ideal)

Let M be a finitely generated R-module. Choose a resolution

$$R^n \xrightarrow{A} R^m \longrightarrow M \longrightarrow 0$$

 $\operatorname{Fitt}_{i}^{R}(M)$ is the ideal generated by the minors of size (m-i) of A.

Fact: Fitting ideals are well defined.

Example

Consider $R = \mathbb{Z}_p$ and $M = \mathbb{Z}_p \times \mathbb{Z}_p/p^3 \times \mathbb{Z}_p/p^2$. A resolution is given by

$$(\mathbb{Z}_p)^3 \xrightarrow{\mu} (\mathbb{Z}_p)^3 \xrightarrow{\varepsilon} M \longrightarrow 0$$

Here ε is the natural map and μ is given by the matrix $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p^3 & 0 \\ 0 & 0 & p^2 \end{pmatrix}$

Selmer tructure

Fitting ideals

pairings Kolyvag

Structure

groups Euler system

Elliptic Curves



systems

Fitting ideals of Selmer group of

Selmer

Fitting ideals

Dualit

Kolyva system

Structur of Selm groups

Euler syster

Ellipti Curve

D... - 4

Let R be a DVR (with maximal ideal $\mathfrak m$ and residue field κ). Then

$$M \cong R^r \times R/\mathfrak{m}^{\alpha_1} \times \cdots \times R/\mathfrak{m}^{\alpha_s}$$

for some non-negative integers $r, s, \alpha_1 \geq \ldots \geq \alpha_s$.



Let R be a DVR (with maximal ideal $\mathfrak m$ and residue field κ). Then

$$M \cong R^r \times R/\mathfrak{m}^{\alpha_1} \times \cdots \times R/\mathfrak{m}^{\alpha_s}$$

for some non-negative integers $r, s, \alpha_1 \geq \ldots \geq \alpha_s$.

Proposition

$$i \in \{0,\ldots,r-1\} \Rightarrow \operatorname{Fitt}_i(M) = (0)$$

Selmer structure

ideals

Duality

Kolyvag systems

Structu of Selm groups

Euler

Elliptic

Droof



Let R be a DVR (with maximal ideal $\mathfrak m$ and residue field κ). Then

$$M \cong R^r \times R/\mathfrak{m}^{\alpha_1} \times \cdots \times R/\mathfrak{m}^{\alpha_s}$$

for some non-negative integers $r, s, \alpha_1 \geq \ldots \geq \alpha_s$.

Proposition

- $i \in \{0,\ldots,r-1\} \Rightarrow \operatorname{Fitt}_i(M) = (0)$
- $lack j \in \{0,\ldots,s-1\} \Rightarrow \mathrm{Fitt}_{r+j} = \prod_{k=i+1}^s \mathfrak{m}^{i_k} = \mathfrak{m}^{\sum_{k=j+1}^s i_k}$

Selmer structure

ideals

pairings

Kolyvag systems

of Selm groups

Euler syste

Ellipti Curve



Let R be a DVR (with maximal ideal $\mathfrak m$ and residue field κ). Then

$$M \cong R^r \times R/\mathfrak{m}^{\alpha_1} \times \cdots \times R/\mathfrak{m}^{\alpha_s}$$

for some non-negative integers $r, s, \alpha_1 \geq \ldots \geq \alpha_s$.

Proposition

- $i \in \{0,\ldots,r-1\} \Rightarrow \operatorname{Fitt}_i(M) = (0)$
- $\mathbf{I} = j \in \{0, \dots, s-1\} \Rightarrow \operatorname{Fitt}_{r+j} = \prod_{k=j+1}^{s} \mathfrak{m}^{i_k} = \mathfrak{m}^{\sum_{k=j+1}^{s} i_k}$
- $i \ge r + s \Rightarrow \operatorname{Fitt}_i(M) = (1).$

Selmer structur

ideals

Kolyvag

Structure of Selme groups

Euler syster

Elliptic Curves

Proof



Let R be a DVR (with maximal ideal $\mathfrak m$ and residue field κ). Then

$$M \cong R^r \times R/\mathfrak{m}^{\alpha_1} \times \cdots \times R/\mathfrak{m}^{\alpha_s}$$

for some non-negative integers $r, s, \alpha_1 \geq \ldots \geq \alpha_s$.

Proposition

- $i \in \{0,\ldots,r-1\} \Rightarrow \operatorname{Fitt}_i(M) = (0)$
- $\mathbf{I} = j \in \{0, \dots, s-1\} \Rightarrow \operatorname{Fitt}_{r+j} = \prod_{k=j+1}^{s} \mathfrak{m}^{i_k} = \mathfrak{m}^{\sum_{k=j+1}^{s} i_k}$
- $i \ge r + s \Rightarrow \operatorname{Fitt}_i(M) = (1).$

Corollary

The Fitting ideals determine i up to isomorphism:

Proofs



Let R be a DVR (with maximal ideal $\mathfrak m$ and residue field κ). Then

$$M \cong R^r \times R/\mathfrak{m}^{\alpha_1} \times \cdots \times R/\mathfrak{m}^{\alpha_s}$$

for some non-negative integers $r, s, \alpha_1 \geq \ldots \geq \alpha_s$.

Proposition

- $i \in \{0,\ldots,r-1\} \Rightarrow \operatorname{Fitt}_i(M) = (0)$
- $lack j \in \{0,\ldots,s-1\} \Rightarrow \operatorname{Fitt}_{r+j} = \prod_{k=i+1}^s \mathfrak{m}^{i_k} = \mathfrak{m}^{\sum_{k=j+1}^s i_k}$
- $i \ge r + s \Rightarrow \text{Fitt}_i(M) = (1).$

Corollary

The Fitting ideals determine *i* up to isomorphism:

r is the minimum i such that $Fitt_i(M) \neq 0$.

Alberto Angurel Andres



Let R be a DVR (with maximal ideal $\mathfrak m$ and residue field κ). Then

$$M \cong R^r \times R/\mathfrak{m}^{\alpha_1} \times \cdots \times R/\mathfrak{m}^{\alpha_s}$$

for some non-negative integers $r, s, \alpha_1 \geq \ldots \geq \alpha_s$.

Proposition

- $i \in \{0,\ldots,r-1\} \Rightarrow \operatorname{Fitt}_i(M) = (0)$
- $\mathbf{I} = j \in \{0, \dots, s-1\} \Rightarrow \operatorname{Fitt}_{r+j} = \prod_{k=j+1}^{s} \mathfrak{m}^{i_k} = \mathfrak{m}^{\sum_{k=j+1}^{s} i_k}$
- $i \ge r + s \Rightarrow \operatorname{Fitt}_i(M) = (1).$

Corollary

The Fitting ideals determine i up to isomorphism:

- r is the minimum i such that $Fitt_i(M) \neq 0$.
- For i > 0, $\alpha_i = \text{Fitt}_{r+i+1}(M)\text{Fitt}_{r+i}(M)^{-1}$.



Definition (dual Galois modules)

- Pontryagin dual: $\mathbf{T}^{\vee} = \operatorname{Hom}(\mathbf{T}, \mathbb{Q}_p/\mathbb{Z}_p)$.
- Cartier dual: $\mathbf{T}^* = \operatorname{Hom}(\mathbf{T}, \mu_{p^{\infty}}).$



Definition (dual Galois modules)

- Pontryagin dual: $\mathbf{T}^{\vee} = \operatorname{Hom}(\mathbf{T}, \mathbb{Q}_p/\mathbb{Z}_p)$.
- Cartier dual: $\mathbf{T}^* = \operatorname{Hom}(\mathbf{T}, \mu_{p^{\infty}})$.

Proposition (local duality)

The cup-product induces a non-degenerate pairing

$$H^1(\mathbb{Q}_\ell,\mathsf{T}) imes H^1(\mathbb{Q}_\ell,\mathsf{T}^*) o H^2(\mathbb{Q}_\ell,\mu_{\mathfrak{p}^\infty})\cong \mathbb{Q}_p/\mathbb{Z}_p$$

Moreover, $H_f^1(\mathbb{Q}_\ell, \mathbf{T})$ and $H_f^1(\mathbb{Q}_\ell, \mathbf{T}^*)$ are exact annihilators of each other.

structu Fitting

ideals

pairing

Kolyvag systems

of Selr groups

system Elliptic

Proof



Definition (dual Galois modules)

- Pontryagin dual: $\mathbf{T}^{\vee} = \operatorname{Hom}(\mathbf{T}, \mathbb{Q}_p/\mathbb{Z}_p)$.
- Cartier dual: $\mathbf{T}^* = \operatorname{Hom}(\mathbf{T}, \mu_{p^{\infty}})$.

Proposition (local duality)

The cup-product induces a non-degenerate pairing

$$H^1(\mathbb{Q}_\ell,\mathsf{T}) imes H^1(\mathbb{Q}_\ell,\mathsf{T}^*) o H^2(\mathbb{Q}_\ell,\mu_{\mathfrak{p}^\infty})\cong \mathbb{Q}_p/\mathbb{Z}_p$$

Moreover, $H_f^1(\mathbb{Q}_\ell, \mathbf{T})$ and $H_f^1(\mathbb{Q}_\ell, \mathbf{T}^*)$ are exact annihilators of each other.

Corollary

$$H^1(\mathbb{Q}_{\ell}, \mathbf{T})^{\vee} \cong H^1(\mathbb{Q}_{\ell}, \mathbf{T}^*)$$

structu Fitting

ideals

Kolyvag

Structur of Selma groups

Euler system

Elliptic Curves



Definition (dual Galois modules)

- Pontryagin dual: $\mathbf{T}^{\vee} = \operatorname{Hom}(\mathbf{T}, \mathbb{Q}_p/\mathbb{Z}_p)$.
- Cartier dual: $\mathbf{T}^* = \operatorname{Hom}(\mathbf{T}, \mu_{p^{\infty}})$.

Proposition (local duality)

The cup-product induces a non-degenerate pairing

$$H^1(\mathbb{Q}_\ell,\mathsf{T}) imes H^1(\mathbb{Q}_\ell,\mathsf{T}^*) o H^2(\mathbb{Q}_\ell,\mu_{\mathfrak{p}^\infty})\cong \mathbb{Q}_p/\mathbb{Z}_p$$

Moreover, $H^1_f(\mathbb{Q}_\ell, \mathsf{T})$ and $H^1_f(\mathbb{Q}_\ell, \mathsf{T}^*)$ are exact annihilators of each other.

Corollary

$$H^1(\mathbb{Q}_\ell,\mathsf{T})^\vee\cong H^1(\mathbb{Q}_\ell,\mathsf{T}^*)$$

$$H^1_f(\mathbb{Q}_\ell,\mathsf{T})^ee\congrac{H^1(\mathbb{Q}_\ell,\mathsf{T}^*)}{H^1_f(\mathbb{Q}_\ell,\mathsf{T}^*)}$$

Fitting

pairings

voiyvagi ystems Structur

or Seim groups Euler system

Elliptic Curves



Dual Selmer structure

Definition (dual Selmer structure)

The dual Selmer structure \mathcal{F}^* is defined by the local conditions

$$\mathit{H}^1_{\mathcal{F}^*}(\mathbb{Q}_\ell,\mathsf{T}^*):=\mathit{Ann}ig(\mathit{H}^1_\mathcal{F}(\mathbb{Q}_\ell,\mathsf{T})ig)\subset \mathit{H}^1(\mathbb{Q}_\ell,\mathsf{T}^*)$$

These are the elements of $H^1(\mathbb{Q}_\ell, \mathbf{T}^*)$ which annihilate $H^1_{\mathcal{T}}(\mathbb{Q}_\ell, T)$ under the local duality pairing.



Dual Selmer structure

Definition (dual Selmer structure)

The dual Selmer structure \mathcal{F}^* is defined by the local conditions

$$\mathcal{H}^1_{\mathcal{F}^*}(\mathbb{Q}_\ell,\mathsf{T}^*):= \mathit{Ann}ig(\mathcal{H}^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathsf{T})ig)\subset \mathcal{H}^1(\mathbb{Q}_\ell,\mathsf{T}^*)$$

These are the elements of $H^1(\mathbb{Q}_\ell, \mathbf{T}^*)$ which annihilate $H^1_{\mathcal{F}}(\mathbb{Q}_\ell, T)$ under the local duality pairing.

Remark (well defined)

The dual Selmer structure is well defined since

$$H^1_f(\mathbb{Q}_\ell,\mathsf{T}^*):=\mathrm{Ann}(H^1_f(\mathbb{Q}_\ell,\mathsf{T}))$$

Selmer structui

Fitting ideals

pairings

Structur of Selm groups

Euler system Elliptic

Curves Proofs



SLIDES

Global duality

Let ${\mathcal F}$ and ${\mathcal G}$ be Selmer structures such that

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T)\subset H^1_{\mathcal{G}}(\mathbb{Q}_\ell,T)\ \forall \ell$$

Introduc

Selmer structur

Fittir

Duality pairings

Kolyva;

Structur of Selm

Euler systen

Elliptic Curves

Droof



Let ${\mathcal F}$ and ${\mathcal G}$ be Selmer structures such that

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T)\subset H^1_{\mathcal{G}}(\mathbb{Q}_\ell,T)\ \forall \ell$$

Then the dual local conditions satisfy the opposite relations

$$H^1_{\mathcal{G}^*}(\mathbb{Q}_\ell, T^*) \subset H^1_{\mathcal{F}^*}(\mathbb{Q}_\ell, T^*) \ orall \ell$$



SLIDES

and Fitting deals of Selmer group of rank 0

Selmer

Fitting ideals

Duality pairing

system:

group:

system Elliptic

Dranfo

Global duality

Let ${\mathcal F}$ and ${\mathcal G}$ be Selmer structures such that

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T)\subset H^1_{\mathcal{G}}(\mathbb{Q}_\ell,T)\ \forall \ell$$

Then the dual local conditions satisfy the opposite relations

$$H^1_{\mathcal{G}^*}(\mathbb{Q}_\ell,T^*)\subset H^1_{\mathcal{F}^*}(\mathbb{Q}_\ell,T^*)\;\forall\ell$$

Clearly,

$$\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q},\mathsf{T})\subset\mathrm{Sel}_{\mathcal{G}}(\mathbb{Q},\mathsf{T}),\quad \mathrm{Sel}_{\mathcal{G}^*}(\mathbb{Q},\mathsf{T}^*)\subset\mathrm{Sel}_{\mathcal{F}^*}(\mathbb{Q},\mathsf{T}^*)$$



Global duality

Let ${\mathcal F}$ and ${\mathcal G}$ be Selmer structures such that

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T)\subset H^1_{\mathcal{G}}(\mathbb{Q}_\ell,T)\ \forall \ell$$

Then the dual local conditions satisfy the opposite relations

$$H^1_{\mathcal{G}^*}(\mathbb{Q}_\ell,T^*)\subset H^1_{\mathcal{F}^*}(\mathbb{Q}_\ell,T^*)\;\forall\ell$$

Clearly,

$$\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},\mathsf{T})\subset\operatorname{Sel}_{\mathcal{G}}(\mathbb{Q},\mathsf{T}),\quad\operatorname{Sel}_{\mathcal{G}^*}(\mathbb{Q},\mathsf{T}^*)\subset\operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q},\mathsf{T}^*)$$

Global duality

$$0 \longrightarrow \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T) \longrightarrow \operatorname{Sel}_{\mathcal{G}}(\mathbb{Q}, T) \longrightarrow \prod_{\ell} \frac{H^{1}_{\mathcal{G}}(\mathbb{Q}_{\ell}, T)}{H^{1}_{\mathcal{F}}(\mathbb{Q}_{\ell}, T)} \longrightarrow \operatorname{Sel}_{\mathcal{F}^{*}}(\mathbb{Q}, T^{*})^{\vee} \longrightarrow 0$$



Assumptions

Selmer

Fittin

Dualit

Kolyvag systems

Structur of Selme groups

groups
Euler

Ellipti Curve

Proofs

■ (H0) Let $\mathfrak{p} \geq 5$ and let **T** be free \mathbb{Z}_p -module of finite rank endowed with a continuous action of $G_{\mathbb{Q}}$, ramifying only at a finite amount of primes.

• (H1) $\rho: G_{\mathbb{Q}} \to \operatorname{Aut}(\mathbf{T})$ is surjective.

• (H2) $H^1(\mathbb{Q}_\ell, \mathbf{T})/H^1_{\mathcal{F}}(\mathbb{Q}_\ell, \mathbf{T})$ is a torsion-free \mathbb{Z}_p -module.



SLIDE

Fix $k \in \mathbb{N}$ and let $T = \mathbf{T}/p^k$. Denote $\pi: \ \mathbf{T} \to T$ to the canonical projection.

Introduct

Selmer

Fittin

Dualit pairing

Kolyvag systems

Structur of Selme groups

Euler syster

Elliptic Curves

D...-6



Fix $k \in \mathbb{N}$ and let $T = \mathbf{T}/p^k$. Denote $\pi : \mathbf{T} \to T$ to the canonical projection.

Definition (propagated local condition)

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell, T) = \pi \left(H^1_{\mathcal{F}}(\mathbb{Q}_\ell, \mathbf{T}) \right)$$

ntroductio

Selmer structure

Fittin

Duali

Kolyvag

Structur of Selm

Euler

Elliptic

D... - 6.



Fix $k \in \mathbb{N}$ and let $T = \mathbf{T}/p^k$. Denote $\pi : \mathbf{T} \to T$ to the canonical projection.

Definition (propagated local condition)

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T)=\pi\left(H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathbf{T})\right)$$

Proposition

Under assumptions (H0), (H1) and (H2), the following equality holds true.

Duality

pairings

systems

of Seln groups

Euler syster

Ellipti Curve

Proof



Fix $k \in \mathbb{N}$ and let $T = \mathbf{T}/p^k$. Denote $\pi : \mathbf{T} \to T$ to the canonical projection.

Definition (propagated local condition)

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T)=\pi\left(H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathbf{T})\right)$$

Proposition

Under assumptions (H0), (H1) and (H2), the following equality holds true.

$$\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*) = \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, \mathbf{T}^*)[p^k]$$

Selmer

Fitting

Duality pairings

Structur of Selme

Euler system

Elliptic Curves

Proof:



Fix $k \in \mathbb{N}$ and let $T = \mathbf{T}/p^k$. Denote $\pi : \mathbf{T} \to T$ to the canonical projection.

Definition (propagated local condition)

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T)=\pi\left(H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathbf{T})\right)$$

Proposition

Under assumptions (H0), (H1) and (H2), the following equality holds true.

$$\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*) = \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, \mathbf{T}^*)[p^k]$$

Remark

A study of $Sel_{\mathcal{F}}(\mathbb{Q}, \mathbf{T}/p^k)$ for all k will determine $Sel_{\mathcal{F}}(\mathbb{Q}, \mathbf{T})$.

Selmer structure

Duality pairings

Kolyvag systems

> of Selm groups Euler

Elliptic Curves



SLIDES

Fitting ideals of Selmer group of

Introducti

Selmer structure

Fittin

Duali

Kolyvag

systems

of Selmogroups

system

Elliptic Curves

Proof

Definition

A prime ℓ is a Kolyvagin prime if



Definition

A prime ℓ is a Kolyvagin prime if

 $\ell \equiv 1 \mod p^k$.

Selmer

Fittir

Dualit pairing

systems

Structur of Selme

Euler

Ellipti

Droof



Definition

A prime ℓ is a Kolyvagin prime if

- $\ell \equiv 1 \mod p^k$.
- $P_{\ell}(1) = \det(1 \operatorname{Frob}_{\ell}|T) = 0.$

Selmer structure

Fittir ideal:

Dualit

Kolyvag systems

Structur of Selme

Euler

Ellipti Curve

Proof



Definition

A prime ℓ is a Kolyvagin prime if

- $\ell \equiv 1 \mod p^k$.
- $P_{\ell}(1) = \det(1 \operatorname{Frob}_{\ell}|T) = 0.$

Notation

P denotes the set of Kolyvagin primes.

Selmer structures

Fittin

Duality

Kolyvag systems

group:

Elliptic

Proof



Definition

A prime ℓ is a Kolyvagin prime if

- $\ell \equiv 1 \mod p^k$.
- $P_{\ell}(1) = \det(1 \operatorname{Frob}_{\ell}|T) = 0.$

Notation

P denotes the set of Kolyvagin primes.

 $\mathcal{N}(\mathcal{P})$ denotes the set of square free products of Kolyvagin primes.

Selmer structure

Fittin ideals

Duality pairing

Struct of Selr

Euler system

Elliptic Curves

Proofs



Definition

A prime ℓ is a Kolyvagin prime if

- $\ell \equiv 1 \mod p^k$.
- $P_{\ell}(1) = \det(1 \operatorname{Frob}_{\ell}|T) = 0.$

Notation

P denotes the set of Kolyvagin primes.

 $\mathcal{N}(\mathcal{P})$ denotes the set of square free products of Kolyvagin primes.

 $\mathcal{N}_i(\mathcal{P})$ denotes the set of square free products of exactly i Kolyvagin primes.



Transverse local condition and finite-singular map

and

Selmer group o rank 0

Introduct

Selmer structure

Fittin

Dualit pairing

Kolyva

Structur of Selm

Euler

Elliptio

D... - 6.

Definition (transverse local condition)

$$H^1_{tr}(\mathbb{Q}_\ell, T) := \operatorname{Im} \left(H^1(\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell, T) \to H^1(\mathbb{Q}_\ell, T) \right)$$



Transverse local condition and finite-singular map

Definition (transverse local condition)

$$H^1_{tr}(\mathbb{Q}_\ell,T):=\mathrm{Im}ig(H^1(\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell,T) o H^1(\mathbb{Q}_\ell,T)ig)$$

Proposition (split of the local cohomology group)

If ℓ is Kolyvagin prime, then

$$H^1(\mathbb{Q}_\ell,T)=H^1_f(\mathbb{Q}_\ell,T)\oplus H^1_{tr}(\mathbb{Q}_\ell,T)\cong \mathbb{Z}/p^k\oplus \mathbb{Z}/p^k$$

Selmer structure

Fitting

Duality

Kolyvag systems

of Selm groups

Euler system

Elliptic Curves

Droof



Transverse local condition and finite-singular map

Definition (transverse local condition)

$$H^1_{tr}(\mathbb{Q}_\ell,T):=\mathrm{Im}ig(H^1(\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell,T) o H^1(\mathbb{Q}_\ell,T)ig)$$

Proposition (split of the local cohomology group)

If ℓ is Kolyvagin prime, then

$$H^1(\mathbb{Q}_\ell, T) = H^1_f(\mathbb{Q}_\ell, T) \oplus H^1_{tr}(\mathbb{Q}_\ell, T) \cong \mathbb{Z}/p^k \oplus \mathbb{Z}/p^k$$

Definition (finite-singular map)

There is a canonical isomorphism

$$\phi_{fs}: H^1_f(\mathbb{Q}_\ell, T) \cong H^1_{tr}(\mathbb{Q}_\ell, T)$$

tructur

Duality

Kolyvag systems

of Seln groups Euler svstem

Elliptic Curves



Let $a, b, c \in \mathbb{N}$ be such that abc is square free.

Assume all primes dividing a, b and c are Kolyvagin primes.

We can define a new Selmer structure $\mathcal{F}_a^b(c)$ by

Selmer

Fittir

ideals

Dualit pairing

systems

of Selme groups

Euler system

Elliptic Curves

Proofs



Let $a, b, c \in \mathbb{N}$ be such that abc is square free.

Assume all primes dividing a, b and c are Kolyvagin primes.

We can define a new Selmer structure $\mathcal{F}_a^b(c)$ by

$$H^1_{\mathcal{F}^b(c)}(\mathbb{Q}_\ell,T):=H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T) \text{ if } \ell
mid abc.$$

structi Fitting

Dualit

Kolyva

Structur

groups

system

Proof



Let $a, b, c \in \mathbb{N}$ be such that abc is square free.

Assume all primes dividing a, b and c are Kolyvagin primes.

We can define a new Selmer structure $\mathcal{F}_a^b(c)$ by

$$lacksymbol{H}^1_{\mathcal{F}^b_a(c)}(\mathbb{Q}_\ell,T):=H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T) ext{ if } \ell
mid ext{abc}.$$

$$\blacksquare \ H^1_{\mathcal{F}^b_2(c)}(\mathbb{Q}_\ell,\,T)=0 \ \text{if} \ \ell\mid a.$$

Selmer structu

Fitting ideals

Duality pairing

Kolyvag systems

of Seln groups Euler

system Elliptic

Proof



Let $a, b, c \in \mathbb{N}$ be such that abc is square free.

Assume all primes dividing a, b and c are Kolyvagin primes.

We can define a new Selmer structure $\mathcal{F}_a^b(c)$ by

$$lacksymbol{H}^1_{\mathcal{F}^b_a(c)}(\mathbb{Q}_\ell,T):=H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T) ext{ if } \ell
mid ext{abc}.$$

$$lacksquare H^1_{\mathcal{F}^b_a(c)}(\mathbb{Q}_\ell,T)=0 \ ext{if} \ \ell\mid a.$$

$$\blacksquare \ H^1_{\mathcal{F}^b_{\sigma}(c)}(\mathbb{Q}_\ell,T)=H^1(\mathbb{Q}_\ell,T) \text{ if } \ell\mid b.$$

Selmer structure

ideals

pairings Kolyvag

> Structure of Selme groups

system
Elliptic

Curve



Let $a, b, c \in \mathbb{N}$ be such that abc is square free.

Assume all primes dividing a, b and c are Kolyvagin primes.

We can define a new Selmer structure $\mathcal{F}_a^b(c)$ by

$$lacksquare H^1_{\mathcal{F}^b_a(c)}(\mathbb{Q}_\ell,T):=H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T) ext{ if } \ell
mid ext{abc}.$$

$$lacksquare H^1_{\mathcal{F}^b_a(c)}(\mathbb{Q}_\ell,T)=0 \ ext{if} \ \ell\mid a.$$

$$lacksquare H^1_{\mathcal{F}^b_{\mathcal{E}}(c)}(\mathbb{Q}_\ell,T)=H^1(\mathbb{Q}_\ell,T) \ ext{if} \ \ell\mid b.$$

•
$$H^1_{\mathcal{F}^b(c)}(\mathbb{Q}_\ell, T) = H^1_{tr}(\mathbb{Q}_\ell, T)$$
 if $\ell \mid c$.

ideals Duality

Kolyvag systems

of Selmo groups Euler

Euler system Elliptic

Curves Proofs



Definition (Kolyvagin system)

A Kolyvagin system is a collection of elements

$$\kappa_n \in \mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \subset H^1(\mathbb{Q}, T)$$

for every $n \in \mathcal{N}(\mathcal{P})$, satisfying the following Kolyvagin conditions.

Selmer structure

Fittin

Dualit

Kolyva system

Structur of Selmi

Euler syster

Ellipti Curve

Droof



Definition (Kolyvagin system)

A Kolyvagin system is a collection of elements

$$\kappa_n \in \mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \subset H^1(\mathbb{Q}, T)$$

for every $n \in \mathcal{N}(\mathcal{P})$, satisfying the following Kolyvagin conditions.

For every $n \in \mathcal{N}(\mathcal{P})$ and $\ell \in \mathcal{P}$ such that $\ell \nmid n$, consider the localization maps at ℓ .

$$\operatorname{loc}_{\ell}(\kappa_n) \in H^1_{\mathcal{F}(n)}(\mathbb{Q}_{\ell},T) = H^1_f(\mathbb{Q}_{\ell},T)$$

$$\operatorname{loc}_{\ell}(\kappa_{n\ell}) \in H^1_{\mathcal{F}(n\ell)}(\mathbb{Q},T) = H^1_{tr}(\mathbb{Q}_{\ell},T)$$

tructu

Fitting deals

Duality pairing

Kolyvag systems

of Selmi groups

Euler systen

Elliptic

Proof



Definition (Kolyvagin system)

A Kolyvagin system is a collection of elements

$$\kappa_n \in \operatorname{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \subset H^1(\mathbb{Q}, T)$$

for every $n \in \mathcal{N}(\mathcal{P})$, satisfying the following Kolyvagin conditions.

For every $n \in \mathcal{N}(\mathcal{P})$ and $\ell \in \mathcal{P}$ such that $\ell \nmid n$, consider the localization maps at ℓ .

$$\operatorname{loc}_{\ell}(\kappa_n) \in H^1_{\mathcal{F}(n)}(\mathbb{Q}_{\ell},T) = H^1_f(\mathbb{Q}_{\ell},T)$$

$$\operatorname{loc}_{\ell}(\kappa_{n\ell}) \in H^1_{\mathcal{F}(n\ell)}(\mathbb{Q},T) = H^1_{\operatorname{tr}}(\mathbb{Q}_{\ell},T)$$

The Kolyvagin condition for $n \in \mathcal{N}(\mathcal{P})$ and $\ell \in \mathcal{P}$ is

$$\phi_{fs}(\operatorname{loc}_{\ell}(\kappa_n)) = \operatorname{loc}_{\ell}(\kappa_{n\ell})$$

Selmer structur

Fitting deals

Duality pairing:

Kolyvag systems

of Selm groups

system Elliptic

Proofs

Definition (Kolyvagin system)

A Kolyvagin system is a collection of elements

$$\kappa_n \in \operatorname{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \subset H^1(\mathbb{Q}, T)$$

for every $n \in \mathcal{N}(\mathcal{P})$, satisfying the following Kolyvagin conditions.

For every $n \in \mathcal{N}(\mathcal{P})$ and $\ell \in \mathcal{P}$ such that $\ell \nmid n$, consider the localization maps at ℓ .

$$\operatorname{loc}_{\ell}(\kappa_n) \in H^1_{\mathcal{F}(n)}(\mathbb{Q}_{\ell},T) = H^1_f(\mathbb{Q}_{\ell},T)$$

$$\operatorname{loc}_{\ell}(\kappa_{n\ell}) \in H^1_{\mathcal{F}(n\ell)}(\mathbb{Q},T) = H^1_{\operatorname{tr}}(\mathbb{Q}_{\ell},T)$$

The Kolyvagin condition for $n \in \mathcal{N}(\mathcal{P})$ and $\ell \in \mathcal{P}$ is

$$\phi_{fs}(\operatorname{loc}_{\ell}(\kappa_n)) = \operatorname{loc}_{\ell}(\kappa_{n\ell})$$

Notation

The module of Kolyvagin systems will be denoted by $KS(T, \mathcal{F})$.

Selmer structur

itting deals

pairings

Structu

groups Euler

Elliptic Curves



Core rank

Definition/proposition (core rank)

There exists a non-negative integer $\chi(\mathcal{F})$ and a non-canonical homomorphism

$$\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},T)\cong \left(\mathbb{Z}/p^k
ight)^{\chi(T)}\oplus \operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q},T^*)$$

(possibly after swapping the roles of T and T^* .)

The integer $\chi(T)$ is called the core rank of T.

structu

Fittin ideals

Dualit pairing

Kolyvag systems

Structure of Selme groups

Euler

Ellipti

Droof



Core rank

Definition/proposition (core rank)

There exists a non-negative integer $\chi(\mathcal{F})$ and a non-canonical homomorphism

$$\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},T)\cong \left(\mathbb{Z}/p^{k}\right)^{\chi(T)}\oplus \operatorname{Sel}_{\mathcal{F}^{*}}(\mathbb{Q},T^{*})$$

(possibly after swapping the roles of T and T^* .)

The integer $\chi(T)$ is called the core rank of T.

Proposition (Sakamoto, 2021)

$$\chi(\mathcal{F}_{\mathsf{a}}^{\mathsf{b}}(\mathsf{c})) = \chi(\mathcal{F}) + \nu(\mathsf{b}) - \nu(\mathsf{a})$$

where $\nu(b)$ and $\nu(a)$ are the number of primes dividing b and a, respectively.

structi

Duality

Kolyvagi systems

Structur of Selme groups

Euler system

Elliptic Curves



Core rank and Kolyvagin systems

Theorem (Mazur-Rubin, 2004)

• If $\chi(\mathcal{F}) = 0$, then $KS(\mathcal{T}, \mathcal{F}) = 0$.

There are no Kolyvagin system to control the Selmer group. We will see a posible solution later in the talk.

miroduci

Selmer

Fittin

Dualit

Kolyva systems

Structur of Selmi

Euler syster

Elliptic Curves

Proofs



Core rank and Kolyvagin systems

Theorem (Mazur-Rubin, 2004)

• If $\chi(\mathcal{F}) = 0$, then $KS(\mathcal{T}, \mathcal{F}) = 0$.

There are no Kolyvagin system to control the Selmer group. We will see a posible solution later in the talk.

• If $\chi(\mathcal{F}) = 1$, then $\mathrm{KS}(\mathcal{T}, \mathcal{F}) \cong \mathbb{Z}/p^k$.

A generator of $KS(T, \mathcal{F})$ is called a primitive Kolyvagin system. We will see next that they carry information to compute all the Fitting ideals of the Selmer group $Sel_{\mathcal{F}^*}(\mathbb{Q}, T^*)$.

Selmer structure

Fitting ideals

pairings Kolyvag

Structur of Selme groups

groups Euler system

Elliptic Curves

Core rank and Kolyvagin systems

Theorem (Mazur-Rubin, 2004)

• If $\chi(\mathcal{F}) = 0$, then $KS(\mathcal{T}, \mathcal{F}) = 0$.

There are no Kolyvagin system to control the Selmer group. We will see a posible solution later in the talk.

• If $\chi(\mathcal{F}) = 1$, then $\mathrm{KS}(\mathcal{T}, \mathcal{F}) \cong \mathbb{Z}/p^k$.

A generator of $\mathrm{KS}(\mathcal{T},\mathcal{F})$ is called a primitive Kolyvagin system. We will see next that they carry information to compute all the Fitting ideals of the Selmer group $\mathrm{Sel}_{\mathcal{F}^*}(\mathbb{Q},\mathcal{T}^*)$.

■ If $\chi(\mathcal{F}) > 1$, then $KS(\mathcal{T}, \mathcal{F})$ is too large.

In order to compute the Selmer group, [Mazur-Rubin, 2016] and [Burns-Sakamoto-Sano, 2025] modified the definition of Kolyvagin system in (biduals of) exterior powers of the Selmer groups.



SLIDE

/vagin

and Fitting ideals o Selmer group o rank 0

Introducti

Selmer structure

Fittin

Duali

Kolyva

Structur of Selme

Euler

Ellipti

Droof

Definition (order of a Kolyvagin element)

$$\operatorname{ord}(\kappa_n) := \max \left\{ j \in \{0, \dots, k\} : \kappa_n \in p^j H^1_{\mathcal{F}(n)}(\mathbb{Q}, T) \right\}$$



Definition (order of a Kolyvagin element)

$$\operatorname{ord}(\kappa_n) := \max \left\{ j \in \{0, \dots, k\} : \kappa_n \in p^j H^1_{\mathcal{F}(n)}(\mathbb{Q}, T) \right\}$$

Proposition

If κ is a primitive Kolyvagin system

$$\operatorname{ord}(\kappa_n) = \min \left\{ k, \operatorname{length}\left(H^1_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*)\right) \right\}$$



Definition (order of a Kolyvagin element)

$$\operatorname{ord}(\kappa_n) := \max \left\{ j \in \{0, \dots, k\} : \kappa_n \in p^j H^1_{\mathcal{F}(n)}(\mathbb{Q}, T) \right\}$$

Proposition

If κ is a primitive Kolyvagin system

$$\operatorname{ord}(\kappa_n) = \min \left\{ k, \operatorname{length} \left(H^1_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*) \right) \right\}$$

Definition

$$\Theta_i := (p)^{\min\{\operatorname{ord}(\kappa_n): n \in \mathcal{N}_i\}}$$

Selmer structure

Fitting ideals

Duality pairings

Kolyvagi systems

of Selm groups

system Elliptic

Curve



Definition (order of a Kolyvagin element)

$$\operatorname{ord}(\kappa_n) := \max \left\{ j \in \{0, \dots, k\} : \kappa_n \in p^j H^1_{\mathcal{F}(n)}(\mathbb{Q}, T) \right\}$$

Proposition

If κ is a primitive Kolyvagin system

$$\operatorname{ord}(\kappa_n) = \min \left\{ k, \operatorname{length} \left(H^1_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*) \right) \right\}$$

Definition

$$\Theta_i := (p)^{\min\{\operatorname{ord}(\kappa_n): n \in \mathcal{N}_i\}}$$

Theorem (Mazur-Rubin, 2004)

When $\chi(\mathcal{F})=1$ and κ is a primitive Kolyvagin system

$$\Theta_i = \operatorname{Fitt}_i \left(\operatorname{Sel}_{\mathcal{F}^*} (\mathbb{Q}, T)^* \right)$$

structure Fitting

Duality

Kolyvagir systems

groups Euler system

Elliptic Curves



SLIDES

Fitting ideals of Selmer

Introduc Selmer

Fittin ideals

Duali pairin

Kolyva system

Structur of Selme groups

Euler syster

Ellipti Curve

Proof

Core rank 0

- We have seen that there are no non-zero Kolyvagin systems.
- Choose a prime ℓ such that $H^1_s(\mathbb{Q}_\ell,T)\cong \mathbb{Z}/p^k$.
- Note that all Kolyvagin primes satisfy the above condition, but we do not restrict to them.
- Then \mathcal{F}^{ℓ} is cartesian and $\chi(\mathcal{F}^{\ell})=1$.



Core rank 0

- We have seen that there are no non-zero Kolyvagin systems.
- Choose a prime ℓ such that $H^1_{\mathfrak{s}}(\mathbb{Q}_{\ell},T) \cong \mathbb{Z}/p^k$.
- Note that all Kolyvagin primes satisfy the above condition, but we do not restrict to them.
- Then \mathcal{F}^{ℓ} is cartesian and $\chi(\mathcal{F}^{\ell}) = 1$.

Definition

Let $\kappa \in \mathrm{KS}(T, \mathcal{F}^{\ell})$. Define

$$\delta_n = \delta_n(\kappa) := \operatorname{loc}_{\ell}(\kappa_n) \in H^1_{\mathfrak{s}}(\mathbb{Q}_{\ell}, T) \cong \mathbb{Z}/p^k$$



9554**6**

Core rank 0

- We have seen that there are no non-zero Kolyvagin systems.
- Choose a prime ℓ such that $H^1_s(\mathbb{Q}_\ell, T) \cong \mathbb{Z}/p^k$.
- Note that all Kolyvagin primes satisfy the above condition, but we do not restrict to them.
- Then \mathcal{F}^{ℓ} is cartesian and $\chi(\mathcal{F}^{\ell}) = 1$.

Definition

Let $\kappa \in \mathrm{KS}(T, \mathcal{F}^{\ell})$. Define

$$\delta_n = \delta_n(\kappa) := \operatorname{loc}_{\ell}(\kappa_n) \in H^1_{\mathfrak{s}}(\mathbb{Q}_{\ell}, T) \cong \mathbb{Z}/p^k$$

Definition (order)

$$\operatorname{ord}(\delta_n) = \max \left\{ j \in \{0, \dots, k\} : \delta^n \in \left(p^j\right) \right\}$$

Fitting ideals

Duality pairings Kolyvag

Structur of Selm groups

Euler system

Elliptic Curves



SSUM

Core rank 0

- We have seen that there are no non-zero Kolyvagin systems.
- Choose a prime ℓ such that $H_s^1(\mathbb{Q}_\ell, T) \cong \mathbb{Z}/p^k$.
- Note that all Kolyvagin primes satisfy the above condition, but we do not restrict to them.
- Then \mathcal{F}^{ℓ} is cartesian and $\chi(\mathcal{F}^{\ell}) = 1$.

Definition

Let $\kappa \in \mathrm{KS}(T, \mathcal{F}^{\ell})$. Define

$$\delta_n = \delta_n(\kappa) := \operatorname{loc}_{\ell}(\kappa_n) \in H^1_{\mathfrak{s}}(\mathbb{Q}_{\ell}, T) \cong \mathbb{Z}/p^k$$

Definition (order)

$$\operatorname{ord}(\delta_n) = \max \left\{ j \in \{0, \dots, k\} : \delta^n \in \left(p^j\right) \right\}$$

Proposition (Kim, 2025)

$$\operatorname{ord}(\delta_n) = \min \left\{ k, \operatorname{length} \left(H^1_{(\mathcal{F}^*)(n)}(\mathbb{Q}, \mathcal{T}^*) \right) \right\}$$



Definition

$$\Theta_i := (p)^{\min\{\operatorname{ord}(\delta_n): n \in \mathcal{N}_i(\mathcal{P})\}} = \langle \{\delta_n: \ n \in \mathcal{N}_i(\mathcal{P})\} \rangle \subset \mathbb{Z}/p^k$$

Introduct

Selmer structure

Fittir

ideals

Duali pairin

Kolyva system

Structure of Selme

Euler

Ellipti

D...-6



Definition

$$\Theta_i := (p)^{\min\{\operatorname{ord}(\delta_n): n \in \mathcal{N}_i(\mathcal{P})\}} = \langle \{\delta_n: \ n \in \mathcal{N}_i(\mathcal{P})\} \rangle \subset \mathbb{Z}/p^k$$

Theorem (A., 2025)

For all i, we have

$$\Theta_i \subset \operatorname{Fitt}_i \left(H^1_{\mathcal{F}^*}(\mathbb{Q}, T^*) \right)$$

Selmer structur

Fittin ideals

Dualit

Kolyva

Structur of Selme

Euler

Ellipti



Definition

$$\Theta_i := (p)^{\min\{\operatorname{ord}(\delta_n): n \in \mathcal{N}_i(\mathcal{P})\}} = \langle \{\delta_n: \ n \in \mathcal{N}_i(\mathcal{P})\} \rangle \subset \mathbb{Z}/p^k$$

Theorem (A., 2025)

For all i, we have

$$\Theta_i \subset \operatorname{Fitt}_i \left(H^1_{\mathcal{F}^*}(\mathbb{Q}, T^*) \right)$$

The equality for some index i holds if any of the following is true:

Duality pairings

Kolyvagi systems

of Selr groups

systen Ellipti

Droof



Definition

$$\Theta_i := (p)^{\min\{\operatorname{ord}(\delta_n): n \in \mathcal{N}_i(\mathcal{P})\}} = \langle \{\delta_n: \ n \in \mathcal{N}_i(\mathcal{P})\} \rangle \subset \mathbb{Z}/p^k$$

Theorem (A., 2025)

For all i, we have

$$\Theta_i \subset \operatorname{Fitt}_i \left(H^1_{\mathcal{F}^*}(\mathbb{Q}, T^*) \right)$$

The equality for some index i holds if any of the following is true:

$$\bullet \Theta_{i-1} \subsetneq \operatorname{Fitt}_{i-1} \left(H^1_{\mathcal{F}^*}(\mathbb{Q}, T^*) \right).$$

Selmer

Fitting

Duality pairing

Kolyvagi systems

Structu of Selm groups

system Elliptic

Curves



Definition

$$\Theta_i := (p)^{\min\{\operatorname{ord}(\delta_n): n \in \mathcal{N}_i(\mathcal{P})\}} = \langle \{\delta_n: \ n \in \mathcal{N}_i(\mathcal{P})\} \rangle \subset \mathbb{Z}/p^k$$

Theorem (A., 2025)

For all i, we have

$$\Theta_i \subset \operatorname{Fitt}_i \left(H^1_{\mathcal{F}^*}(\mathbb{Q}, T^*) \right)$$

The equality for some index i holds if any of the following is true:

$$\bullet \Theta_{i-1} \subsetneq \operatorname{Fitt}_{i-1} \left(H^1_{\mathcal{F}^*}(\mathbb{Q}, T^*) \right).$$

$$\bullet \Theta_{i-1} = \operatorname{Fitt}_{i-1} \left(H^1_{\mathcal{F}^*}(\mathbb{Q}, T^*) \right) = 0.$$

Curves



Definition

$$\Theta_i := (p)^{\min\{\operatorname{ord}(\delta_n): n \in \mathcal{N}_i(\mathcal{P})\}} = \langle \{\delta_n: \ n \in \mathcal{N}_i(\mathcal{P})\} \rangle \subset \mathbb{Z}/p^k$$

Theorem (A., 2025)

For all i, we have

$$\Theta_i\subset \mathrm{Fitt}_i\left(H^1_{\mathcal{F}^*}(\mathbb{Q},\,T^*)\right)$$

The equality for some index i holds if any of the following is true:

$$\bullet \Theta_{i-1} \subsetneq \operatorname{Fitt}_{i-1} \left(H^1_{\mathcal{F}^*}(\mathbb{Q}, T^*) \right).$$

$$\bullet \Theta_{i-1} = \operatorname{Fitt}_{i-1} \left(H^1_{\mathcal{T}^*}(\mathbb{Q}, T^*) \right) = 0.$$

Remark

When $T = \mathbf{T}/p^k$ for some \mathbb{Z}_{p} -module \mathbf{T} and k is large enough, the ideals Θ_i determine $H^1_T(\mathbb{Q}, \mathbf{T}^*)$ up to isomorphism



Galois representations which are not residually self-dual

Theorem (A., 2025)

Under the following assumption on non self-duality,

■ (N1) $T/p \ncong T^*[p]$.



Galois representations which are not residually self-dual

Theorem (A., 2025)

Under the following assumption on non self-duality,

$$\blacksquare (N1) \ T/p \ncong T^*[p].$$

Then for all $i \in \mathbb{Z}_{\geq 0}$, we have the equality

$$\Theta_i = \operatorname{Fitt}_i \left(H^1_{\mathcal{F}}(\mathbb{Q}, T^*) \right)$$

tructui

Fittin, ideals

Duality

Kolyva

Structur of Selmo

Euler system

Elliptic Curves



Connection to Euler systems

Kolyvaş systen and Fittin

Assume that we have an Euler system z.

Introduct

Selmer

Fittin

Dualit

Kolyva

Structur of Selm

Euler

Elliptio Curves

ь с



Connection to Euler systems

- Assume that we have an Euler system z.
- The Kolyvagin derivative operator produces a Kolyvagin system for \mathbf{T}/p^k for all k and the canonical Selmer structure, defined as

$$\begin{cases} H^1_{\mathcal{F}^{\mathrm{can}}}(\mathbb{Q}_\ell,\mathsf{T}/p^k) = H^1_f(\mathbb{Q}_\ell,\mathsf{T}/p^k) \text{ if } \ell \neq p,\infty \\ H^1_{\mathcal{F}^{\mathrm{can}}}(\mathbb{Q}_p,\mathsf{T}/p^k) = H^1(\mathbb{Q}_p,\mathsf{T}/p^k) \\ H^1_{\mathcal{F}^{\mathrm{can}}}(\mathbb{R},\mathsf{T}/p^k) = H^1(\mathbb{R},\mathsf{T}/p^k) \end{cases}$$

This is also known as *relaxed at p*. Its dual Selmer structure will be called *restricted at p*.

Selmer tructur

Fittin

Dualit

Kolyva system

of Selm groups

Euler

Ellipti Curve



Connection to Euler systems

- Assume that we have an Euler system z.
- The Kolyvagin derivative operator produces a Kolyvagin system for \mathbf{T}/p^k for all k and the canonical Selmer structure, defined as

$$\begin{cases} H^1_{\mathcal{F}^{\mathrm{can}}}(\mathbb{Q}_\ell,\mathsf{T}/\rho^k) = H^1_f(\mathbb{Q}_\ell,\mathsf{T}/\rho^k) \text{ if } \ell \neq \rho,\infty \\ H^1_{\mathcal{F}^{\mathrm{can}}}(\mathbb{Q}_\rho,\mathsf{T}/\rho^k) = H^1(\mathbb{Q}_\rho,\mathsf{T}/\rho^k) \\ H^1_{\mathcal{F}^{\mathrm{can}}}(\mathbb{R},\mathsf{T}/\rho^k) = H^1(\mathbb{R},\mathsf{T}/\rho^k) \end{cases}$$

This is also known as *relaxed at p*. Its dual Selmer structure will be called *restricted at p*.

Theorem (Kolyvagin, 1995)

$$\operatorname{ord}(z_{\mathbb{Q}}) \geq \operatorname{ord}(\kappa_1) \geq \operatorname{length}\left(H^1_{(\mathcal{F}^{\operatorname{can}})^*}(\mathbb{Q},\mathsf{T}^*)\right)$$

Selmer structur

Duality pairings

> Colyvagin ystems Structure

Euler system

> Elliptic Curves Proofs



system

We apply the results to $\mathbf{T} = T_p \mathbf{E} \otimes \chi$.

Introduct

Selmer structure

Fittin

ideals

pairing

Kolyva; system:

of Selm

Euler systen

Elliptio

D....6



We apply the results to $\mathbf{T} = T_p \mathbf{E} \otimes \chi$. We assume the following:

- **(E0)** E is defined over \mathbb{Q} .
- \bullet (χ 1) The conductor of χ is not divisible by p or any bad prime of E.
- (χ^2) The order of χ is prime to p.

Selmer

Fittin

Dualit

Kolyva

Structu

group

Ellipti



We apply the results to $\mathbf{T} = T_p \mathbf{E} \otimes \chi$. We assume the following:

- **(E0)** E is defined over \mathbb{Q} .
- \bullet (χ 1) The conductor of χ is not divisible by p or any bad prime of E.
- **(\chi2)** The order of χ is prime to p.

Modularity There exists a modular form $f_{\chi} = \sum \chi(n) a_n q^n$ such that

$$T_{f_\chi} = T_p E \otimes \chi$$

struct

Fittin ideals

Duality pairing

Kolyvag systems

of Seli group:

Euler syste

Elliptic Curves



We apply the results to $\mathbf{T} = T_p \mathbf{E} \otimes \chi$. We assume the following:

- **(E0)** E is defined over \mathbb{Q} .
- \bullet (χ 1) The conductor of χ is not divisible by p or any bad prime of E.
- **(\chi2)** The order of χ is prime to p.

Modularity There exists a modular form $f_{\chi} = \sum \chi(n) a_n q^n$ such that

$$T_{f_\chi} = T_p E \otimes \chi$$

Kato constructed an Euler system for this representation.

structi Fitting

ideals

pairings Kolyvag

systems Structur

group Euler system

Elliptic Curves



Elliptic curves: Bloch-Kato Selmer structure

Bloch-Kato Selmer structure

The classical local conditions are defined by Bloch-Kato condition

$$\begin{cases} H^1_{\mathcal{F}_{BK}}(\mathbb{Q}_\ell, T) = H^1_f(\mathbb{Q}_\ell, T) & \forall \ell \neq p \\ H^1_{\mathcal{F}_{BK}}(\mathbb{Q}_p, T) = \ker \left(H^1(\mathbb{Q}_p, T) \to H^1(\mathbb{Q}_p, T \otimes_{Z_p} \mathcal{B}_{\operatorname{crys}}) \right) \end{cases}$$

Selmer

Fittir

ideal

Duali

Kolyva system

of Selm groups

Euler

Ellipti

Droof



Elliptic curves: Bloch-Kato Selmer structure

Bloch-Kato Selmer structure

The classical local conditions are defined by Bloch-Kato condition

$$\begin{cases} H^1_{\mathcal{F}_{BK}}(\mathbb{Q}_\ell, T) = H^1_f(\mathbb{Q}_\ell, T) & \forall \ell \neq \rho \\ H^1_{\mathcal{F}_{BK}}(\mathbb{Q}_p, T) = \ker \left(H^1(\mathbb{Q}_p, T) \to H^1(\mathbb{Q}_p, T \otimes_{Z_p} \mathcal{B}_{\operatorname{crys}}) \right) \end{cases}$$

Assume the following:

E1) $\rho: G_{\mathbb{O}} \to \operatorname{Aut}(\mathbf{T})$ is surjective.



Elliptic curves: Bloch-Kato Selmer structure

Bloch-Kato Selmer structure

The classical local conditions are defined by Bloch-Kato condition

$$\begin{cases} H^1_{\mathcal{F}_{BK}}(\mathbb{Q}_\ell, \mathsf{T}) = H^1_f(\mathbb{Q}_\ell, \mathsf{T}) & \forall \ell \neq p \\ H^1_{\mathcal{F}_{BK}}(\mathbb{Q}_p, \mathsf{T}) = \ker \left(H^1(\mathbb{Q}_p, \mathsf{T}) \to H^1(\mathbb{Q}_p, \mathsf{T} \otimes_{Z_p} \mathcal{B}_{\operatorname{crys}}) \right) \end{cases}$$

Assume the following:

■ (E1) $\rho: G_{\mathbb{O}} \to \operatorname{Aut}(\mathbf{T})$ is surjective.

Proposition

Assuming (E1), \mathcal{F}_{BK} satisfies all the assumptions (H0), (H1) and (H2) and $\chi(\mathcal{F}_{BK})=0$.



Kolyvagin derivative

The Kolyvagin derivative operator applied to Kato's Euler system, produces a Kolyvagin system for $\mathcal{F}^{\operatorname{can}} = (\mathcal{F}_{BK})^p$.

Selmer structure

Fittir

Dualit

Kolyva

Structu of Selm

Euler

Elliptic Curves

Droof



Kolyvagin derivative

The Kolyvagin derivative operator applied to Kato's Euler system, produces a Kolyvagin system for $\mathcal{F}^{\operatorname{can}} = (\mathcal{F}_{BK})^p$.

Denote K_{χ} to the fixed field of χ . Assume:

• $(\chi 3)$ $E((K_{\chi})_{\mathfrak{p}})[p] = \{O\}$ for every prime \mathfrak{p} above p.

ideals Dualit

pairin

system

of Seln groups

Euler

Ellipti



Kolyvagin derivative

The Kolyvagin derivative operator applied to Kato's Euler system, produces a Kolyvagin system for $\mathcal{F}^{\operatorname{can}} = (\mathcal{F}_{BK})^p$.

Denote K_{χ} to the fixed field of χ . Assume:

• $(\chi 3)$ $E((K_{\chi})_{\mathfrak{p}})[p] = \{O\}$ for every prime \mathfrak{p} above p.

Proposition

Assuming (χ 3), the Selmer structure (\mathcal{F}_{BK}) p satisfies all the assumptions (H0), (H1) and (H2) and $\chi((\mathcal{F}_{BK})^p)=1$.

elmer tructur

itting deals

Duality pairing:

Kolyva; system: Structi

of Seli group:

Euler syster

Elliptic Curves

Kolyvagin derivative

The Kolyvagin derivative operator applied to Kato's Euler system, produces a Kolyvagin system for $\mathcal{F}^{\operatorname{can}} = (\mathcal{F}_{BK})^p$.

Denote K_{χ} to the fixed field of χ . Assume:

• $(\chi 3)$ $E((K_{\chi})_{\mathfrak{p}})[p] = \{O\}$ for every prime \mathfrak{p} above p.

Proposition

Assuming (χ 3), the Selmer structure (\mathcal{F}_{BK}) p satisfies all the assumptions (H0), (H1) and (H2) and $\chi((\mathcal{F}_{BK})^p)=1$.

Assume further:

- $(\chi 4)$ The Tamagawa numbers of E over K_{χ} are prime to p.
- $(\chi 5)$ Iwasawa main conjecture (in the sense of Kato) holds for f_{χ} .

Proposition

The Kolyvagin derivative produced from Kato's Euler system is a primitive Kolyvagin system.



SLIDES

and Fitting ideals of Selmer group of

Introduct

Selmer structure

Fittir

Dualit

Kolyva

Structu of Selm

Euler syster

Elliptic

Droof

Elliptic curves: Kurihara numbers

The final goal is to compute $\delta_{n,\chi} = \log_p(\kappa_n)$. Assume

(E2) The Manin constant is prime to p.



Elliptic curves: Kurihara numbers

The final goal is to compute $\delta_{n,\chi} = \log_p(\kappa_n)$. Assume

■ (E2) The Manin constant is prime to p.

Proposition (Kurihara numbers)

Let n be a square-free product of Kolyvagin systems for T/p^k .



SLIDES

and
Fitting
deals of
Selmer
roup of
rank 0

Selmer structure

Fittin ideals

Pairing

Structi of Seln

Euler

Ellipti

Elliptic curves: Kurihara numbers

The final goal is to compute $\delta_{n,\chi} = \log_p(\kappa_n)$. Assume

(E2) The Manin constant is prime to p.

Proposition (Kurihara numbers)

Let *n* be a square-free product of Kolyvagin systems for T/p^k .

$$\delta_{n,\chi} = \sum_{a \in (\mathbb{Z}/nc)^*} \chi(a) \left(\left[\frac{a}{cn} \right]^+ + \left[\frac{a}{cn} \right]^- \right) \prod_{\ell \mid n} \log_{\eta_\ell}(a) \in \mathbb{Z}_p[\chi]/p^k$$



CLIDES

and
Fitting
deals of
Selmer
roup of

Selmer structure

Fitting ideals

Kolyva

Structu of Selm

Euler

Ellipti Curve

Elliptic curves: Kurihara numbers

The final goal is to compute $\delta_{n,\chi} = \log_p(\kappa_n)$. Assume

(E2) The Manin constant is prime to p.

Proposition (Kurihara numbers)

Let *n* be a square-free product of Kolyvagin systems for T/p^k .

$$\delta_{n,\chi} = \sum_{a \in (\mathbb{Z}/nc)^*} \chi(a) \left(\left[\frac{a}{cn} \right]^+ + \left[\frac{a}{cn} \right]^- \right) \prod_{\ell \mid n} \log_{\eta_\ell}(a) \in \mathbb{Z}_p[\chi]/p^k$$

where

lacksquare c is the conductor of χ .



CLIDE

Seimer group of rank 0

Selmer structure

Fitting ideals

Kolyvag systems

Kolyvag systems Structur

Euler

Elliptic Curves

Elliptic curves: Kurihara numbers

The final goal is to compute $\delta_{n,\chi} = \log_p(\kappa_n)$. Assume

(E2) The Manin constant is prime to p.

Proposition (Kurihara numbers)

Let n be a square-free product of Kolyvagin systems for \mathbf{T}/p^k .

$$\delta_{n,\chi} = \sum_{a \in (\mathbb{Z}/nc)^*} \chi(a) \left(\left[\frac{a}{cn} \right]^+ + \left[\frac{a}{cn} \right]^- \right) \prod_{\ell \mid n} \log_{\eta_\ell}(a) \in \mathbb{Z}_p[\chi]/p^k$$

where

- lacksquare c is the conductor of χ .
- $\left[\frac{a}{cn}\right]^{\pm}$ are the real and imaginary part of the modular symbols of E.



011050

ntroduct

Fitting ideals

pairings Kolyvagi

lolyvagi ystems tructur

Euler syster

Elliptio Curves

Elliptic curves: Kurihara numbers

The final goal is to compute $\delta_{n,\chi} = \log_p(\kappa_n)$. Assume

E2) The Manin constant is prime to p.

Proposition (Kurihara numbers)

Let n be a square-free product of Kolyvagin systems for \mathbf{T}/p^k .

$$\delta_{n,\chi} = \sum_{a \in (\mathbb{Z}/nc)^*} \chi(a) \left(\left[\frac{a}{cn} \right]^+ + \left[\frac{a}{cn} \right]^- \right) \prod_{\ell \mid n} \log_{\eta_\ell}(a) \in \mathbb{Z}_p[\chi]/p^k$$

where

- lacksquare c is the conductor of χ .
- $\left[\frac{a}{cn}\right]^{\pm}$ are the real and imaginary part of the modular symbols of E.
- η_{ℓ} is a primitive root of $(\mathbb{Z}/\ell)^{\times}$ and $\log_{\eta_{\ell}}(a)$ is the image of the logarithm under the projection $(\mathbb{Z}/\ell)^{\times} \cong \mathbb{Z}/(\ell-1) \twoheadrightarrow Z/p^k$.



Elliptic curves: Kurihara numbers

The final goal is to compute $\delta_{n,\chi} = \text{loc}_{p}(\kappa_{n})$. Assume

E2) The Manin constant is prime to p.

Proposition (Kurihara numbers)

Let *n* be a square-free product of Kolyvagin systems for \mathbf{T}/p^k .

$$\delta_{n,\chi} = \sum_{a \in (\mathbb{Z}/nc)^*} \chi(a) \left(\left[\frac{a}{cn} \right]^+ + \left[\frac{a}{cn} \right]^- \right) \prod_{\ell \mid n} \log_{\eta_\ell}(a) \in \mathbb{Z}_p[\chi]/p^k$$

where

- lacksquare c is the conductor of χ .
- η_ℓ is a primitive root of $(\mathbb{Z}/\ell)^{\times}$ and $\log_{\eta_\ell}(a)$ is the image of the logarithm under the projection $(\mathbb{Z}/\ell)^{\times} \cong \mathbb{Z}/(\ell-1) \twoheadrightarrow Z/p^k$.

Remark

If K/\mathbb{Q} is an abelian extension such that all the characters of $\mathrm{Gal}(K/\mathbb{Q})$ satisfy $(\chi 1)$ - $(\chi 5)$, then the twisted Kurihara numbers determine $\mathrm{Sel}(K, E[p^{\infty}])$ up to isomorphism of $\mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q})]$ -modules.



ES

Proposition

Consider the exact sequence of (\mathbb{Z}/p^k) -modules

$$0 \longrightarrow C \longrightarrow M \xrightarrow{\phi} (\mathbb{Z}/p^k)^i$$

Then

$$(p)^{\operatorname{length}(\mathcal{C})} \subset \operatorname{Fitt}_i(M)$$

Selmer

structur

Fittir ideal:

Dualit

Kolyva

Structur of Selm

Euler

Ellipti Curve



Proposition

Consider the exact sequence of (\mathbb{Z}/p^k) -modules

$$0 \longrightarrow C \longrightarrow M \xrightarrow{\phi} (\mathbb{Z}/p^k)^i$$

Then

$$(p)^{\operatorname{length}(C)} \subset \operatorname{Fitt}_i(M)$$

If we choose ϕ and C maximizing the image of ϕ , the equality holds.

Selmer

Fittin; ideals

Dualit

Kolyva

Structu of Selm

Euler syster

Ellipti Curve



Proposition

Consider the exact sequence of (\mathbb{Z}/p^k) -modules

$$0 \longrightarrow C \longrightarrow M \xrightarrow{\phi} (\mathbb{Z}/p^k)^i$$

Then

$$(p)^{\operatorname{length}(C)} \subset \operatorname{Fitt}_i(M)$$

If we choose ϕ and C maximizing the image of ϕ , the equality holds.

Theorem (first inequality)

$$\Theta_i \subset \operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)^{\vee})$$

structur Fitting

itting deals

Duality pairings

Kolyvag systems

of Selr groups

system



Proposition

Consider the exact sequence of (\mathbb{Z}/p^k) -modules

$$0 \longrightarrow C \longrightarrow M \xrightarrow{\phi} (\mathbb{Z}/p^k)^i$$

Then

$$(p)^{\operatorname{length}(C)} \subset \operatorname{Fitt}_i(M)$$

If we choose ϕ and C maximizing the image of ϕ , the equality holds.

Theorem (first inequality)

$$\Theta_i \subset \operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)^{\vee})$$

Proof Apply the proposition to

$$0 \longrightarrow \operatorname{Sel}_{(\mathcal{F}^*)_n}(\mathbb{Q}, T) \longrightarrow \operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*) \longrightarrow \prod_{\ell \mid n} H^1_f(\mathbb{Q}_\ell, T^*) \cong (\mathbb{Z}/p^k)^{\nu(n)}$$

Fitting ideals

Duality pairings Kolyvagi

systems Structure of Selme groups

Euler system Elliptic Curves



Proposition

Consider the exact sequence of (\mathbb{Z}/p^k) -modules

$$0 \longrightarrow C \longrightarrow M \xrightarrow{\phi} (\mathbb{Z}/p^k)^i$$

Then

$$(p)^{\operatorname{length}(C)} \subset \operatorname{Fitt}_i(M)$$

If we choose ϕ and C maximizing the image of ϕ , the equality holds.

Theorem (first inequality)

$$\Theta_i \subset \operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)^{\vee})$$

Proof Apply the proposition to

$$0 \longrightarrow \operatorname{Sel}_{(\mathcal{F}^*)_n}(\mathbb{Q}, T) \longrightarrow \operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*) \longrightarrow \prod_{\ell \mid n} H^1_f(\mathbb{Q}_\ell, T^*) \cong (\mathbb{Z}/p^k)^{\nu(n)}$$

Then

$$(p)^{\operatorname{length}(\operatorname{Sel}(\mathcal{F}^*)_{(n)}(\mathbb{Q},\mathcal{T}^*))} \subset (p)^{\operatorname{length}(\operatorname{Sel}(\mathcal{F}^*)_n(\mathbb{Q},\mathcal{T}^*))} \subset \operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q},\mathcal{T}))$$



Proofs: first inequality

Proposition

Consider the exact sequence of (\mathbb{Z}/p^k) -modules

$$0 \longrightarrow C \longrightarrow M \xrightarrow{\phi} (\mathbb{Z}/p^k)^i$$

Then

$$(p)^{\operatorname{length}(C)} \subset \operatorname{Fitt}_i(M)$$

If we choose ϕ and C maximizing the image of ϕ , the equality holds.

Theorem (first inequality)

$$\Theta_i \subset \operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)^{\vee})$$

Proof Apply the proposition to

$$0 \longrightarrow \operatorname{Sel}_{(\mathcal{F}^*)_n}(\mathbb{Q}, T) \longrightarrow \operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*) \longrightarrow \prod_{\ell \mid n} H^1_f(\mathbb{Q}_\ell, T^*) \cong (\mathbb{Z}/p^k)^{\nu(n)}$$

Then

$$(p)^{\operatorname{length}(\operatorname{Sel}(\mathcal{F}^*)(n)(\mathbb{Q},T^*))} \subset (p)^{\operatorname{length}(\operatorname{Sel}(\mathcal{F}^*)_n(\mathbb{Q},T^*))} \subset \operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q},T))$$

The proof is completed by taking the minimum over all $n \in \mathcal{N}_i(\mathcal{P})$.

Alberto Angurel Andres



■ The localization map

$$\operatorname{loc}_{\ell}: \ \operatorname{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*) \to H^1_f(\mathbb{Q}_{\ell}, T^*)$$

has the largest possile image.

Introduct

Selmer structure

Fittir

Duali

Kolyva

Structu of Selm

Euler

Ellipti

D.---6-



■ The localization map

$$\operatorname{loc}_{\ell}:\ \operatorname{Sel}_{(\mathcal{F}^{*})(n)}(\mathbb{Q},\,T^{*})\to H^{1}_{f}(\mathbb{Q}_{\ell},\,T^{*})$$

has the largest possile image.

This can be achieved using Chebotarev density theorem.

Introduct

structur

Fittir ideal:

Duali

Kolyva system

Structur of Selm

Euler syste

Ellipti Curve



The localization map

$$\operatorname{loc}_{\ell}: \operatorname{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*) \to H^1_f(\mathbb{Q}_{\ell}, T^*)$$

has the largest possile image.

This can be achieved using Chebotarev density theorem.

■ We want to choose $n \in \mathcal{N}(\mathcal{P})$ such that

$$\operatorname{Sel}_{(\mathcal{F}^*)_n}(\mathbb{Q}, T^*) = \operatorname{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*)$$

Introduct

Fittin

ideals

Dualit

Kolyva system

Structu of Selm

Euler syster

Ellipti Curve



The localization map

$$\operatorname{loc}_{\ell}: \ \operatorname{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*) \to H^1_f(\mathbb{Q}_{\ell}, T^*)$$

has the largest possile image.

This can be achieved using Chebotarev density theorem.

■ We want to choose $n \in \mathcal{N}(\mathcal{P})$ such that

$$\mathrm{Sel}_{(\mathcal{F}^*)_n}(\mathbb{Q}, T^*) = \mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*)$$

Proposition

Assume the following localization map is surjective

$$\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T) \to H^1_f(\mathbb{Q}_\ell,T)$$

Then

$$\operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) = \operatorname{Sel}_{(\mathcal{F}^*)_{\ell}(n)}(\mathbb{Q}, T^*)$$

Fitting

Duality pairings Kolyvagir

Structur of Selme groups

Euler system Elliptic Curves



Proposition

Assume the following localization map is surjective

$$\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T) \to H^1_f(\mathbb{Q}_\ell,T)$$

Then

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) = \mathrm{Sel}_{(\mathcal{F}^*)_{\ell}(n)}(\mathbb{Q}, T^*)$$

Proof

$$0 \longrightarrow \operatorname{Sel}_{\mathcal{F}_{\ell}(n)}(\mathbb{Q}, T) \longrightarrow \operatorname{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \longrightarrow H^1_f(\mathbb{Q}_{\ell}, T)$$

$$\overbrace{\operatorname{Sel}_{(\mathcal{F}^*)^\ell(n)}}(\mathbb{Q},T^*)^\vee \longrightarrow \operatorname{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q},T^*)^\vee \longrightarrow 0$$

Selmer structure

Duality

systems

of Selme groups Fuler

Elliptic

Droof



Proposition

Assume the following localization map is surjective

$$\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T) \to H^1_f(\mathbb{Q}_\ell,T)$$

Then

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) = \mathrm{Sel}_{(\mathcal{F}^*)_{\ell}(n)}(\mathbb{Q}, T^*)$$

$$0 \longrightarrow \operatorname{Sel}_{\mathcal{F}_{\ell}(n)}(\mathbb{Q}, T) \longrightarrow \operatorname{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \longrightarrow H_{f}^{1}(\mathbb{Q}_{\ell}, T)$$

$$\hookrightarrow \operatorname{Sel}_{(\mathcal{F}^{*})^{\ell}(n)}(\mathbb{Q}, T^{*})^{\vee} \longrightarrow \operatorname{Sel}_{(\mathcal{F}^{*})(n)}(\mathbb{Q}, T^{*})^{\vee} \longrightarrow 0$$

$$\blacksquare \operatorname{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*) = \operatorname{Sel}_{(\mathcal{F}^*)^{\ell}(n)}(\mathbb{Q}, T^*)$$



Proposition

Assume the following localization map is surjective

$$\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T) \to H^1_f(\mathbb{Q}_\ell,T)$$

Then

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) = \mathrm{Sel}_{(\mathcal{F}^*)_{\ell}(n)}(\mathbb{Q}, T^*)$$

Proof

$$0 \longrightarrow \operatorname{Sel}_{\mathcal{F}_{\ell}(n)}(\mathbb{Q}, T) \longrightarrow \operatorname{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \longrightarrow H^{1}_{f}(\mathbb{Q}_{\ell}, T)$$

$$\hookrightarrow \operatorname{Sel}_{(\mathcal{F}^{*})^{\ell}(n)}(\mathbb{Q}, T^{*})^{\vee} \longrightarrow \operatorname{Sel}_{(\mathcal{F}^{*})(n)}(\mathbb{Q}, T^{*})^{\vee} \longrightarrow 0$$

$$\blacksquare \operatorname{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*) = \operatorname{Sel}_{(\mathcal{F}^*)^{\ell}(n)}(\mathbb{Q}, T^*)$$

$$\blacksquare \operatorname{Sel}_{(\mathcal{F}^*)_{\ell}(n)} = \operatorname{Sel}_{(\mathcal{F}^*)(n)} \cap \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}$$

Proofs



Proposition

Assume the following localization map is surjective

$$\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T) \to H^1_f(\mathbb{Q}_\ell,T)$$

Then

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) = \mathrm{Sel}_{(\mathcal{F}^*)_{\ell}(n)}(\mathbb{Q}, T^*)$$

Proof

$$0 \longrightarrow \operatorname{Sel}_{\mathcal{F}_{\ell}(n)}(\mathbb{Q}, T) \longrightarrow \operatorname{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \longrightarrow H^{1}_{f}(\mathbb{Q}_{\ell}, T)$$

$$\hookrightarrow \operatorname{Sel}_{(\mathcal{F}^{*})^{\ell}(n)}(\mathbb{Q}, T^{*})^{\vee} \longrightarrow \operatorname{Sel}_{(\mathcal{F}^{*})(n)}(\mathbb{Q}, T^{*})^{\vee} \longrightarrow 0$$

$$\blacksquare \operatorname{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*) = \operatorname{Sel}_{(\mathcal{F}^*)^{\ell}(n)}(\mathbb{Q}, T^*)$$

$$\blacksquare \operatorname{Sel}_{(\mathcal{F}^*)_{\ell}(n)} = \operatorname{Sel}_{(\mathcal{F}^*)(n)} \cap \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)} = \operatorname{Sel}_{(\mathcal{F}^*)^{\ell}(n)} \cap \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}$$



Proposition

Assume the following localization map is surjective

$$\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T) \to H^1_f(\mathbb{Q}_\ell,T)$$

Then

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) = \mathrm{Sel}_{(\mathcal{F}^*)_{\ell}(n)}(\mathbb{Q}, T^*)$$

Proof

$$0 \longrightarrow \operatorname{Sel}_{\mathcal{F}_{\ell}(n)}(\mathbb{Q}, T) \longrightarrow \operatorname{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \longrightarrow H^{1}_{f}(\mathbb{Q}_{\ell}, T)$$

$$\hookrightarrow \operatorname{Sel}_{(\mathcal{F}^{*})^{\ell}(n)}(\mathbb{Q}, T^{*})^{\vee} \longrightarrow \operatorname{Sel}_{(\mathcal{F}^{*})(n)}(\mathbb{Q}, T^{*})^{\vee} \longrightarrow 0$$

$$\blacksquare \operatorname{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*) = \operatorname{Sel}_{(\mathcal{F}^*)^{\ell}(n)}(\mathbb{Q}, T^*)$$

$$\blacksquare \operatorname{Sel}_{(\mathcal{F}^*)_{\ell}(n)} = \operatorname{Sel}_{(\mathcal{F}^*)(n)} \cap \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)} = \operatorname{Sel}_{(\mathcal{F}^*)^{\ell}(n)} \cap \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)} = \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}$$



Proposition

$$\mathrm{Fitt}_{\mathit{i}}(\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q},\mathit{T}^*)^{\vee}) = \Theta_{\mathit{i}} := (p)^{min\{\mathrm{ord}(\kappa_{\mathit{n}}): \mathit{n} \in \mathcal{N}_{\mathit{i}}(\mathcal{P})\}}$$



olyvag

Proposition

$$\operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)^{\vee}) = \Theta_i := (p)^{\min\{\operatorname{ord}(\kappa_n) : n \in \mathcal{N}_i(\mathcal{P})\}}$$

Proof Inductively, assume we have constructed some n_i such that

$$\operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q}, T^*)^{\vee}) = p^{\operatorname{ord}(\kappa_{n_i})}$$

Selmer

Fittir

Duali

pairin

system:

of Selm groups

Euler syste

Ellipti Curve



Proposition

$$\operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)^{\vee}) = \Theta_i := (p)^{\min\{\operatorname{ord}(\kappa_n): n \in \mathcal{N}_i(\mathcal{P})\}}$$

Proof Inductively, assume we have constructed some n_i such that

$$\operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q}, T^*)^{\vee}) = p^{\operatorname{ord}(\kappa_{n_i})}$$

Since the core rank is one,

$$\mathrm{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q},T)\cong (\mathbb{Z}/p^k)\oplus \mathrm{Sel}_{\mathcal{F}^*(n_i)}(\mathbb{Q},T^*)$$

Then there exists a surjective map $\operatorname{Sel}_{\mathcal{F}(p_i)}(\mathbb{Q},T) \to \mathbb{Z}/p^k$.

Selmer structur

Fitting ideals

Duality pairing

Kolyva; system:

of Selr groups

systen



Proposition

$$\operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)^{\vee}) = \Theta_i := (p)^{\min\{\operatorname{ord}(\kappa_n): n \in \mathcal{N}_i(\mathcal{P})\}}$$

Proof Inductively, assume we have constructed some n_i such that

$$\operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q}, T^*)^{\vee}) = p^{\operatorname{ord}(\kappa_{n_i})}$$

Since the core rank is one,

Proofs: equality in rank one

$$\mathrm{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q},T)\cong (\mathbb{Z}/p^k)\oplus \mathrm{Sel}_{\mathcal{F}^*(n_i)}(\mathbb{Q},T^*)$$

Then there exists a surjective map $\mathrm{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q},T) \to \mathbb{Z}/p^k$.

By Chebotarev density theorem, we can find a prime ℓ_{i+1} such that

- $lacksquare{1}$ loc $_{\ell_{i+1}}$: Sel $_{\mathcal{F}(n_i)}(\mathbb{Q},T) \to H^1_f(\mathbb{Q}_\ell,T)$ is surjective.
- $\blacksquare \operatorname{loc}_{\ell_{i+1}} : \operatorname{Sel}_{\mathcal{F}^*(\eta_i)}(\mathbb{Q}, T^*) \to H^1_f(\mathbb{Q}_\ell, T^*)$ has maximal image.

Elliptic Curves



Proposition

$$\operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},\, T^*)^\vee) = \Theta_i := (p)^{\min\{\operatorname{ord}(\kappa_n): n \in \mathcal{N}_i(\mathcal{P})\}}$$

Proof Inductively, assume we have constructed some n_i such that

$$\operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q}, T^*)^{\vee}) = p^{\operatorname{ord}(\kappa_{n_i})}$$

Since the core rank is one,

$$\mathrm{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q},T)\cong (\mathbb{Z}/p^k)\oplus \mathrm{Sel}_{\mathcal{F}^*(n_i)}(\mathbb{Q},T^*)$$

Then there exists a surjective map $\mathrm{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q},T) \to \mathbb{Z}/p^k$.

By Chebotarev density theorem, we can find a prime ℓ_{i+1} such that

- $\blacksquare \operatorname{loc}_{\ell_{i+1}} : \operatorname{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q}, T) \to H^1_{\mathfrak{f}}(\mathbb{Q}_{\ell}, T)$ is surjective.
- $\blacksquare \operatorname{loc}_{\ell_{i+1}} : \operatorname{Sel}_{\mathcal{F}^*(n_i)}(\mathbb{Q}, T^*) \to H^1_{\mathfrak{f}}(\mathbb{Q}_{\ell}, T^*)$ has maximal image.

For $n_{i+1} := n_i \ell_{i+1}$, we get that

$$\mathrm{Sel}_{(\mathcal{F}^*)(n_{i+1})}(\mathbb{Q},T^*) = \mathrm{Sel}_{(\mathcal{F}^*)_{\ell_{i+1}}(n_i)}(\mathbb{Q},T^*)$$

Moreover.

$$(p)^{\operatorname{ord}(\kappa_{n_{i+1}})} = (p)^{\operatorname{length}(\operatorname{Sel}_{(\mathcal{F}^*)(n_{i+1})})} = \operatorname{Fitt}_{i+1}(\operatorname{Sel}_{(\mathcal{F}^*)}(\mathbb{Q}, T))$$

Alberto Angurel Andres



Proofs: equality in rank zero: characteristic reduction

When $\chi(\mathcal{F}) = 0$,

$$\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T)\cong\mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q},T)\ \forall n\in\mathcal{N}(\mathcal{P})$$

It might not exist a surjective map $\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T) \to \mathbb{Z}/p^k$.



Proofs: equality in rank zero: characteristic reduction

When $\chi(\mathcal{F}) = 0$,

$$\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \cong \mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T) \ \forall n \in \mathcal{N}(\mathcal{P})$$

It might not exist a surjective map $\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T) o \mathbb{Z}/p^k$.

By the structure theorem,

$$\mathrm{Sel}_{\mathcal{F}(n)} = \mathbb{Z}/p^{e_1} \times \cdots \times \mathbb{Z}/p^{e_s}$$

for some $e_1 > \cdots > e_s$.

Trick Swap T by $T_{e_1} := T/p^{e_1}$.



Proofs: equality in rank zero: characteristic reduction

When $\chi(\mathcal{F}) = 0$,

$$\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T)\cong\mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q},T)\ \forall n\in\mathcal{N}(\mathcal{P})$$

It might not exist a surjective map $\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T) o \mathbb{Z}/p^k$.

By the structure theorem,

$$\mathrm{Sel}_{\mathcal{F}(n)} = \mathbb{Z}/p^{e_1} \times \cdots \times \mathbb{Z}/p^{e_s}$$

for some $e_1 > \cdots > e_s$.

Trick Swap T by $T_{e_1} := T/p^{e_1}$.

Similarly, we can find a prime ℓ such that the maps

- \blacksquare Sel_{$\mathcal{F}(n)$}(\mathbb{Q}, T_{e_1}) $\to H^1_f(\mathbb{Q}_\ell, T_{e_1})$.
- $\blacksquare \operatorname{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, (T_{e_1})^*) \to H^1_f(\mathbb{Q}_\ell, (T_{e_1})^*).$

are surjective. We obtain the following for the Selmer group over T_{e_1} .

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q},(T_{e_1})^*) = \mathrm{Sel}_{(\mathcal{F}^*)_{\ell}(n)}(\mathbb{Q},(T_{e_1})^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$



What information can we deduce from this to the Selmer group over \mathcal{T} ?

Introduct

Selmer structure

Fittin

Duali

Kolyva

Structur of Selm

Euler

Elliptio



What information can we deduce from this to the Selmer group over T?

$$\blacksquare \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*)[p^{e_1}] \cong \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, (T_{e_1})^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}.$$

Introduct

Selmer structur

Fittir

- Ideai

pairin

system

Structu of Selm

Euler

Ellipt



What information can we deduce from this to the Selmer group over T?

$$\blacksquare \ \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q},\, T^*)[p^{e_1}] \cong \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q},(\, T_{e_1})^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}.$$

■ If either $e_1 = k$ or $e_1 > e_2$, we can conclude that

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, \mathcal{T}^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$

Selmer structure

Fittir

Duali

Kolyva

Structu of Selm

Euler syster

Ellipti Curve



What information can we deduce from this to the Selmer group over T?

$$\blacksquare \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*)[p^{e_1}] \cong \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, (T_{e_1})^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}.$$

■ If either $e_1 = k$ or $e_1 > e_2$, we can conclude that

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$

■ When $e_1 = e_2$, we only know that

$$Sel_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) \subset Sel_{(\mathcal{F}^*)^{\ell}(n)}(\mathbb{Q}, T^*) \cong$$

$$\mathbb{Z}/p^k \times Sel_{\mathcal{F}_e(n)}(\mathbb{Q}, T) \cong \mathbb{Z}/p^k \times \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_5}$$

Selmer

Fittin:

Dualit

Kolyva;

Structur of Selme

Euler

Ellipti

What information can we deduce from this to the Selmer group over T?

- $\blacksquare \ \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q},\, T^*)[p^{e_1}] \cong \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q},(\, T_{e_1})^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}.$
- If either $e_1 = k$ or $e_1 > e_2$, we can conclude that

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$

■ When $e_1 = e_2$, we only know that

$$\begin{split} & \mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) \subset \mathrm{Sel}_{(\mathcal{F}^*)^{\ell}(n)}(\mathbb{Q}, T^*) \cong \\ & \mathbb{Z}/p^k \times \mathrm{Sel}_{\mathcal{F}_{\ell}(n)}(\mathbb{Q}, T) \cong \mathbb{Z}/p^k \times \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_5} \end{split}$$

■ The structure theorem implies that

$$\mathrm{Sel}_{(\mathcal{F}^*)(p\ell)}(\mathbb{Q}, \mathcal{T}^*) \cong \mathbb{Z}/p^{f_2} \times \mathbb{Z}/p^{e_3} \times \cdots \times \mathbb{Z}/p^{e_s}$$

for some $f_2 \geq e_2$.

Fitting ideals

Duality pairings

systems Structur

groups Euler svstem

Elliptic Curves



lmer up of nk 0

Introduct

Selmer structu

Fittir ideal:

Duali

Kolyva

Structu of Selm

Euler syster

Ellipti

D...-6

We start with $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q},T)$ and choose a prime ℓ_1 such that the localization maps for T_{e_1} and $T_{e_1}^*$ are surjective, and minimizing f_2 .

We have two cases



We start with $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, \mathcal{T})$ and choose a prime ℓ_1 such that the localization maps for \mathcal{T}_{e_1} and $\mathcal{T}_{e_1}^*$ are surjective, and minimizing f_2 .

We have two cases

• If $f_2 = e_2$, then $\Theta_1 = \operatorname{Fitt}_1(\operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*)^{\vee})$.

Selmer

Fittin

ideals

pairin

Kolyva system

of Selm groups

Euler systen

Ellipti Curve

D---6



We start with $\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},T)$ and choose a prime ℓ_1 such that the localization maps for T_{e_1} and $T_{e_1}^*$ are surjective, and minimizing f_2 .

We have two cases

- If $f_2 = e_2$, then $\Theta_1 = \operatorname{Fitt}_1(\operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*)^{\vee})$.
- If $f_2 > e_2$, then $\Theta_1 \subsetneq \operatorname{Fitt}_1(\operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*)^{\vee})$.

Selmer

Fittin

Duality

Kolyva; system:

of Selme groups

Euler systen

Ellipti Curve



We start with $\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T)$ and choose a prime ℓ_1 such that the localization maps for T_{e_1} and $T_{e_1}^*$ are surjective, and minimizing f_2 .

We have two cases

- If $f_2 = e_2$, then $\Theta_1 = \text{Fitt}_1 \left(\text{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*)^{\vee} \right)$.
- If $f_2 > e_2$, then $\Theta_1 \subseteq \operatorname{Fitt}_1(\operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*)^{\vee})$. In this case.

$$\mathrm{Sel}_{\mathcal{F}(\ell_1)} \cong \mathbb{Z}/p^{f_2} \times Z/p^{e_3} \times \cdots \times \mathbb{Z}/p^{e_s}$$

Since $f_2 > e_3$, we can choose a prime ℓ_2 in a way such that $f_3 = e_3$, so

$$\Theta_2 = \operatorname{Fitt}_2 \left(\operatorname{Sel}_{\mathcal{F}^*} (\mathbb{Q}, T^*)^{\vee} \right)$$



Non self-dual representations

Chebotarev density theorem is stronger in this case:

For every pair of subgroups $C \subset \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T)$ and $D \subset \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)$ such that the quotients $\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T)/C$ and $\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)/D$ are cyclic, we can find a prime ℓ such that the kernels of the localization maps are C and D.

Selmer structures

Fittir ideal:

Duali

Kolyva

Structu of Selm groups

Euler syste

Ellipti Curve



Non self-dual representations

Chebotarev density theorem is stronger in this case:

For every pair of subgroups $C \subset \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},T)$ and $D \subset \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},T^*)$ such that the quotients $\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},T)/C$ and $\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},T^*)/D$ are cyclic, we can find a prime ℓ such that the kernels of the localization maps are C and D.

■ If $\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T) \cong \mathbb{Z}/p^{e_1} \times \cdots \times \mathbb{Z}/p^{e_s}$, a technical argument constructs a prime ℓ such that

$$\mathrm{Sel}_{\mathcal{F}(\ell)}\cong \mathbb{Z}/p^{e_2} imes\cdots imes\mathbb{Z}/p^{e_5}$$

ntroduc Selmer structure

Fittin

ideals

pairings

system:

of Seli groups

Euler systen

Ellipti Curve

Chebotarev density theorem is stronger in this case:

For every pair of subgroups $C \subset \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},T)$ and $D \subset \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},T^*)$ such that the quotients $\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},T)/C$ and $\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},T^*)/D$ are cyclic, we can find a prime ℓ such that the kernels of the localization maps are C and D.

• If $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q},T)\cong \mathbb{Z}/p^{e_1}\times \cdots \times \mathbb{Z}/p^{e_s}$, a technical argument constructs a prime ℓ such that

$$\mathrm{Sel}_{\mathcal{F}(\ell)} \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$

■ Therefore, the equality $\Theta_i = \operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)^{\vee})$ holds for all i.



