Posgrado de Ingeniería de Sistemas

Ejercicios Procesos de ramificación

Alberto Benavides
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Exercise 1 (P. 392, e. 1). *Let $Z_1, Z_2, \ldots, Z_n$ describe a branching process in which each parent has j offspring with probability p_j. Find the probability d that the process eventually dies out if*

(a) $p_0 = 1/2, p_1 = 1/4, p_2 = 1/4$.

Para este caso, el número esperado de hijos es $m = h'(1) = p_1 + 2p_2 = 1/4 + 2(1/4) = 3/4 \leq 1$, por lo que por el teorema 10.2, $d = 1$ así que el proceso o herencia o apellido se acabará.

(b) $p_0 = 1/3, p_1 = 1/3, p_2 = 1/3$.

Igual que el inciso anterior, $m = 1/3 + 2(1/3) = 1 \leq 1$, así que $d = 1$.

(c) $p_0 = 1/3, p_1 = 0, p_2 = 2/3$.

Aquí, $m = 0 + 2(2/3) = 4/3 > 1$, pero como $p_0 < p_2$ se puede obtener $d = p_0/p_2 = \frac{1/3}{2/3} = 1/2$.

(d) $p_j = 1/2^{j+1}$, for $j = 0, 1, 2, \ldots$

En este inciso,

$$\begin{aligned} h(z) &= p_0 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \\ &= 1/2^{0+1} + 1/2^{1+1} z + 1/2^{2+1} z^2 + 1/2^{3+1} z^3 + \ldots \\ &= 1/2^1 + 1/2^2 z + 1/2^3 z^2 + 1/2^4 z^3 + \ldots \\ &= \frac{1}{2}(1/2^1 + 1/2^2 z + 1/2^3 z^2 + 1/2^4 z^3 + \ldots)/\frac{1}{2} \\ &= \frac{1}{2}(1 + 1/2^1 z + 1/2^2 z^2 + 1/2^3 z^3 + \ldots) \\ &= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}z}\right) \\ &= \frac{1}{2-z}. \end{aligned}$$

Si esto es verdad, entonces

$$\begin{aligned} h'(z) &= \frac{d}{dz}\left(\frac{1}{2-z}\right) \\ &= \frac{-\frac{d}{dz}(2-z)}{(2-z)^2} \\ &= \frac{1}{(2-z)^2} \end{aligned}$$

por lo que, como $m = h'(1) = \frac{1}{(2-1)^2} = 1 \leq 1$, $d = 1$.

(e) $p_j = (1/3)(2/3)^j$, for $j = 0, 1, 2, \ldots$

De manera análoga al inciso precedente,

$$\begin{aligned} h(z) &= p_0 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \\ &= \frac{1}{3}\left(\frac{2}{3}\right)^0 + \frac{1}{3}\left(\frac{2}{3}\right)^1 z^1 + \frac{1}{3}\left(\frac{2}{3}\right)^2 z^2 + \frac{1}{3}\left(\frac{2}{3}\right)^3 z^3 \ldots \\ &= \frac{1}{3}\left[1 + \left(\frac{2}{3}\right)^1 z^1 + \left(\frac{2}{3}\right)^2 z^2 + \left(\frac{2}{3}\right)^3 z^3 \ldots\right] \\ &= \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}z}\right) \\ &= \frac{1}{3-2z} \end{aligned}$$

de donde

$$\begin{aligned} h'(z) &= \frac{d}{dz}\left(\frac{1}{3-2z}\right) \\ &= \frac{-\frac{d}{dz}(3-2z)}{(3-2z)^2} \\ &= \frac{2}{(3-2z)^2} \end{aligned}$$

por lo que $m = h'(1) = \frac{2}{(3-2)^2} = \frac{2}{(1)^2} = 2$ y $d < 1$ cuando $z \neq 1$. Para calcular esta d se obtienen las raíces a partir de igualar $z = h(z)$, así que

$$z = \frac{1}{3 - 2z}$$

$$2z^2 - 3z + 1 = 0$$

de donde $z_1 = 1$ (ya conocida) y $z_2 = 1/2 = d$.

(f) $p_j = e^{-2}2^j/j!$, for $j = 0, 1, 2, \ldots$ (estimate d numerically).

Finalmente, $d = 0.2032$. Esto se obtiene mediante el código 1

Código 1: Aproximación

```
p = function(j){
  return( exp(-2) * 2 ** j / factorial(j) )
}
d = p(0)
for (m in 1:1000){
  sum = 0
  for (j in 0:100){
    sum = sum + p(j) * (d ** j)
  }
  d = sum
}
d
# 0.2031878699799799
```

Exercise 2 (P. 392, e. 3)**.** *In the chain letter problem (see Example 10.14) find your expected profit if*

(a) $p_0 = 1/2, p_1 = 0, p_2 = 1/2$.

Como $m = p_1 + 2p_2 = 0 + 2(1/2) = 1$, entonces se espera una ganancia de $50(1 + 1^{12}) - 100 = 0$.

(b) $p_0 = 1/6, p_1 = 1/2, p_2 = 1/3$.

Aquí $m = p_1 + 2p_2 = 1/2 + 2(1/3) = 7/6$, entonces se espera una ganancia de $50(7/6 + (7/6)^{12}) - 100 \approx 276.26$.

Show that if $p_0 > 1/2$, you cannot expect to make a profit.

Exercise 3 (P. 401, e. 1)**.** *Let X be a continuous random variable with values in* $[0, 2]$ *and density f_X. Find the moment generating function $g(t)$ for X if*

(a) $f_X(x) = \frac{1}{2}$.

$$\begin{aligned}
g_X(t) &= \int_0^2 e^{tx} \cdot \frac{1}{2} dx \\
&= \int_0^2 \frac{e^{tx}}{2} dx; u = tx \to \frac{du}{dx} = t \to dx = \frac{du}{t} \\
&= \frac{1}{2t} \int_0^2 e^u du \\
&= \frac{1}{2t}\ e^t x \Big|_0^2 = \frac{1}{2t}(e^{2t} - e^0) \\
&= \frac{e^{2t} - 1}{2t}.
\end{aligned}$$

(b) $f_X(x) = \frac{1}{2}x$.

$$\begin{aligned}
g_X(t) &= \int_0^2 e^{tx} \cdot \frac{1}{2} x dx \\
&= \frac{1}{2} \int_0^2 x e^{tx} dx \\
&= \frac{1}{2} \left[\frac{xe^{tx}}{t} - \int_0^2 \left((1) \frac{e^{tx}}{t} \right) \right] dx; u = tx \to \frac{du}{dx} = t \to dx = \frac{du}{t} \\
&= \frac{1}{2} \left[\frac{xe^{tx}}{t} - \int_0^2 \frac{e^u}{t^2} du \right] \\
&= \frac{1}{2} \left[\frac{xe^{tx}}{t} - \frac{1}{t^2} \int_0^2 e^{tx} dx \right] \\
&= \frac{1}{2} \left[\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2} \Big|_0^2 \right] \\
&= \frac{1}{2} \left[\frac{txe^{tx} - e^{tx}}{t^2} \Big|_0^2 \right] \\
&= \frac{1}{2} \left[\frac{e^{tx}(tx - 1)}{t^2} \Big|_0^2 \right] \\
&= \frac{1}{2} \left[\frac{e^{2t}(2t - 1)}{t^2} - \frac{(1)(-1)}{t^2} \right] \\
&= \frac{e^{2t}(2t - 1) + 1}{2t^2}.
\end{aligned}$$

(c) $f_X(x) = 1 - \frac{1}{2}x$.

$$\begin{aligned}
g_X(t) &= \int_0^2 e^{tx} \cdot (1 - \frac{1}{2}x) dx \\
&= -\frac{1}{2} \int_0^2 (x-2) e^{tx} dx \\
&= -\frac{1}{2} \left[(x-2)\frac{e^{tx}}{t} - \int_0^2 (1) \frac{e^{tx}}{t} dx \right] ; u = tx \to \frac{du}{dx} = t \to dx = \frac{du}{t} \\
&= -\frac{1}{2} \left[\frac{(x-2)e^{tx}}{t} - \frac{1}{t^2} \int_0^2 e^u du \right] \\
&= -\frac{1}{2} \left[\frac{(x-2)e^{tx}}{t} - \frac{e^{tx}}{t^2} \Big|_0^2 \right] \\
&= -\frac{1}{2} \left[\frac{(x-2)te^{tx} - e^{tx}}{t^2} \Big|_0^2 \right] \\
&= -\frac{1}{2} \left[\frac{e^{tx}[t(x-2)-1]}{t^2} \Big|_0^2 \right] \\
&= -\frac{1}{2} \left[\left(\frac{e^{2t}[t(2-2)-1]}{t^2} \right) - \left(\frac{e^{0t}[t(0-2)-1]}{t^2} \right) \right] \\
&= -\frac{1}{2} \left[\left(\frac{-e^{2t}}{t^2} \right) - \left(\frac{-2t-1}{t^2} \right) \right] \\
&= -\frac{1}{2} \left[\frac{-e^{2t} + 2t + 1}{t^2} \right] \\
&= \frac{e^{2t} - 2t - 1}{2t^2}.
\end{aligned}$$

(d) $f_X(x) = |1-x|$.

$$\begin{aligned}
g_X(t) &= \int_0^2 e^{tx}|1-x|dx \\
&= \int_0^1 (1-x)\,e^{tx}dx + \int_1^2 (-1+x)\,e^{tx}dx \\
&= \frac{e^t-1}{t} - \frac{e^t}{t} + \frac{e^t}{t^2} - \frac{1}{t^2} - \frac{e^{2t}-e^t}{t} + \frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} - \frac{e^t}{t} + \frac{e^t}{t^2} \\
&= \frac{t(e^t-1) - t(e^t) + e^t - 1 - t(e^{2t}-e^t) + t(2e^{2t}) - e^{2t} - t(e^t) + e^t}{t^2} \\
&= \frac{te^t - t - te^t + e^t - 1 - te^{2t} + te^t + 2te^{2t} - e^{2t} - te^t + e^t}{t^2} \\
&= \frac{-t + 2e^t - 1 + te^{2t} - e^{2t}}{t^2} \\
&= \frac{2e^t - t - 1 + e^{2t}(t-1)}{t^2}.
\end{aligned}$$

(e) $f_X(x) = \frac{3}{8}x^2$.

$$\begin{aligned}
g_X(t) &= \int_0^2 e^{tx}(\frac{3}{8}x^2)dx \\
&= \frac{3}{8}\cdot\int_0^2 e^{tx}x^2dx \\
&= \frac{3}{8}\left[\frac{e^{tx}x^2}{t} - \int \frac{2e^{tx}x}{t}dx\right]_0^2 \\
&= \frac{3}{8}\left[\frac{e^{tx}x^2}{t} - \frac{2}{t}\left(\frac{e^{tx}x}{t} - \frac{e^{tx}}{t^2}\right)\right]_0^2 \\
&= \frac{3}{8}\left(\frac{4e^{2t}}{t} - \frac{2}{t}\left(\frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2}\right) - \frac{2}{t^3}\right) \\
&= \frac{3}{8}\left(\frac{4e^{2t}}{t} - \frac{4e^{2t}}{t^2} + \frac{2e^{2t}}{t^3} - \frac{2}{t^3}\right) \\
&= \frac{3}{8}\left(\frac{4t^2e^{2t} - 4te^{2t} + 2e^{2t} - 2}{t^3}\right) \\
&= \frac{3}{8}\left(\frac{2e^{2t}(2t^2-2t+1) - 2}{t^3}\right) \\
&= \frac{3}{4}\left(\frac{e^{2t}(2t^2-2t+1) - 1}{t^3}\right).
\end{aligned}$$

Exercise 4 (P. 402, e. 6)**.** *Let X be a continuous random variable whose characteristic function $k_X(\tau)$ is $k_X(\tau) = e^{-|\tau|}$, $-\infty < \tau < \infty$. Show directly that the density f_X of X is*

$$f_X(x) = \frac{1}{\pi(1+x^2)}.$$

$$
\begin{aligned}
f_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\tau}e^{-|\tau|}d\tau \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-ix\tau-(-\tau)}d\tau + \int_{0}^{\infty} e^{-ix\tau-\tau}d\tau\right] \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{\tau-ix\tau}d\tau + \int_{0}^{\infty} e^{-ix\tau-\tau}d\tau\right] \\
&= \frac{1}{2\pi}\left[\frac{1}{1-ix}\int_{-\infty}^{0} e^{u}du - \frac{1}{1+ix}\int_{0}^{\infty} e^{v}dv\right] \\
&= \frac{1}{2\pi}\left[\frac{1}{1-ix} - \frac{1}{1+ix}\right] \\
&= \frac{1}{2\pi}\left[\frac{1+ix}{i^2x^2-1^2} - \frac{1-ix}{1^2-i^2x^2}\right] \\
&= \frac{1}{2\pi}\left[\frac{ix+1-ix+1}{1^2-i^2x^2}\right] \\
&= \frac{1}{2\pi}\left[\frac{2}{1+x^2}\right] \\
&= \frac{1}{\pi(1+x^2)}.
\end{aligned}
$$

Exercise 5 (P. 403, e. 10)**.** *Let $X_1, X_2, \ldots, X_n$ be an indepentent trials process with density*

$$f(x) = \frac{1}{2}e^{-|x|}, -\infty < x < +\infty.$$

1. *Find the mean and variance of $f(x)$.*
2. *Find the moment generating function for X_1, S_n, A_n, and S_n^*.*
3. *What can you say about the moment generating function of S_n^* as $n \to \infty$.*
4. *What can you say about the moment generating function of A_n as $n \to \infty$.*