UNIVERSITÄT HEIDELBERG

MASTER THESIS

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# 1 Introduction

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convection is very important in astrophysics

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convection is very important in geophysics too!!!

### 1.1.3 Convection Somewhere Else

And somewhere else for sure!!!

# 2 Underlying Physics

## 2.1 Hydrodynamics

### 2.1.1 Derivation of the Equations of Ideal Hydrodynamics

As all the most beautiful and successful laws of Physics, Hydrodynamics is derived from some conserved quantities.
Let's consider for instance a monoatomic gas in a box. A fundamental assumption in ideal hydrodynamic is that the mean free path of particles is infinitesimal. Particles might be equally distributed in it (like in a bottle) or might not (maybe because the box is so big that we get a stratification because of gravity like the one in the atmosphere). Same story holds for the velocity distribution, it might be Maxwellian or it might not be. In any

case we can define a **distribution function in phase space** $f(\vec{x^{\mu}}, \vec{p^{\mu}})$ such that

$$N = \int d^4x \; d^4p \; f(x^{\mu}, p^{\mu}) \; \delta_D[(p^0)^2 - \vec{p}^2 + m^2c^2] \; \Theta(p^0)$$

where $N$ is the total number of particle, $\delta_D$ is the Dirac $\delta$ function that selects in the integrations only the hypersurfaces physically allowed by the energy-momentum relation of General Relativity, and finally $\Theta$ makes sure that we are taking into account only positive momenta. Keeping in mind this, we can define the first and second momentum of our distribution function

$$J^{\alpha} = c \int \frac{dp^3}{E} \; f(x^{\mu}, p^{\mu}) p^{\alpha} \qquad T^{\alpha\beta} = c \int \frac{dp^3}{E} \; f(x^{\mu}, p^{\mu}) p^{\alpha} p^{\beta} \tag{1}$$

namely the **current density** $J$ and the **energy-momentum tensor** $T$. Carrying out the calculation the components read

$$J = \frac{n(t, \vec{x})}{c} \begin{pmatrix} c \\ \langle \dot{x} \rangle \\ \langle \dot{y} \rangle \\ \langle \dot{z} \rangle \end{pmatrix} \quad T = \rho(t, \vec{x}) \begin{pmatrix} c^2 \langle \gamma \rangle & c \langle \gamma \dot{x} \rangle & c \langle \gamma \dot{y} \rangle & c \langle \gamma \dot{z} \rangle \\ c \langle \gamma \dot{x} \rangle & \langle \gamma \dot{x}^2 \rangle & \langle \gamma \dot{x} y \rangle & \langle \gamma \dot{x} \dot{z} \rangle \\ c \langle \gamma \dot{y} \rangle & \langle \gamma \dot{y} \dot{x} \rangle & \langle \gamma \dot{y}^2 \rangle & \langle \gamma \dot{y} \dot{z} \rangle \\ c \langle \gamma \dot{z} \rangle & \langle \gamma \dot{z} \dot{x} \rangle & \langle \gamma \dot{z} \dot{y} \rangle & \langle \gamma \dot{z}^2 \rangle \end{pmatrix}$$

where $n(t, \vec{x})$ is the number density, $c$ the speed of light, $\langle a(t, \vec{x}) \rangle$ means the average of the quantity $a$ over time and space in a neighborhood of $(t, \vec{x})$, $\gamma$ is the well known general relativistic parameter.
By integrating the first and second momentum of Boltzmann Equation, one could show that $J$ and $T$ are divergenceless, meaning in the non-relativistic case

$$\frac{\partial J^{\mu}}{\partial^{\mu}} = 0 \qquad\qquad \frac{\partial T^{\mu\nu}}{\partial^{\mu}} = 0 \;\; \forall \nu = 0, .., 3 \tag{2}$$

This means that we have to strike $J$ and $T$ with the operator $(c^{-1}\partial_t, \partial_x, \partial_y, \partial_z)$ from upside down. Doing so with the density current we obtain the **continuity equation**

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{3}$$

We apply the same procedure to the energy-momentum tensor.
We might choose to do it in the first column ($\nu = 0$), and that would lead us to the **energy conservation equation**

$$\partial_t \epsilon + \vec{\nabla} \cdot (\epsilon \vec{v}) + P \vec{\nabla} \cdot \vec{v} = 0 \tag{4}$$

where $\epsilon$ is the internal energy and $P$ the pressure. These two variables, that were not esplecialy included in $T$, arise naturally by splitting up the microscopic velocity of the fluid $\langle \dot{x} \rangle$ into a mean macroscopic velocity $\vec{v}$ and a random velocity in the neighborhood of the mean one $\vec{u}$

$$\langle \dot{x} \rangle = \vec{v} + \vec{u}$$

and recalling that

$$\epsilon = \frac{\rho}{2} \langle u^2 \rangle = \frac{3}{2} n k_B T = \frac{P}{2}$$

When we strike on the other three columns of $T$, we obtain the **momentum conservation equations** in the three spatial dimensions.

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} + \frac{\vec{\nabla} P}{\rho} = 0 \tag{5}$$

These are often called **Euler equations**.
Note that the operator on the left hand side is nothing but the total time derivative

$$\partial_t + \vec{v} \cdot \vec{\nabla} = \frac{d}{dt}$$

so what we are actually writing is Newton's equation per unit mass

$$\rho \vec{a} = \vec{\nabla} P$$

In case other macroscopic forces like gravity are present, we simply add them on the right hand side as we would do in classical mechanics.
So far we have obtained 5 equations in 6 unknowns, namely the internal energy $\epsilon$, the momentum $\vec{p}$, the pressure $P$ and the density $\rho$. If we want to have at least a chance of integrating we're missing one equation, the **equation of state**, that relates the thermodynamic variables.

### 2.1.2 Viscous Hydrodynamics

As stated at the beginning of the previous subsection, one fundamental assumption of ideal Hydrodynamics is that particles have an infinitesimal mean free path. What actually happens in nature might be very different. Particle have a finite mean free path, and hence they can transport local property of the fluid by diffusion only: transport of energy generates heat conduction, transport of momentum friction. The consequence is that we end up with one additional term in the energy-momentum tensor representing the diffusive processes

$$T^{ij} \to T^{ij} + T_d^{ij}$$

while the density current remains unchanged because mass is always conserved. As in the previous subsection, the new $T$ needs to be struck with $\nabla$ and set equal to zero in order to obtain our modified equations in the viscous case. We will not carry out all the calculations, rather simply quote the result for the new QUOTE and QUOTE.
The new energy conservation equation reads

$$\partial_t \epsilon + \vec{\nabla} \cdot (\epsilon \vec{v}) + P \vec{\nabla} \cdot \vec{v} = \vec{\nabla} \cdot (k \vec{\nabla} T) + v_{ij} T_d^{ij}$$

where $v_{ij}$ is the symmetrised velocity gradient tensor

$$v_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$$

This equation shows that the internal energy in a certain region of the fluid can change either if there is a temperature gradient or if the fluid is moving with a velocity field with a non vanishing symmetrised gradient (non solid body rotation like) because of friction. The new momentum conservation equations read

$$\rho \left( \partial_t + \vec{v} \cdot \vec{\nabla} \right) v^j + \partial^j P = \eta \vec{\nabla}^2 v^j + \left( \xi + \frac{\eta}{3} \right) \partial^j \vec{\nabla} \cdot \vec{v}$$

These are often called **Navier-Stokes Equations**, and fully show the non-linearity of Hydrodynamic.

### 2.1.3 Hydrostatic Equilibrium in Stars

We shall now consider the static configuration of QUOTE with gravitational potential. This reads

$$\vec{\nabla} P = -\rho \vec{\nabla} \Phi \tag{6}$$

This shows that the pressure stratification is adjusted only by the gravitational potential. We can say more by taking the curl of this equation and we get

$$\vec{\nabla} \rho \times \vec{\nabla} \Phi = 0$$

which shows that the gradient of the gravitational potetial and of the density are parallel. Surfaces of constant density are surfaces of constant gravitational potential.
Taking the divergence of **??** and substituting Poisson equation we can find the relation between pressure and density

$$\vec{\nabla} \cdot \left( \frac{\vec{\nabla} P}{\rho} \right) = -4\pi G \rho$$

provided an equation of state.
In a spherical symmetry configuration exploiting the $\vec{\nabla}$ we obtain the **Lane-Emden Equation**. If we choose a polytropic equation of state, which means that the pressure is a function of the density only and not of the temperature, we obtain stable solutions for adiabatic indices up to 4/3 only. This EoS is used for instance for modeling white dwarfs, where the pressure doesn't depend on the temperature.

## 2.2 Transport of Energy by Radiation

: In deep interiors of stars energy in generated through nuclear reactions. In the sun for instance all the energy is provided by the so called p-p chain, where because of temperatures of the order of $10^9 K$, thermal motions allows protons to overcome the electrostatic potential barrier and interact through the nuclear force, giving rise to the reaction

$$6\mathrm{H}^1 \rightarrow \mathrm{He}_2^4 + 2\mathrm{H}^1 + 2\gamma + 2\nu \tag{7}$$

Where on the right hand side we have photons and neutrinos, that carries away energy.
Neutrinos, interacting only via the weak force, have a very low cross section and hence leave the core of the sun without scattering, draining energy very efficiently. We need to look at very massive systems like Core Collapse Supernovae and neutron stars creation in order to see a tangible interaction between the neutrino flux and the envelope. In fact those astronomical objects are excellent neutrino laboratories gently provided by mother nature.
Photons instead, interacting electromagnetically, scatter multiple times before reaching the *photosphere*, the region where statistically they scatter for the last time before ending up in our eyes or our telescopes.
Let's now make a rough estimate of the mean free path of a photon in the sun.

$$l_{p,p} = \frac{1}{k\rho} \tag{8}$$

where $k$ is the mean cross section (averaged over all frequencies) per unit mass. For ionized hydrogen in stellar medium a rough average is 1 $cm^2/g$.
The average density of the sun is $\bar{\rho}_\odot = 3M_\odot/4\pi R_\odot^3 = 1.4\mathrm{g/cm}^3$.
Plugging in the numbers we get $\bar{l}_{p,p\odot} = 2$ cm.
At this point we need to answer a fundamental question: does two sequential scatterings happen at thermodinamic equilibrium? When one photon scatter, and then scatters again after a frew millimeters or centimeters, does

it encounter the same thermodinamic conditions? In order to evaluate that, let's consider a mean temperature gradient between the center of the sun and the photosphere

$$\frac{\Delta T}{\Delta r} = \frac{10^7 \text{ K} - 10^4 \text{ K}}{R_\odot} \simeq 1.4 \times 10^{-4} \text{ K cm}^{-1}$$

which means that after 2cm on the radial direction a photon sees a difference in temperature of $\Delta T = l_{p,p} \times 1.4 \times 10^{-4}$ K cm$^{-1}$ $= 3 \times 10^{-4}$ K. The relative difference of radiation energy density can be easly computed since $u \sim T^4$, hence in the center of the sun $\Delta u/u = 4\Delta T/T \sim 10^{-10}$. This little anisotropy that we neglect is the actual cause of the luminosity of the sun, because it generates a non-vanishing net flux.

#### 2.2.1 Diffusion of radiative energy

We have seen that the mean free path $l_{p,p}$ of photons is much smaller that the charatteristic length of our physical system, hence we are allowed to treat the scattering processes statistically. This leads us to describe energy transfer as a diffusion process.
The diffusive flux $j$ of a certain species of particles (number of particles migrating per unit area and unit time) is given by

$$\vec{j} = -D\nabla n \tag{9}$$

where $n$ is the species density. $D$ is the diffusion coefficient

$$D = \frac{1}{3} v l_p \tag{10}$$

function of the mean free path and of the mean velocity. What we are actually intrested in is not the diffusive flux of the particles $j$, but rahter in the diffusive flux of their energy $\vec{F}$. The density needs to be substituted with the energy density $u = aT^4$, $\bar{v}$ with $c$ and$l_p$ with $l_p, p$. $a$ is the radiation density constant. If we assume spherical symmetry the gradient of $u$ is simply the radial component

$$\frac{\partial u}{\partial r} = 4aT^3 \frac{\partial T}{\partial r} \tag{11}$$

So recalling **??** we have immediately that

$$F = -\frac{4ac}{3} \frac{T^3}{k\rho} \frac{\partial T}{\partial r} \tag{12}$$

which can be written down in a more compact way as

$$\vec{F} = -k_{rad} \nabla T \tag{13}$$

after having properly defined $k_{rad}$ that is the conduction coefficient. Now we use the well known relation between luminosity and flux and solve it for the temperature gradient

$$\frac{\partial T}{\partial r} = -\frac{3}{16\pi ac}\frac{k\rho l}{r^2 T^3} \tag{14}$$

which in lagrangian coordinate reads

$$\frac{\partial T}{\partial m} = -\frac{3}{64\pi^2 ac}\frac{kl}{r^4 T^3} \tag{15}$$

We shall never forget the assumptions we did when deriving this equation. When we approach the photosphere for instance this doesn't hold any longer, because scattering becomes less likely and hence diffusion is no longer the way heat is transfered.
Another form of heat transfer is conduction, but in stellar physics it becomes relevant only in cores of red giants or white dwarfs and it is fully neglected in our simulations.

### 2.2.2 The astrophysical $\nabla$

Assuming hydrostatic equilibrium we can now divide REF by REF and obtain

$$\frac{\partial T/\partial m}{\partial P/\partial m} = \frac{3}{16\pi acG}\frac{kl}{mT^3} \tag{16}$$

This is nothing but the derivative of the temperature with respect to the pressure, that increasing monotonically inward is representative of the radious. We furthermore rewrite this quantity as

$$\nabla_{rad} = \left(\frac{d\ln T}{d\ln P}\right)_{rad} = \frac{3}{16\pi acG}\frac{klP}{mT^4} \tag{17}$$

This quantity will be of fundamental importance when we will discuss convective stability. Recall that we took into account only radiative transfer, but defining $k$ properly we could take into account also heat conduction.

## 2.3 Fundamental Equations of Stellar Structure

At this point we are able to write down a system of equation that we will call **fundamental equations of stellar structure**, which describe an oversimplified thus very useful model. This will hold in a static, stable and spherically symmetric case.

**Mass continuity**
Here $m(r)$ is the mass contained inside the sphere of radius $r$

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{18}$$

**Hydrostatic equilibrium**
We have already encountered the hydrostatic equilibrium equation

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho \tag{19}$$

**Equation of State**
So far we have written down two equations in three variables, namely $m$, $\rho$, and $P$. If we want to have at lest a chance to solve them we need another equation, namely the equation of state. Possible choices are:

- $\rho = const$ this would be the case for liquids.
- $P \propto \rho^\gamma$ i. e. the density and pressure are not a function of the temperature. This is the case, as already stated in the previous section for white dwarfs and leads to the Lane-Edmen equation and its solutions.
- $P = \frac{\mu}{R}\frac{P}{T}$ the equation of state for perfect gases and it's the case for the sun. It has the inconvenient that it introduces another variable, namely the temperature $T$, and hence it doesn't solve our original problem. We need at least another equation to solve the system.

**Energy conservation**
In case we have an energy source (such as nuclear reactions), this needs to be conserved

$$\frac{dL}{dr} = 4\pi r^2 \epsilon \tag{20}$$

**Energy transfer by radiation**
Energy can be radiated through a medium, depending on the opacity $k$.

$$L(r) = -\frac{c(4\pi r^2)^2}{3k}\frac{d(aT^4)}{dr} \tag{21}$$

## 2.4 Convection

Until now we have assumed a perfect spherical symmetry, meaning that all dynamical and thermodynamical quantities were a function of the radius only. Obviously this is not a realistic case. For a number of reasons stars have small perturbations, that may eventually grow and give rise to macroscopic phenomena. A classic example is convection.
We are relaxing the spherically symmetric case, but this doesn't mean that our fundamental equations of stellar structure are useless. As explained in the introduction, if we could section stars we would see that convection appears in concentric regions, hence we can still treat the problem as spherically symmetric defining dynamycam and thermodynamical quantities as averages on a proper region.
We shall now understand when, given thermodynamic variables, we obtain a dynamically stable or unstable layer.

### 2.4.1 Dynamical instability

Let's consider a perfect density stratification like the one in a star or in the atmosphere and let's break the symmetry of the system by adding a little perturbation in the termodynamic variables. For any given quantity $A(r, \theta)$ (from now on $A_e$, because it's a function of the mass element we are considering), we compute $A_s$ (which is an average at given $r$ on the surrounding material). We shall furthermore assume that the fluid moves adiabatically, thus the timescale for heating transfer is much smaller than the timescale for convection turnover.
We define a local property of the fluid

$$DA = A_e - A_s$$

For instance we could imagine that a little region of a star is slightly hotter than the surroundings, hence we have in that region $DT > 0$. Note that because of the assumption we have made $DP = 0$ always, because when these is a pressure spike, the gas expands at the sound velocity $c_s$ which is way higher than the low hydro regime we observe in stellar inertia.
If we have a $DT > 0$, since in a perfect gas the equation of state reads $\rho \sim P/T$, we obtain $D\rho < 0$. With a lower density, that mass element will be lifted by buoyancy force. In a non adiabatic workframe it might happen that heat is exchanged so quickly that temperature differences vanish immediately, but we assumed adiabatic processes. The question we are trying to answer is if the mass element, after a little upward movement, will

still be buoyant and give rise to macroscopic convection, or if it will bounce back. The answer lyes obviously in the temperature gradient, i. e. if once lifted of a little bit the new $DT$ will still be in favor to buoy it to the next layer, and so on.

Let's approach the problem from another point of view. Let's assume that we have a stable layer without perturbations, and we lift a mass element upward of $\Delta r$. The density difference now is

$$D\rho = \left[ \left(\frac{d\rho}{dr}\right)_e - \left(\frac{d\rho}{dr}\right)_s \right] \Delta r \tag{22}$$

where the first derivative tells us how much the density of the mass element changes when lifted, the second one tells us how the surrounding density changes along the radial direction.

We call the buoyancy force per unit volume

$$f_b = -g\ D\rho$$

which points upward if $D\rho < 0$, which is the unstable configuration. If instead $D\rho > 0$ the mass element sinks back to its original position and no macroscopic motion appears. As a consequence $D\rho < 0$ is our **condition for stability**.

The problem with this criterion is that very often the density gradient is not known, since it does not appear in the fundamental equations for stellar structure. In order to proceed let's turn our gradient in spacial coordinate into a gradient in thermodynamic coordinates. As previously stated, our transformations are adiabatic, hence no exchange of energy occurs. This is very close to reality for stellar intertia. To begin let's write down the equation of state $\rho = \rho(P, T, \mu)$ in a differential form

$$\frac{d\rho}{\rho} = \alpha \frac{dP}{P} - \delta \frac{dT}{T} + \phi \frac{d\mu}{\mu} \tag{23}$$

where $d\mu$ represents the change in the chemical composition, including ionization processes.

Substituting REF in REF we obtain

$$\left(\frac{\alpha}{P}\frac{dp}{dr}\right)_e - \left(\frac{\delta}{T}\frac{dT}{dr}\right)_e - \left(\frac{\alpha}{P}\frac{dp}{dr}\right)_s + \left(\frac{\delta}{T}\frac{dT}{dr}\right)_s - \left(\frac{\phi}{\mu}\frac{d\mu}{\mu dr}\right)_s > 0 \tag{24}$$

The two terms containing the pressure gradient cancel each other out because as previously stated $DP = 0$. Let's multiply the remaining terms by the so calles **pressure scale height** $H_P$

$$H_P = -\frac{dr}{d\ln P} = -P\frac{dr}{dP} \tag{25}$$

$H_P$ has the dimension of a length, being the characteristic distance over which the pressure changes. We finally obtain our condition for stability

$$\left(\frac{d\ln T}{d\ln P}\right)_s < \left(\frac{d\ln T}{d\ln P}\right)_e + \frac{\phi}{\mu}\left(\frac{d\ln \mu}{d\ln P}\right)_s \tag{26}$$

We have already encountered the first term which is $\nabla_{rad}$ of **??**. Let's define two new quantities

$$\nabla_e = \left(\frac{d\ln T}{d\ln P}\right)_e \quad \nabla_\mu = \left(\frac{d\ln \mu}{d\ln P}\right)_s \tag{27}$$

Recall that since pressure is always decreasing at increasing radius we can measure the temperature as a function of the pressure, and hence its gradient. Furthermore, since our mass element buoys adiabatically, we call $\nabla_e = \nabla_{\text{ad}}$. With this new notation we can write down our stability criterion in a more compact way as

$$\nabla_{\text{rad}} < \nabla_{\text{ad}} + \frac{\phi}{\delta}\nabla_\mu \tag{28}$$

And obtain the **Ledoux criterion**.
In our simulations we used a monoatomic ideal gas equation of state, hence $\nabla_\mu = 0$. What we obtain is the **Schwarzschild criterion** for dynamic stability

$$\nabla_{\text{rad}} < \nabla_{\text{ad}} \tag{29}$$

If the left hand side is bigger than the right hand side, it means that our stratification is not able to transport all the energy generated only through radiation, and it is hence required to rely on convection.

### 2.4.2 Mixing length theory

A fundamental problem in Stellar Astrophysics is to understand how much energy is transported by convection. Before the era of Computational Physics, when only analytics solutions were available, the most common tool used to model convective energy trasnport was the so called **mixing length theory**.
The total energy flux $l/4\pi r^2$ at given radius is given by the radiative flux $F_{\text{rad}}$ (which may also include the conductive flux), and the convective flux $F_{\text{con}}$. Their summ is according to what defines the $\nabla_{\text{rad}}$ that would be necessary to transport away all the energy. But part of it is ultimately trasnported by convection, at the most $\nabla_{\text{rad}}$ can equal $\nabla$. In order to quantify then

$F_{\text{con}}$, let's consider a blob with a $DT$ over its surroundings. Its motion will be mainly radial wich a velocity $v$ and with $DP = 0$ for reasons already explained. We can write down the energy flux as

$$F_{\text{con}} = \rho v c_P DT \tag{30}$$

Of course we should average $v$ and $DT$ over an imaginary sphere and hence obtain an equation that holds statistically over the solud angle at a given radius and not locally. One can tell since now the approximations and limits of this theory.

Bulbs have an initial velocity that equals zero, and the same holds for $DT$. Over time it will migrate of a discance $l_m$, namely the *mixing length* before dissolving via turbulent cascade. This means that on average a bulb will move of a distance $l_m/2$ where it accelerates, and of another $l_m/2$ where it decelerates of the same amount to stop at the end of the convective zone. Bulbs will definitelly also change shape during the migration, which makes it pretty difficult to strictly define temperature and velocity, but in a very rough approximation we claim that

$$\frac{DT}{T} = \frac{1}{T}\frac{\partial(DT)}{\partial r}\frac{l_m}{2} = (\nabla - \nabla_e)\frac{l_m}{2}\frac{1}{H_P} \tag{31}$$

which is simply a taylor espansion to determine $DT$ after $l_m/2$. Recall that the buoyancy force per unit mass is $f_b = -g \cdot D\rho/\rho$ and that $D\rho/\rho is -\delta DT/T$. Let's pretend on average that half of this force acted on the blob during its migration of $l_m$, hence the work done is

$$\frac{1}{2}k_r\frac{l_m}{2} = g\delta(\nabla - \nabla_e)\frac{l_m^2}{8H_P} \tag{32}$$

Let's furthermore suppose that only half of this work ends up in kinetc energy, because the other half has to go in the work necessary for moving the surroundings. Then the average velocity of an element half way is

$$v^2 = g\delta(\nabla - \nabla_e)\frac{l_m^2}{8H_p} \tag{33}$$

and plugging this result into

$$F_{\text{con}} = \rho c_P T\sqrt{g\delta}\frac{L_m^2}{4\sqrt{2}}H_P^{-3/2}(\nabla - \nabla_e)^{3/2} \tag{34}$$

### 2.4.3 Bulk Richardson

So far we got to the point where we have a convective layer and a stable layer above or beneath it. What happens over time is that the convective boundary accrets mass from the stable layer by ingestion, because turbulent eddies capture stable fluid that is consequentially dragged and entrained in the turbulent layer. A fundamental problem in stellar evolution is to understand the dynamics of this boundary: which is its velocity (if we are in an eulerian workframe), or how much mass from the stable layer is entrained in unit time (in a lagrangian workframe).
One fo the methods used to model this migration is the **Bulk-Richardson number**

$$\mathrm{Ri}_B = \frac{\Delta b L}{\sigma^2} \tag{35}$$

which is a dimensionelss parameter that compares the strenth of the boundary to the one of the turbulence. $L$ is the length scale of the turbulence (which is of course model dependent), $\sigma$ is the rms of its velocity, $\Delta b$ is the buoyancy jump between unstable and stable layer. This is defined by considering the **Brunt-Väisääla frequency**

$$N^2(r) = -g\left(\frac{\partial \ln \rho}{\partial r} - \frac{\partial \ln \rho}{\partial r}\Big|_s\right) \tag{36}$$

and integrating it over the boundary

$$\Delta b = \int_{r_i}^{r_f} N^2 dr \tag{37}$$

Of course we need a proper definition of the boundary limits $r_i$ and $r_f$, hence also $\Delta b$ is model dependent.
Let's furthermore define the **entrainment coefficient** $E$ as the boundary migration speed $u_b$ over the turbulence rms $\sigma$. Previous works have found that

$$E = A\mathrm{Ri}_B^{-n} \tag{38}$$

Where $A$ and $n$ are coefficients that need to be determined, and this will be the primary goal of our simulations. If this relation is proven to be true, it would be possible to know the boundary migration speed $u_b$ given only $L$, $\Delta b$ and $\sigma$, and consequentially map this phenomenon into 1D stellar models. In the hope that these two free parameters exist and are universal (i. e. not system- code- or resolution-dependent) the goal of this work is to determine them.

### 2.5 Il r'Hamiltoniana

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## 3 Recent Works

The only way to study thoroughly the macroscopic effects of convection like the boundary mixing problem is through multidimensional simulations. Only in the last decade computational power allowed astrophysics groups to run 3D simulations with a satisfying resolution.
The first extensive work was carried out by C. Meaking and D. Arnett (2007) who simulated the oxygen shell burning and the hydrogen core burning in a $23M_{\odot}$ star both in 2D and in 3D. They used the 1D stellar evolution code TYCHO that was then mapped into PROPMI, a multidimensional parallelized hydro code that solves euler equations implementing PPM (piecewise parabolic method) with a nuclear reaction network. They reported $\log A = 0.027 \pm 0.38$ and $n = 1.05 \pm 0.21$ for the three dimensional case.
Another recent study has been carried out by A. Cristini et al, 2016. They simulated a Carbon burning shell in a $15M_{\odot}$ star in four runs with different resolutions (from $128^3$ to $1024^3$). They used the code PROMPI which solves euler equations (ideal Hydrodynamics) in an eulerian framework, using as well the PPM method, parallelization with MPI, a nuclear reaction network, heat transfer (not used in their case) and self gravity in the Crowling approximation necessary to describe deep interiors of stars. They reported parametric values of $A = 0.06(+0.27/-0.04)$ and $n = 0.81(+0.38/-0.28)$. Although the second one agrees pretty well with the previous study, the first one is not even in the same order of magnitude, hence the motivation for this work.

### 3.1 Il r e l'Hamiltoniana

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## 4 Code Description

### 4.1 ciao

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# 5 Results

## 5.1 ciao

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