Assignment

Homework HW2

Alberto Doimo 10865196

Riccardo Iaccarino 10868500

Musical Acoustics

POLITECNICO
MILANO 1863

October 20, 2023

Introduction

A square thin plate (thickness $h = 1\ mm$) with clamped edges and side $a = 0.15\ m$ is made with aluminum ($E = 69\ GPa$, $\rho = 2700\ Kg/m^3$, $\nu = 0.334$).

a) Propagation speed of quasi-longitudinal and longitudinal waves

When the material slightly expands and contracts as the wave propagates in the plate, the wave is called quasi-longitudinal and propagation speed is greater than longitudinal waves in bars since $\frac{1}{1-\nu^2} > 1$.

We computed the *quasi-longitudinal* wave velocity by using the following expression:

$$c_L = \sqrt{\frac{E}{\rho(1-\nu^2)}} = 5363.24\ [m/s] \tag{1}$$

While for the *longitudinal* wave velocity we use the following expression:

$$c_L' = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} = 6199.16\ [m/s] \tag{2}$$

b) Propagation speed of bending waves

The equation of motion for bending waves in plates can be written as:

$$\frac{\partial^2 z}{\partial t^2} + \frac{Eh^2}{12\rho(1-\nu^2)}\nabla^4 z = 0 \tag{3}$$

where z is the displacement and E, h, ρ, ν values can be found in the introduction. Assuming an harmonic solution and solving the equation of motion, we can obtain the wavenumber as a function of ω:

$$k^2 = \frac{\sqrt{12}\omega}{c_L h} \tag{4}$$

Now, recalling that $k = \frac{\omega}{v(f)}$, the velocity can be obtained by the following expression:

$$v(f) = \frac{\omega}{k} = \sqrt{1.8 f h c_L} \tag{5}$$

We can notice looking at Fig. 1 that the velocity grows as the square root of the frequency. Since the trend is clearly visible we decided to plot the function only up to 1.5 KHz so that all the six modes of the plate are included in this range.

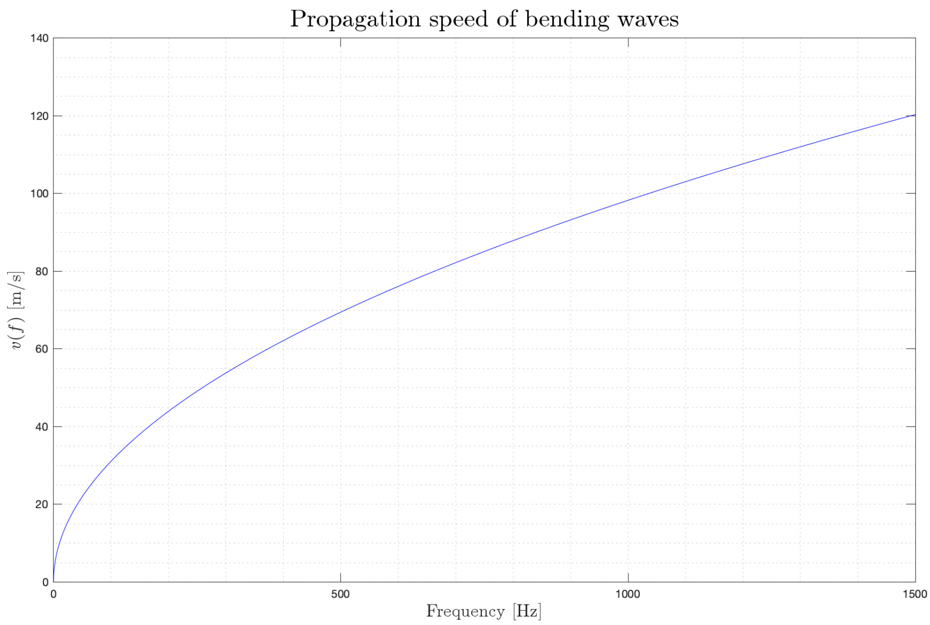

Figure 1: Propagation speed of bending waves

c) Modal frequencies of the plate

The fundamental mode (0,0) of a square plate with clamped edges is observed at the following frequency:

$$f_{00} = \frac{1.654 c_L h}{L^2} = 394.26\ [Hz] \tag{6}$$

Successive modal frequencies can be obtained by multiplying the fundamental frequency f_{00} by a constant as indicated in Tab.1. In a rectangular plate characterized by L_x and L_y dimensions, when $\frac{L_x}{L_y} \gg 1$, vibrational modes resembles those of a bar. As we approach $L_x \approx L_y$ the nodal lines start to influence each other to the point in which they become one single line. In a square plate $L_x = L_y$, this means that mode five is a composition of (2,0)-(0,2) modes that create the so called *X* mode. Similarly the sixth mode or ***Ring*** mode is originated from (2,0)+(0,2) modes.

Modes	Multiplicative constant	Frequency
(0,0)	1.00	394.26 Hz
(1,0)	2.04	804.28 Hz
(0,1)	2.04	804.28 Hz
(1,1)	3.01	1186.72 Hz
(2,0)-(0,2)	3.66	1442.98 Hz
(2,0)+(0,2)	3.67	1446.93 Hz

Table 1: Modal frequencies of the plate with corresponding multiplicative factors

Qualitative visual cues on the shape of the modes are shown in Fig.2, in which the nodal lines are drawn on the plate. The main differences with the free edges boundary conditions are the presence of the mode (0,0) and the circumference of the ***Ring*** mode closer to the center of the plate.

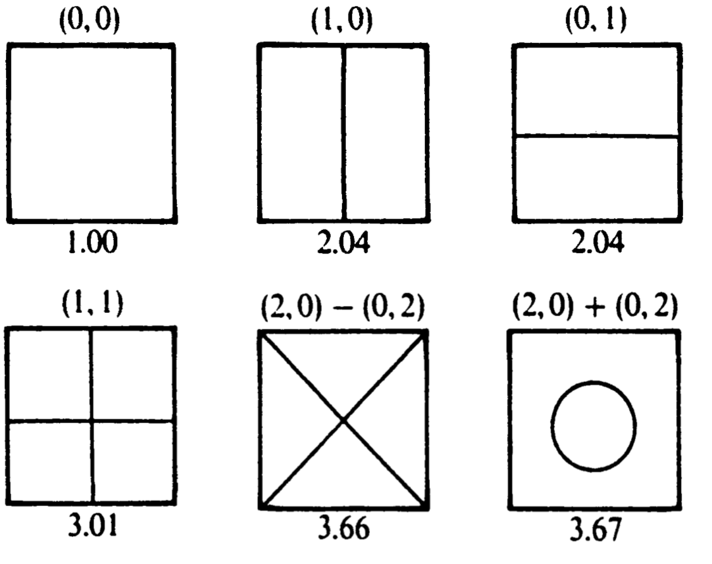

Figure 2: First six bending modes of a square plate with clamped edges

d) Sitka spruce scenario

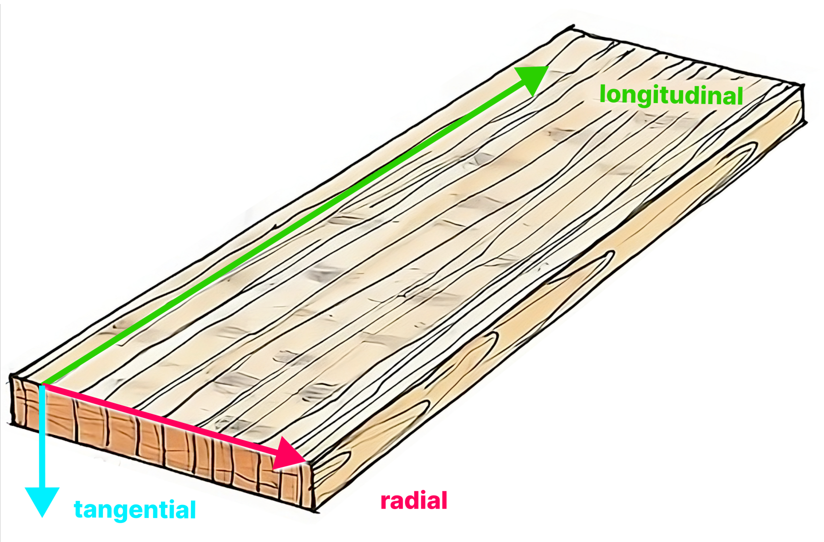

Figure 3: Quarter-cut scheme

Now we consider a plate of Sitka spruce wood, realized using the quarter-cut scheme. The plate is defined by the longitudinal and radial direction of the wood (Fig.3), respectively of length L_x and $a = L_y = 0.15\ m$. Those directions are also associated to different Young Moduli, that we assumed to be as $E_L = 12\ [GPa]$ and $E_R = 0.9\ [GPa]$ since those are indicated as typical values for Sitka spruce wood (source: The Physics of Musical Instruments). Differently from the case of isotropic material, wooden square plates would not exhibit the modes of Fig.2 (and so X and $Ring$ modes). In order to observe them we need to impose the ratio of the two main dimensions to be equal to a constant that depends on the material's properties:

$$\frac{L_x}{L_y} = \left(\frac{E_L}{E_R}\right)^{1/4} \tag{7}$$

from which we derive the longitudinal length:

$$L_x = \left(\frac{E_L}{E_R}\right)^{1/4} \cdot L_y = 0.28\ m \tag{8}$$

e) Tension of the string

We now consider that a string is attached to the plate, with its fundamental mode tuned to the frequency of the first mode of the plate (in Tab.2 are listed the frequencies up to the sixth mode of the string). The string has length $L = 0.45\ m$ and a circular cross-section of radius $r = 0.0011\ m$; moreover we also know its density $\rho = 5000\ kg/m^3$ which is of course associated with its material (iron). In order to compute the tension applied to the string to match the two modes, we first need to recall its propagation speed (assuming neglectable stiffness):

$$c_s = \sqrt{\frac{T}{\mu_s}} \tag{9}$$

where T is the tension and μ_S the linear density of the string. The latter can be easily computed starting from the volumetric density and the section of the string: $\mu_s = \rho_s \pi r^2$. Subsequently, we can impose the expression of the fundamental mode of standing waves in strings, which features both the propagation speed and the frequency: $2\pi f_0 = c_s \pi / L$. Putting everything together we can obtain the final expression for the tension:

$$T = 4{f_0}^2 L^2 \mu_s = 2393.05\ N \tag{10}$$

Modes	**String**	**Soundboard**
1	394.26 Hz	394.26 Hz
2	788.51 Hz	804.28 Hz
3	1182.77 Hz	804.28 Hz
4	1577.03 Hz	1186.72 Hz
5	1971.29 Hz	1442.98 Hz
6	2365.54 Hz	1446.93 Hz

Table 2: First six modes of string and plate

f) Modes of the coupled system

Connecting the string to the plate creates a new composed system whose behaviour generally differs from that of the individual ones. In particular we can observe two forms of coupling depending on the considered mode and on the characteristics of the involved structures. We can differentiate between weak and strong coupling as follows:

- weak coupling arises for $m/n^2M < \pi^2/(4Q^2)$
- strong coupling arises for $m/n^2M > \pi^2/(4Q^2)$

in which m and M are the masses of string and plate and n is the mode number of the string, whereas the merit factor of the plate is given and it is equal to $Q = 25$. We can immediately compute the two masses as:

$$m = \rho_s L\pi r^2 \qquad\qquad M = \rho a^2 h \tag{11}$$

We will refer instead to the second term of the inequality as the threshold of the two cases, which is an adimensional number of value 0.0039. Analyzing the first six modes of the string we can observe that up to the fifth one they all are effected by a condition of strong coupling, while the last one is related to weak coupling since the ratio of the masses overcomes the threshold. In both cases of coupling the damping factor will be altered, whereas the frequencies of the modes of the two structures will be altered just in the strong one.

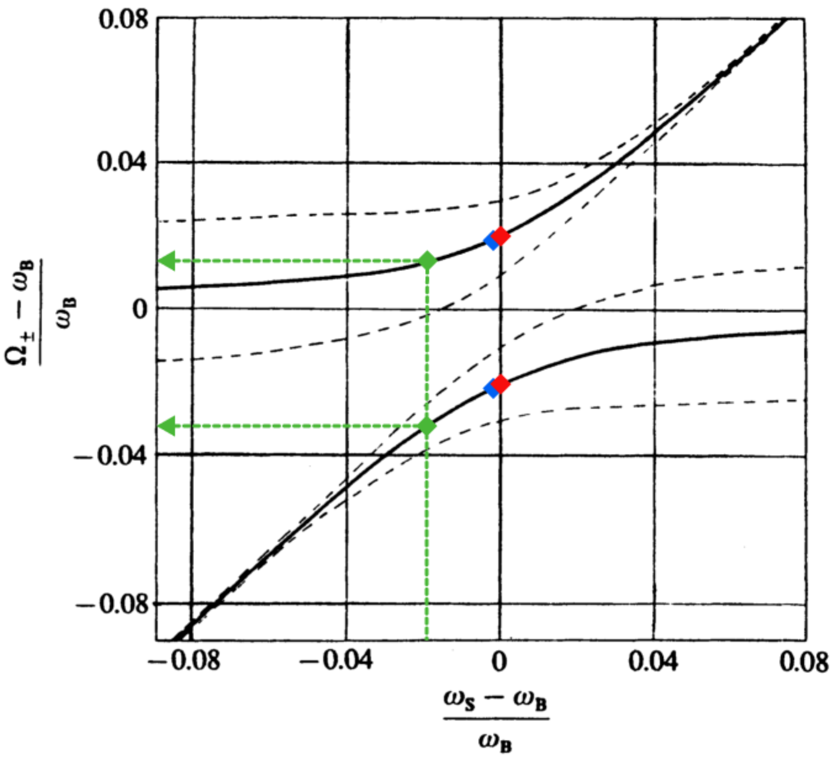

Figure 4: Normal mode frequencies of a string coupled to a plate as function of their uncoupled frequencies ω_B and ω_S for strong coupling

At this point we can compare the resonances of string and board to then exploit the curves in Fig.4 to get the new pairs of frequencies. To do so, we organized in Tab.3 the percentage of difference between the frequencies of the modes of the string (S_i) and those of the board (B_i).

Modes	S_1	S_2	S_3	S_4	S_5	S_6
B_1	0	100.0000	200.0000	300.0000	400.0000	500.0000
B_2	-50.9804	-1.9608	47.0588	96.0784	145.0980	194.1176
B_3	-50.9804	-1.9608	47.0588	96.0784	145.0980	194.1176
B_4	-66.7774	-33.5548	-0.3322	32.8904	66.1130	99.3355
B_5	-72.6776	-45.3552	-18.0328	9.2896	36.6120	63.9344
B_6	-72.7520	-45.5041	-18.2561	8.9918	36.2398	63.4877

Table 3: Percentage of difference between the frequencies of the modes of the string (S_i) and those of the board (B_i)

As highlighted from the colours, just a few combinations of modes return interesting scenarios: indeed, as we can also appreciate from Fig.4 when the resonances of the single systems are further away, the resulting ones are pretty much unaltered. The last step to compute the new set of frequencies is to derive their deviation from the plate's resonances directly from the graph. The trivial combination is the one that features the first mode of both string and plate (since we tuned one after the other) for which we will have a discrepancy of $\pm 2\%$ resulting in the new couple of resonances being $f^{-}_{1,1} = 386.37\ Hz$ and $f^{+}_{1,1} = 402.14\ Hz$. The second peculiar result is the combination between the modes 2 and 3 of the plate (at same frequency) and mode 2 of the string, for which we spotted a -3.2% and $+1.3\%$ deviances which translate in the couple $f^{-}_{2-3,2} = 778.55\ Hz$, $f^{+}_{2-3,2} = 814.74\ Hz$. Finally we have the combination between third mode of the string and fourth of the board, which appear to be very close in frequency: for this reason we chose to adopt the same correction coefficients of the first case, that gave us $f^{-}_{4,3} = 1162.98\ Hz$ and $f^{+}_{4,3} = 1210.45\ Hz$.