Kolyvagin systems and Fitting ideals of Selmer group of rank 0

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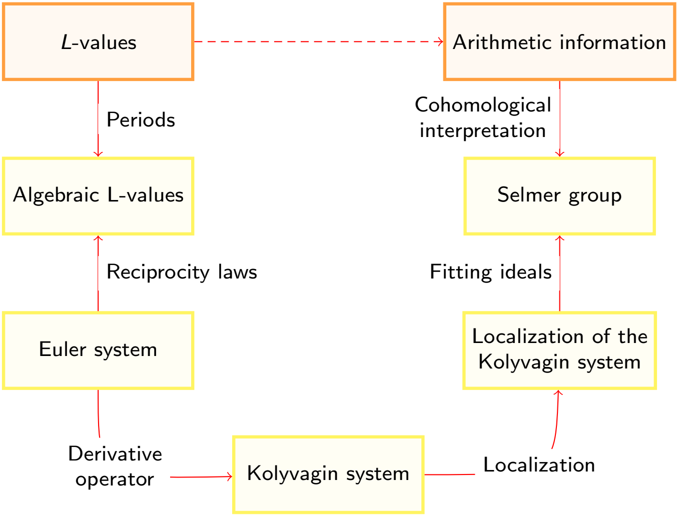
General picture



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- The general theory of Kolyvagin systems only describes the structure of the Selmer group *restricted at p*.

- We extend this theory to Selmer groups of rank zero by considering Kolyvagin systems over an auxiliary Selmer structure.

Setting and assumptions

- (H0) Let $p \geq 5$ and let $\mathbf{T}$ be free $\mathbb{Z}_p$-module of finite rank endowed with a continuous action of $G_{\mathbb{Q}}$, ramifying only at a finite amount of primes.
- (H1) $\rho : \ G_{\mathbb{Q}} \to \mathrm{Aut}(\mathbf{T})$ is surjective.
- (H2) (will appear later)

Selmer pre-structures

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Definition (Selmer group)

The Selmer group for $\mathcal{F}$ is defined as

$$\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, \mathbf{T}) := \ker \left(H^1(\mathbb{Q}, \mathbf{T}) \to \bigoplus_{\ell} \frac{H^1(\mathbb{Q}_\ell, \mathbf{T})}{H^1_{\mathcal{F}}(\mathbb{Q}_\ell, \mathbf{T})} \right)$$

Definition (finite cohomology)

$$H^1_f(\mathbb{Q}_\ell, \mathbf{T}) := \ker\left(H^1(\mathbb{Q}, \mathbf{T}) \to H^1(I_\ell, \mathbf{T} \otimes \mathbb{Q}_p)\right)$$

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A Selmer structure is a Selmer pre-structure such that there is a finite set of primes Σ such that

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Proposition (Selmer groups)

If $\mathbb{Q}_\Sigma$ denotes the maximal extension of $\mathbb{Q}$ unramified outside Σ, we have that

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Corollary

$\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, \mathbf{T}) \subset H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathbf{T})$ is a finitely generated $\mathbb{Z}_p$-module.

Fitting ideals

Definition (Fitting ideal)

Let M be a finitely generated R-module. Choose a resolution

$$R^n \xrightarrow{A} R^m \longrightarrow M \longrightarrow 0$$

$\mathrm{Fitt}_i^R(M)$ is the ideal generated by the minors of size $(m-i)$ of A.

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$Fitt_0(M) = (0)$,	$\mathrm{Fitt}_1(M) = (p^5)$,
$\mathrm{Fitt}_2(M) = (p^2) + (p^3) = (p^2)$	$\mathrm{Fitt}_i(M) = (1)\ \forall i \geq 3$

Fitting ideals over Discrete Valuation Rings

Let R be a DVR (with maximal ideal $\mathfrak{m}$ and residue field κ). Then

$$M \cong R^r \times R/\mathfrak{m}^{\alpha_1} \times \cdots \times R/\mathfrak{m}^{\alpha_s}$$

for some non-negative integers $r, s, \alpha_1 \geq \ldots \geq \alpha_s$.

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- For $i \geq 0$, $\alpha_i = \mathrm{Fitt}_{r+i+1}(M)\mathrm{Fitt}_{r+i}(M)^{-1}$.

Local duality

Definition (dual Galois modules)

- Pontryagin dual: $\mathbf{T}^{\vee} = \mathrm{Hom}(\mathbf{T}, \mathbb{Q}_p/\mathbb{Z}_p)$.
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The cup-product induces a non-degenerate pairing

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Moreover, $H^1_f(\mathbb{Q}_\ell, \mathbf{T})$ and $H^1_f(\mathbb{Q}_\ell, \mathbf{T}^*)$ are exact annihilators of each other.

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Dual Selmer structure

Definition (dual Selmer structure)

The dual Selmer structure $\mathcal{F}^*$ is defined by the local conditions

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell, \mathbf{T}^*) := Ann\big(H^1_{\mathcal{F}}(\mathbb{Q}_\ell, \mathbf{T})\big) \subset H^1(\mathbb{Q}_\ell, \mathbf{T}^*)$$

These are the elements of $H^1(\mathbb{Q}_\ell, \mathbf{T}^*)$ which annihilate $H^1_{\mathcal{F}}(\mathbb{Q}_\ell, T)$ under the local duality pairing.

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Remark (well defined)

The dual Selmer structure is well defined since

$$H^1_f(\mathbb{Q}_\ell, \mathbf{T}^*) := \mathrm{Ann}\big(H^1_f(\mathbb{Q}_\ell, \mathbf{T})\big)$$

Global duality

Let $\mathcal{F}$ and $\mathcal{G}$ be Selmer structures such that

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell, T) \subset H^1_{\mathcal{G}}(\mathbb{Q}_\ell, T) \ \forall \ell$$

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Clearly,

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$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, T) \arrow[r] & \mathrm{Sel}_{\mathcal{G}}(\mathbb{Q}, T) \arrow[r] & \prod_\ell \dfrac{H^1_{\mathcal{G}}(\mathbb{Q}_\ell, T)}{H^1_{\mathcal{F}}(\mathbb{Q}_\ell, T)} \arrow[dll, rounded corners] \\
& \mathrm{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*)^\vee \arrow[r] & \mathrm{Sel}_{\mathcal{G}^*}(\mathbb{Q}, T^*)^\vee \arrow[r] & 0
\end{tikzcd}$$

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- (H2) $H^1(\mathbb{Q}_\ell, \mathbf{T})/H^1_{\mathcal{F}}(\mathbb{Q}_\ell, \mathbf{T})$ is a torsion-free $\mathbb{Z}_p$-module.

Propagation to positive characteristic

Fix $k \in \mathbb{N}$ and let $T = \mathbf{T}/p^k$. Denote $\pi : \mathbf{T} \to T$ to the canonical projection.

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Remark

A study of $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, \mathbf{T}/p^k)$ for all k will determine $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, \mathbf{T})$.

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Notation

$\mathcal{P}$ denotes the set of Kolyvagin primes.

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$\mathcal{P}$ denotes the set of Kolyvagin primes.

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$\mathcal{N}_i(\mathcal{P})$ denotes the set of square free products of exactly i Kolyvagin primes.

Transverse local condition and finite-singular map

Definition (transverse local condition)

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If ℓ is Kolyvagin prime, then

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Modified Selmer structures

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Kolyvagin systems

Definition (Kolyvagin system)

A Kolyvagin system is a collection of elements

$$\kappa_n \in \mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \subset H^1(\mathbb{Q}, T)$$

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Notation

The module of Kolyvagin systems will be denoted by $\mathrm{KS}(T, \mathcal{F})$.

Core rank

Definition/proposition (core rank)

There exists a non-negative integer $\chi(\mathcal{F})$ and a non-canonical homomorphism

$$\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, T) \cong \left(\mathbb{Z}/p^k\right)^{\chi(T)} \oplus \mathrm{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*)$$

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Proposition (Sakamoto, 2021)

$$\chi(\mathcal{F}_a^b(c)) = \chi(\mathcal{F}) + \nu(b) - \nu(a)$$

where $\nu(b)$ and $\nu(a)$ are the number of primes dividing b and a, respectively.

Theorem (Mazur-Rubin, 2004)

- If $\chi(\mathcal{F}) = 0$, then $\mathrm{KS}(T, \mathcal{F}) = 0$.

  There are no Kolyvagin system to control the Selmer group. We will see a posible solution later in the talk.

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- If $\chi(\mathcal{F}) > 1$, then $\mathrm{KS}(T, \mathcal{F})$ is too large.

  In order to compute the Selmer group, [Mazur-Rubin, 2016] and [Burns-Sakamoto-Sano, 2025] modified the definition of Kolyvagin system in (biduals of) exterior powers of the Selmer groups.

Definition (order of a Kolyvagin element)

$$\mathrm{ord}(\kappa_n) := \max\left\{j \in \{0, \dots, k\} : \kappa_n \in p^j H^1_{\mathcal{F}(n)}(\mathbb{Q}, T)\right\}$$

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- We have seen that there are no non-zero Kolyvagin systems.
- Choose a prime ℓ such that $H^1_f(\mathbb{Q}_\ell, T) \cong \mathbb{Z}/p^k$.
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Remark

When $T = \mathbf{T}/p^k$ for some $\mathbb{Z}_p$-module $\mathbf{T}$ and k is large enough, the ideals Θ_i determine $H^1_{\mathcal{F}}(\mathbb{Q}, \mathbf{T}^*)$ up to isomorphism

Galois representations which are not residually self-dual

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Then for all $i \in \mathbb{Z}_{\geq 0}$, we have the equality

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This is also known as *relaxed at p*. Its dual Selmer structure will be called *restricted at p*.

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- We call $\mathrm{ord}(z_{\mathbb{Q}}) = \sup\left\{j \in \mathbb{N} : z_{\mathbb{Q}} \in p^j H^1(\mathbb{Q}, \mathbf{T})_{/\mathrm{tors}}\right\}$.

Theorem (Kolyvagin, 1995)

$$\mathrm{ord}(z_{\mathbb{Q}}) \geq \mathrm{ord}(\kappa_1) \geq \mathrm{length}\left(H^1_{(\mathcal{F}^{\mathrm{can}})^*}(\mathbb{Q}, \mathbf{T}^*)\right)$$

Elliptic curves: construction of the Euler system

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Kato constructed an Euler system for this representation.

Elliptic curves: Bloch-Kato Selmer structure

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The classical local conditions are defined by Bloch-Kato condition

$$\begin{cases} H^1_{\mathcal{F}_{BK}}(\mathbb{Q}_\ell, \mathbf{T}) = H^1_f(\mathbb{Q}_\ell, \mathbf{T}) \quad \forall \ell \neq p \\ H^1_{\mathcal{F}_{BK}}(\mathbb{Q}_p, \mathbf{T}) = \ker\left(H^1(\mathbb{Q}_p, \mathbf{T}) \to H^1(\mathbb{Q}_p, \mathbf{T} \otimes_{Z_p} B_{\mathrm{crys}})\right) \end{cases}$$

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Proposition

Assuming (E1), $\mathcal{F}_{BK}$ satisfies all the assumptions (H0), (H1) and (H2) and $\chi(\mathcal{F}_{BK}) = 0$.

Ellliptic curves: construction of the Kolyvagin system

Kolyvagin derivative

The Kolyvagin derivative operator applied to Kato's Euler system, produces a Kolyvagin system for $\mathcal{F}^{\mathrm{can}} = (\mathcal{F}_{BK})^p$.

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Assuming $(\chi 3)$, the Selmer structure $(\mathcal{F}_{BK})^p$ satisfies all the assumptions (H0), (H1) and (H2) and $\chi\big((\mathcal{F}_{BK})^p\big) = 1$.

Ellliptic curves: construction of the Kolyvagin system

Kolyvagin derivative

The Kolyvagin derivative operator applied to Kato's Euler system, produces a Kolyvagin system for $\mathcal{F}^{\mathrm{can}} = (\mathcal{F}_{BK})^p$.

Denote K_χ to the fixed field of χ. Assume:

- $(\chi 3)$ $E((K_\chi)_\mathfrak{p})[p] = \{O\}$ for every prime $\mathfrak{p}$ above p.

Proposition

Assuming $(\chi 3)$, the Selmer structure $(\mathcal{F}_{BK})^p$ satisfies all the assumptions (H0), (H1) and (H2) and $\chi\big((\mathcal{F}_{BK})^p\big) = 1$.

Assume further:

- $(\chi 4)$ The Tamagawa numbers of E over K_χ are prime to p.
- $(\chi 5)$ Iwasawa main conjecture (in the sense of Kato) holds for f_χ.

Proposition

The Kolyvagin derivative produced from Kato's Euler system is a primitive Kolyvagin system.

Elliptic curves: Kurihara numbers

The final goal is to compute $\delta_{n,\chi} = \mathrm{loc}_p(\kappa_n)$. Assume

- (E2) The Manin constant is prime to p.

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Remark

If $K/\mathbb{Q}$ is an abelian extension such that all the characters of $\mathrm{Gal}(K/\mathbb{Q})$ satisfy $(\chi 1)$-$(\chi 5)$, then the twisted Kurihara numbers determine $\mathrm{Sel}(K, E[p^\infty])$ up to isomorphism of $\mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q})]$-modules.

Proofs: first inequality

Proposition

Consider the exact sequence of $(\mathbb{Z}/p^k)$-modules

$$0 \longrightarrow C \longrightarrow M \xrightarrow{\phi} (\mathbb{Z}/p^k)^i$$

Then

$$(p)^{\mathrm{length}(C)} \subset \mathrm{Fitt}_i(M)$$

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$$0 \longrightarrow \mathrm{Sel}_{(\mathcal{F}^*)_n}(\mathbb{Q}, T) \longrightarrow \mathrm{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*) \longrightarrow \prod_{\ell | n} H^1_f(\mathbb{Q}_\ell, T^*) \cong (\mathbb{Z}/p^k)^{\nu(n)}$$

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Then

$$(p)^{\mathrm{length}(\mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*))} \subset (p)^{\mathrm{length}(\mathrm{Sel}_{(\mathcal{F}^*)_n}(\mathbb{Q}, T^*))} \subset \mathrm{Fitt}_i(\mathrm{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T))$$

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Then

$$(p)^{\mathrm{length}(\mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*))} \subset (p)^{\mathrm{length}(\mathrm{Sel}_{(\mathcal{F}^*)_n}(\mathbb{Q}, T^*))} \subset \mathrm{Fitt}_i(\mathrm{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T))$$

The proof is completed by taking the minimum over all $n \in \mathcal{N}_i(\mathcal{P})$.

Proofs: what is needed for the equality?

- The localization map

$$\mathrm{loc}_\ell : \mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*) \to H^1_f(\mathbb{Q}_\ell, T^*)$$

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- We want to choose $n \in \mathcal{N}(\mathcal{P})$ such that

$$\mathrm{Sel}_{(\mathcal{F}^*)_n}(\mathbb{Q}, T^*) = \mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*)$$

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Assume the following localization map is surjective

$$\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \to H^1_f(\mathbb{Q}_\ell, T)$$

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Proof

- $$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}_{\mathcal{F}_\ell(n)}(\mathbb{Q}, T) \arrow[r] & \mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \arrow[r] & H^1_f(\mathbb{Q}_\ell, T) \arrow[dll, bend left] \\
 & \mathrm{Sel}_{(\mathcal{F}^*)^\ell(n)}(\mathbb{Q}, T^*)^\vee \arrow[r] & \mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*)^\vee \arrow[r] & 0
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Proof

- $$0 \longrightarrow \mathrm{Sel}_{\mathcal{F}_\ell(n)}(\mathbb{Q}, T) \longrightarrow \mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \longrightarrow H^1_f(\mathbb{Q}_\ell, T) \longrightarrow \mathrm{Sel}_{(\mathcal{F}^*)^\ell(n)}(\mathbb{Q}, T^*)^\vee \longrightarrow \mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*)^\vee \longrightarrow 0$$
- $\mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*) = \mathrm{Sel}_{(\mathcal{F}^*)^\ell(n)}(\mathbb{Q}, T^*)$

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- $$\begin{array}{l} 0 \longrightarrow \mathrm{Sel}_{\mathcal{F}_\ell(n)}(\mathbb{Q}, T) \longrightarrow \mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \longrightarrow H^1_f(\mathbb{Q}_\ell, T) \longrightarrow \\ \qquad \longrightarrow \mathrm{Sel}_{(\mathcal{F}^*)^\ell(n)}(\mathbb{Q}, T^*)^\vee \longrightarrow \mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*)^\vee \longrightarrow 0 \end{array}$$

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Proofs: equality in rank one

Proposition

$$\mathrm{Fitt}_i(\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)^\vee) = \Theta_i := (p)^{\min\{\mathrm{ord}(\kappa_n):n\in\mathcal{N}_i(\mathcal{P})\}}$$

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$$\mathrm{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q}, T) \cong (\mathbb{Z}/p^k) \oplus \mathrm{Sel}_{\mathcal{F}^*(n_i)}(\mathbb{Q}, T^*)$$

Then there exists a surjective map $\mathrm{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q}, T) \to \mathbb{Z}/p^k$.

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Then there exists a surjective map $\mathrm{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q}, T) \to \mathbb{Z}/p^k$.

By Chebotarev density theorem, we can find a prime ℓ_{i+1} such that

- $\mathrm{loc}_{\ell_{i+1}} : \mathrm{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q}, T) \to H^1_f(\mathbb{Q}_\ell, T)$ is surjective.
- $\mathrm{loc}_{\ell_{i+1}} : \mathrm{Sel}_{\mathcal{F}^*(n_i)}(\mathbb{Q}, T^*) \to H^1_f(\mathbb{Q}_\ell, T^*)$ has maximal image.

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For $n_{i+1} := n_i \ell_{i+1}$, we get that

$$\mathrm{Sel}_{(\mathcal{F}^*)(n_{i+1})}(\mathbb{Q}, T^*) = \mathrm{Sel}_{(\mathcal{F}^*)_{\ell_{i+1}}(n_i)}(\mathbb{Q}, T^*)$$

Moreover,

$$(p)^{\mathrm{ord}(\kappa_{n_{i+1}})} = (p)^{\mathrm{length}(\mathrm{Sel}_{(\mathcal{F}^*)(n_{i+1})})} = \mathrm{Fitt}_{i+1}(\mathrm{Sel}_{(\mathcal{F}^*)}(\mathbb{Q}, T))$$

Proofs: equality in rank zero: characteristic reduction

When $\chi(\mathcal{F}) = 0$,

$$\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \cong \mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T)\ \forall n \in \mathcal{N}(\mathcal{P})$$

It might not exist a surjective map $\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \to \mathbb{Z}/p^k$.

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It might not exist a surjective map $\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \to \mathbb{Z}/p^k$.

By the structure theorem,

$$\mathrm{Sel}_{\mathcal{F}(n)} = \mathbb{Z}/p^{e_1} \times \cdots \times \mathbb{Z}/p^{e_s}$$

for some $e_1 \geq \cdots \geq e_s$.

Trick Swap T by $T_{e_1} := T/p^{e_1}$.

Proofs: equality in rank zero: characteristic reduction

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Trick Swap T by $T_{e_1} := T/p^{e_1}$.

Similarly, we can find a prime ℓ such that the maps

- $\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T_{e_1}) \to H^1_f(\mathbb{Q}_\ell, T_{e_1})$.
- $\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, (T_{e_1})^*) \to H^1_f(\mathbb{Q}_\ell, (T_{e_1})^*)$.

are surjective. We obtain the following for the Selmer group over T_{e_1}.

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, (T_{e_1})^*) = \mathrm{Sel}_{(\mathcal{F}^*)_\ell(n)}(\mathbb{Q}, (T_{e_1})^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$

Proofs: equality in rank zero: recover information

What information can we deduce from this to the Selmer group over T?

Proofs: equality in rank zero: recover information

What information can we deduce from this to the Selmer group over T?

- $\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*)[p^{e_1}] \cong \mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, (T_{e_1})^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$.

Proofs: equality in rank zero: recover information

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- If either $e_1 = k$ or $e_1 > e_2$, we can conclude that

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$

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What information can we deduce from this to the Selmer group over T?

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$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$

- When $e_1 = e_2$, we only know that

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) \subset \mathrm{Sel}_{(\mathcal{F}^*)^{\ell}(n)}(\mathbb{Q}, T^*) \cong$$
$$\mathbb{Z}/p^k \times \mathrm{Sel}_{\mathcal{F}_\ell(n)}(\mathbb{Q}, T) \cong \mathbb{Z}/p^k \times \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$

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What information can we deduce from this to the Selmer group over T?

- $\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*)[p^{e_1}] \cong \mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, (T_{e_1})^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$.
- If either $e_1 = k$ or $e_1 > e_2$, we can conclude that

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$

- When $e_1 = e_2$, we only know that

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) \subset \mathrm{Sel}_{(\mathcal{F}^*)^{\ell}(n)}(\mathbb{Q}, T^*) \cong$$
$$\mathbb{Z}/p^k \times \mathrm{Sel}_{\mathcal{F}_\ell(n)}(\mathbb{Q}, T) \cong \mathbb{Z}/p^k \times \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$

- The structure theorem implies that

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) \cong \mathbb{Z}/p^{f_2} \times \mathbb{Z}/p^{e_3} \times \cdots \times \mathbb{Z}/p^{e_s}$$

  for some $f_2 \geq e_2$.

Proofs: equality in rank zero: inductive step

We start with $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, T)$ and choose a prime ℓ_1 such that the localization maps for T_{e_1} and $T^*_{e_1}$ are surjective, and minimizing f_2.

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  In this case,
  $$\mathrm{Sel}_{\mathcal{F}(\ell_1)} \cong \mathbb{Z}/p^{f_2} \times Z/p^{e_3} \times \cdots \times \mathbb{Z}/p^{e_s}$$
  Since $f_2 > e_3$, we can choose a prime ℓ_2 in a way such that $f_3 = e_3$, so
  $$\Theta_2 = \mathrm{Fitt}_2\left(\mathrm{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*)^\vee\right)$$

Non self-dual representations

- Chebotarev density theorem is stronger in this case:

  For every pair of subgroups $C \subset \mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, T)$ and $D \subset \mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)$ such that the quotients $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, T)/C$ and $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)/D$ are cyclic, we can find a prime ℓ such that the kernels of the localization maps are C and D.

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- Therefore, the equality $\Theta_i = \mathrm{Fitt}_i(\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)^{\vee})$ holds for all i.

Thank you for your attention!

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