Assignment

Homework HW3

Alberto Doimo 10865196

Riccardo Iaccarino 10868500

Musical Acoustics

POLITECNICO
MILANO 1863

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Introduction

The goal of this assignment is to synthesize the vibrational field that can be measured on the body of a guitar (in particular we will refer to a Martin D28) by combining the following models:

- **Bridge impedance** Z: this first component can be modeled as part of a the two-mass system that defines a guitar body, referring to the scheme depicted in Fig.1. Specifically we can assume rigid back and ribs to divide the body of the instrument in just the top plate (of mass m_p and stiffness K_p) and the mass of air m_h in the soundhole, coupled with the volume V of the cavity that acts as a spring.

- **Transfer function from the plucking point to the bridge** $H_{E,B}$: the second component of the system is the FRF given by the ratio between the force F exerted by the strings at the bridge and the displacement X at the excitation point.

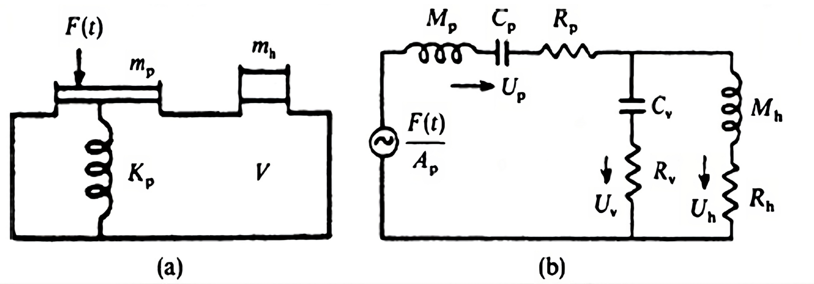

Figure 1: Mechanical and electric analog of a guitar

1) Bridge impedance

The following parameters are derived for the Martin D28 guitar:

- Top plate: Stiffness $K_p = 1.41 * 10^5$ $[N/m]$; Effective Mass $m_p = 0.128 * 0.385$ $[Kg]$; Area $A_p = 0.0375 * 0.385$ $[m^2]$; Resistance $R_p = 32$ $[Nm/kg/s]$

- Soundhole: Air piston mass $m_h = 8.04 * 10^{-4}$ $[Kg]$; Air piston area $A_p = 0.00785$ $[m^2]$; Resistance $R_h = 30$ $[N/m]$

- Air cavity: Volume $V = 0.0172$ $[m^3]$; Resistance $R_v \approx 0$ $[N/m]$

We decided to derive the value of the bridge impedance by simulating in Simscape the mechanical system of Fig.1. The following electrical quantities are then defined:

- Top plate: Inertance $M_p = m_p/A_p^2$ $[kg/m^4]$; Compliance $C_p = A_p^2/K_p$ $[N/m^5]$; Loss $R_p = 32$ $[Nm/kg/s]$

- Soundhole: Inertance of air $M_h = m_h/A_h^2$ $[kg/m^4]$; Radiation loss $R_h = 30$ $[N/m]$

- Air cavity: Compliance $C_v = V/rho * c^2$ $[N/m^5]$; Loss: $R_v \approx 0$ $[N/m]$

(rho = 1.204 $[Kg/m^3]$ =air density, c = 343 $[m/s]$ = speed of sound in air)
In Tab.1 all the values of the elements of the circuit are listed.

Component	R_p	R_h	R_v	C_p	C_v	M_p	M_h
Value	$32[\Omega]$	$30[\Omega]$	$0[\Omega]$	$1.4783[nF]$	$0.012143[nF]$	$236.42[H]$	$13.047[H]$

Table 1: Electrical analog components and associated values

The bridge impedance can be obtained from the electric analog as the ratio between the voltage V (electric analogue of an acoustic pressure F/A_p) and the current I (equivalent of the acoustic volume flow on the bridge U_p). The two time trends extracted from the simulation are then processed through an FFT so that we can obtain the impedance as a function of frequency.

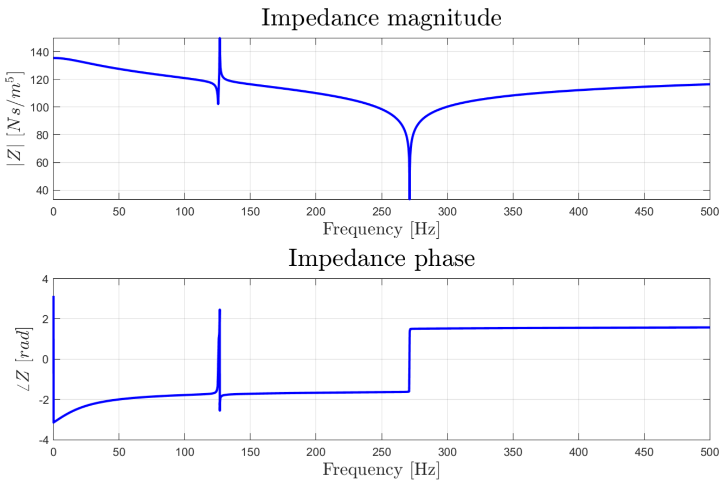

Figure 2: Martin D28 bridge impedance

The bridge impedance can be appreciated in Fig.2, as magnitude and phase: as expected we can recognize two resonances separated by an anti-resonance (placed immediately after the first peak). In particular the first eigenfrequency can be found at around 125 Hz, whereas the second one is placed at 271 Hz ca. Due to the adopted model (Fig.1), these will be the only resonances present in the impedance under investigation, even considering a wider range that exceeds 500 Hz.

An equivalent procedure to get the impedance of the bridge without the help of a simulation software implies its derivation starting from the impedances of the different components. In particular we can express our target quantity as the combination of series and parallel connections between the impedances of:

- Resistors: $Z_A^{res} = R_A$
- Inductors: $Z_A^{ind} = j\omega M_A$
- Capacitors: $Z_A^{cap} = \frac{1}{j\omega C_A}$

Taking into account that a series of two components (Z_1, Z_2) returns $Z_{ser} = Z_1 + Z_2$, whereas the parallel would be $Z_{par} = (1/Z_1 + 1/Z_2)^{-1}$, we obtain the bridge impedance as:

$$Z = Z_p + \frac{Z_h Z_v}{Z_h + Z_v} \tag{1}$$

in which $Z_p = j\omega M_p + \frac{1}{j\omega C_p} + R_p$ is the impedance related to the top plate, $Z_h = j\omega M_h + R_h$ is associated to the soundhole and $Z_v = \frac{1}{j\omega C_v}$ is the contribution of the air cavity.

2) Transfer function $H_{E,B}$

The second part of our study focuses on deriving the transfer function $H_{E,B}(\omega) = \frac{F(\omega)}{X(\omega)}$, where F_j identifies the force exerted by the strings at the bridge and X_j is the acceleration at the excitation point. Starting from the impedance of the bridge, we can exploit the *dual delay-line waveguide model for a plucked string* exposed in [1].

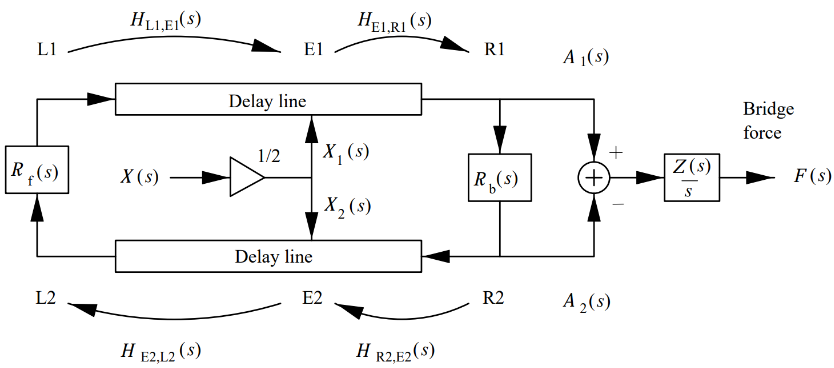

Figure 3: Dual delay-line waveguide model for a plucked string with output at the bridge

As can be noticed from the scheme of Fig.3, the model involves the use of two parallel and opposite delay lines that are associated to the components X_1 and X_2 of the initial excitation. Two additional blocks (R_f and R_b) are then placed at the extremities to include the effects of nut and bridge on the propagating waves. In particular they have been both considered as filters that invert the phase (with a minus sign) of the incident wave, in addition to a slight dampening effect. We adopted $R_f = -0.995$ and $R_b = -0.99$, hence assuming that the nut has a stiffer response. In order to compute the filter some parameters had to be calculated. First of all, the fundamental frequencies of the guitar strings are defined; then to properly design our delay lines, we compute the number of samples N_s that will characterize each string as:

$$N_s = \left\lfloor \frac{F_s}{2f_0} \right\rfloor \tag{2}$$

Note that the F_s is the sampling frequency and the N_s is the number of points needed to correctly describe each string in samples, halved since the fundamental mode of a guitar is an half wavelength. Additionally, the plucking position at one fifth of the string length is considered by the parameters $g = 5$ and $\beta = 1/g$ and the nut and bridge distances are defined in samples as:

$$N_{nut} = \lfloor \beta N_s \rfloor; \quad N_{bridge} = N_s - N_{nut} \tag{3}$$

String	E2	A2	D3	G3	B3	E4
Fundamental frequency f_0 $[Hz]$	82.41	110	146.83	196	246.94	329.63
N_s	267	200	150	112	89	66

Table 2: Guitar string properties

The filter was evaluated in the z-domain after computing the single parts of the delay line, following the aforementioned study [1]:

$$H_{E,B}(z) = \frac{1}{2}[1 + H_{E_2,R_1}(z)]\frac{H_{E_1,R_1}}{1 - H_{loop}(z)}\frac{Z(z)}{z}[1 - R_b(z)] \tag{4}$$

The following components are present:

- The transfer function from point E_2 to point R_1 and from E_1 to R_1 is generally described by the notation $H_{initial\ point,arriving\ point}$ and it has been derived as:

$$H_{initial\ point,arriving\ point} = z^{-N} \tag{5}$$

  where N is the delay in samples from the *initial point* to the *arriving point*.

- The transfer function when the signal is circulated once around the loop:

$$H_{loop}(z) = R_b(z)H_{R_2,E_2}(z)H_{E_2,E_1}(z)H_{E_1,R_1}(z) \tag{6}$$

- The bridge impedance Z that has been derived at point **1)**, but after normalization.

Note that in general the transfer function can contain also reflection coefficients such as R_f, as for example in the case of H_{E_1,R_1}. Referring to the Fig.3 we can also notice how the delay line from a generic point A to B is composed of several successive blocks that are multiplied together, as it happens in H_{loop}.

The filter $H_{E,B}$ is plotted in Fig.4 in the range $0 - 500[Hz]$. The first point to be made about the magnitude of the six transfer functions is that their first resonance is aligned with the fundamental frequencies of the strings. Speaking of resonances, we can also appreciate how harmonics are evident and linearly spaced in frequency, except for the fifth one of the E2 string (highlighted in red) which is significantly attenuated. As a matter of fact this behaviour is also present in the $H_{E,B}$ of the other strings, but it does not show due to our restricted frequency range of observation. This phenomenon occurs when a string is excited in correspondence of a node: in this case we are plucking at $1/g$ of the length with $g = 5$, so all the harmonics integer multiples of g will be ideally absent (applies for any $g \in \mathbb{N}$). Lastly we should also notice the presence of the impedance of the bridge, which does not depend on the string under investigation because of the way it is included in the computation of the transfer function (4).

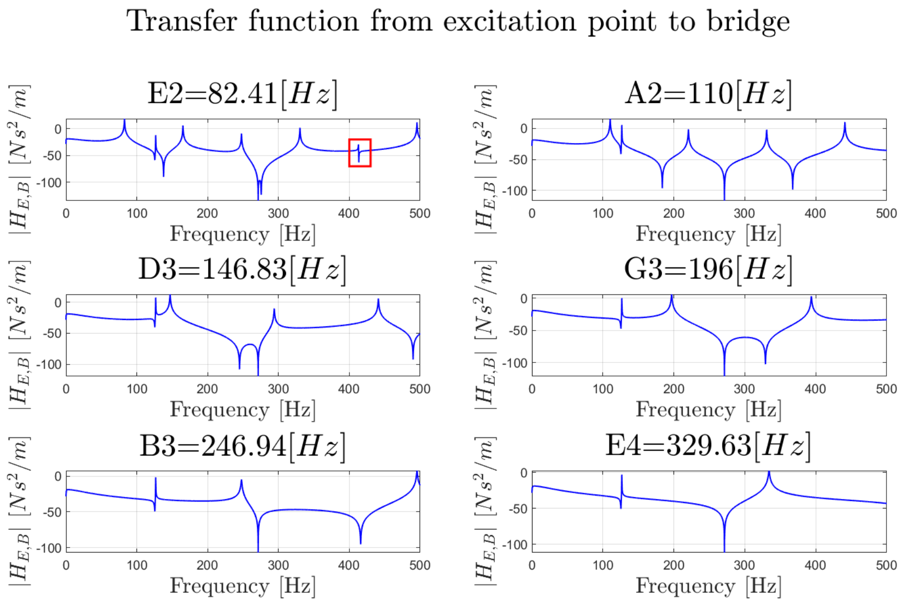

Figure 4: Filter $H_{E,B}$ for the six strings of a Martin D28 guitar

3) Time Response

This final section is dedicated to the analysis of the time domain response of the system to a plucking excitation. In particular, we will consider the case of a pluck at time $t_0 = 0$ $[s]$ that displaces the string of $d_0 = 0.003$ $[m]$ at 1/5 of its length (considered as distance from the bridge). In order to get the response, we will rely of course on the transfer function $H_{E,B}$ computed in the previous section, but we first have to understand which should be the input. Since we are dealing with a triangular displacement, we can derive it in time, obtaining a step function that is associated with the transverse velocity of the wave. Repeating this process we finally obtain the acceleration that takes the form of an impulse, which turns to be a very convenient input for our purposes:

$$x(t) = A\delta(t) \tag{7}$$

The displacement has been calculated by implementing a proper MATLAB function following the analytical formulation presented in [2] and has been derived to obtain the transverse velocity. The amplitude of the input is then calculated by taking the maximum value of the transverse velocity function, since its derivative will simply be a $\delta(t)$ multiplied by A. This computation allows to obtain a more reliable amplitude value since it avoids peaks derived by the step discontinuities of the velocity function. At this point we just need to transform $x(t)$ in the frequency domain, so that we can exploit the property of LTI systems and compute the output of the system as the product between input and transfer function.

Hence we can compute the time response exploiting the Signal Processing Toolbox library of MATLAB as:

$$F(t) = IFFT(A \cdot H_{E,B}) \tag{8}$$

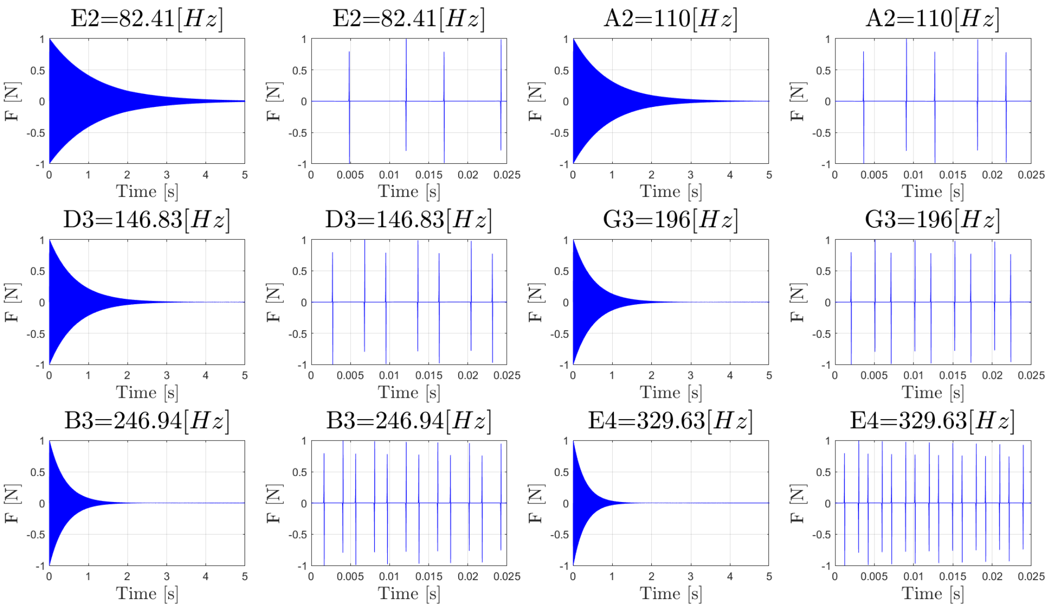

Figure 5: Time domain response of the system

In Fig.5 the time responses of the six strings of our guitar are plotted both completely and with a zoom on the first 25 ms. The first thing that stands out is the decaying envelope that comes from R_f and R_b being smaller than 1; the slope associated to the attenuation can be changed by modifying their values. Comparing the six strings complete motion we can notice how lower resonant frequencies are associated with larger decay times: this behaviour comes from the fact that their wavefronts travel slower, hence they will undergo fewer reflections in the same time interval. The input signal decays in amplitude as it continues to be reflected and attenuated by the bridge and the nut, but we can see that these impulses arrive linearly spaced in time. Since the waves have different propagation velocity along the six strings, in the $E2$ string we can seen only few periods ,whereas for the $E4$ we can appreciate almost ten in the same time interval of 25 ms. Interestingly a close look at the waveforms reveals an up and down trend of the force around zero, that can be found in [3] as the theoretical shape of the transverse force F_T on a guitar in a similar excitation configuration. However the amplitude of these oscillations about zero is very small and the overall trend is dominated by the impulses that are generated when the force inverts sign, (i.e. when the propagating waves arrive to the bridge and invert their phase).

References

[1] M. Karjalainen, V. Välimäki, and T. Tolonen *Plucked string models: From the Karplus-Strong algorithm to digital waveguides and beyond*, Computer Music Journal, vol. 22, pp. 17-32, Fall 1998.

[2] `https://www.acs.psu.edu/drussell/Demos/Pluck-Fourier/Pluck-Fourier.html`, *Motion of the Plucked String* chapter.

[3] N. H. Fletcher and T. D. Rossing, *The Physics of Musical Instruments*, 2nd Edition, Springer-Verlag, New York, 1998. pp. 241-244.