Assignment

Homework HL3

Alberto Doimo 10865196

Riccardo Iaccarino 10868500

Musical Acoustics

POLITECNICO
MILANO 1863

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1 Piano string-hammer interaction

In the first part of this report we will analyze the interaction between the hammer and the string of a piano with the help of a finite differences model. Following the FD method we will discretize the continuous string in both space and time in order to move from the continuous equations that describe its behaviour to a set of finite differences equations. The last step will be to solve these equations iteratively to obtain the motion of the string as a function of the striking position of the hammer and the observation position.

a) Hammer

The first component that we want to analyze is the hammer, since it is responsible for the only external force acting on the string. Specifically, we will consider just its contribution meaning that we are neglecting the attenuation brought by the damper (Fig.1).

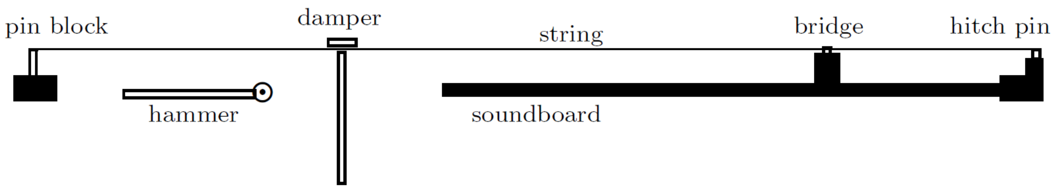

Figure 1: Simplified mechanism of a piano string

Moreover we omit also the whole motion mechanism the drives the hammer, hence focusing just on the final part of the system that will hit the string. As we can appreciate from Fig.2, the tip of the hammer is composed of a hard wood core, which is covered by two different layers of wool felt. The thicknesses of these layers usually vary depending on the note associated to that particular hammer.

Figure 2: Example of a piano hammer

The felt is responsible for the non-linear hardening of the hammer, in the sense that its hardness varies with the velocity in a direct proportional (yet non-linear) way. In particular we can model the felt as a non-linear spring whose stiffness increases with compression, which is attached to a mass. On the basis of these hypothesis we can write the hammer force as:

$$F_H = K\xi^p \tag{1}$$

where K is the hammer stiffness, p describes how the stiffness changes with the force and ξ is the felt compression. The latter can be expanded as the hammer vertical position w.r.t.

the string's stroked point x_0:

$$\xi(t) = |\eta(t) - y(x_0, t)| \tag{2}$$

where $\eta(t)$ is the hammer displacement w.r.t. the string equilibrium position and $y(x_0, t)$ is the string position at instant t. The interaction between the two systems will be modeled with the force exerted by the hammer on the point x_0 of the string as follows:

$$M_H \frac{d^2\eta}{dt^2} = -F_H(t) - b_H \frac{d\eta}{dt} \tag{3}$$

in which the last term is related to air damping. The influence of the hammer on the string movement is valid until:

$$\eta(t) < y(x_0, t) \tag{4}$$

Some useful parameters related to the hammer modelling are listed:

- $M_H = 4.9 \cdot 10^{-3}$ Mass of the hammer
- $p = 2.3$ Stiffness exponent
- $b_H = 1 \cdot 10^{-4}$ air damping coefficient
- $K = 4 \cdot 10^8$ Hammer felt stiffness

b) String

The analytical wave equation for the stiff and lossy string is the product of the combination of different studies and can be expressed as follows:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \kappa^2 \frac{\partial^4 y}{\partial x^4} - 2b_1 \frac{\partial y}{\partial t^2} + 2b_2 \frac{\partial^3 y}{\partial x^2 \partial t} + \rho^{-1} f(x, x_0, t) \tag{5}$$

where f, which represents the excitation of the string by the hammer and has the form of a force density, can be computed as:

$$f(x, x_0, t) = f_H(t) g(x, x_0) \quad \text{with} \quad f_H(t) = F_H(t) \left(\int_{x_0 - \delta x}^{x_0 + \delta x} g(x, x_0)\, dx \right)^{-1} \tag{6}$$

Specifically this passage is fundamental in order to take into account the distribution of the force applied from the hammer to the string. For this exact reason $g(x, x_0)$ is an adimensional Hanning window centered in the striking position (12% of the total length of the string) of width $w = 0.2$ $[m]$. It is possible to appreciate the distribution function g in Fig.3.

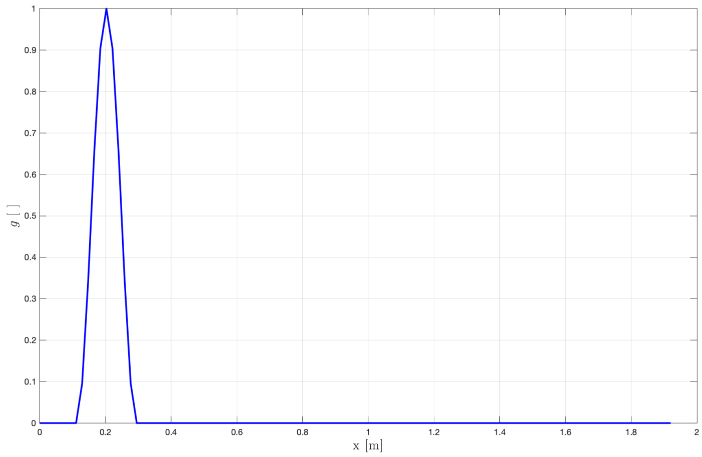

Figure 3: Distribution function shape

Once we have the equation, we need to impose boundary conditions to it, which will concern the two extremities of the string. At the bridge (right end) the string cannot move, yet it produces a force F_b on the bridge so we can write:

$$F_b = R_b \frac{\partial y}{\partial t} \; ; \qquad F_b = -T_e \frac{\partial y}{\partial x} + \kappa^2 \rho \frac{\partial^3 y}{\partial x^3} \; ; \qquad \frac{\partial^2 y}{\partial x^2}|_{x_M, t_n} = 0 \tag{7}$$

Combining the three equations and calling x_m the coordinate of the string at this position we get:

$$\frac{\partial^3 y}{\partial x^3} = -\frac{c^2}{\kappa^2} \frac{\partial y}{\partial x} + \frac{\zeta_b c}{\kappa^2} \frac{\partial y}{\partial t} \; ; \qquad \zeta_b = \frac{R_b}{\rho c} \tag{8}$$

with R_b being the impedance of the bridge and ζ_b its normalization. Similarly, at the left hinged end we should impose:

$$\frac{\partial^3 y}{\partial x^3} = -\frac{c^2}{\kappa^2} \frac{\partial y}{\partial x} + \frac{\zeta_l c}{\kappa^2} \frac{\partial y}{\partial t} \; ; \qquad \zeta_l = \frac{R_l}{\rho c} \; ; \qquad \left. \frac{\partial^2 y}{\partial x^2} \right|_{x_0, t_n} = 0 \tag{9}$$

with R_l being the impedance of the left end and ζ_l its normalization. Finally, we list the parameters needed to model the string with their values:

- $f_1 = 65.4$[Hz] Fundamental frequency (C2)
- $L = 1.92$[m] Length
- $M_s = 35 \cdot 10^{-3}$[Kg] Mass
- $\rho = \frac{M_s}{L} = 18.2 \cdot 10^{-3}$[Kg/m] Linear density
- $b_1 = 0.5 \left[s^{-1}\right]$ Air damping coefficient
- $b_2 = 6.25 \cdot 10^{-9}[s]$ Air damping coefficient
- $\kappa = \epsilon$ Stiffness coefficient

- $\epsilon = 7.5 \cdot 10^{-6}$ Stiffness parameter
- $\zeta_l = 1 \cdot 10^{20}$ Left end normalized impedance
- $\zeta_b = 1000$ Right end (bridge) normalized impedance
- $T = 4 \cdot L^2 \rho f_1^2 = 1149.7$[N] Tension applied to the string
- $c = \sqrt{\frac{T}{\rho}} = 251.136$[m/s] Wave propagation velocity

c) Finite difference

Firstly we discuss about the stability of the algorithm. In particular, the sampling frequency F_s and the coefficient $\gamma = F_s/2f_1$ are defined. We want now to subdivide the string in M finite small segments of length X , so that L = M X . By defining the Courant number $\lambda = cT/X$ where the time-sample length $T = 1/F_s$, we can then calculate the Courant-Friedrichs-Lewy (CFL) condition to be respected:

$$\lambda = \frac{cT}{X} \leq 1 \tag{10}$$

In order to have the best spatial resolution we can consider the Courant number $\lambda = 1$ so that we obtain the minimum length of the spatial step X_{min} and the resulting maximum number of spatial samples M_{max} defined as follows:

$$X_{min} = cT \Rightarrow M_{max,CFL} = \left\lfloor \frac{L}{X_{min}} \right\rfloor = 122 \tag{11}$$

The maximum number of spatial samples M_{max} has been defined for this case as in [1] as follows:

$$M_{max} = \sqrt{\frac{-1 + \sqrt{1 + 16\epsilon\gamma^2}}{8\epsilon}} = 105.83 \tag{12}$$

and so M has been rounded as:

$$M = \lfloor M_{max} \rfloor = 105 \tag{13}$$

The obtained value is valid since its corresponding Courant number is $\lambda = 0.85 \leq 1$ and $M_{max} \leq 122$. To proceed, we introduce some useful parameters that will be required in later equations:

$$\mu = k^2/(c^2X^2) = 2.67 \cdot 10^{-12} \tag{14}$$

$$\nu = 2b_2T/X^2 \tag{15}$$

In order to compute the string displacement this finite difference equation is used:

$$y_m^{n+1} = a_1\left(y_{m+2}^n + y_{m-2}^n\right) + a_2\left(y_{m+1}^n + y_{m-1}^n\right) + a_3 y_m^n + a_4 y_m^{n-1} + a_5\left(y_{m+1}^{n-1} + y_{m-1}^{n-1}\right) + a_F F_m^n \tag{16}$$

where m is the evaluation spatial index, n is the evaluation time index and the following coefficients are used:

- $a_1 = \frac{-\lambda^2\mu}{1+b_t 1}$

- $a_2 = \frac{\lambda^2+4\lambda^2\mu+\nu}{1+b_t1}$
- $a_3 = \frac{2-2\lambda^2+6\lambda^2\mu+2\nu}{1+b_t1}$
- $a_4 = \frac{-1+b_1T+2\nu}{1+b_t1}$
- $a_5 = \frac{-\nu}{1+b_t1}$
- $a_F = \frac{T^2/\rho}{1+b_1T}$

Since the previous equation is valid only in the interior of the string some additional formulation are needed in order to use the finite difference. In particular, four boundary conditions need to be imposed: two for the left side ($m = 0, m = 1$) and two for the right side ($m = M, m = M - 1$).
In **m=0**:

$$y_m^{n+1} = b_{L1}y_m^n + b_{L2}y_{m+1}^n + b_{L3}y_{m+2}^n + b_{L4}y_m^{n-1} + b_{LF}F_m^n \tag{17}$$

where:

- $b_{L1} = \frac{2-2\lambda^2\mu-2\lambda^2}{1+b_1T+\zeta_l}$
- $b_{L2} = \frac{4\lambda^2\mu+2\lambda^2}{1+b_1T+\zeta_l\lambda}$
- $b_{L3} = \frac{-2\lambda^2\mu}{1+b_1T+\zeta_l\lambda}$
- $b_{L4} = \frac{-1-b_1T+\zeta_l\lambda}{1+b_1T+\zeta_l\lambda}$
- $b_{LF} = \frac{T^2/\rho}{1+b_1T+\zeta_l\lambda}$

In **m=1**:

$$y_m^{n+1} = a_1\left(y_{m+2}^n - y_m^n + y_{m-1}^n\right) + a_2\left(y_{m+1}^n + y_{m-1}^n\right) + a_3y_m^n + a_4y_m^{n-1} + a_5\left(y_{m+1}^{n-1} + y_{m-1}^{n-1}\right) + a_FF_m^n \tag{18}$$

In **m=M**:

$$y_m^{n+1} = b_{R1}y_m^n + b_{R2}y_{m-1}^n + b_{R3}y_{m-2}^n + b_{R4}y_m^{n-1} + b_{RF}F_m^n \tag{19}$$

where the coefficients b_{L1}, b_{L2}, b_{L3}, b_{L4}, b_{LF} are computed by substituting ζ_l with ζ_b.
In **m=M-1**:

$$y_m^{n+1} = a_1\left(2y_{m+1}^n - y_m^n + y_{m-2}^n\right) + a_2\left(y_{m+1}^n + y_{m-1}^n\right) + a_3y_m^n + a_4y_m^{n-1} + a_5\left(y_{m+1}^{n-1} + y_{m-1}^{n-1}\right) + a_FF_m^n \tag{20}$$

d) Results

The initial conditions are now applied to the system.
At t=0 (n=0):

- The string is not moving so the speed is zero.
- The string is in its equilibrium position.

- The hammer is moving so its speed is $V_{H0} = 2.5$ $[m/s]$ and it is set to be in contact with the string therefore the displacement $\eta(0) = 0$
- The hammer force is null since $\eta(0) = 0$ and $y(m, 0) = 0$

At n=1:

- The hammer displacement depends on its speed V_{H0} and on the temporal resolution between n and $n+1(\Delta T)$ since $\eta(1) = V_{H0}(\Delta T)$. The formula for the hammer displacement is valid only from $n = 2$

Finally, the string displacement is obtained by running the FD simulation. In Fig.4 the time response of eight seconds is plotted. The corresponding sound has been computed by averaging the displacement over 12 spatial samples (5 before and 6 after) centered on the specular position w.r.t the striking one, so around $x = L - x_0$.

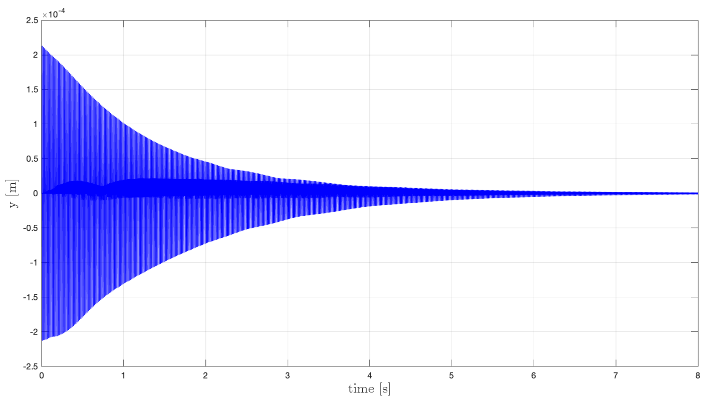

Figure 4: Time oscillation of a C2 piano string averaged over 12 spatial samples

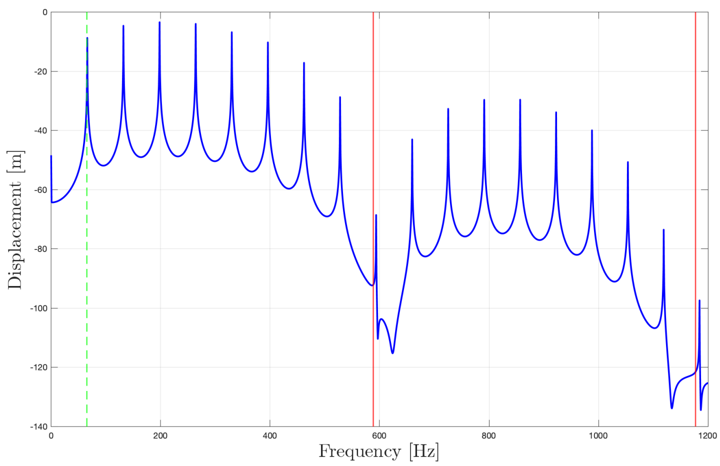

Figure 5: Frequency representation of the C2 piano string sound

Additionally the frequency response of the exported sound is presented in Fig.5. The resonance at $f_1 = 65.4$ $[Hz]$ is indicated with a green dashed vertical line, whereas in red the position of the 9th and 18th harmonics are marked. A minimum at ninth harmonic and its multiples is present since the string is excited at $0.12 \approx 1/9$ of its length. This result in the frequency domain, shows that this model properly simulates a piano string.

The displacement of the string is represented in the attached GIF and some time frames are plotted in Fig.6.

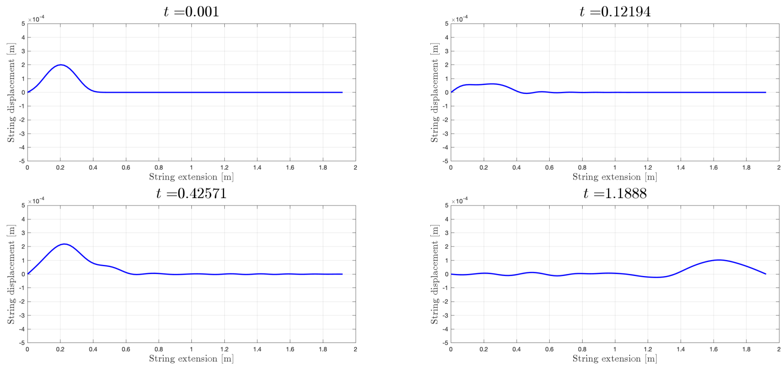

Figure 6: String displacement at different time instants

2 Acoustic guitar modeling

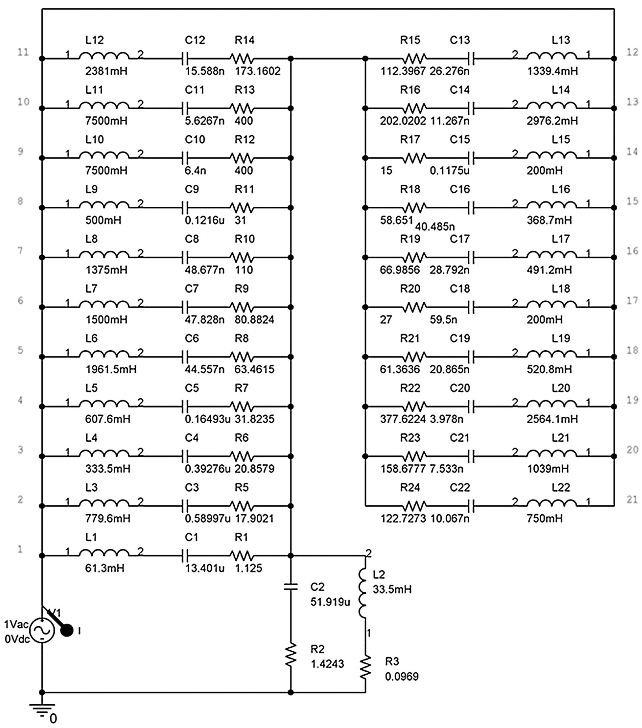

Figure 7: Complete model of the body of the guitar

In this section we will model instead an acoustic guitar through its electric analog, and in particular we will focus on the plucking of the E1 string. At first, we will exploit a simple input signal that consists in a damped square wave, whereas in the second part of the study we will employ the model of a plucked string in order to improve the final result. In both cases we adopt the complete guitar model that includes 20 resonances thanks to the parallel of the same amount of L-C-R series, as depicted in Fig.7. Referring to this scheme, we can identify the bottom right part of the circuit (R2, R3, C2 and L2) as responsible for the sound box resonance, whereas the series L1, C1, R1 models the first resonance of the top plate; all the other elements in parallel handle the higher ones.

a) Damped square wave input

As already mentioned, in this first approach we will excite the model with a more approximate signal that will drive the voltage generator following the time trend:

$$V(t) = sgn(sin(2\pi f_0 t))e^{-\beta t} \tag{21}$$

in which $f_0 = 329.6$ $[Hz]$ is the fundamental frequency of the E1 string and $\beta = 3$ is the term responsible for the decay. In Fig.8 it is showed the Simulink implementation of the input, how it is connected to the rest of the model and how the current is measured and then sent to the MATLAB workspace (`out.simout`).

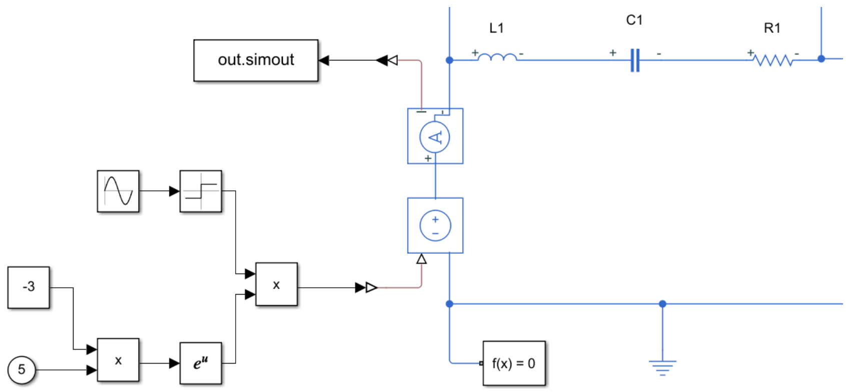

Figure 8: Zoom of the circuit on the damped square wave input

Then we plotted the results in Fig.9: as expected the decay resembles that of an acoustic guitar, and also we could try to validate the fundamental frequency by counting the number of periods present in the zoomed signal (33 ca. in 0.1 s). Despite this, when we play the signal with `soundsc`, the timbre results to be way off the peculiar sound of an acoustic guitar.

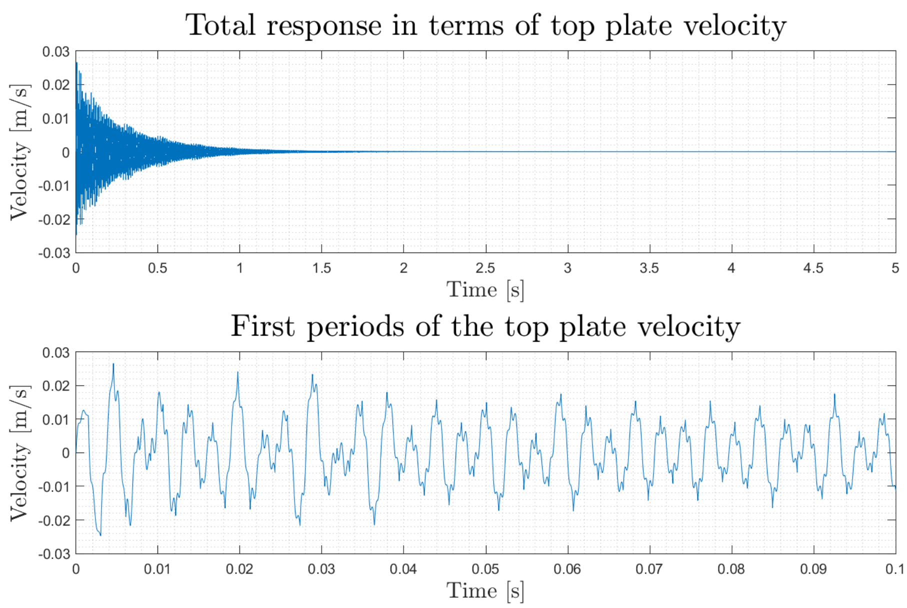

Figure 9: Top plate velocity for a damped square wave input (with zoom)

In particular this effect can be legitimated once again by studying the time response of Fig.9, in which we can see that the waveform is close to a triangular one (similar to the audio contribution), with a few harmonics acting on it. Finally, if we transform in the frequency domain the output signal, we can appreciate not only the fundamental f_0 but also some of its harmonics. Specifically we noticed that only the odd multiples are present, which means that with this kind of input we are in a similar condition to a string plucking happening at its middle point. In Fig.10 the resonance at f_0 is indicated with a dashed vertical line, whereas the odd harmonics are in correspondence of the whole lines.

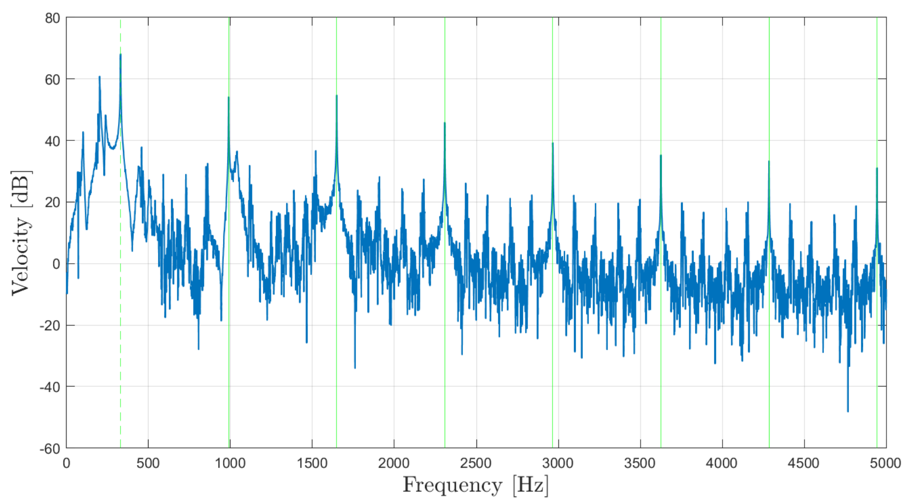

Figure 10: Spectrum of the top plate velocity for a damped square wave input

All the remaining peaks that are present refer to the modeling of the guitar body through the circuit first presented in Fig.7.

b) String model

The second approach takes a more complex path since it exploits as input the model of a plucked guitar string. With this method we are also able to model strings of arbitrary fundamental frequency plucked at an arbitrary plucking point. For a matter of consistency we will choose the E1 string, but since we can select the plucking position, this time we go for a more realistic one, placing it at 1/5 of the length of the string. Referring to the circuit of Fig.11 we can see that the model of the string is in some way specular w.r.t. the transmission line. Namely, we want to feed both generators with a triangle of amplitude 1 *mV* and with a duration that is related with the target frequency as $d = 1/2f_0$ which happens to be, in our case, 1.515 *ms*. In order to take into account the plucking position, we can instead work on the position in time of the maximum of that signal: the generator on the left will have it at 1/5th of its total duration, and symmetrically the other one will have it at 4/5ths.

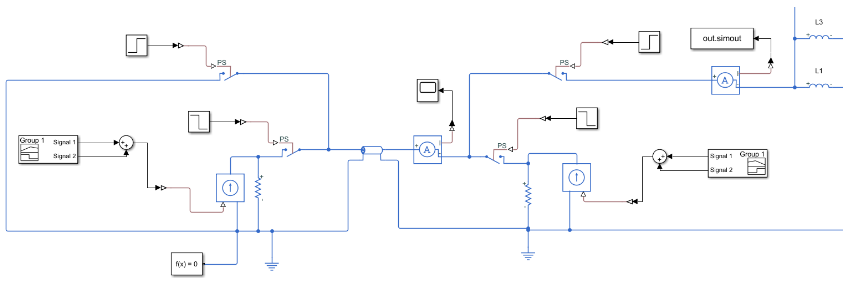

Figure 11: Zoom of the new circuit on the plucked string model

The transmission line is responsible for the generation of the reflections given by nut and bridge and so it has been given the same delay value of 1.515 *ms*, along with a characteristic impedance of 2000 *Ω* (which is the same of the two resistors). The final elements of the circuit are the switches that are designed so that the generators get disconnected from the transmission line right after the end of the triangle excitation. In this way the signal passes through the circuit of the body and gets attenuated at each cycle, without the introduction of any unwanted noise. As for the previous approach, the output is the current measured in correspondence of the top plate (analog of its acoustic velocity), which is sent directly to the MATLAB workspace.

We can look at the time response of the system right away from Fig.12. From the complete trend we can immediately notice a slower decay rate w.r.t. the damped square wave scenario, in fact in this case the signal does not extinguish by the end of the 5 seconds simulation. Moreover the signal appears way smoother, feature that pops out in the zoomed plot as well, and that will have its consequences in the output audio.

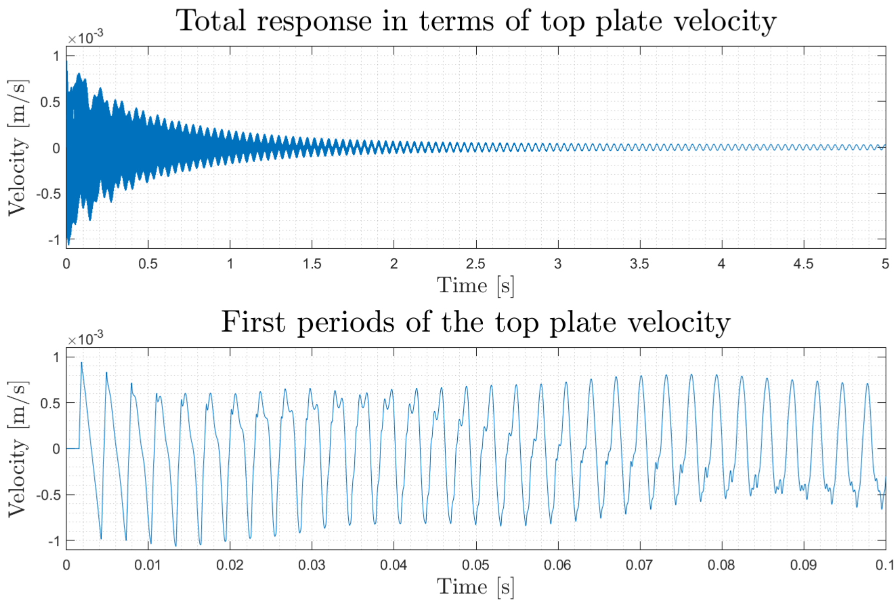

Figure 12: Top plate velocity for the plucked string model input (with zoom)

Keeping an eye on the first periods of the top plate velocity, we can count 32 of them (and not 33) due to an initial period of silence: this happens because of design of the circuit that initially cuts out the amperometer branch. When playing the audio the improvement is evident and the overall result is very accurate from all points of view. Finally we analyze the frequency content of the output in the range $[0, 5000]$ Hz in Fig.13. In this interval we should be able to observe 15 resonances, but we can notice that some of them are missing: in fact, as expected, due to the plucking position all the harmonics at integer multiples of $5f_0$ are attenuated. In the graph the fundamental frequency is indicated with a dashed green line while the missing harmonics are marked with a red one. Overall the spectrum related to this advanced model is also clearer than the previous one, characteristic that justifies the better acoustic outcome.

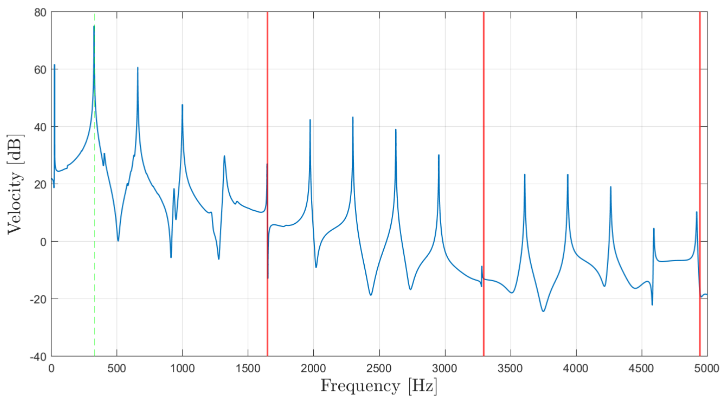

Figure 13: Spectrum of the top plate velocity for a damped square wave input

References

[1] Chaigne, Antoine Askenfelt, Anders.*Numerical simulations of piano strings. I. A physical model for a struck string using finite difference methods*. Journal of The Acoustical Society of America - J ACOUST SOC AMER. 95.1994 1112-1118. 10.1121/1.408459.