

## Ejercicios Procesos de ramificación

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**Exercise 1** (P. 392, e. 1). *Let  $Z_1, Z_2, \dots, Z_n$  describe a branching process in which each parent has  $j$  offspring with probability  $p_j$ . Find the probability  $d$  that the process eventually dies out if*

(a)  $p_0 = 1/2, p_1 = 1/4, p_2 = 1/4$ .

Para este caso, el número esperado de hijos es  $m = h'(1) = p_1 + 2p_2 = 1/4 + 2(1/4) = 3/4 \leq 1$ , por lo que por el teorema 10.2,  $d = 1$  así que el proceso o herencia o apellido se acabará.

(b)  $p_0 = 1/3, p_1 = 1/3, p_2 = 1/3$ .

Igual que el inciso anterior,  $m = 1/3 + 2(1/3) = 1 \leq 1$ , así que  $d = 1$ .

(c)  $p_0 = 1/3, p_1 = 0, p_2 = 2/3$ .

Aquí,  $m = 0 + 2(2/3) = 4/3 > 1$ , pero como  $p_0 < p_2$  se puede obtener  $d = p_0/p_2 = \frac{1/3}{2/3} = 1/2$ .

(d)  $p_j = 1/2^{j+1}$ , for  $j = 0, 1, 2, \dots$

En este inciso,

$$\begin{aligned}
h(z) &= p_0 + p_1 z + p_2 z^2 + p_3 z^3 + \dots \\
&= 1/2^{0+1} + 1/2^{1+1} z + 1/2^{2+1} z^2 + 1/2^{3+1} z^3 + \dots \\
&= 1/2^1 + 1/2^2 z + 1/2^3 z^2 + 1/2^4 z^3 + \dots \\
&= \frac{1}{2} (1/2^1 + 1/2^2 z + 1/2^3 z^2 + 1/2^4 z^3 + \dots) / \frac{1}{2} \\
&= \frac{1}{2} (1 + 1/2^1 z + 1/2^2 z^2 + 1/2^3 z^3 + \dots) \\
&= \frac{1}{2} \left( \frac{1}{1 - \frac{1}{2} z} \right) \\
&= \frac{1}{2 - z}.
\end{aligned}$$

Si esto es verdad, entonces

$$\begin{aligned}
h'(z) &= \frac{d}{dz} \left( \frac{1}{2 - z} \right) \\
&= \frac{-\frac{d}{dz} (2 - z)}{(2 - z)^2} \\
&= \frac{1}{(2 - z)^2}
\end{aligned}$$

por lo que, como  $m = h'(1) = \frac{1}{(2-1)^2} = 1 \leq 1$ ,  $d = 1$ .

(e)  $p_j = (1/3)(2/3)^j$ , for  $j = 0, 1, 2, \dots$

De manera análoga al inciso precedente,

$$\begin{aligned}
h(z) &= p_0 + p_1 z + p_2 z^2 + p_3 z^3 + \dots \\
&= \frac{1}{3} \left( \frac{2}{3} \right)^0 + \frac{1}{3} \left( \frac{2}{3} \right)^1 z^1 + \frac{1}{3} \left( \frac{2}{3} \right)^2 z^2 + \frac{1}{3} \left( \frac{2}{3} \right)^3 z^3 \dots \\
&= \frac{1}{3} \left[ 1 + \left( \frac{2}{3} \right)^1 z^1 + \left( \frac{2}{3} \right)^2 z^2 + \left( \frac{2}{3} \right)^3 z^3 \dots \right] \\
&= \frac{1}{3} \left( \frac{1}{1 - \frac{2}{3} z} \right) \\
&= \frac{1}{3 - 2z}
\end{aligned}$$

de donde

$$\begin{aligned}
h'(z) &= \frac{d}{dz} \left( \frac{1}{3 - 2z} \right) \\
&= \frac{-\frac{d}{dz} (3 - 2z)}{(3 - 2z)^2} \\
&= \frac{2}{(3 - 2z)^2}
\end{aligned}$$

por lo que  $m = h'(1) = \frac{2}{(3-2)^2} = \frac{2}{(1)^2} = 2$  y  $d < 1$  cuando  $z \neq 1$ . Para calcular esta  $d$  se obtienen las raíces a partir de igualar  $z = h(z)$ , así que

$$z = \frac{1}{3 - 2z}$$

$$2z^2 - 3z + 1 = 0$$

de donde  $z_1 = 1$  (ya conocida) y  $z_2 = 1/2 = d$ .

(f)  $p_j = e^{-2}2^j/j!$ , for  $j = 0, 1, 2, \dots$  (estimate  $d$  nummerically).

Finalmente,  $d = 0.2032$ . Esto se obtiene mediante el código 1

Código 1: Aproximación

```
1 p = function(j){
2   return( exp(-2) * 2 ** j / factorial(j) )
3 }
4 d = p(0)
5 for (m in 1:1000){
6   sum = 0
7   for (j in 0:100){
8     sum = sum + p(j) * (d ** j)
9   }
10  d = sum
11 }
12 d
13 # 0.2031878699799799
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**Exercise 2** (P. 392, e. 3). *In the chain letter problem (see Example 10.14) find your expected profit if*

(a)  $p_0 = 1/2, p_1 = 0, p_2 = 1/2$ .

Como  $m = p_1 + 2p_2 = 0 + 2(1/2) = 1$ , entonces se espera una ganancia de  $50(1 + 1^{12}) - 100 = 0$ .

(b)  $p_0 = 1/6, p_1 = 1/2, p_2 = 1/3$ .

Aquí  $m = p_1 + 2p_2 = 1/2 + 2(1/3) = 7/6$ , entonces se espera una ganancia de  $50(7/6 + (7/6)^{12}) - 100 \approx 276.26$ .

Show that if  $p_0 > 1/2$ , you cannot expect to make a profit.

**Exercise 3** (P. 401, e. 1). *Let  $X$  be a continuous random variable with values in  $[0, 2]$  and density  $f_X$ . Find the moment generating function  $g(t)$  for  $X$  if*

(a)  $f_X(x) = \frac{1}{2}$ .

$$\begin{aligned}
 g_X(t) &= \int_0^2 e^{tx} \cdot \frac{1}{2} dx \\
 &= \int_0^2 \frac{e^{tx}}{2} dx; u = tx \rightarrow \frac{du}{dx} = t \rightarrow dx = \frac{du}{t} \\
 &= \frac{1}{2t} \int_0^2 e^u du \\
 &= \frac{1}{2t} e^u \Big|_0^2 = \frac{1}{2t} (e^{2t} - e^0) \\
 &= \frac{e^{2t} - 1}{2t}.
 \end{aligned}$$

(b)  $f_X(x) = \frac{1}{2}x$ .

$$\begin{aligned}
 g_X(t) &= \int_0^2 e^{tx} \cdot \frac{1}{2} x dx \\
 &= \frac{1}{2} \int_0^2 x e^{tx} dx \\
 &= \frac{1}{2} \left[ \frac{x e^{tx}}{t} - \int_0^2 \left( (1) \frac{e^{tx}}{t} \right) dx; u = tx \rightarrow \frac{du}{dx} = t \rightarrow dx = \frac{du}{t} \right] \\
 &= \frac{1}{2} \left[ \frac{x e^{tx}}{t} - \int_0^2 \frac{e^u}{t^2} du \right] \\
 &= \frac{1}{2} \left[ \frac{x e^{tx}}{t} - \frac{1}{t^2} \int_0^2 e^{tx} dx \right] \\
 &= \frac{1}{2} \left[ \frac{x e^{tx}}{t} - \frac{e^{tx}}{t^2} \right]_0^2 \\
 &= \frac{1}{2} \left[ \frac{t x e^{tx} - e^{tx}}{t^2} \right]_0^2 \\
 &= \frac{1}{2} \left[ \frac{e^{tx} (tx - 1)}{t^2} \right]_0^2 \\
 &= \frac{1}{2} \left[ \frac{e^{2t} (2t - 1)}{t^2} - \frac{(1)(-1)}{t^2} \right] \\
 &= \frac{e^{2t} (2t - 1) + 1}{2t^2}.
 \end{aligned}$$

(c)  $f_X(x) = 1 - \frac{1}{2}x$ .

$$\begin{aligned}
g_X(t) &= \int_0^2 e^{tx} \cdot \left(1 - \frac{1}{2}x\right) dx \\
&= -\frac{1}{2} \int_0^2 (x-2)e^{tx} dx \\
&= -\frac{1}{2} \left[ (x-2) \frac{e^{tx}}{t} - \int_0^2 (1) \frac{e^{tx}}{t} dx \right]; u = tx \rightarrow \frac{du}{dx} = t \rightarrow dx = \frac{du}{t} \\
&= -\frac{1}{2} \left[ \frac{(x-2)e^{tx}}{t} - \frac{1}{t^2} \int_0^2 e^u du \right] \\
&= -\frac{1}{2} \left[ \frac{(x-2)e^{tx}}{t} - \frac{e^{tx}}{t^2} \right]_0^2 \\
&= -\frac{1}{2} \left[ \frac{(x-2)te^{tx} - e^{tx}}{t^2} \right]_0^2 \\
&= -\frac{1}{2} \left[ \frac{e^{tx}[t(x-2) - 1]}{t^2} \right]_0^2 \\
&= -\frac{1}{2} \left[ \left( \frac{e^{2t}[t(2-2) - 1]}{t^2} \right) - \left( \frac{e^{0t}[t(0-2) - 1]}{t^2} \right) \right] \\
&= -\frac{1}{2} \left[ \left( \frac{-e^{2t}}{t^2} \right) - \left( \frac{-2t-1}{t^2} \right) \right] \\
&= -\frac{1}{2} \left[ \frac{-e^{2t} + 2t + 1}{t^2} \right] \\
&= \frac{e^{2t} - 2t - 1}{2t^2}.
\end{aligned}$$

(d)  $f_X(x) = |1 - x|$ .

$$\begin{aligned}
g_X(t) &= \int_0^2 e^{tx} |1 - x| dx \\
&= \int_0^1 (1 - x) e^{tx} dx + \int_1^2 (-1 + x) e^{tx} dx \\
&= \frac{e^t - 1}{t} - \frac{e^t}{t} + \frac{e^t}{t^2} - \frac{1}{t^2} - \frac{e^{2t} - e^t}{t} + \frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} - \frac{e^t}{t} + \frac{e^t}{t^2} \\
&= \frac{t(e^t - 1) - t(e^t) + e^t - 1 - t(e^{2t} - e^t) + t(2e^{2t}) - e^{2t} - t(e^t) + e^t}{t^2} \\
&= \frac{te^t - t - te^t + e^t - 1 - te^{2t} + te^t + 2te^{2t} - e^{2t} - te^t + e^t}{t^2} \\
&= \frac{-t + 2e^t - 1 + te^{2t} - e^{2t}}{t^2} \\
&= \frac{2e^t - t - 1 + e^{2t}(t - 1)}{t^2}.
\end{aligned}$$

(e)  $f_X(x) = \frac{3}{8}x^2$ .

$$\begin{aligned}
g_X(t) &= \int_0^2 e^{tx} \left(\frac{3}{8}x^2\right) dx \\
&= \frac{3}{8} \cdot \int_0^2 e^{tx} x^2 dx \\
&= \frac{3}{8} \left[ \frac{e^{tx} x^2}{t} - \int \frac{2e^{tx} x}{t} dx \right]_0^2 \\
&= \frac{3}{8} \left[ \frac{e^{tx} x^2}{t} - \frac{2}{t} \left( \frac{e^{tx} x}{t} - \frac{e^{tx}}{t^2} \right) \right]_0^2 \\
&= \frac{3}{8} \left( \frac{4e^{2t}}{t} - \frac{2}{t} \left( \frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} \right) - \frac{2}{t^3} \right) \\
&= \frac{3}{8} \left( \frac{4e^{2t}}{t} - \frac{4e^{2t}}{t^2} + \frac{2e^{2t}}{t^3} - \frac{2}{t^3} \right) \\
&= \frac{3}{8} \left( \frac{4t^2 e^{2t} - 4te^{2t} + 2e^{2t} - 2}{t^3} \right) \\
&= \frac{3}{8} \left( \frac{2e^{2t}(2t^2 - 2t + 1) - 2}{t^3} \right) \\
&= \frac{3}{4} \left( \frac{e^{2t}(2t^2 - 2t + 1) - 1}{t^3} \right).
\end{aligned}$$

**Exercise 4** (P. 402, e. 6). Let  $X$  be a continuous random variable whose characteristic function  $k_X(\tau)$  is  $k_X(\tau) = e^{-|\tau|}$ ,  $-\infty < \tau < \infty$ . Show directly that the density  $f_X$  of  $X$  is

$$f_X(x) = \frac{1}{\pi(1 + x^2)}.$$

$$\begin{aligned}
f_X(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\tau} e^{-|\tau|} d\tau \\
&= \frac{1}{2\pi} \left[ \int_{-\infty}^0 e^{-ix\tau - (-\tau)} d\tau + \int_0^{\infty} e^{-ix\tau - \tau} d\tau \right] \\
&= \frac{1}{2\pi} \left[ \int_{-\infty}^0 e^{\tau - ix\tau} d\tau + \int_0^{\infty} e^{-ix\tau - \tau} d\tau \right] \\
&= \frac{1}{2\pi} \left[ \frac{1}{1 - ix} \int_{-\infty}^0 e^u du - \frac{1}{ix + 1} \int_0^{\infty} e^v dv \right] \\
&= \frac{1}{2\pi} \left[ \frac{1}{1 - ix} - \frac{1}{ix + 1} \right] \\
&= \frac{1}{2\pi} \left[ \frac{ix + 1}{i^2 x^2 - 1^2} - \frac{1 - ix}{i^2 x^2 - 1^2} \right] \\
&= \frac{1}{2\pi} \left[ \frac{ix + 1 - 1 + ix}{1^2 - i^2 x^2} \right] \\
&= \frac{1}{2\pi} \left[ \frac{2}{1 + x^2} \right] \\
&= \frac{1}{\pi(1 + x^2)}.
\end{aligned}$$

**Exercise 5** (P. 403, e. 10). Let  $X_1, X_2, \dots, X_n$  be an independent trials process with density

$$f(x) = \frac{1}{2} e^{-|x|}, -\infty < x < +\infty.$$

1. Find the mean and variance of  $f(x)$ .
2. Find the moment generating function for  $X_1, S_n, A_n$ , and  $S_n^*$ .
3. What can you say about the moment generating function of  $S_n^*$  as  $n \rightarrow \infty$ .
4. What can you say about the moment generating function of  $A_n$  as  $n \rightarrow \infty$ .