Assignment

Homework HW4

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Introduction

This report discusses the design of a recorder treble flute in its main components: the main bore and the finger holes. The input impedance is then discussed in the last part.

1) Bore dimensioning

The design of the flute's resonator is made with the help of the following parameters: the shape of the resonator has to be a truncated cone with a conical semiangle $\alpha = 0.75°$; the length is L = 0.45 m; the instrument is modelled with two finger holes only. Firstly, the head and foot diameters are computed in order to define the cone geometry of the resonator. The flute is requested to produce an E4 (329.63 Hz) when all the finger holes are closed. The recorder's acoustic behaviour is associated to the one of an Helmholtz resonator, therefore we considered the bore as an acoustic volume in order to find the value of r_{in} and r_{out} (Fig.1) that guarantee a minimum on the input impedance, which is the working condition of the recorder. Knowing the length of the resonator L and conical semi-angle α, we can express the radius at the mouth r_{in} in terms of the radius at the foot r_{out}:

$$r_{in} = r_{out} + L \cdot \tan(\alpha) \tag{1}$$

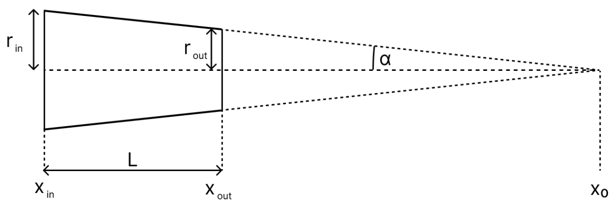

Figure 1: Cone geometry

In order to find the correct value of r_{out}, assuming the radiation load impedance $Z_{Load} = 0$ we compute the input impedance as:

$$Z_{in} = \frac{j\rho c}{S_{in}} \cdot \frac{\sin(k_0 L_{cone}) \sin(k_0 \theta_{in})}{\sin(k_0 (L_{cone} + \theta_{in}))} + j\omega_0 M \tag{2}$$

where:

- $\rho = 1.225 kg/m^3$ is the air density and $c = 343 m/s$ air propagation speed.
- $L = 0.45m$ is the bore length; $L_{cone} = L + 0.85 \cdot r_{out}$ is the length of the bore considering the end correction (the baffled end correction is used since in a recorder the resonator foot is baffled); $S_{in} = \pi r_{in}^2$ is the input surface.
- $\omega_0 = 2\pi f_0 = 2071 rad/s$; $k_0 = \omega_0 / c = 6.03831/m$ is the wavenumber.
- $j\omega_0 M$ is the impedance which models the air volume at the mouth of the resonator, with $M = \frac{\Delta L \rho}{S_{in}}$ inertance of the mouth and $\Delta L = 0.04m$ is the typical value assumed for an alto recorder [1]

The angle θ_{in} is related to the distance x_{in} from the open end to the truncated apex of the cone and can be computed as $\theta_{in} = \tan^{-1}(k_0 x_{in})$ where $x_{in} = r_{in}/\tan(\alpha)$ [1].
Finally we can retrieve the value of the smaller radius as the minimum of the input impedance, which we plotted as a function of r_{out} in Fig.2: the minima is located at $r_{out} = 3.7cm$ and consequently is calculated to be $r_{in} = 4.3cm$. The two diameters results to be:

$$d_{out} = 7.4[cm] \qquad d_{in} = 8.6[cm]$$

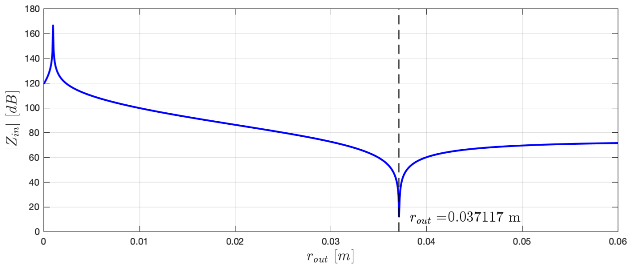

Figure 2: Impedance of the resonator as function of the radius r_{out}

2) First hole position

We aim now to insert a finger hole along the cone in order to produce a F#4 (349.23 Hz) when open. Since a new acoustic component is inserted in the system, the acoustic behaviour of the resonator changes and therefore some approximations are necessary:

- $S_{hole} = S_{out}$: the hole surface is the same as the foot surface of the recorder;
- $l = \Delta$: the acoustic length of the hole is equal to the virtual elongation of the foot resonator.

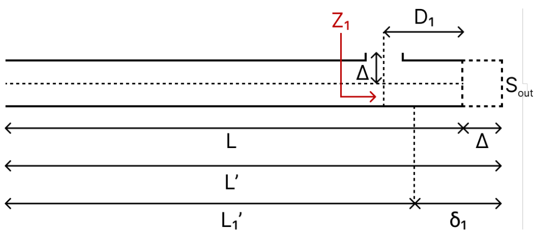

Figure 3: First hole parameters on the recorder

First of all we want to study the behaviour of the last part of the resonator: since the variation of the radius is negligible we can approximate the cone with a straight pipe. As visible in Fig.3, we can study the impedance Z_1 at the hole position (at distance D_1 from the foot of the resonator) when it is closed:

$$Z_1^{(close)} = j\frac{\rho c}{S_1}\tan\left(k_1\left(D_1 + \Delta\right)\right) \tag{3}$$

where $k_1 = \omega_1/c,\ \omega_1 = 2\pi f_1,\ f_1 = 349.23Hz$. If the hole is open, the impedance Z_1 changes and can be computed as the combination of the impedance of the hole Z_{H1} and that of the final portion of pipe Z_{P1}. Since in this point the two parts share the same pressure and the volume flow is split, the impedance Z_1 is computed as the parallel configuration of the two mentioned impedances: $Z_1^{(open)} = Z_{H1}//Z_{P1}$ Now, in order to simplify the computations, it is convenient to switch to the admittance notation. The admittances are computed as:

$$Y_{H1} = -j\frac{S_{hole}}{\rho c}\cot(k_1\Delta) \qquad\qquad Y_{P1} = -j\frac{S_{hole}}{\rho c}\cot(k_1(D1+\Delta)) \tag{4}$$

By summing these two we can obtain the admittance of the parallel

$$Y_1^{(open)} = -j\frac{S_{hole}}{\rho c}\left[\cot(k_1\Delta) + \cot(k_1(D_1+\Delta))\right] \tag{5}$$

Since $k(D_1+\Delta) << \pi/2$ we can approximate the cotangent function as the inverse of its argument, obtaining the admittance, that can be inversed to get the impedance:

$$Y_1^{(open)} \approx -j\frac{S_{hole}}{\rho c k_1}\left[\frac{D_1+2\Delta}{D_1+\Delta}\right] \qquad\qquad Z_1^{(open)} \approx j\frac{\rho c k_1}{S_{hole}}\left[\frac{D_1+\Delta}{D_1+2\Delta}\right] = j\frac{\rho c}{S_1}\Delta' \tag{6}$$

Since we know that in a pipe the impedance can be approximately computed as in equation (6), where Δ' is the acoustic length, we can study how much the acoustic length of the resonator reduces after the opening of the fingerhole. Now, recalling equations (3) and (6), we study the acoustic virtual reduction δ_1 defined as:

$$\delta_1 = L' - \Delta' = D_1 + \frac{\Delta^2}{D_1+2\Delta} \tag{7}$$

To represent the new acoustic length when the hole is open, we introduce a new notation that refers to Fig.3: $L_1' = L' - \delta_1$. From this new acoustic length we can now compute the impedance of the cone of length L_1' with mouth surface S_2 as:

$$Z_{in,1}^{(open)} = j\frac{\rho c}{S_{in}}\frac{\sin(k_1L_1')\sin(k_1\theta_1')}{\sin(k_1(L_1'+\theta_1'))} + j\omega_1 M \tag{8}$$

Where $\theta_1' = \tan^{-1}(k_1x')/k_1$ for which x' is the distance of the end of the modified acoustic length from the apex of the cone, and it can be computed as $x' = x_1 + \delta_1 + \Delta$.

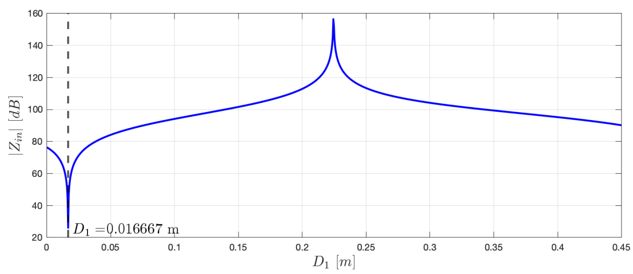

Figure 4: Impedance of the resonator as function of D_1

Since all of these formulas seen above can be written as function of D_1, we iterated the computation of the impedance $Z_{in}^{(open)}$ in order to find the value of D_1 for which the resonator presents a minimum, and therefore a resonance at the desired frequency. The impedance behaviour as function of D_1 can be seen in Fig.4. We found the optimum value for the distance D_1 of the hole from the foot to be $D_1 = 1.67[cm]$, from which we can compute the distance of the first hole from mouth of the resonator: $x_{H1} = L - D_1 = 43.3[cm]$

3) Second hole position

In this section in order to find the position of a second hole, we will re-iterate the process seen in section 2) but now to produce a G#4(392 Hz) when both are opened.

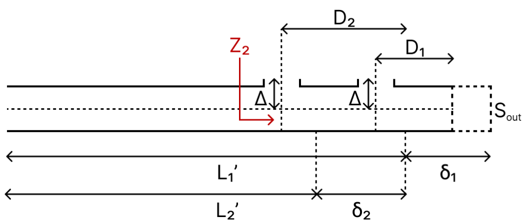

Figure 5: Second hole parameters on the recorder

As we seen in the previous section, when H_1 is open the acoustic length of the resonator is reduced to L_1' so now we can consider to have an acoustic pipe of that length and calculate the impedance formula of Z_2 at the hole position when H_2 is closed:

$$Z_2^{(closed)} = j\frac{\rho c}{S_{hole}} \tan(k_2 D_2) \tag{9}$$

where $k_2 = \omega_2/c$, $\omega_2 = 2\pi f_2$, $f_2 = 392.3Hz$ and D_2 is the distance of H_2 from the foot of the acoustic length L_1' previously defined. Now, as already done for the first hole, we study the impedance starting from the admittance of the same pipe section when H_2 gets opened:

$$Y_{H2} = -j\frac{S_{hole}}{\rho c} \cot(k_2 \Delta) \qquad Y_{P2} = -j\frac{S_{hole}}{\rho c} \cot(k_2 D_2) \tag{10}$$

The total admittance and therefore the impedance are:

$$Y_2^{(open)} \approx -j\frac{S_{hole}}{\rho c k_2}\left[\frac{\Delta + D_2}{\Delta D_2}\right] \qquad Z_2^{(open)} \approx j\frac{\rho c k_2}{S_{hole}}\left[\frac{\Delta D_2}{\Delta + D_2}\right] = j\frac{\rho c k_2}{S_1}\Delta'' \tag{11}$$

At this point, looking at the acoustic lengths in equations (9) and (11) we can study the acoustic length reduction that occurs in the resonator when H_2 is open:

$$\delta_2 = D_2 - \Delta'' = \frac{D_2^2}{\Delta + D_2} \tag{12}$$

Starting from the acoustic length reduction, we can retrieve the new acoustic length of the equivalent cone when both holes are open: $L_2' = L_1' - \delta_2$. Then we can put it in the equation of the input impedance of the resonator:

$$Z_{in,2}^{(open)} = j\frac{\rho c}{S_{in}} \frac{\sin\left(k_2 L_2'\right)\sin\left(k_2\theta_2'\right)}{\sin\left(k_2\left(L_2' + \theta_2'\right)\right)} + j\omega_2 M \tag{13}$$

where $\theta_2' = \tan^{-1}\left(k_2 x''\right)/k_2$, and $x'' = x' + \delta_2$ is the distance of the foot of L_2' from the apex of the cone. As done before, since all these equations can be written as function of D_2, we iterated the computation of the impedance over the variation of D_2 in order to find the value which allows the existence of a minima in the impedance (Fig.6). The optimum value for the distance D_2 is $D_2 = 7.56[cm]$, therefore the coordinate of the second hole with respect to the resonator length will be $x_{H2} = L_1' - D_2 = 37.4[cm]$.

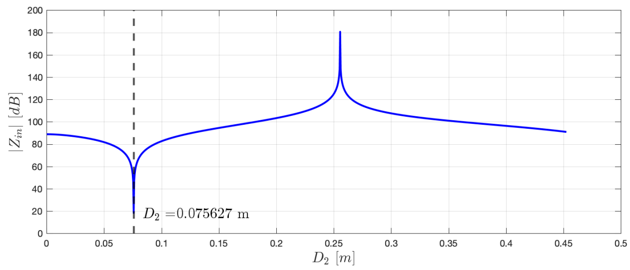

Figure 6: Impedance of the resonator as function of the distance D_1 of the hole from the foot

We can now make some comments about the final configuration of the recorder flute obtained by imposing the frequencies f_1 and f_2, that is schematically depicted in Fig.7. It's clearly visible that the two holes are superimposed but they are also placed too far towards the end recorder's foot. This non-realizable geometry derives from the simplification made at an earlier stage of the design, i.e. the adoption of holes that have the same surface of the resonator foot.

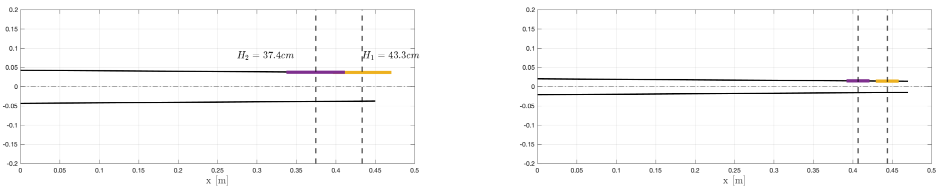

Figure 7: Resulting recorders with L=0.45 m (left) and L=0.47 m (right)

This approximation implies the use of finger holes with independent diameters w.r.t. the geometry of the resonator, so that they can fit properly along the resonator without overlapping or exceeding its length. As visible in Fig.7 the geometry can be alternatively fixed simply by acting on its total length: an increase of 2 cm of the resonator so that $L = 47[cm]$ implies a reduction of the outer diameter and consequently of the holes that can now fit on the resonator.

4) Input impedance

Now that the geometry is complete we can analyze the total input impedance, modelled with a transmission line, in three different configurations: all finger holes closed, only the last finger hole open, both finger holes open. In order to do so we divided the main bore into three conical portions, that are interrupted by the two holes. The equivalent electrical scheme is then composed of:

a) the conical portions, analytically described by the input impedance of a conical horn [1]:

$$Z_{in} = \frac{\rho c}{S_1}\left[\frac{jZ_L\frac{\sin(kL-\theta_2)}{\sin\theta_2} + \left(\frac{\rho c}{S_2}\right)\sin(kL)}{Z_L\frac{\sin(kL+\theta_1-\theta_2)}{\sin\theta_1\sin\theta_2} - \left(\frac{j\rho c}{S_2}\right)\frac{\sin(kL+\theta_1)}{\sin\theta_1}}\right] \tag{14}$$

where L is the length of each considered part, $\theta = tan^{-}1(kx)$ and the indexes 1 and 2 refer for each conical portion to the left and right part as in Fig.8.

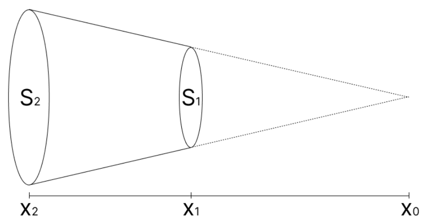

Figure 8: Reference scheme for the conical portions

b) the holes' impedance that are instead modelled as in Fig.9 by the combination of Z_a and Z_s here described:

$$Z_a = \left(\frac{\rho c}{\pi a^2}\right)\left(\frac{a}{b}\right)^2 \times jkt_a \ (closed\ or\ open) \tag{15}$$

$$Z_s = \left(\frac{\rho c}{\pi a^2}\right)\left(\frac{a}{b}\right)^2 \times \begin{cases} -jcot(kt) \ (closed) \\ jkt_e \ (open) \end{cases} \tag{16}$$

where t_e, t_a, t have been calculated as described in [2], a is the radius of the resonator calculated in correspondence to the middle of the hole, b is the holes' radius.

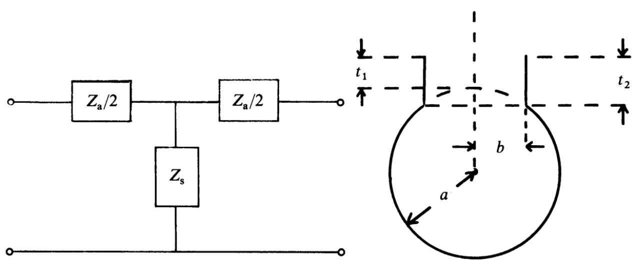

Figure 9: Finger holes electric equivalent and geometry

In order to evaluate the input impedance at the mouth, the first segment considered is the last conical section. Recalling that the load impedance considered to be $Z_L = 0$, equation

(14) can be simplified as:

$$Z_{in} = \frac{j\rho c}{S_1} \cdot \frac{\sin(kL)\sin(k\theta_1)}{\sin(k(L+\theta_1))} \tag{17}$$

Once obtained, the evaluated input impedance is imposed as load impedance of the first hole, so it faces the configuration in Fig.9 since it can be modelled as a single impedance attached to the right.

Repeating this process for the central conical part, the second hole and the last cone that terminates at the mouth of the resonator ($+j\omega M$), we can evaluate the approximated Z_{in} as a function of the frequency ω. Since we want to evaluate one single case at a time, we can reiterate the same process varying the values of Z_s for the closed and open configurations of the holes.

frequency	L=0.45 m	L=0.47 m
f_0 = 329.63 Hz	369.9 Hz	331.5 Hz
f_1 = 349.23 Hz	373.6 Hz	347.7 Hz
f_2 = 392 Hz	419.8 Hz	395.6 Hz

Table 1: Comparison of target resonance frequencies and computed ones

Despite the effort in trying to model the instrument correctly, the results don't match the expected frequency values of 329.63 Hz (both finger holes are closed), 349.23 Hz (first finger hole is open, second is closed) and 392 Hz (both finger holes are open) as we can see from Tab.1. However, an interesting solution comes from the already mentioned adjustment of point 3), where we proposed an elongation of the tube of 2 cm. In this case as we can see in Fig.10 and Tab.1 the results match almost perfectly with the expected ones, for all the configurations analyzed. Moreover the harmonic behaviour is the one expected since the minima of the impedance correspond to the fundamental note and its multiples. We would like to highlight the fact that the overall trend of the impedance is shared between the two cases of L, which is the reason why we just plot the more accurate one.

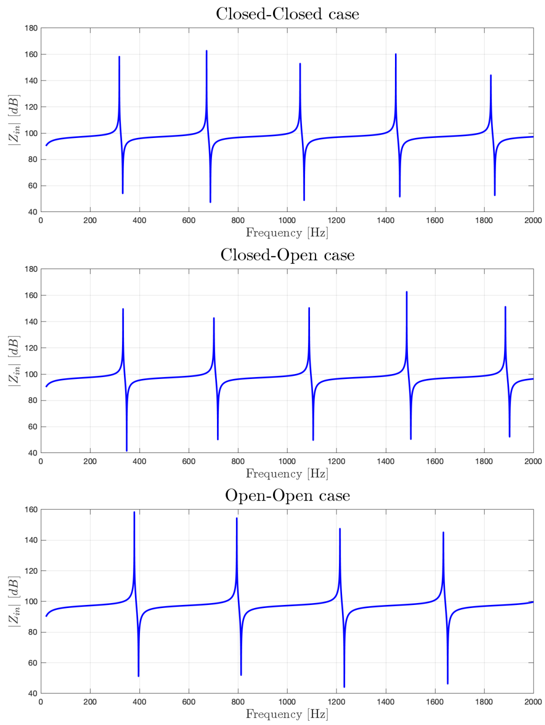

Figure 10: Input impedances in the three configurations

References

[1] N. H. Fletcher and T. D. Rossing, *The Physics of Musical Instruments*, 2nd Edition, Springer-Verlag, New York, 1998.

[2] Douglas H. Keefe, Theory of the single woodwind tone hole, Journal of the Acoustical Society of America, Sep 1982.