Assignment

Homework HW1

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Musical Acoustics

POLITECNICO
MILANO 1863

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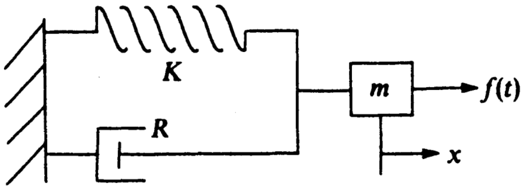

Figure 1: Resonator system scheme

a) Resonance frequency

Resonance frequency of the system in Fig.1 can be computed as:

$$\omega_0 = \sqrt{\frac{K}{m}} \tag{1}$$

Taking into account the proposed values of K and m, the following frequency is obtained:

$$\omega_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{25300}{0.1}} = 502.99\ [rad/s] \tag{2}$$

$$f_0 = \frac{\omega_0}{2\pi} = 80.05\ [Hz] \tag{3}$$

b) Decay time

The motion of the system decays by 5 dB in a time $t_{-5} = 0.576$. Considering the ratio between the displacement at time t_{-5} : $x(t_{-5})$ and at time $t = 0$: $x(0)$, the decibel value can be computed:

$$10\ log_{10}\left(\frac{x(t_{-5})}{x(0)}\right) = -5\ [dB] \tag{4}$$

where the displacement is considered in general as:

$$x(t) = Ce^{-\alpha t}cos(\omega_d t + \phi) \tag{5}$$

In particular only the exponential term is meaningful, so the ratio simplifies to $e^{-\alpha t_{-5}}$ Now, using the logarithmic properties, the decay time τ is computed as:

$$\tau = \frac{-t_{-5}}{ln(10^{-1/2})} = 0.503\ [s] \tag{6}$$

The value of the damped frequency is computed as:

$$\omega_d = \sqrt{{\omega_0}^2 - \alpha^2} = 502.98\ [rad/s] \tag{7}$$

in which:

$$\alpha = \frac{1}{\tau} = 1.9988\ [1/s] \tag{8}$$

c) Quality factor

The quality factor is determined as:

$$Q = \frac{\omega_0}{2\alpha} = 125.82 \tag{9}$$

Technically speaking, there should be the damping frequency at the numerator but, as we have already seen, we are in the case of low damping and so the two frequencies almost coincide. We can think of Q as the ratio between the central frequency and the bandwidth of the receptance $H(\omega)$, i.e. the ratio between displacement and applied force. We can easily derive this function starting from the characteristic quantities of the resonator (m, R, K).

$$H(\omega) = \frac{\frac{1}{K}}{1 - \frac{m}{K}\omega^2 + j\omega\frac{R}{K}} \tag{10}$$

$H(\omega)$ will also be useful in the computation of the forced response.

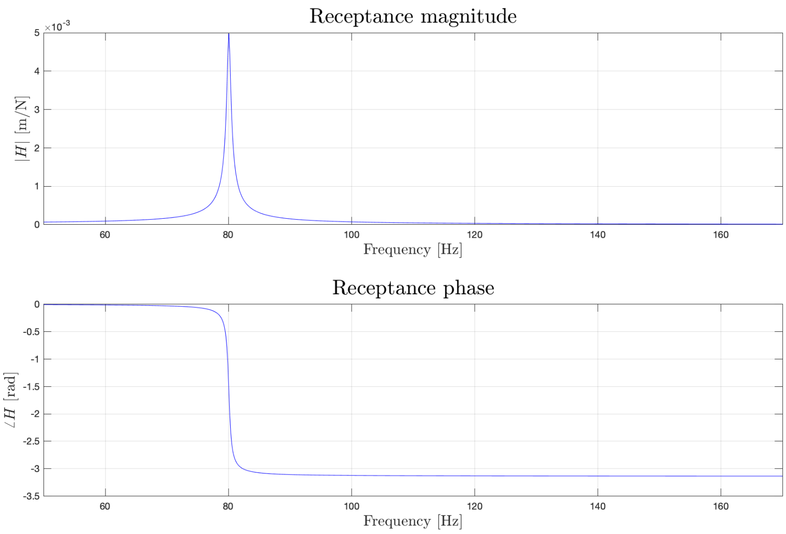

Figure 2: Magnitude and phase of the receptance H

d) Resistance

We now compute the mechanical resistance, which can be seen as the frictional coefficient of the system.

$$R = \frac{2m}{\tau} = 0.3998 \ [Ns/m] \tag{11}$$

Notice that R is also the real part of the impedance Z and it is constant in frequency.

e) -3dB bandwidth

The -3dB bandwidth gives an insight of the efficiency of the system, indeed it is strictly related to the quality factor Q. Depending on the measurement unit of our frequency axis we can compute it in the following two ways:

$$B = 2\alpha = 3.9975\ [rad/s] = 0.63622\ [Hz] \tag{12}$$

We can visualize the bandwidth on the plot of the receptance in Fig.2. The -3dB band is characterized by a ratio between the amplitude of the maximum and the edges equal to $1/\sqrt{2}$.

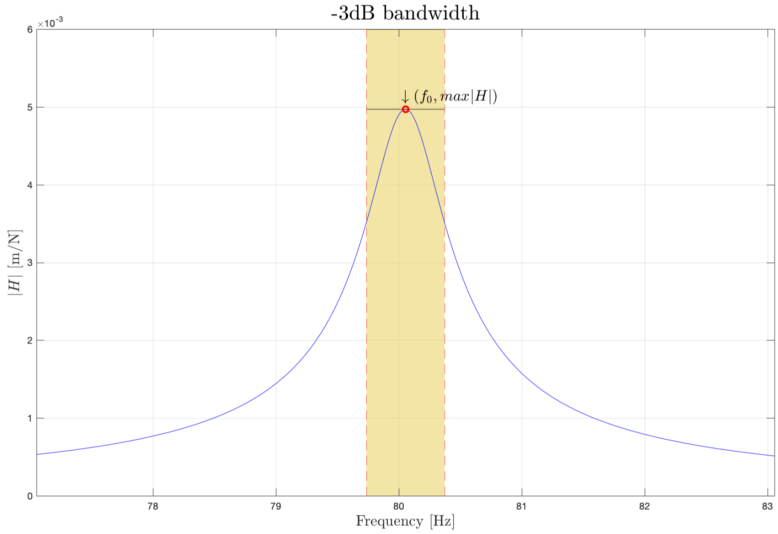

Figure 3: Bandwidth on receptance plot, centered in f_0.

f) Admittance

In order to compute the admittance Y of the system we first need to calculate the impedance Z, which is defined as the ratio between the applied force and the relative output velocity, but can also be derived as follows:

$$Z(\omega) = R + j(\omega m - K/\omega) = R + jX(\omega) \tag{13}$$

where R is the resistance and X is the reactance. Differently from the former, that has been already analyzed in point d), the latter depends on frequency and it will have a null for $\omega = \omega_0$.

We can observe from the plots that the impedance has a minimum in magnitude (not a zero since the system is damped) at the resonance frequency, whereas the phase is null.

We can now compute the admittance, which we remind is just the inverse of the impedance:

$$Y(\omega) = 1/Z(\omega) = G(\omega) + jB(\omega) \tag{14}$$

where G is the conductance and B is the susceptance. In this case, both G and B depend on frequency and they will present a maximum and a null respectively in correspondence of ω_0. Moreover, we can observe from the phase plots how Y is related to Z (inverted trend)

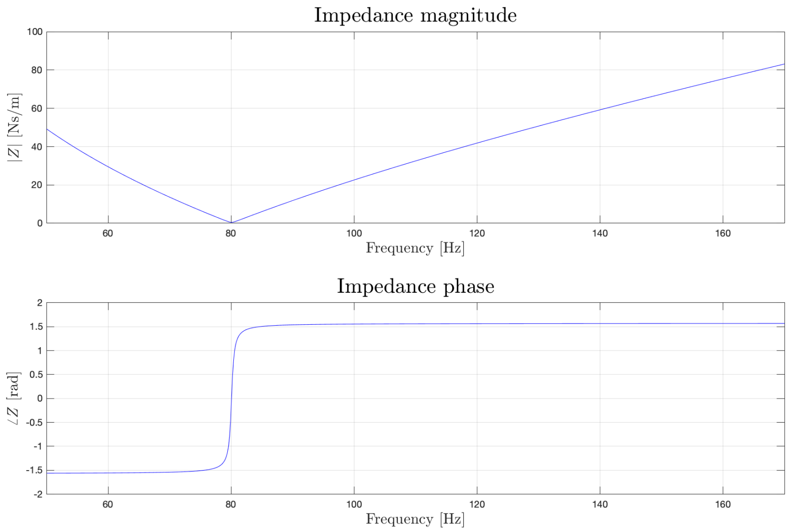

Figure 4: Magnitude and phase of the impedance Z.

and to the receptance (the derivation introduces a 90 degrees phase shift).

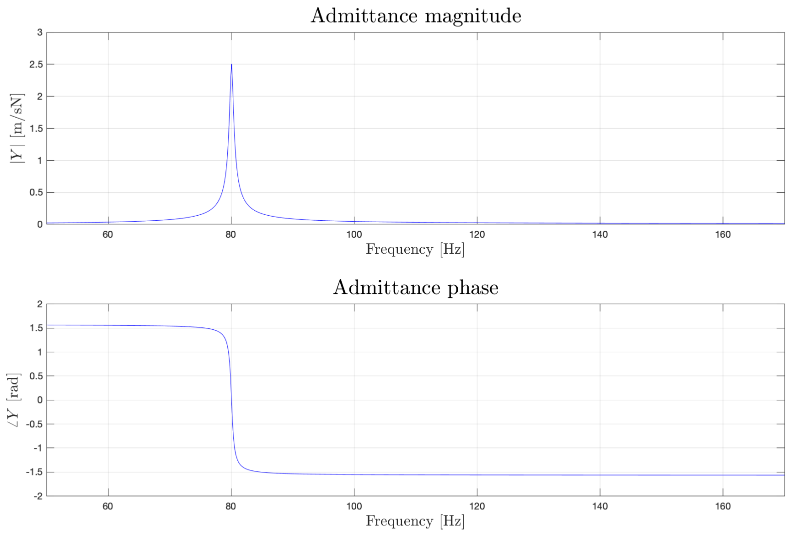

Figure 5: Magnitude and phase of the Admittance Y.

g) Time responses

We now want to compute the time responses of the system when it is subject to the external force $F(t) = 0.1sin(2\pi f_i t)$ $[N]$, with $f_i = [60, 80, 100, 120, 140, 160]$ $[Hz]$. The general expression for the time response of a forced linear system can be written in the following way:

$$x(t) = e^{-\alpha t}[Acos(\omega_d t) + Bsin(\omega_d t)] + |H(\omega)|F_0 sin(\omega t + \beta + \angle H(\omega)) \tag{15}$$

in which we take into account the response of the system to the force, but also the transient generated from the discontinuity between the resting position and the steady-state.

Given $F_0 = 0.1, \beta = 0$ and ω that varies depending on the considered force, we can already compute the second term of (15), i.e. the response to the applied force. In order to get the full expression of the transient, we need to compute the values of A and B, which can be found by applying the initial conditions. The choice of their values is trivial: since the system was at rest before the application of $F(t)$, we must impose that both displacement and velocity are null for $t = 0$:

$$\begin{cases} x_0 = x(0) = 0 \\ v_0 = \dot{x}(0) = 0 \end{cases} \tag{16}$$

At this point we compute $x(0)$ and $\dot{x}(0)$ starting from the definition given in (15) to then isolate the coefficients A and B:

$$\begin{cases} A = x_0 - |H(\omega)|F_0 cos(\beta + \angle H(\omega)) \\ B = \frac{v_0 + \alpha x_0 + |H(\omega)|F_0[\alpha sin(\beta + \angle H(\omega)) - \omega cos(\beta + \angle H(\omega))]}{\omega_d} \end{cases} \tag{17}$$

Notice that not only x_0 and v_0 are null but also β, which represents the phase of the external force. Nonetheless, we would like to remind that the displacement of the system can be obtained also through the equivalent formula (18). In that case for the transient there will be a single cosinusoid whose amplitude and phase are determined by the initial conditions, whereas for the forced response the impedance will be involved:

$$x(t) = Ce^{-\alpha t} cos(\omega_d t + \phi) + \frac{F_0}{\omega |Z(\omega)|} sin(\omega t + \beta + \angle Z(\omega)) \tag{18}$$

At this point we have everything needed to compute the total time responses of our system, for different frequencies of the external force $F(t)$. As we can appreciate from the plots of Fig.6, the most interesting time response is that to the 80 Hz force: not only in terms of magnitude (it is the largest one out of the six analyzed), but also regarding the trend and the product of the interference between transient and forced response. This happens because the frequency of the force almost matches the natural one of the system, and so the superposition of the two terms results in a single sinusoid modulated in amplitude by the fading exponential of the transient. As concerns the remaining plots, being the excitation frequencies far away from the resonance, they all will have a smaller amplitude, in particular the further we get from f_0, the weakest will be the output (predictable behaviour given the receptance of Fig.2). The most evident common point instead is the validation of the initial conditions: all the time responses have null displacement and null velocity in $t = 0$. Finally, a peculiar phenomenon can be observed in particular in the 100 Hz plot, that is the presence of slight beatings given by the interaction of sinusoidal components close in frequency. Indeed we can easily count the 20 periods in the first second of the response, accordingly to the expected frequency of the beat $f_b = f_3 - f_0 = 20Hz$.

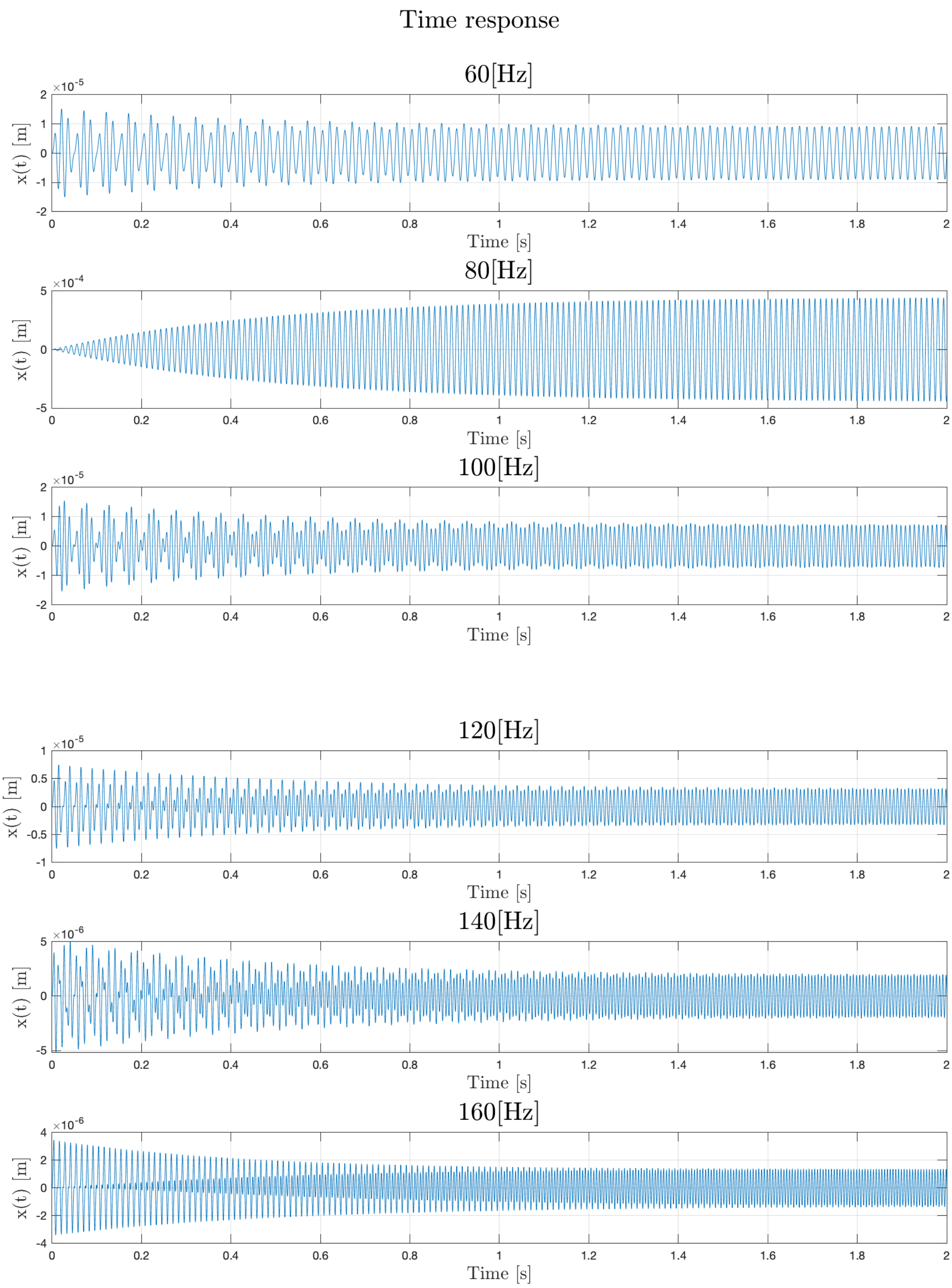

Figure 6: Time responses for each f_i