Vibration Analysis and Vibroacoustic

Assignment 1: Analysis of a 1-dof System

Homework 1 Report

Students

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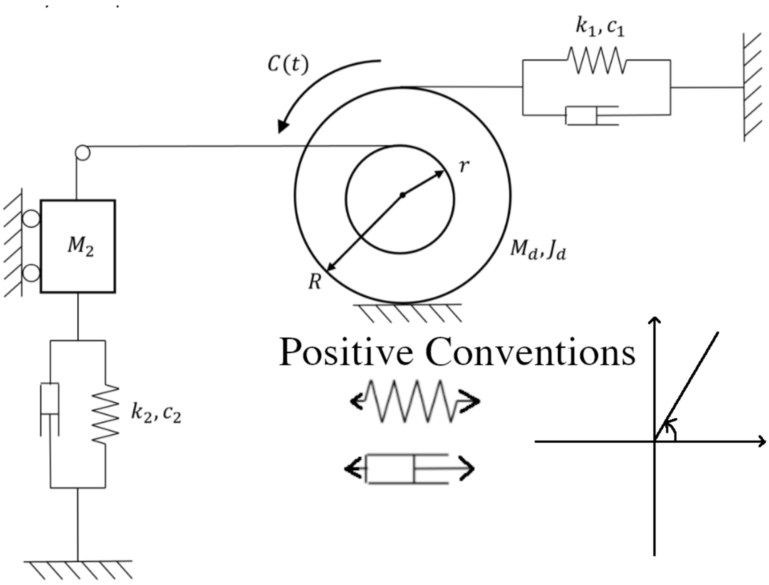

Figure 1: Physical System and Positive Conventions

1. Equation of Motion

1.1 Preliminary passages

The aim of this section is to define the equations that describe movements around the equilibrium position of the system described in figure 1. To achieve this goal we need to define some conventions which will be necessary in order to correctly define the equations of motion. As represented in the picture we define as positive the elongation of a spring or of a damper or an angle taken anti-clockwise from the equilibrium state.
In order to correctly analyse the system it is needed to first determine the total number of Degrees of Freedom (DOF). To achieve this result two steps are required:

- Determine the Degrees of Constraint (DOC) of the system
- Subtract the DOC from the total possible amount of DOF to get the effective number of DOF of the system

1.2.1 Determination of the Degrees of Constraint

To correctly characterize the system it is needed to determine how the constraints act in a mechanical system. This system in particular has three types of constraint: inextensible strings, contact points and trolleys.

- A string does not allow free motion to the object in the same direction of the string itself thus adding 1 DOC to the system
- A contact point (given that the object in question cannot slip on the surface) prevents the object from translating along both axis and only allows rotation around the point itself. Two DOC are introduced
- Trolleys only allow motion along the surface they roll against. This results in 2 DOC being added to the system

Constraint	DOC
String	1
Trolley	2
Contact Point	2

Table 1: DOC for each element in the system

These results are summarised in Table 1
Having gathered all the necessary information it is now possible to determine the total amount of DOC introduced in the system

$$n_{DOC} = 1 * 1_{string} + 1 * 2_{trolley} + 1 * 2_{\text{contact point}} = 5$$

1.2.2 Determination of the Degrees of Freedom

In a 2D system like the one taken into consideration in this report an object is free to move along the x and y axis and to rotate. This means that an object without any constraint introduces 3 DOF into the system. By taking this into consideration, along with the results above it is possible to determine the effective amount of DOF of this system

$$n_{DOF} = 3 * 2_{objects} - n_{DOC} = 1$$

1.3.1 Choice of Independent Variable

There are different possible choices in this regard but in this case the angle θ of counter-clockwise rotation of the disc from its equilibrium position was chosen.

1.3.2 Derivation of Parameters

All other parameters of the system need to be expressed as a function of the independent variable

$\Delta\ell_1$	$\Delta\ell_2$	x_2	$\delta\theta$	$\dot{\Delta\ell_1}$	$\dot{\Delta\ell_2}$	$\dot{x_2}$
$2\theta R$	$-(R+r)\theta$	$-(R+r)\theta$	θ	$2\dot{\theta}R$	$-(R+r)\dot{\theta}$	$-(R+r)\dot{\theta}$

Now that the system is fully characterized only by functions of the independent variable it is possible to proceed with the computation of the Equation of Motion

1.4.1 Equation of Motion

The derivation of the Equation of Motion (EOM) will be done through an energy approach using the Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\partial E_k}{\partial \dot{\theta}}\right) - \frac{\partial E_k}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} + \frac{\partial V}{\partial \theta} = Q_\theta$$

Computing the EOM requires first to determine the functions for the kinetic and potential energy, the energy dissipation function and the virtual work function.

1.4.2 Kinetic Energy

Starting from the kinetic energy it is needed to consider both translation and rotation of the bodies. In this system the disc will both translate and rotate at the same time (a motion which is called rototranslation) and the mass M_2 is only able to translate along the y axis. These considerations will be used to determine the Kinetic Energy Function.

$$E_k = \frac{1}{2}J_d\dot{\theta}^2 + \frac{1}{2}M_dR^2\dot{\theta}^2 + \frac{1}{2}M_2(R+r)^2\dot{\theta}^2 = \frac{1}{2}\left(J_d + M_dR^2 + M_2(R+r)^2\right)\dot{\theta}^2$$

1.4.3 Potential Energy

The potential energy is made of two main components: gravitational and elastic energy. Being that the system is assumed in its equilibrium position we assume the gravitational potential energy of the mass M_2 to be zero and since the disc is in contact on a plane we get that the potential energy function will be only comprised of the elastic part

$$V = \frac{1}{2}k_1\Delta\ell_1^2 + \frac{1}{2}k_2\Delta\ell_2^2 = \frac{1}{2}(4k_1R^2 + k_2(R+r^2))\theta^2$$

1.4.4 Dissipative Energy

In this system the dissipative function only has contributions from the dampers c_1 and c_2. This results in the following expression

$$D = \frac{1}{2}c_1\dot{\Delta\ell}_1^2 + \frac{1}{2}c_2\dot{\Delta\ell_2}^2 = \frac{1}{2}(4c_1R^2 + c_2(R+r)^2))\dot{\theta}^2$$

1.4.5 Virtual Work

By applying a torque $C(t)$ the work applied is evaluated in the following expression.

$$\delta w = c(t)\delta\theta$$

1.4.6 Derivation of the Equation of Motion

It is now necessary to individually deriive all the elements of the Lagrange equation:

$$\frac{1}{\partial t}\frac{\partial E_k}{\partial \dot{\theta}} = (J_d + M_dR^2 + M_2(r+R)^2)\ddot{\theta}$$

$$\frac{\partial E_k}{\partial \theta} = 0$$

$$\frac{\partial V}{\partial \theta} = (4k_1R^2 + k_2(R+r)^2)\theta$$

$$\frac{\partial D}{\partial \dot{\theta}} = (4c_1R^2 - c_2(R+r)^2\dot{\theta}$$

by substituting into the main equation the result is:

$$(J_d + M_dR^2 + M_2(R+r)^2)\ddot{\theta} + (4c_1R^2 + c_2(R+r)^2)\dot{\theta} + (4k_1R^2 + k_2(R+r)^2)\theta = C(t)$$

with the data of the system the resulting equation is:

$$3.05\ddot{\theta} + 2.48\dot{\theta} + 179.5 = C(t)$$

1.5.1 Natural Frequency of the System

Now that the EOM has been found it is possible to extrapolate a number of different kind of information about the system. The first is the natural frequency or resonant frequency of the system. To achieve this result it is necessary to resolve the EOM as a differential equation and in particular the free motion solution is of interest.

$$3.05\lambda^2 + 2.48\lambda + 179.5 = 0$$

which gives the result $f_0 = 1,24hz, \omega_0 = 7.67rad/s$
Also, by rewriting the EOM in the form:

$$\lambda^2 + 2\alpha\lambda + \omega_0^2$$

it is possible to achieve the following result

$$\alpha = \frac{C_{eq}}{2M_{eq}} = 0.4$$

whith α being the damping coefficient

1.5.2 Adimensional Damping Ratio and Damped Frequency

At this point it is desirable to compute the effect of the damping on the natural frequency of the system and then adjusting it accordingly to obtain a better representation overall. To achieve this the adimensional damping ratio of the system is computed first

$$h = \frac{c}{2M_{eq}\omega_0} = 0.053$$

Being h very small we do not expect the damped natural frequency to differ a lot from the original natural frequency found at point 1.5.1.

$$f_{damped} = f_0\sqrt{1 - h^2} = 1.238hz = 7.66s^-1$$

It is clearly seen, as expected that the two results only differ by $0.01hz$

2.0 Free Motion of the System

By considering the system as linear and time-invariant and being it a 1 DOF system with damping it was possible, as in chapter 1, to determine the EOM. The solution of the equation is the following:

$$x(t) = C_1 e^{\lambda 1t} + C_2 e^{\lambda_2 t}$$

where C_1 and C_2 are constants to be determined by applying the opportune boundary conditions for the system. When the boundary conditions are set as follows this is the result achieved.

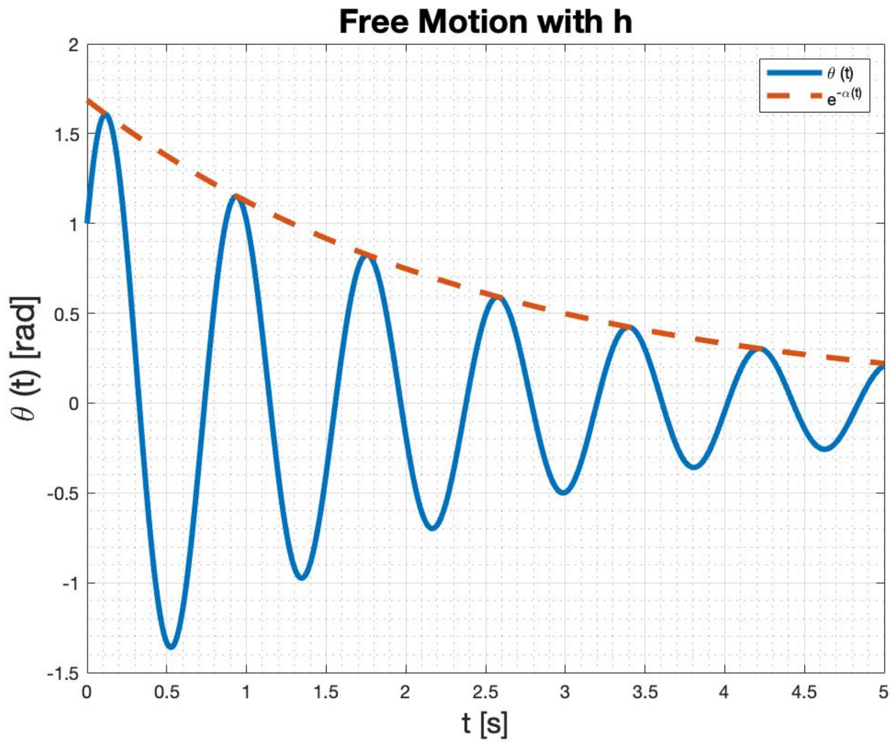

Figure 2: Plot of Free motion with arbitrary initial conditions

This plot perfectly represents the expectations for the system. Since the system in question is a lightly damped system the free motion of the system is supposed to be a sinusoid at the natural frequency of the system which will slowly fade out in time following the envelope of a decaying exponential function. This is because the losses of the system slowly dissipate the energy of the system since no restoring force is applied and it tends to a steady state.

2.1 Free Motion of the System (with $h_1 = 4h_0$)

By quadrupling the adimensional damping ratio, the response will decay faster than the before example at point 2.0. This is due to the fact that λ shrinks for higher values of h so the result is a faster-decaying exponential which multiplies the sinusoid. These considerations are perfectly represented in the following plot

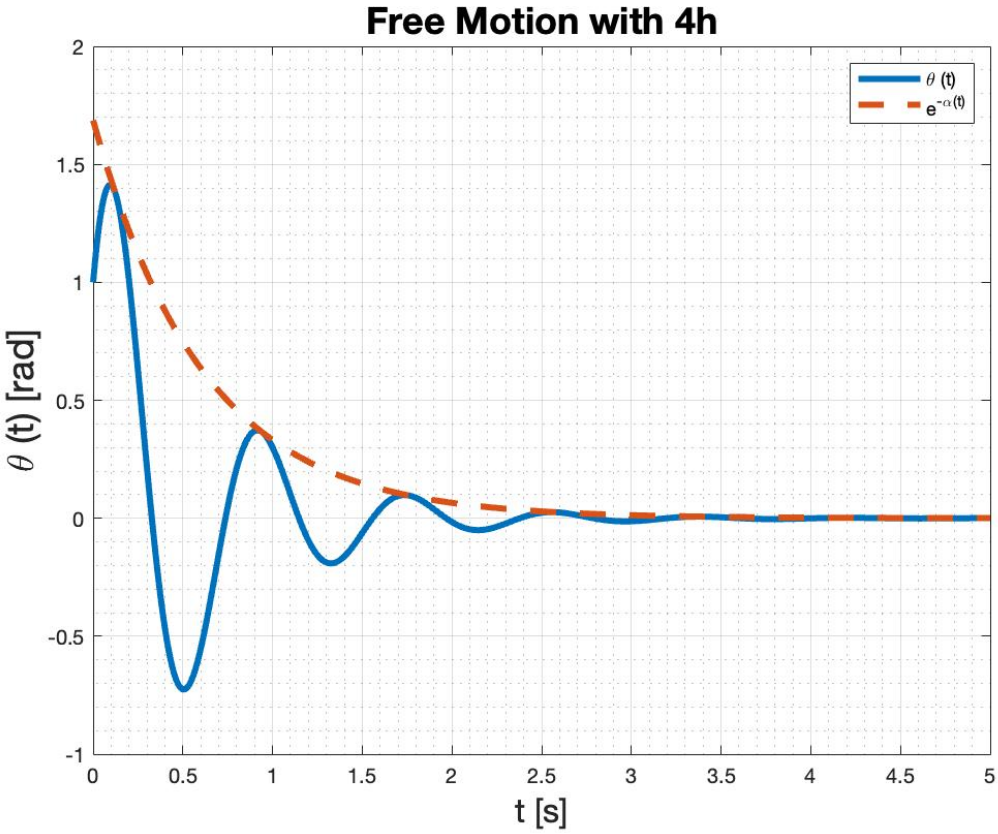

Figure 3: Plot of Free motion with four times the adimensional damping ratio

2.2 Free Motion of the System (with $h_1 = 25h_0$)

The considerations made for the previous point still apply here. The result is even clearer as the damping ratio is now way bigger.

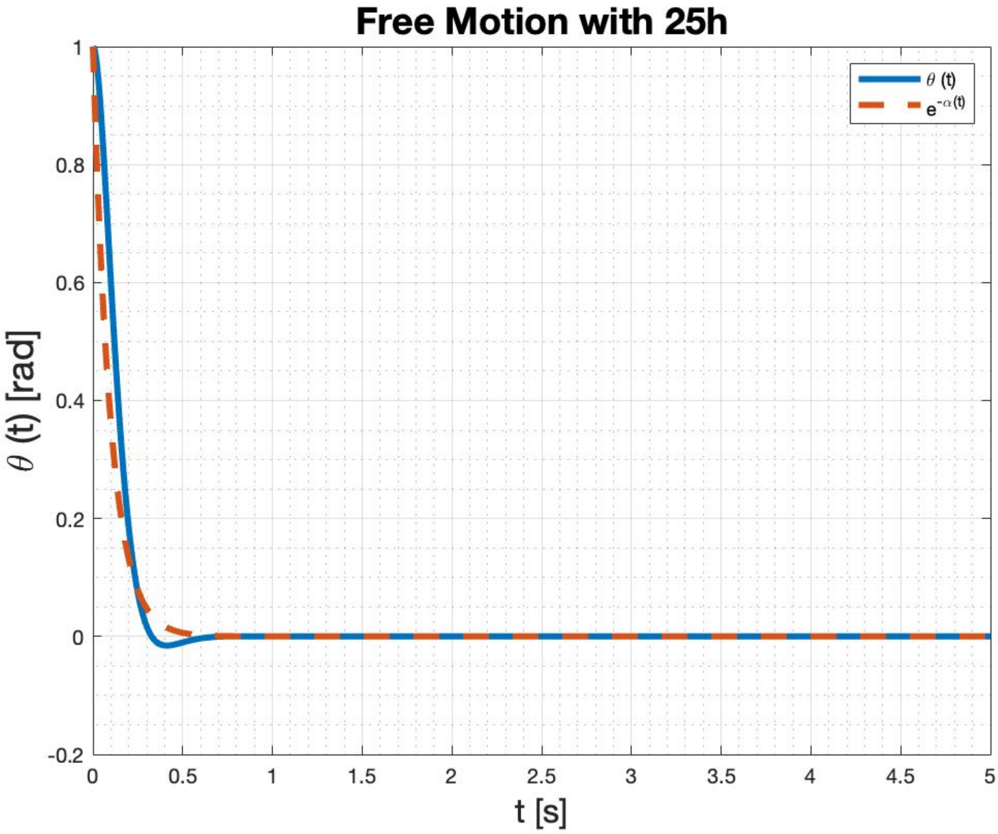

Figure 4: Plot of Free motion with twenty-five times the adimensional damping ratio

3.1 Forced Motion of the System: Frequency Response Function

The frequency response function (FRF) is a common way to define a linear time-invariant system since it describes how the system reacts to an harmonic input in terms of amplification (or attenuation) of certain frequencies and in terms of phase shift of those same frequencies. This is a useful way to describe a system and it can give different information in different scenarios. A FRF is computes as follows

$$H(\Omega) = \frac{x_0}{F_0}$$

where x_0 represents the movement of the system with respect to the applied harmonic force F_0. From this representation it is possible to extract the magnitude (amplitude response) of the system by taking the absolute value of the function and the phase response by taking the angle of function. In the specific case of this system the FRF is the following:

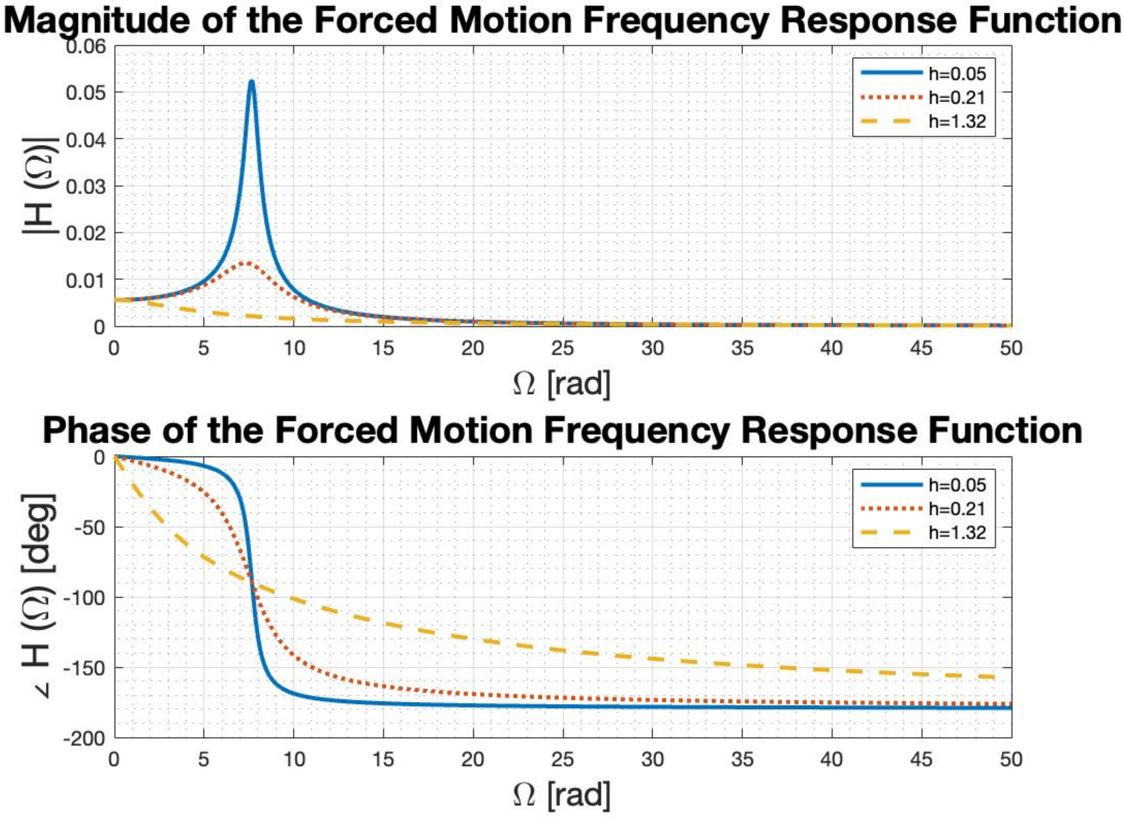

Figure 5: Plot of the Frequency Response Function with different values of h

3.2 Complete Time Response to a Harmonic Torque

It is now required to evaluate the response of the system to an harmonic torque

$$C(t) = Acos(2\pi f_1 + \varphi)$$

with $A = 2.5N$, $f_1 = 0.35hz$ and $\varphi = \frac{\pi}{3}$. The following plot shows the amplitude and phase response of the system

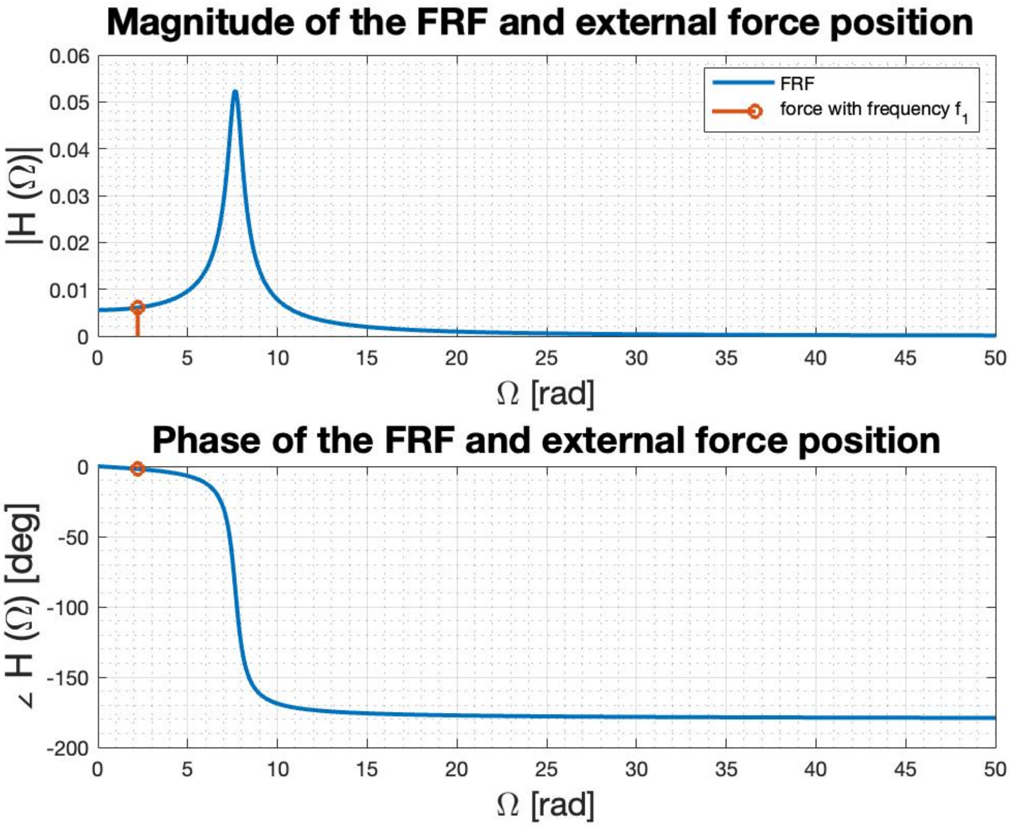

Figure 6: Representation of the torque C(t) in the FRF plot

It is now possible to compute the complete response of the system. As it is easily foreseeable the input force will not have much effect of the system as the FRF at that frequency has less than a 0.01 amplification factor. To achieve better visual results plots for both individual responses (free and forced) are given and also a comparison plot to visually show how small the forced response really is.

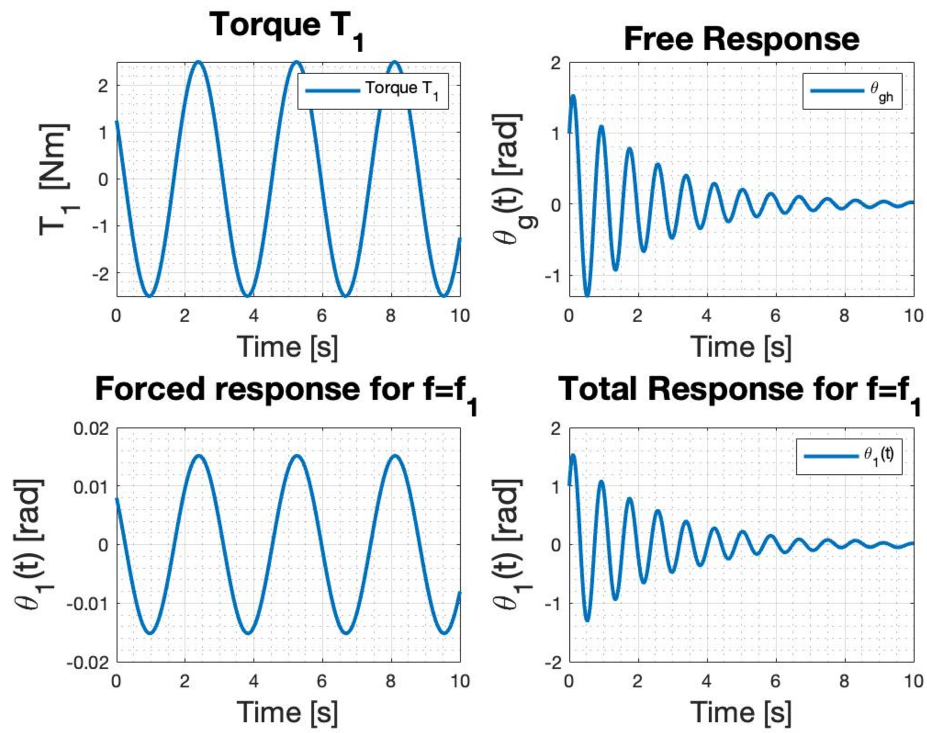

Figure 7: From top-left to bottom right: torque applied, forced response, free response, total response

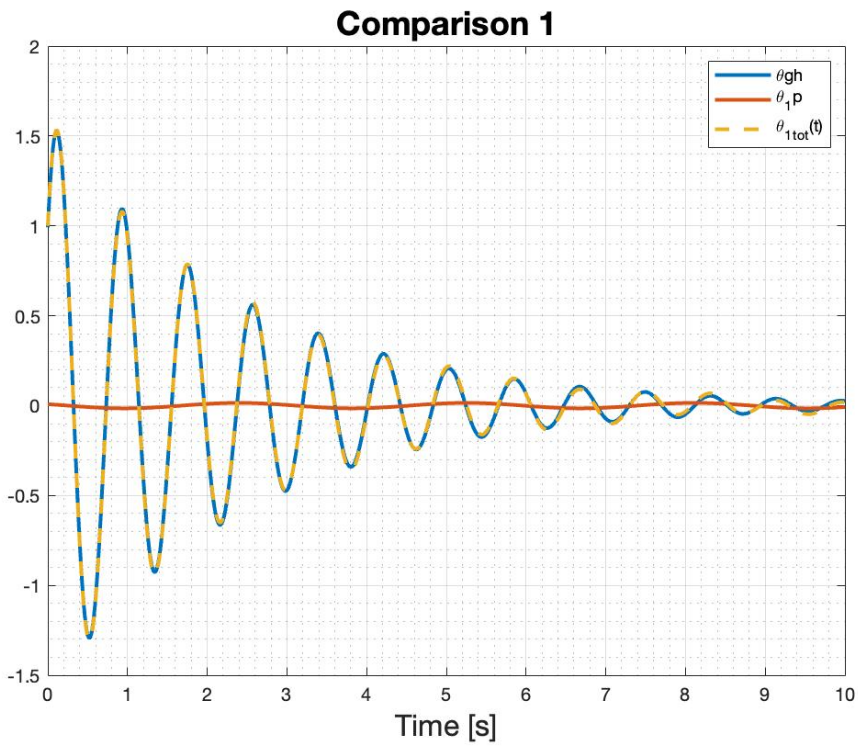

Figure 8: Visual comparison of the free and forced responses

3.3 Different cases of response of the system

First the same calculations as the previous point are done with a torque which has a frequency of $10hz$. The same considerations apply as also this frequency lies way beyond the peak of the FRF

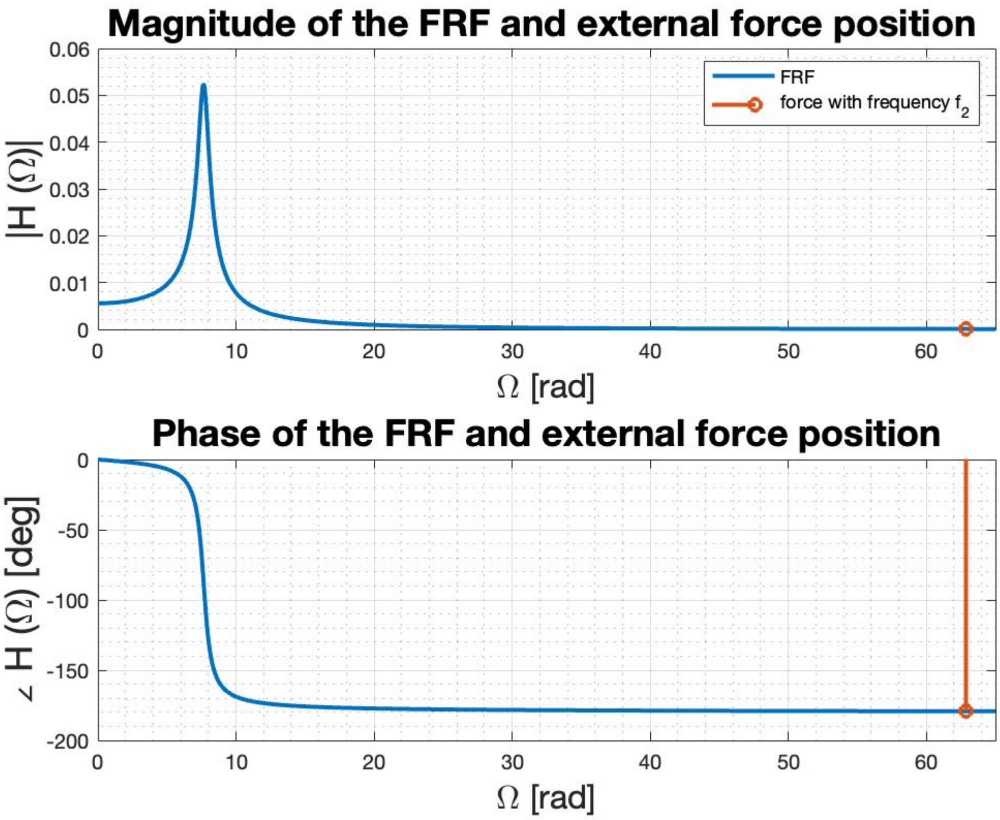

Figure 9: Representation of the torque C(t) in the FRF plot

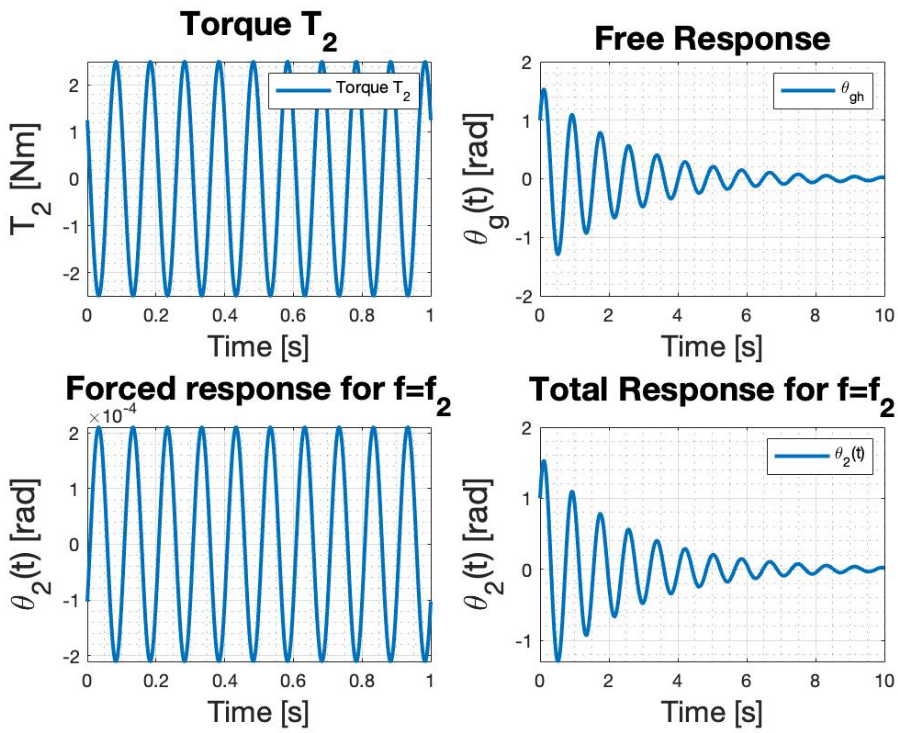

Figure 10: From top-left to bottom right: torque applied, forced response, free response, total response

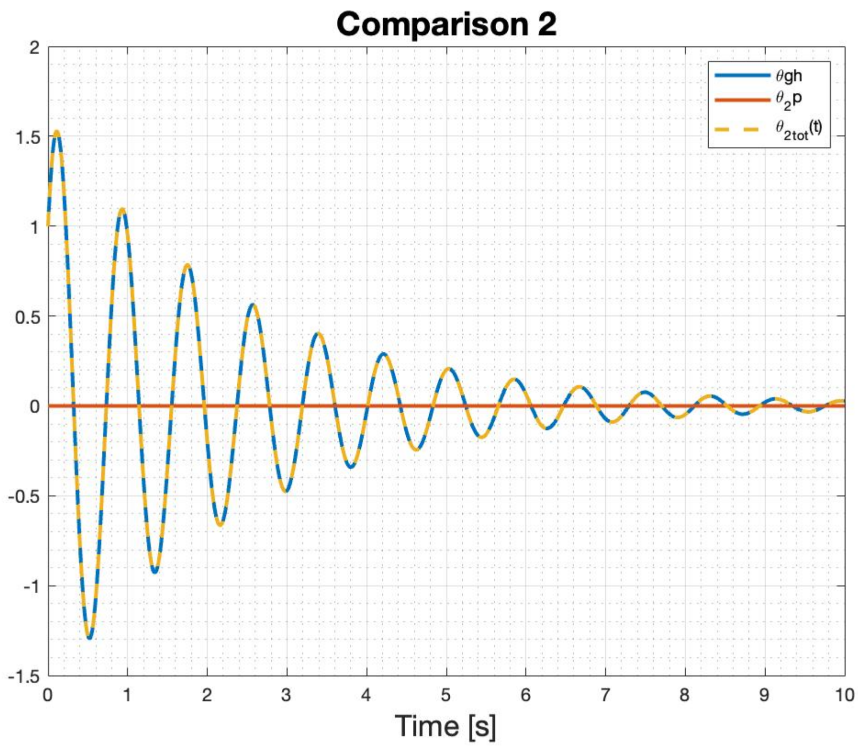

Figure 11: Visual comparison of the free and forced responses

Statically-applied torque

It is now considered the case of a statically-applied torque such as the frequency of the force is $0hz$ with the same amplitude as the previous examples. The following plots represent the results achieved

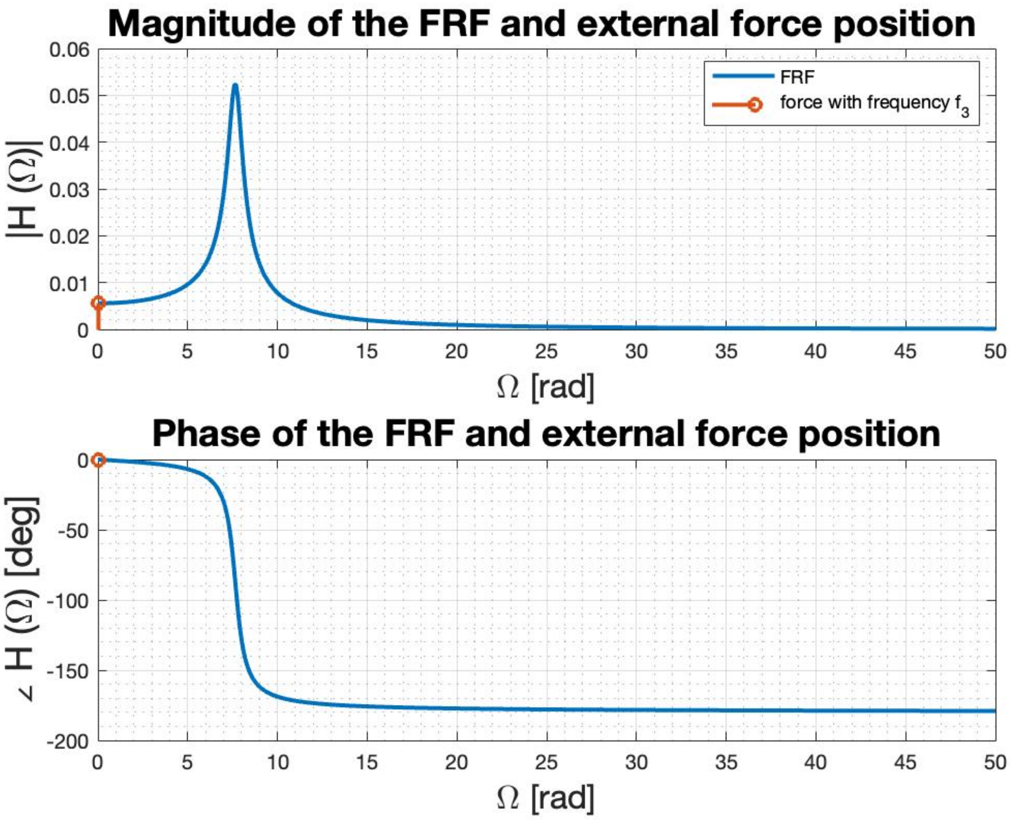

Figure 12: Representation of the constant torque in the FRF plot

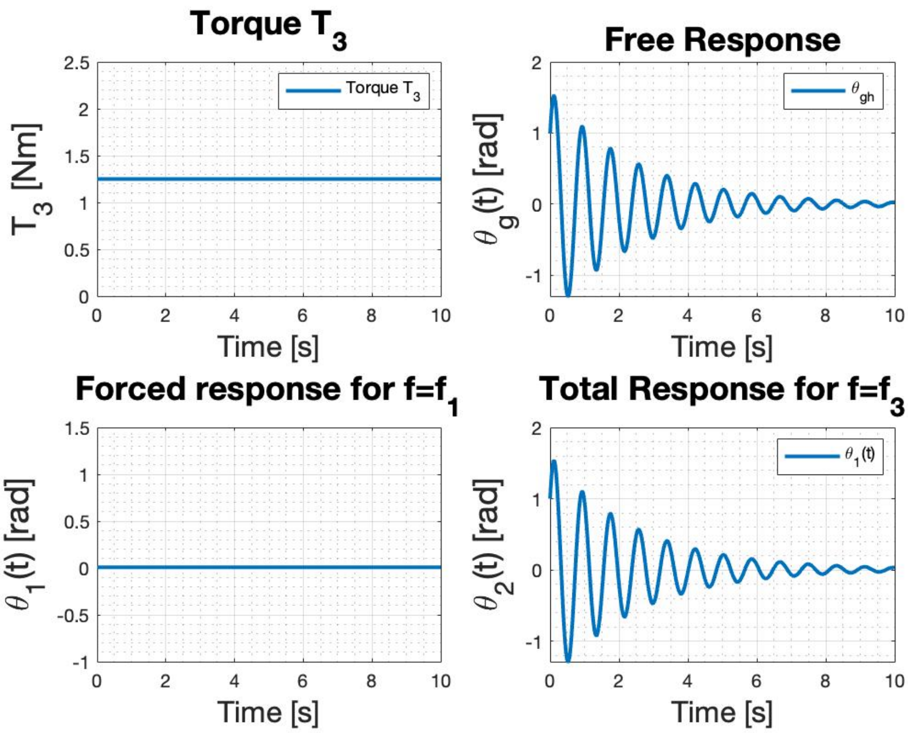

Figure 13: From top-left to bottom right: torque applied, free response, forced response, total response

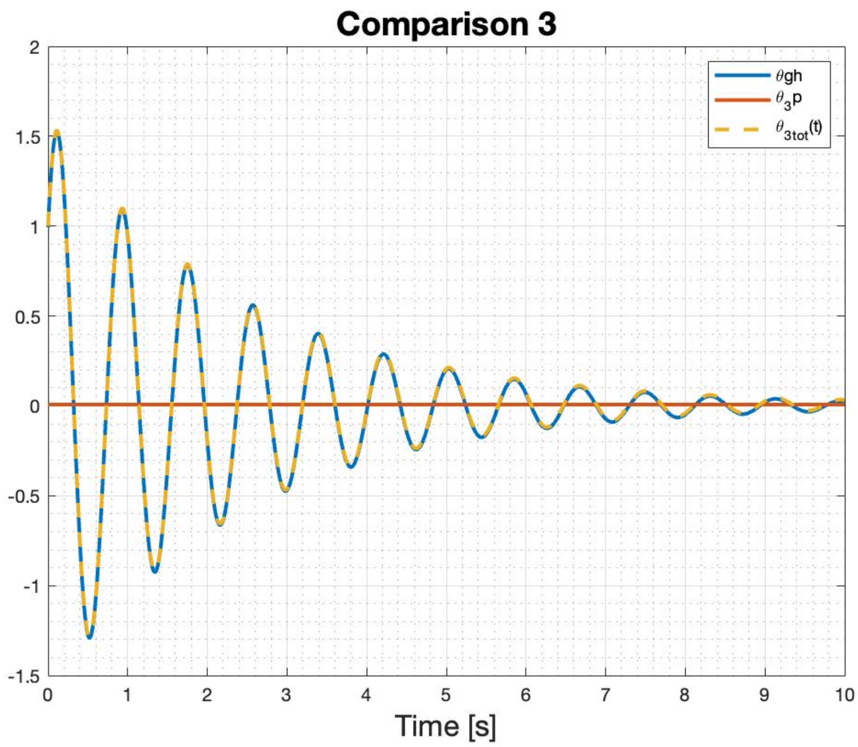

Figure 14: Visual comparison of the free and forced responses

3.4 Response to a multi-harmonic torque

This time the superposition of three different harmonic torques is applied to the system. The total torque has the following form:

$$C(t) = \sum_{k=1}^{3} B_k \cos(2\pi f_k t + \beta_k)$$

with B_k, f_k and β_k having the following values: $B_1 = 1.2$, $B_2 = 0.5$, $B_3 = 5$, $f_1 = 0.35hz$, $f_2 = 2.5hz$, $f_3 = 10hz$, $\beta_1 = \frac{\pi}{4}$, $\beta_2 = \frac{\pi}{5}$ and $\beta_3 = \frac{\pi}{6}$ The total torque is represented in the following plot:

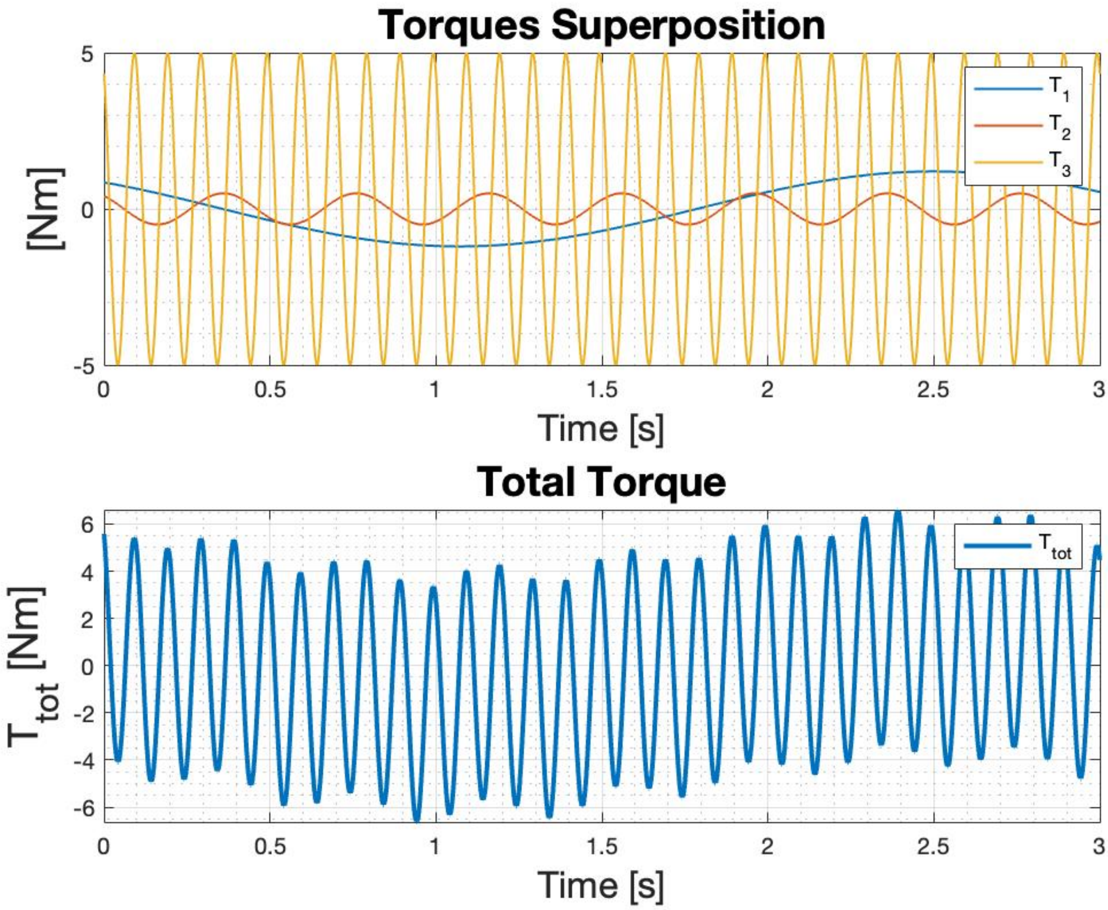

Figure 15: Total torque applied (superposition of 3 harmonic torques)

Since the system in study is linear it is possible to apply the superposition of effects to compute the total output of the system. This means that the motion will be the sum of the motion for each of the three harmonic torques that make the total torque $C(t)$. The following plot will show the result of these computations

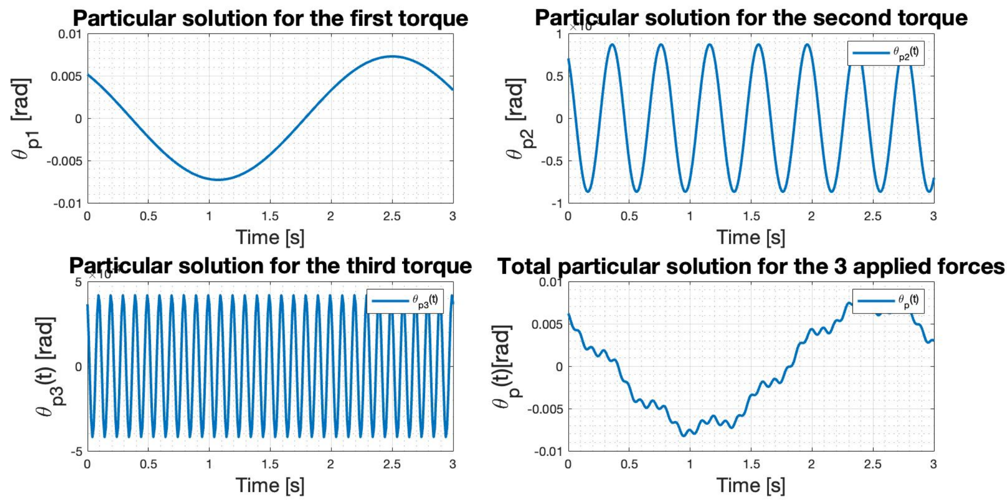

Figure 16: Forced response of the system to the multi-harmonic torque

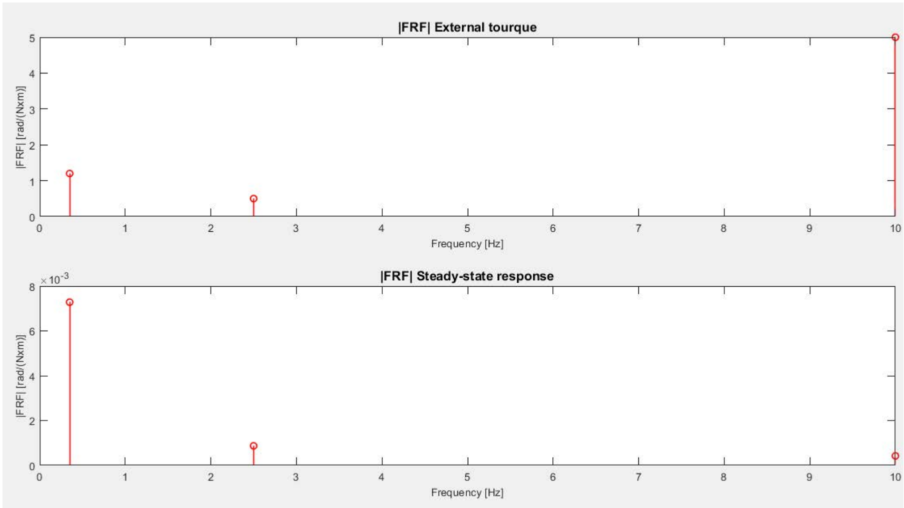

Figure 17: |FRF| of C(t) and of the motion of the system

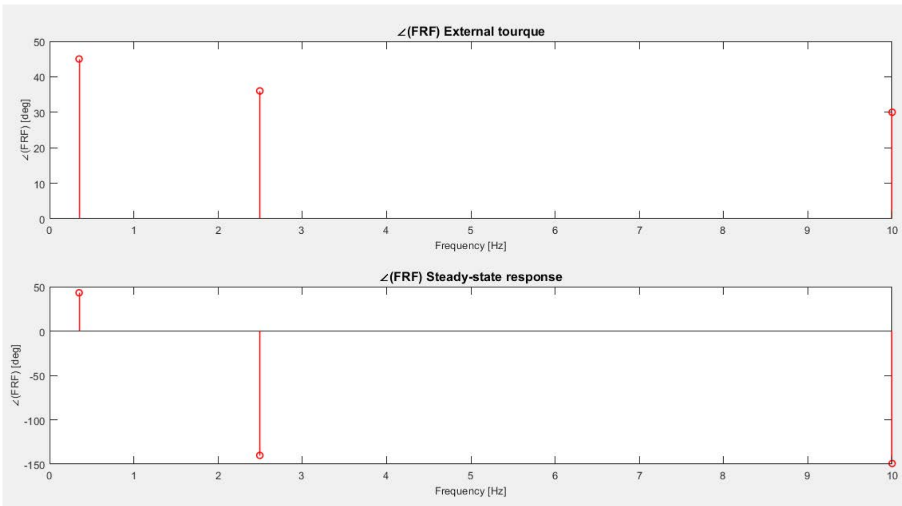

Figure 18: Phse of the FRF of C(t) and of the motion of the system

Vibration Analysis and Vibroacoustic

Assignment 2: Analysis of a n-dof System

Homework 2 Report

Students

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1. Equations of Motion

1.0.1 Determination of the Degrees of Constraint

As stated in report 1 it is paramount to correctly analyze a system to determine its Degrees Of Freedom. To get to that result it is first needed to determine the total amount of DOC of the system. By following the same path the result is the following:

$$n_{DOC} = 2 * 2_{cart} + 2 * 1_{hinge} + 1 * 1_{roller} + 2 * 1_{string} = 9$$

1.0.2 Degrees of Freedom

Having now concluded the number of DOC of the system, and by remembering that in a 2D system each body has a maximum of 3 DOF it is easy to conclude that the system has the following number of DOF:

$$n_{DOF} = 3 * 4_{bodies} - n_{DOC} = 3$$

The system has 3 DOF

1.0.3 Choice of Independent Variable

Since the system has 3 DOF it is first needed to find 3 independent variables that describe the system in a way that all other variables can be expressed as a linear combination of the independent ones. For this purpose the displacement of mass M1 and the angles θ_2 and θ_3 respectively the angular displacement of the discs M2 and M3 with respect to their equilibrium position.

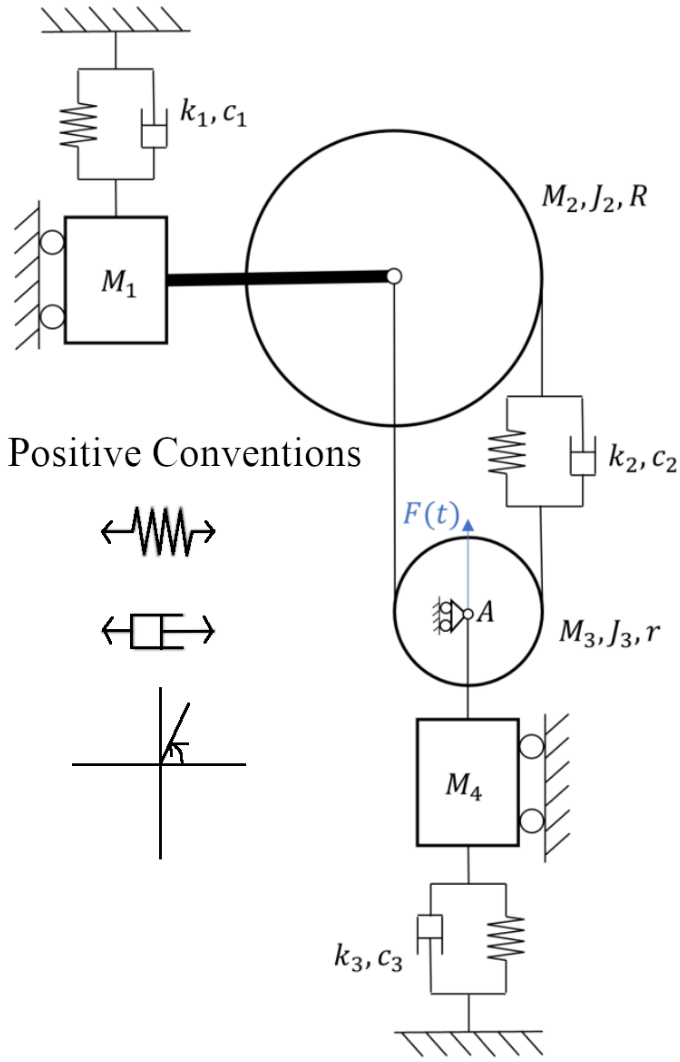

Figure 1: Representation of the system and positive conventions

1.0.4 Derivation of Parameters

All other parameters of the system need to be expressed as a function of the independent variables

$\Delta\ell_1$	$-y_1$
$\Delta\ell_2$	$y_1 - r\theta_3 + R\theta_2$
$\Delta\ell_3$	$y_1 + r\theta_3$
y_4	$\Delta\ell_3$
y_4	y_3
$\dot{\Delta\ell_1}$	$-\dot{y_1}$
$\dot{\Delta\ell_2}$	$\dot{y_1} - r\dot{\theta_3} + R\dot{\theta_2}$
$\dot{\Delta\ell_3}$	$\dot{y_1} + r\dot{\theta_3}$
y_2	y_1

1.1.0 Derivation of the Equations of Motion

The general formulation of the equations of motion can be computed by means of the Lagrange's equation in its vectorial form:

$$\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \underline{\dot{x}}}\right)\right\}^T - \left\{\frac{\partial T}{\partial \underline{x}}\right\}^T + \left\{\frac{\partial D}{\partial \underline{\dot{x}}}\right\}^T + \left\{\frac{\partial V}{\partial \underline{x}}\right\}^T = \underline{Q}$$

where T is the total kinetic energy, D is the total dissipation function, V is the total potential energy and $\underline{Q}$ is the lagrangian component of the external forces applied to the system. Every element will now be individually computed. For the sake of simplicity the three independent variables will be grouped in a column vector as follows:

$$\underline{x} = \begin{Bmatrix} y_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

1.1.1 Kinetic Energy

The total kinetic energy is a sum of individual translation and rotation components of the different bodies present in the system

$$T = \frac{1}{2}M_1 v_1^2 + \frac{1}{2}M_2 v_2^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}M_3 v_3^2 + \frac{1}{2}J_3\omega_3^2 + \frac{1}{2}M_4 v_4^2$$

where v_i is the vertical velocity of the mass M_i, while ω_i is the angular velocity of the mass M_i with moment of inertia J_i.

To write the kinetic energy as a function of the chosen independent variables the vector $\underline{y}$ is introduced such that each of the variables that characterize the motion of the bodies of the system can be considered.

$$\underline{y} = \begin{Bmatrix} y_1 \\ y_2 \\ \theta_2 \\ y_3 \\ \theta_3 \\ y_4 \end{Bmatrix}$$

From this it is possible to derive $\underline{\dot{y}}$

$$\underline{\dot{y}} = \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_2 \\ \dot{y}_3 \\ \dot{\theta}_3 \\ \dot{y}_4 \end{Bmatrix} \equiv \begin{Bmatrix} v_1 \\ v_2 \\ \omega_2 \\ v_3 \\ \omega_3 \\ v_4 \end{Bmatrix}$$

Now the previous equation can be rewritten as $T = \frac{1}{2}\underline{\dot{y}}^T[M]\underline{\dot{y}}$, where

$$[M] = \begin{bmatrix} M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_4 \end{bmatrix}$$

The vector $\underline{\dot{y}}$ can be reformulated considering the dependencies of its coordinates on the independent variables:

$$\underline{\dot{y}} = \left(\frac{\partial \underline{y}}{\partial \underline{x}}\right)\underline{\dot{x}} = [\Lambda_m]\underline{\dot{x}}$$

where the Jacobian matrix $[\Lambda_m]$ contains the partial derivatives of the vector $\underline{y}$ with respect to the vector $\underline{x}$ of the independent variables. Since

	$\dot{y}_1$	$\dot{\theta}_2$	$\dot{\theta}_3$
v_1	1	0	0
v_2	1	0	0
ω_2	0	1	0
v_3	1	0	r
ω_3	0	0	1
v_4	1	0	r

Which results in the following Jacobian matrix

$$[\Lambda_m] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & r \\ 0 & 0 & 1 \\ 1 & 0 & r \end{bmatrix}$$

Now the total kinetic energy can be written as a function of the independent variables:

$$T = \frac{1}{2}\underline{\dot{y}}^T[M]\underline{\dot{y}} \equiv \frac{1}{2}\underline{\dot{x}}^T[\Lambda_m]^T[M][\Lambda_m]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{x}}^T[M^*]\underline{\dot{x}}$$

In the what has been obtained is the following:

$$[M^*] = [\Lambda_m]^T[M][\Lambda_m] \equiv \begin{bmatrix} M_1 + M_2 + M_3 + M_4 & 0 & r(M_3 + M_4) \\ 0 & J_2 & 0 \\ r(M_3 + M_4) & 0 & r^2(M_3 + M_4) + J_3 \end{bmatrix}$$

1.1.2 Potential Energy

Just as in the previous paragraph the potential energy is computed as a sum of the individual potential energies of the single bodies

$$V_{el} = \frac{1}{2}k_1\Delta l_1^2 + \frac{1}{2}k_2\Delta l_2^2 + \frac{1}{2}k_3\Delta l_3^2$$

As the EOM is computed for small vibrations around the point of equilibrium of the system it is safe to assume that the gravitational potential energy will not have an impact on our computations so it is assumed 0 at all times.
As done previously a vector is introduced to help in the computations

$$\underline{\Delta l} = \begin{Bmatrix} \Delta l_1 \\ \Delta l_2 \\ \Delta l_3 \end{Bmatrix}$$

And a matrix to keep track of all the springs is created

$$[K] = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}$$

Just as before the jacobian matrix is computed for this case:

	y_1	θ_2	θ_3
Δl_1	-1	0	0
Δl_2	0	R	$-R$
Δl_3	1	0	r

Which results in the following matrix

$$[\Lambda_k] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & R & -R \\ 1 & 0 & r \end{bmatrix}$$

By substitution it is obtained:

$$V_{el} = \frac{1}{2}\underline{\Delta l}^T[K]\underline{\Delta l} \equiv \frac{1}{2}\underline{x}^T[\Lambda_k]^T[K][\Lambda_k]\underline{x} = \frac{1}{2}\underline{x}^T[K^*]\underline{x}$$

where, in conclusion,

$$[K^*] = [\Lambda_k]^T[K][\Lambda_k] \equiv \begin{bmatrix} k_1 + k_2 & 0 & rk_3 \\ 0 & R^2k_2 & -R^2k_2 \\ rk_3 & -R^2k_2 & R^2k_2 + r^2k_3 \end{bmatrix}$$

1.1.3 Dissipation Function

Again, the total dissipation function D is given by

$$D = \frac{1}{2}c_1\dot{\Delta l_1}^2 + \frac{1}{2}c_2\dot{\Delta l_2}^2 + \frac{1}{2}c_3\dot{\Delta l_3}^2$$

As done for the elastic potential energy, the vector $\underline{\dot{\Delta l}}$ is introduced so that the dissipation function can be first rewritten in a vectorial form by means of the diagonal damping matrix $[C]$, and then can be written as a function of the vector of the independent coordinates thanks to a jacobian matrix $[\Lambda_c]$.
Since the springs and the dampers in the system are all in parallel, then the Jacobian matrix $[\Lambda_c]$ is equal to the jacobian matrix $[\Lambda_k]$. Hence, for

$$[C] = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

$$\underline{\dot{\Delta l}} = [\Lambda_c]\underline{\dot{x}} \equiv [\Lambda_k]\underline{\dot{x}}$$

$$D = \frac{1}{2}\dot{\underline{\Delta l}}^T[C]\dot{\underline{\Delta l}} \equiv \frac{1}{2}\dot{\underline{x}}^T[\Lambda_c]^T[C][\Lambda_c]\dot{\underline{x}} = \frac{1}{2}\dot{\underline{x}}^T[C^*]\dot{\underline{x}}$$

$$[C^*] = [\Lambda_c]^T[C][\Lambda_c] \equiv \begin{bmatrix} c_1 + c_2 & 0 & rc_3 \\ 0 & R^2c_2 & -R^2c_2 \\ rc_3 & -R^2c_2 & R^2c_2 + r^2c_3 \end{bmatrix}$$

1.1.4 Lagrangian Component

The Lagrangian component

$$\underline{Q} = \left\{\frac{\partial^* W}{\partial \underline{x}}\right\}^T$$

is referred to the virtual work $\partial^* W$ of the external force applied to the system in the point A. Since $\partial^* W = F(t)\delta^*$ and the virtual displacement δ^* can be written as a function of the independent variables by means of a proper jacobian matrix $[\Lambda_f]$ as follows $\delta^* = [\Lambda_f]\delta\underline{x}$, then $\partial^* W \equiv F(t)[\Lambda_f]\delta\underline{x}$ and so

$$\underline{Q} = \left\{\frac{\partial^* W}{\partial \underline{x}}\right\}^T \equiv F(t)[\Lambda_f]^T$$

Since

	y_1	θ_2	θ_3
δ	1	0	r

$$\Rightarrow [\Lambda_f] = \begin{bmatrix} 1 & 0 & r \end{bmatrix}$$

then, in conclusion,

$$\underline{Q} = F(t)[\Lambda_f]^T = \begin{Bmatrix} F(t) \\ 0 \\ rF(t) \end{Bmatrix}$$

1.1.5 Derivation of the equations of motion

Now that all preliminary passages have been made it is possible to compute the EOM of the system

$$\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\underline{x}}}\right)\right\}^T = \left\{\frac{d}{dt}(\dot{\underline{x}}^T[M^*])\right\}^T = [M^*]^T\ddot{\underline{x}} \equiv [M^*]\ddot{\underline{x}}$$

$$\left\{\frac{\partial T}{\partial \underline{x}}\right\}^T \equiv \underline{0} \text{ (The system is linear)}$$

$$\left\{\frac{\partial D}{\partial \dot{\underline{x}}}\right\}^T = \{\dot{\underline{x}}^T[C^*]\}^T = [C^*]^T\dot{\underline{x}} \equiv [C^*]\dot{\underline{x}}$$

$$\left\{\frac{\partial V_{el}}{\partial \underline{x}}\right\}^T = \{\underline{x}^T[K^*]\}^T = [K^*]^T\underline{x} \equiv [K^*]\underline{x}$$

then the equation becomes

$$[M^*]\ddot{\underline{x}} + [C^*]\dot{\underline{x}} + [K^*]\underline{x} = \underline{Q}$$

which is a system of 3 linear differential equations.

1.2 Eigenvalues and Eigenvectors

It is now possible to further analyse the system by computing its eigenvalues and eigenvectors

1.2.1 Free Undamped Case

The free undamped system is characterized by $\underline{Q} = 0$ and $[C^*] = 0$. In this case the equations of motion (EOM) are simply the following:

$$[M^*]\underline{\ddot{x}} + [K^*]\underline{x} = \underline{0}$$

The solution of the EOM is of the form $\underline{x} = \underline{X}e^{\lambda t}$. Since $\underline{\ddot{x}} = \lambda^2 \underline{X}e^{\lambda t}$, then it is possible to rewrite the equation as

$$(\lambda^2[M^*] + [K^*])\underline{X}e^{\lambda t} = 0$$

Since $e^{\lambda t} \neq 0 \ \forall t \in R$ and the trivial solution is not of interest $\underline{X} = 0$, then the previous equation is satisfied if and only if

$$\lambda^2[M^*] + [K^*] = 0$$

Since $\det([M^*]) \neq 0$ then $\exists \ [M^*]^{-1}$ and so it is possible to rearrange the latter equation in the following way:

$$\lambda^2[I] + [M^*]^{-1}[K^*] = 0$$

where $[I]$ is the 3×3 identity matrix.

The different λ_i are the roots of the determinant of $\lambda^2[I] + [M^*]^{-1}[K^*]$. Hence, by imposing

$$\det(\lambda^2[I] + [M^*]^{-1}[K^*]) = 0$$

We can easily solve the problem through a MATLAB code, that gives us the following results:

$$\lambda^2_{1,2} = -0.8623, \qquad \lambda^2_{3,4} = -25.0586, \qquad \lambda^2_{5,6} = -67.0143$$

$$\lambda_{1,2} = \pm\ i\ \omega_{01}, \qquad \lambda_{3,4} = \pm\ i\ \omega_{02}, \qquad \lambda_{5,6} = \pm\ i\ \omega_{03}$$

$$\omega_{01} = 0.9286\ rad/s, \qquad \omega_{02} = 5.0059\ rad/s, \qquad \omega_{03} = 8.1862\ rad/s$$

$$\underline{X}^{(I)} = \begin{Bmatrix} -0.0677 \text{ m} \\ 0.7151 \text{ rad} \\ 0.6958 \text{ rad} \end{Bmatrix}, \ \underline{X}^{(II)} = \begin{Bmatrix} 0.9894 \text{ m} \\ -0.1417 \text{ rad} \\ -0.0307 \text{ rad} \end{Bmatrix}, \ \underline{X}^{(III)} = \begin{Bmatrix} 0.1600 \text{ m} \\ 0.6659 \text{ rad} \\ -0.7287 \text{ rad} \end{Bmatrix}$$

which can be normalized with respect to one of their coordinates. For simplicity, for each of them, the first coordinate was chosen which represents the amplitude of the vertical motion of M_1.

$$\underline{X}^{(I)} = \begin{Bmatrix} 1.00 \text{ m} \\ -10.5598 \text{ rad} \\ -10.2753 \text{ rad} \end{Bmatrix}, \ \underline{X}^{(II)} = \begin{Bmatrix} 1.00 \text{ m} \\ -0.1432 \text{ rad} \\ -0.0311 \text{ rad} \end{Bmatrix}, \ \underline{X}^{(III)} = \begin{Bmatrix} 1.00 \text{ m} \\ 4.1612 \text{ rad} \\ -4.5532 \text{ rad} \end{Bmatrix}$$

1.2.2 Free Damped Case

The free damped system is characterized by $\underline{Q} = 0$ and $[C^*] \neq 0$. In this case the equations of motion (EOM) are the following:

$$[M^*]\underline{\ddot{x}} + [C^*]\underline{\dot{x}} + [K^*]\underline{x} = \underline{0}$$

To compute the general solution it is noted that

$$\begin{cases} [M^*]\underline{\ddot{x}} + [C^*]\underline{\dot{x}} + [K^*]\underline{x} = \underline{0} \\ [M^*]\underline{\dot{x}} - [M^*]\underline{\dot{x}} = \underline{0} \end{cases} \Leftrightarrow \begin{bmatrix} [M^*] & [0] \\ [0] & [M^*] \end{bmatrix} \begin{Bmatrix} \underline{\ddot{x}} \\ \underline{\dot{x}} \end{Bmatrix} + \begin{bmatrix} [C^*] & [K^*] \\ -[M^*] & [0] \end{bmatrix} \begin{Bmatrix} \underline{\dot{x}} \\ \underline{x} \end{Bmatrix} = \underline{0} \Leftrightarrow [A]\underline{\dot{z}} + [B]\underline{z} = \underline{0}$$

for

$$\underline{z} = \begin{Bmatrix} \underline{\dot{x}} \\ \underline{x} \end{Bmatrix}, \qquad [A] = \begin{bmatrix} [M^*] & [0] \\ [0] & [M^*] \end{bmatrix}, \qquad [B] = \begin{bmatrix} [C^*] & [K^*] \\ -[M^*] & [0] \end{bmatrix}$$

Since $[M^*]$ is invertible, so is $[A]$ and then the EOM is simply

$$\underline{\dot{z}} - [\Lambda_z]\underline{z} = \underline{0} \text{ for } [\Lambda_z] = -[A]^{-1}[B]$$

which solution is of the form $\underline{z} = \underline{Z}e^{\lambda t}$.

To compute the different eigenvalues λ_i the trivial solution is excluded $\underline{z}_0 = 0$ and so it is desirable to look for the roots of the determinant of $\lambda[I]_{3\times 6} - [\Lambda_z]$. These solutions happen to be of the form $\lambda_i = -\alpha_i + \omega_i$, where ω_i are quite the same eigenvalues found previously for the undamped system

$$\underline{\lambda} = \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{Bmatrix} = \begin{Bmatrix} (-0.0335 + 0.9286i) \text{ rad/s} \\ (-0.0335 - 0.9286i) \text{ rad/s} \\ (-0.3995 + 4.9870i) \text{ rad/s} \\ (-0.3995 - 4.9870i) \text{ rad/s} \\ (-0.1124 + 8.1846i) \text{ rad/s} \\ (-0.1124 - 8.1846i) \text{ rad/s} \end{Bmatrix} = \begin{Bmatrix} \lambda_I \\ \lambda_I^* \\ \lambda_{II} \\ \lambda_{II}^* \\ \lambda_{III} \\ \lambda_{III}^* \end{Bmatrix}$$

The corresponding eigenvectors $\underline{Z}^{(i)}$ are the one that solve

$$(\lambda_i[I]_{6\times 6} - [\Lambda_z])\underline{Z}^{(i)} = 0, \ \forall \ i = 1, \dots, 6$$

Discarding the 3 first elements of each vector (which are the ones related to velocity),the following eigenvectors are obtained

$$\underline{X}_I^{(1,2)} = \begin{Bmatrix} (-0.0496 \mp 0.0032i) \text{ m} \\ (0.5238 \pm 0.0000i) \text{ rad} \\ (0.5097 \mp 0.0010i) \text{ rad} \end{Bmatrix}$$

$$\underline{X}_{II}^{(3,4)} = \begin{Bmatrix} (0.0155 \pm 0.1930i) \text{ m} \\ (-0.0121 \mp 0.0267i) \text{ rad} \\ (-0.0057 \mp 0.0046i) \text{ rad} \end{Bmatrix}$$

$$\underline{X}_{III}^{(5,6)} = \begin{Bmatrix} (0.0017 \pm 0.0196i) \text{ m} \\ (0.0017 \pm 0.0807i) \text{ rad} \\ (-0.0012 \mp 0.0883i) \text{ rad} \end{Bmatrix}$$

which can be normalized with respect to one of their coordinates. As done before, for each of them, the first coordinate are chosen

$$\underline{X}_I^{(1,2)} = \begin{Bmatrix} (1.0000 \pm 0.0000i)\ \text{m} \\ (-10.5210 \pm 0.6702i)\ \text{rad} \\ (-10.2366 \pm 0.6718i)\ \text{rad} \end{Bmatrix}$$

$$\underline{X}_{II}^{(3,4)} = \begin{Bmatrix} (1.0000 \mp 0.0000i)\ \text{m} \\ (-0.1421 \pm 0.0511i)\ \text{rad} \\ (-0.0262 \pm 0.0277i)\ \text{rad} \end{Bmatrix}$$

$$\underline{X}_{III}^{(5,6)} = \begin{Bmatrix} (1.0000 \pm 0.0000i)\ \text{m} \\ (4.1062 \pm 0.2576i)\ \text{rad} \\ (-4.4893 \mp 0.3168i)\ \text{rad} \end{Bmatrix}$$

1.3 Rayleigh Damping

It is now required to compute the two constants α and β that allow to approximate the generalized damping matrix $[C^*]$ through the Rayleigh proportional damping formula:

$$[C^*] \approx \alpha[M^*] + \beta[K^*]$$

In this particular case, the system of equations related to the Rayleigh proportional damping formula is analytically unsolvable: we have a system of nine equations and 2 unknowns to determine. Hence, to compute α and β we decide to apply the least square method (LSM), which is based on the computation of the values α and β that minimise the sum of the residuals squared. In particular, assuming to re-shape the 3×3 matrices $[M^*]$, $[C^*]$, $[K^*]$ of the system in such a way that their coefficients can be stored in a 9×1 array, the LSM assumes that the best values for α and β are the ones that minimize the following function

$$S = \sum_{i,j=1}^{3} (C^*_{i,j} - \alpha M^*_{i,j} - \beta K^*_{i,j})^2 = \sum_{k=1}^{9} (C^*_k - \alpha M^*_k - \beta K^*_k)^2$$

where the $C^*_k - \alpha M^*_k - \beta K^*_k$ are the so called residuals. Hence, by imposing

$$\begin{cases} \dfrac{\partial S}{\partial \alpha} = 0 \\ \dfrac{\partial S}{\partial \beta} = 0 \end{cases}$$

it is found that the two equations that allow to compute α and β. In conclusion:

$$\alpha = 0.725\ 1/\text{s}, \ \beta = -0.002\ \text{s}$$

Lastly, assuming that $[C^*] = \alpha[M^*] + \beta[K^*]$, we compute again the eigenvalues and the eigenvectors for the damped system.

Repeating the same steps presented earlier, we find the following eigenvalues and eigenvectors:

$$\underline{\lambda}_{appr} = \begin{Bmatrix} \lambda^{(1,2)}_{I,appr} \\ \lambda^{(3,4)}_{II,appr} \\ \lambda^{(5,6)}_{III,appr} \end{Bmatrix} = \begin{Bmatrix} (-0.3617 \pm 0.8553i)\ \text{rad/s} \\ (-0.3392 \pm 4.9943i)\ \text{rad/s} \\ (-0.3002 \pm 8.1807ii)\ \text{rad/s} \end{Bmatrix}$$

$$\underline{X}^{(1,2)}_{I,appr} = \begin{Bmatrix} 1.00 \text{ m} \\ -10.5598 \text{ rad} \\ -10.2753 \text{ rad} \end{Bmatrix}, \; \underline{X}^{(3,4)}_{II,appr} = \begin{Bmatrix} 1.0000 \text{ m} \\ -0.1432 \text{ rad} \\ -0.0311 \text{ rad} \end{Bmatrix}, \; \underline{X}^{(5,6)}_{III,appr} = \begin{Bmatrix} 1.0000 \text{ m} \\ 4.1612 \text{ rad} \\ -4.5532 \text{ rad} \end{Bmatrix}$$

As expected the eigenvectors are real and have a null imaginary part. This property is a consequence of the Rayleigh proportional damping formula.

2. Free Motion of the System

2.1 Free Motion from given Initial Conditions

We want to solve the EOM in the damped case considering Rayleigh damping for a particular set of initial conditions. In other words, we want to solve the following *Cauchy problem*:

$$\begin{cases} [M^*]\underline{\ddot{x}} + [C^*_{appr}]\underline{\dot{x}} + [K^*]\underline{x} = \underline{0} \\ [C^*_{appr}] = \alpha[M^*] + \beta[K^*] \\ \underline{x}(t=0) = \underline{x}_0 \\ \underline{\dot{x}}(t=0) = \underline{\dot{x}}_0 \end{cases}$$

The solution is given by the following relation:

$$\underline{x}(t) = \sum_{i=1}^{6} a_i \underline{X}^{(i)}_{appr} e^{\lambda^{(i)}_{appr} t}$$

where $\lambda^{(i)}_{appr}$ and $\underline{X}^{(i)}_{appr}$ are the estimated eigenvalues and eigenvectors presented earlier, while the a_i coefficients are the ones specifically related to the chosen initial conditions. In particular:

$$\underline{a} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{Bmatrix} = \begin{bmatrix} \underline{X}^{(1)}_{appr} & \underline{X}^{(2)}_{appr} & \dots & \underline{X}^{(6)}_{appr} \\ \lambda^{(1)}_{appr}\underline{X}^{(1)}_{appr} & \lambda^{(2)}_{appr}\underline{X}^{(2)}_{appr} & \dots & \lambda^{(6)}_{appr}\underline{X}^{(6)}_{appr} \end{bmatrix}^{-1} \begin{Bmatrix} \underline{x}_0 \\ \underline{\dot{x}}_0 \end{Bmatrix}$$

Assuming that

$$\underline{x}_0 = \begin{Bmatrix} 0.1 \text{ m} \\ \pi/12 \text{ rad} \\ -\pi/12 \text{ rad} \end{Bmatrix}, \; \underline{\dot{x}}_0 = \begin{Bmatrix} 1.0 \text{ m/s} \\ 0.5 \text{ rad/s} \\ 2.0 \text{ rad/s} \end{Bmatrix}$$

we are able to determine and plot a particular solution

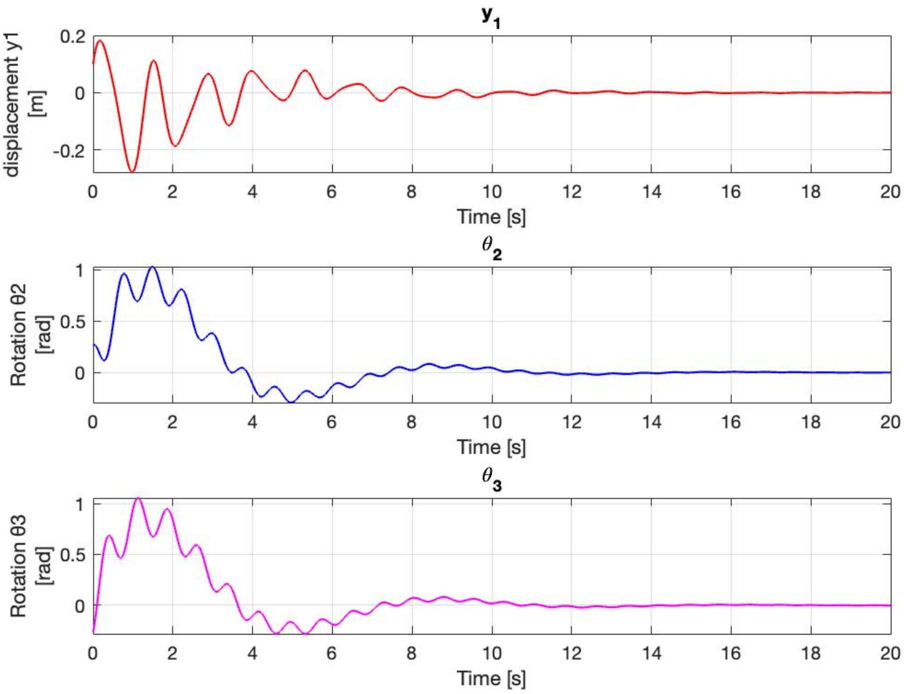

Figure 2: Free Motion from given initial conditions

2.2 Single mode initial conditions

In order to have a set of initial conditions that ensure that only one mode contributes to the free motion of the system, it is necessary to impose that, for a certain i, $\underline{x}_0 \equiv \underline{X}^{(i)}_{appr}$ and $\underline{\dot{x}}_0 = \underline{0}$. Under these conditions it is assured that there are no contributions, due to any initial energy, which could enhance different modes from the selected one. An equivalent way to reach the same results is to put as initial velocity a vector given by $\underline{\dot{x}} = \lambda^{(i)}_{appr} X^{(i)}_{appr}$.

It was decided to compute and display the motion correspondent to all the normalized modes taken one by one as given in Equation

$$\underline{x}_0 = \underline{X}^{(i)}_{appr},\ \underline{\dot{x}}_0 = \underline{0}\ where\ i = 1, 2, 3, ...$$

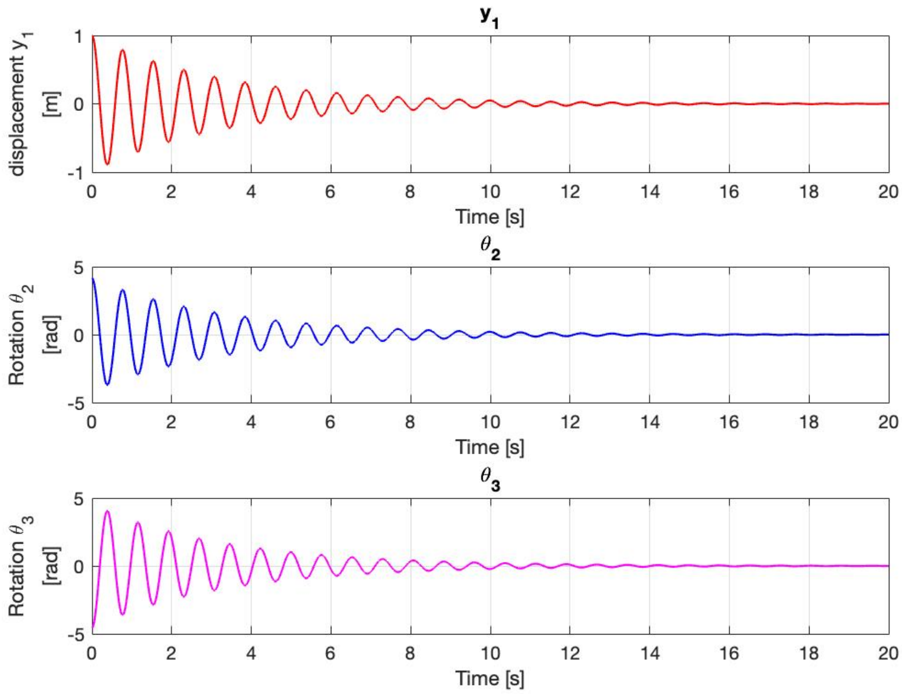

Figure 3: Free Motion with single mode

Forced Motion of the System

3.1 Frequency Response Matrix

It is now required to solve the EOM considering Rayleigh damping for a particular set of initial conditions and for a given external force $F(t)$. In other words, it is desirable to solve the following *Cauchy problem*:

$$\begin{cases} [M^*]\underline{\ddot{x}} + [C^*_{appr}]\underline{\dot{x}} + [K^*]\underline{x} = F(t)[\Lambda_f]^T \\ [C^*_{appr}] = \alpha[M^*] + \beta[K^*] \\ \underline{x}(t=0) = \underline{x}_0 \\ \underline{\dot{x}}(t=0) = \underline{\dot{x}}_0 \end{cases}$$

The complete time response (or general integral) of the linear system is given by the superposition of two solutions:

$$\underline{x}(t) = \underline{x}_g(t) + \underline{x}_p(t)$$

where $\underline{x}_g(t)$ is the general solution related to the free homogeneous problem, while $\underline{x}_p(t)$ is the so called steady state response or particular solution (since it is associated to the external force). The former solution has already been computed. Hence, we now only the general expression of $\underline{x}_p(t)$ will be computed.
The steady state response is the one that becomes dominant after the initial transient given by the general solution and that directly reflects the presence of an external force. Hence, assuming

to excite the system with an harmonic force $F(t) = F_0 e^{i\Omega t}$, the steady state response is given by the following expression:

$$\underline{x}_p = \underline{X}_{p,0} e^{i\Omega t}$$

$$\underline{\dot{x}}_p = i\Omega \underline{X}_{p,0} e^{i\Omega t}, \ \underline{\ddot{x}}_p = -\Omega^2 \underline{X}_{p,0} e^{i\Omega t}$$

and then EOM can be rewritten as follows:

$$(- \Omega^2 [M^*] + i\Omega [C^*_{appr}] + [K^*]) \underline{X}_{p,0} e^{i\Omega t} = \underline{F}_0 e^{i\Omega t}$$

where $\underline{F}_0 = F_0 [\Lambda_f]^T$.

If we introduce in the equation the matrix $[D^*(\Omega)] = -\Omega^2 [M^*] + i\Omega [C^*_{appr}] + [K^*]$ and simplify all the time dependencies, then we get the following expression

$$[D^*(\Omega)] \underline{X}_{p,0} = \underline{F}_0$$

Since $[D^*(\Omega)]$ is a matrix with complex coefficients then its determinant is never null and then we are able to compute the *Frequency Response Matrix* (FRM) of the system: $[H^*(\Omega)] = [D^*(\Omega)]^{-1}$. Thanks to $[H^*(\Omega)]$, it is possible to derive the amplitude of the steady state response:

$$\underline{X}_{p,0} = [D^*]^{-1} \underline{F}_0 = [H^*(\Omega)] \underline{F}_0$$

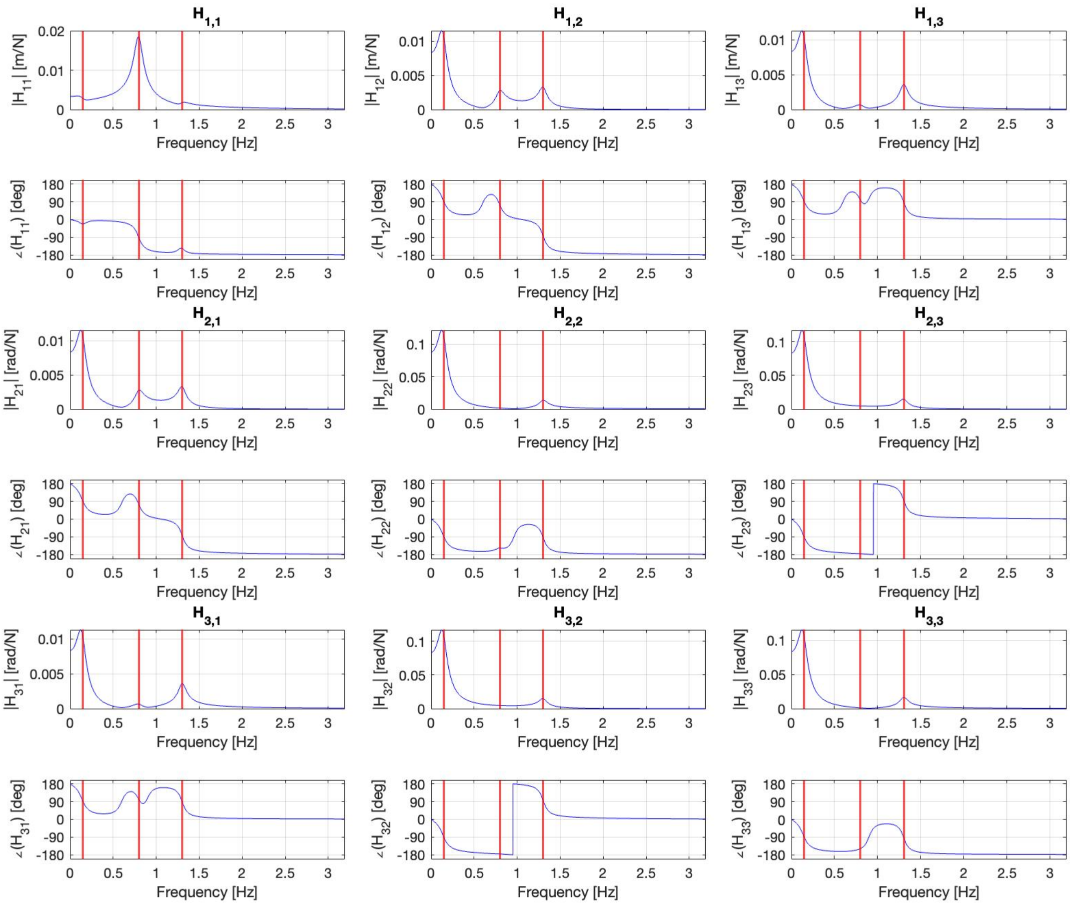

Figure 4: Frequency Response Matrix plot

The elements of the matrix $[H^*(\Omega)]$ tell us what is the steady state behaviour of the independent coordinates under the effect of an external harmonic force with frequency Ω. In particular, the element $[H^*(\Omega)]_{j,k}$ is the *Frequency Response Function* that describes how the j-th independent coordinate of the system responses if the force acts on the k-th independent coordinate.

Furthermore, each element $[H^*(\Omega)]_{j,k}$ shows which is the behaviour of the coordinates when the external force has a frequency similar to the natural frequencies ω_I, ω_{II}, ω_{III} of the system. In fact, as can be seen in the previous figures, all the FRM elements show a certain behaviour in correspondence of all the three computed natural frequencies. We can generally say that in the amplitude's graphs the presence or absence of peaks in correspondence of the natural frequencies is correlated to the modeshapes.

3.2 Co-Located FRF at point A

Thanks to an appropriate matrix transformation, it is possible to understand, by starting from the original independent coordinates, the steady state behaviour of new independent coordinates under the effect of the same external harmonic force.

Supposing the FRM $[H'(\Omega)]$ related to the new set of independent coordinates $\underline{x}'$ and calling $[\Phi']$ the transformation matrix. In this way

$$\underline{x}' = [\Phi']\underline{x} \text{ and } \underline{x} = [\Phi']^{-1}\underline{x}'$$

Since $\underline{\dot{x}} = [\Phi']^{-1}\underline{\dot{x}}'$, $\underline{\ddot{x}} = [\Phi']^{-1}\underline{\ddot{x}}'$ and $\delta\underline{x} = [\Phi']^{-1}\delta\underline{x}'$, then, without loss of generality, it is possible to compute the system's total kinetic energy T, dissipation function D, potential energy V and lagrangian component $\underline{Q}'$ by means of $\underline{x}'$, hence obtaining the following new set of equations of motion:

$$[M']\underline{\ddot{x}}' + [C']\underline{\dot{x}}' + [K']\underline{x}' = \underline{Q}'$$

where

$$\begin{cases} [M'] = ([\Phi']^{-1})^T[M^*][\Phi']^{-1} \\ [C'] = ([\Phi']^{-1})^T[C^*_{appr}][\Phi']^{-1} \\ [K'] = ([\Phi']^{-1})^T[K^*][\Phi']^{-1} \\ \underline{Q}' = F(t)([\Phi']^{-1})^T[\Lambda_f]^T \end{cases}$$

From this it is possible to study the steady state response of the system and hence compute the *co-located* Frequency Response Matrix $[H'(\Omega)]$ with the same previously shown relation:

$$[H'(\Omega)] = [-\Omega^2[M'] + i\Omega[C'] + [K']]^{-1}$$

It is now required to compute the co-located Frequency Response Function (FRF) related to the displacement of the point A when the force acts on A itself. To this end, we introduce the new set of independent coordinates

$$\underline{x}' = \begin{Bmatrix} x_1 \\ \theta_2 \\ x_3 \end{Bmatrix}$$

where x_1 and θ_2 are still the same independent coordinates, while x_3 is the vertical absolute motion in the positive direction of the centre of the disk of mass M_3 (point A) about its static equilibrium position. For this particular case the transformation matrix $[\Phi']$ is such that

$$[\Phi'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & r \end{bmatrix}$$

Now it is possible to compute $[H'(\Omega)]$ for this particular case and display the co-located FRF of point A at the centre of the disk 3, which corresponds to the element $[H'(\Omega)]_{3,3}$ (both displacement and force seen and applied in A).

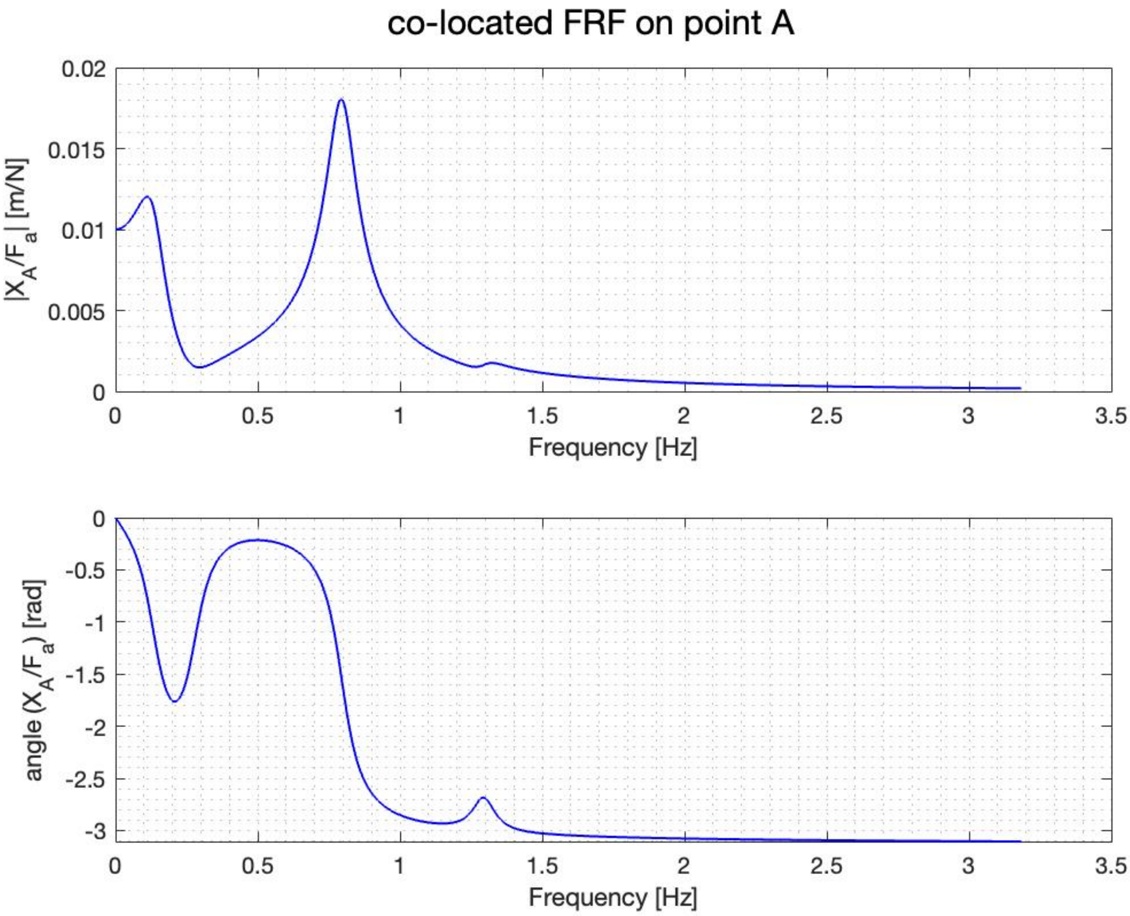

Figure 5: Co-Located FRF on point A

3.3 Co-Located FRF between the rotation of $M3$ and torque applied to the same object

Now it is required to compute the co-located FRF between the rotation of the disk of mass M_3 and the torque applied onto the disk itself. Since the rotation of the disk of mass M_3 is described by the independent coordinate θ_3 and the torque applied onto the disk is exactly given by the correspondent component of $F(t)[\Lambda_f]^T$, then it is possible to avoid defining a new set of coordinates and directly computing and displaying the co-located FRF, which corresponds to the element $[H(\Omega)]_{3,3}$ of the original FRM.

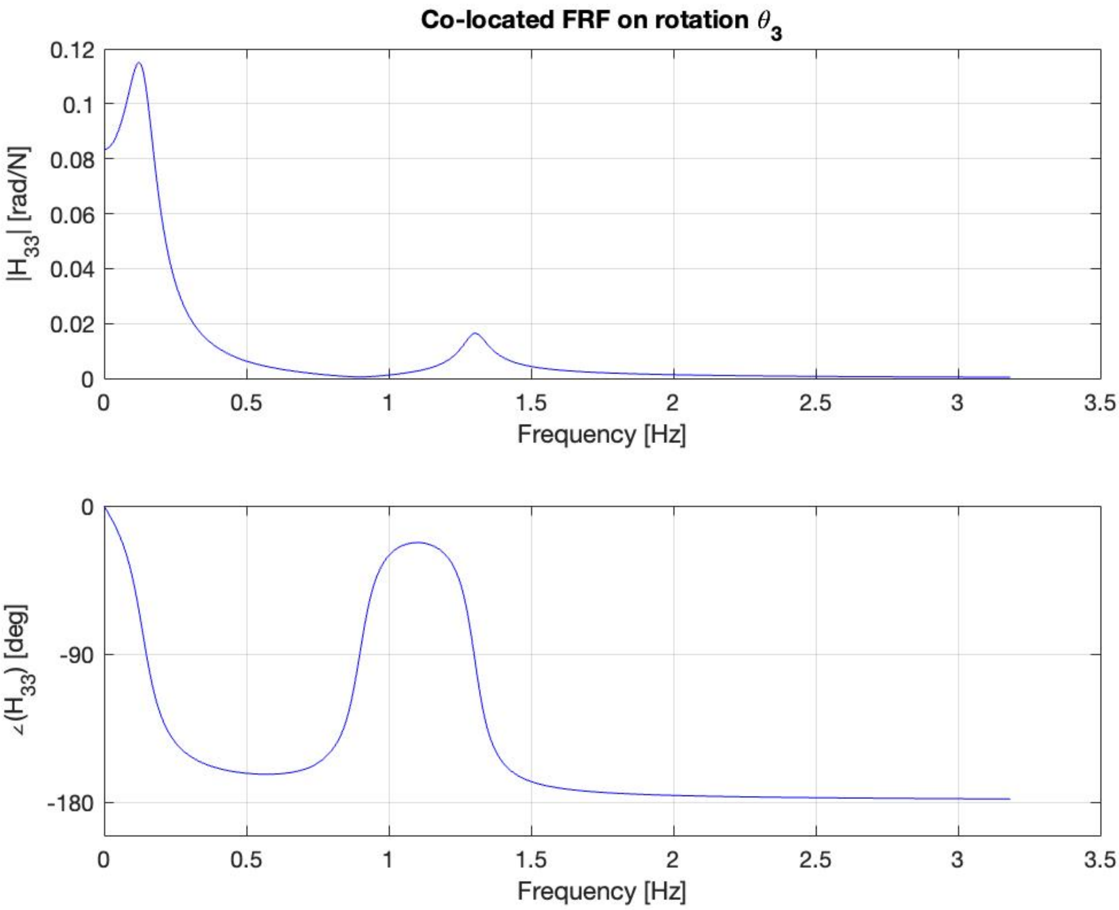

Figure 6: Co-Located FRF on rotation θ_3

3.4 Complete response to a bi-harmonic force

It is requested to compute, assuming Rayleigh damping the complete time solution of the EOM assuming that on the system acts the following bi-harmonic force:

$$F(t) = A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t)$$

The complete time solution of the system is given by the superposition of two different contributions: the general solution and the steady state response, both of which were previously discussed. Now it is possible to write as follows:

$$\underline{x}_p(t) = \Re e\left\{ \sum_{k=1,2} \underline{\tilde{x}}_{p,k,0} e^{i\Omega_k t} \right\} \equiv \Re e\left\{ \sum_{k=1,2} A_k [H^*(\Omega_k)][\Lambda_f]^T e^{i\Omega_k t} \right\}$$

$$\underline{x}(t) = \underline{x}_g(t) + \underline{x}_p(t)$$

where $\Omega_k \equiv 2\pi f_k$.

Using again the relations, we are able to compute the a_i coefficients.

$$\underline{a} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{Bmatrix} = \begin{bmatrix} \underline{X}^{(1)}_{appr} & \underline{X}^{(2)}_{appr} & \dots & \underline{X}^{(6)}_{appr} \\ \lambda^{(1)}_{appr}\underline{X}^{(1)}_{appr} & \lambda^{(2)}_{appr}\underline{X}^{(2)}_{appr} & \dots & \lambda^{(6)}_{appr}\underline{X}^{(6)}_{appr} \end{bmatrix}^{-1} \begin{Bmatrix} \underline{x}_0 - \sum\limits_{k=1,2} A_k[H^*(\Omega_k)][\Lambda_f]^T \\ \underline{\dot{x}}_0 - i \sum\limits_{k=1,2} \Omega_k A_k[H^*(\Omega_k)][\Lambda_f]^T \end{Bmatrix}$$

Hence, setting in conclusion the same initial conditions used earlier

$$\underline{x}_0 = \begin{Bmatrix} 0.1 \text{ m} \\ \pi/12 \text{ rad} \\ -\pi/12 \text{ rad} \end{Bmatrix}, \ \underline{\dot{x}}_0 = \begin{Bmatrix} 1.0 \text{ m/s} \\ 0.5 \text{ rad/s} \\ 2.0 \text{ rad/s} \end{Bmatrix}$$

it is now possible to compute the plots for this case

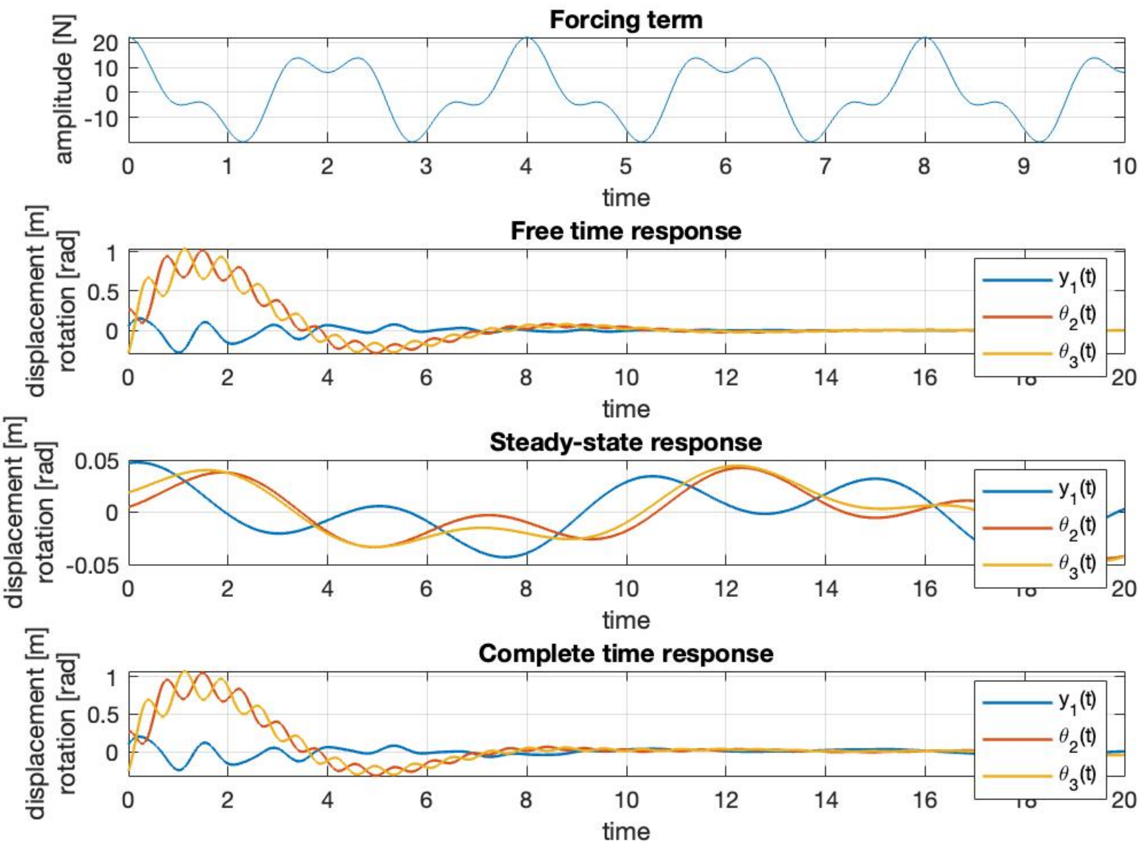

Figure 7: Response to a bi-harmonic force

3.5 Response to a force described by a triangular wave

Lastly, it is required to evaluate only the steady-state response of the vertical displacement of point A with respect to the vertical force applied in A considering that the force is a periodic triangular wave, with fundamental frequency f_0, of the form

$$F(t) = \frac{8}{\pi^2} \sum_{k=0}^{4} (-1)^k \frac{\sin\left((2k+1)2\pi f_0 t\right)}{(2k+1)^2}$$

Since it is required to compute the steady-state response of the vertical displacement of point A, it is necessary to compute the solution for the vector $\underline{x}$ and then apply the same coordinate transformation used to deduce the co-located FRF of point A at the centre of M_3 disk, so that we are able to correctly plot the asked solution.

First of all, the expression for the triangular wave is rewritten with a complex notation:

$$F(t) = \frac{8}{\pi^2}\sum_{k=0}^{4}(-1)^k \frac{\cos\left((2k+1)2\pi f_0 t - \pi/2\right)}{(2k+1)^2} = \Re e\left\{\sum_{k=0}^{4}\left[\frac{8}{\pi^2}\frac{(-1)^k e^{-i\pi/2}}{(2k+1)^2}\right]e^{i(2k+1)2\pi f_0 t}\right\} =$$
$$= \Re e\left\{\sum_{k=0}^{4}\tilde{F}_{k,0}e^{i\Omega_k t}\right\}$$

where the amplitudes $\tilde{F}_{k,0} = \frac{8}{\pi^2}\frac{(-1)^k}{(2k+1)^2}e^{-i\pi/2}$ and the set of frequencies $\Omega_k = (2k+1)2\pi f_0$ are introduced. In this way, the steady state response is simply given by

$$\underline{x}_p(t) = \Re e\left\{\sum_{k=0,4}\tilde{F}_{k,0}[H^*(\Omega_k)][\Lambda_f]^T e^{i\Omega_k t}\right\}$$

Then the solution is applied to the same transformation matrix $[\Phi']$ used earlier for the problem of the co-located FRF. Since $\underline{x}' = [\Phi']\underline{x}$ and $[\Phi']$ is real

$$\underline{x}'_p(t) = \Re e\left\{\sum_{k=0,4}\tilde{F}_{k,0}[\Phi'][H^*(\Omega_k)][\Lambda_f]^T e^{i\Omega_k t}\right\}$$

from which it is easy to extract and display the third component, which is exactly the steady state response of the vertical displacement of point A to the vertical force applied in A. Furthermore another way to find directly the displacement of the point A is to substitute $[\Phi']$ with $[\Lambda_f]$ in the previous formula as follows

$$y_{A_p(t)} = \Re e\left\{\sum_{k=0,4}\tilde{F}_{k,0}[\Lambda_f][H^*(\Omega_k)][\Lambda_f]^T e^{i\Omega_k t}\right\}$$

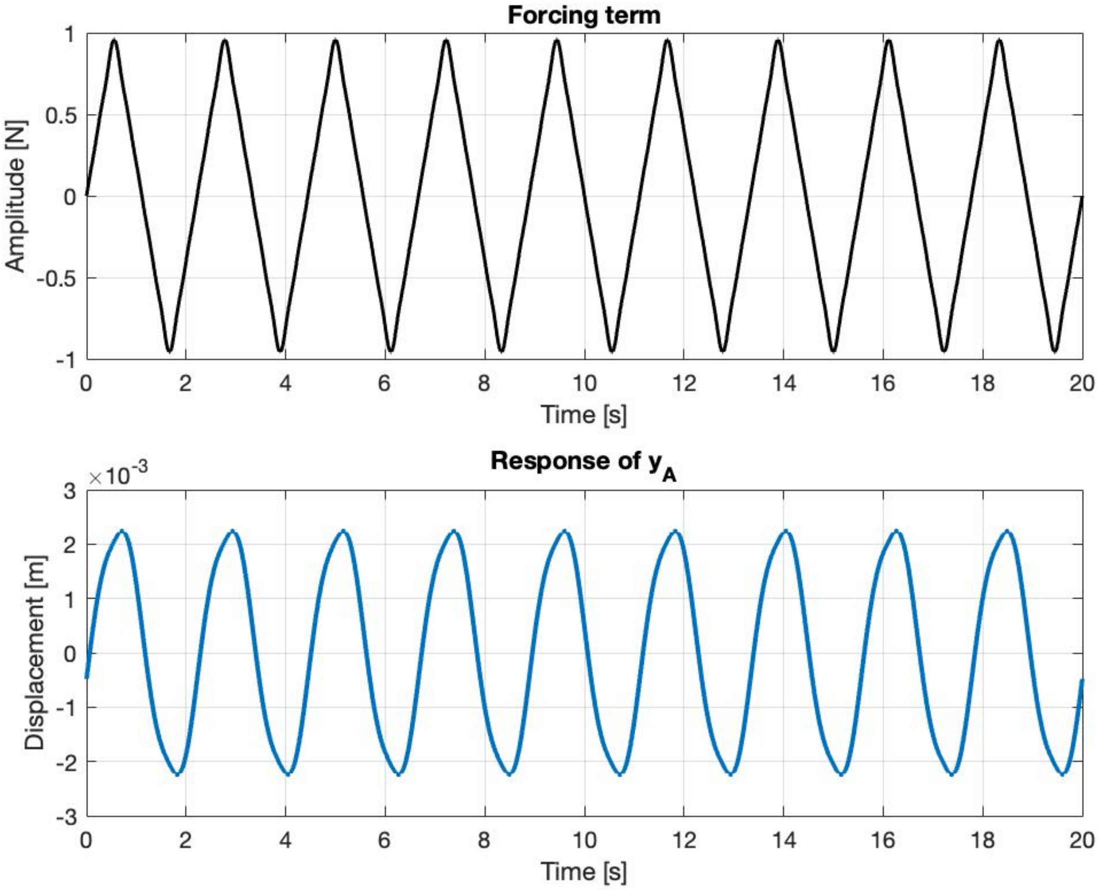

Figure 8: Response to a triangular force

Modal approach considering Rayleigh damping

4.1 Derivation of the EOM in modal coordinates

The modal approach is based on the following transformation of coordinates:

$$\underline{x} = [\Phi]\underline{q}$$

where $\underline{q}$ is the vector that contains the so called modal coordinates and $[\Phi]$ is the matrix containing the mode shapes computed in the first section considering the Rayleigh damping

$$[\Phi] = [\underline{X}_{I,appr} \quad \underline{X}_{II,appr} \quad \underline{X}_{III,appr}]$$

Since $\underline{\dot{x}} = [\Phi]\underline{\dot{q}}$, $\underline{\ddot{x}} = [\Phi]\underline{\ddot{q}}$ and $\delta\underline{x} = [\Phi]\delta\underline{q}$, then, without loss of generality, the system's total kinetic energy T, dissipation function D, potential energy V and lagrangian component $\underline{Q}_q$ can be computed by means of $\underline{q}$, hence obtaining the following new set of equations of motion:

$$[M_q]\underline{\ddot{q}} + [C_q]\underline{\dot{q}} + [K_q]\underline{q} = \underline{Q}_q$$

where

$$\begin{cases} [M_q] = [\Phi]^T[M^*][\Phi] \\ [C_q] = [\Phi]^T[C^*_{appr}][\Phi] \\ [K_q] = [\Phi]^T[K^*][\Phi] \\ \underline{Q}_q = F(t)[\Phi]^T[\Lambda_f]^T \end{cases}$$

This can be shown starting from the Lagrange's equation written in modal coordinates:

$$\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \underline{\dot{q}}}\right)\right\}^T - \left\{\frac{\partial T}{\partial \underline{q}}\right\}^T + \left\{\frac{\partial D}{\partial \underline{\dot{q}}}\right\}^T + \left\{\frac{\partial V}{\partial \underline{q}}\right\}^T = \underline{Q}_q$$

Using the relations $\underline{\dot{x}} = [\Phi]\underline{\dot{q}}$, $\underline{\ddot{x}} = [\Phi]\underline{\ddot{q}}$ and $\delta\underline{x} = [\Phi]\delta\underline{q}$ it is possible to deduce

$$T = \frac{1}{2}\underline{\dot{x}}^T[M^*]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{q}}^T[\Phi]^T[M][\Phi]\underline{\dot{q}} = \frac{1}{2}\underline{\dot{q}}^T[M_q]\underline{\dot{q}}$$

$$V = \frac{1}{2}\underline{x}^T[K*]\underline{x} = \frac{1}{2}\underline{q}^T[\Phi]^T[K][\Phi]\underline{q} = \frac{1}{2}\underline{q}^T[K_q]\underline{q}$$

$$D \approx \frac{1}{2}\underline{\dot{x}}^T[C^*_{appr}]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{q}}^T[\Phi]^T[C^*_{appr}][\Phi]\underline{\dot{q}} = \frac{1}{2}\underline{\dot{q}}^T[C_q]\underline{\dot{q}}$$

Taking into account that $\partial^*W = F(t)[\Lambda_f]\delta\underline{x} = F(t)[\Lambda_f][\Phi]\delta\underline{q}$, lastly the result is

$$\underline{Q}_q = \left\{\frac{\partial^* W}{\partial \underline{q}}\right\}^T \equiv F(t)[\Phi]^T[\Lambda_f]^T$$

from which it is possible to compute all the different contribution of the Lagrange's equation in modal coordinates

$$\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \underline{\dot{q}}}\right)\right\}^T = \left\{\frac{d}{dt}(\underline{\dot{q}}^T[M_q])\right\}^T = [M_q]^T\underline{\ddot{q}} \equiv [M_q]\underline{\ddot{q}}, \quad \left\{\frac{\partial T}{\partial \underline{q}}\right\}^T \equiv \underline{0}$$

$$\left\{\frac{\partial D}{\partial \underline{\dot{q}}}\right\}^T = \{\underline{\dot{q}}^T[C_q]\}^T = [C_q]^T\underline{\dot{q}} \equiv [C_q]\underline{\dot{q}}, \quad \left\{\frac{\partial V}{\partial \underline{q}}\right\}^T = \{\underline{q}^T[K_q]\}^T = [K_q]^T\underline{q} \equiv [K_q]\underline{q}$$

and finally getting the set of equations of motion as previously presented
Starting from the modal EOM we are able to compute the Frequency Response Modal Matrix $[H_q(\Omega)]$ with the relation:

$$[H_q(\Omega)] = [-\Omega^2[M_q] + i\Omega[C_q] + [K_q]]^{-1}$$

Since $[M_q]$, $[K_q]$ and, thanks to Rayleigh damping, also $[C_q]$ are all diagonal matrices, then, also $[H_q(\Omega)]$ is a diagonal matrix.

It is now possible to write the following expression:

$$\begin{aligned} \underline{x}_0 =& [\Phi]\underline{q}_0 = [\Phi] \cdot [H_q(\Omega)]\underline{F}_{q,0} = [\Phi][H_q(\Omega)]([\Phi]^T[\Lambda_f]^T F_0) \\ =& ([\Phi][H_q(\Omega)][\Phi]^T)[\Lambda_f]^T F_0 \equiv [H^*(\Omega)][\Lambda_f]^T F_0 \end{aligned}$$

from which it is clear that $[H^*(\Omega)] = [\Phi][H_q(\Omega)][\Phi]^T$. This equation tells that the FRF $[H^*(\Omega)]$ is a linear superposition of the elements of $[H^*(\Omega)]$.

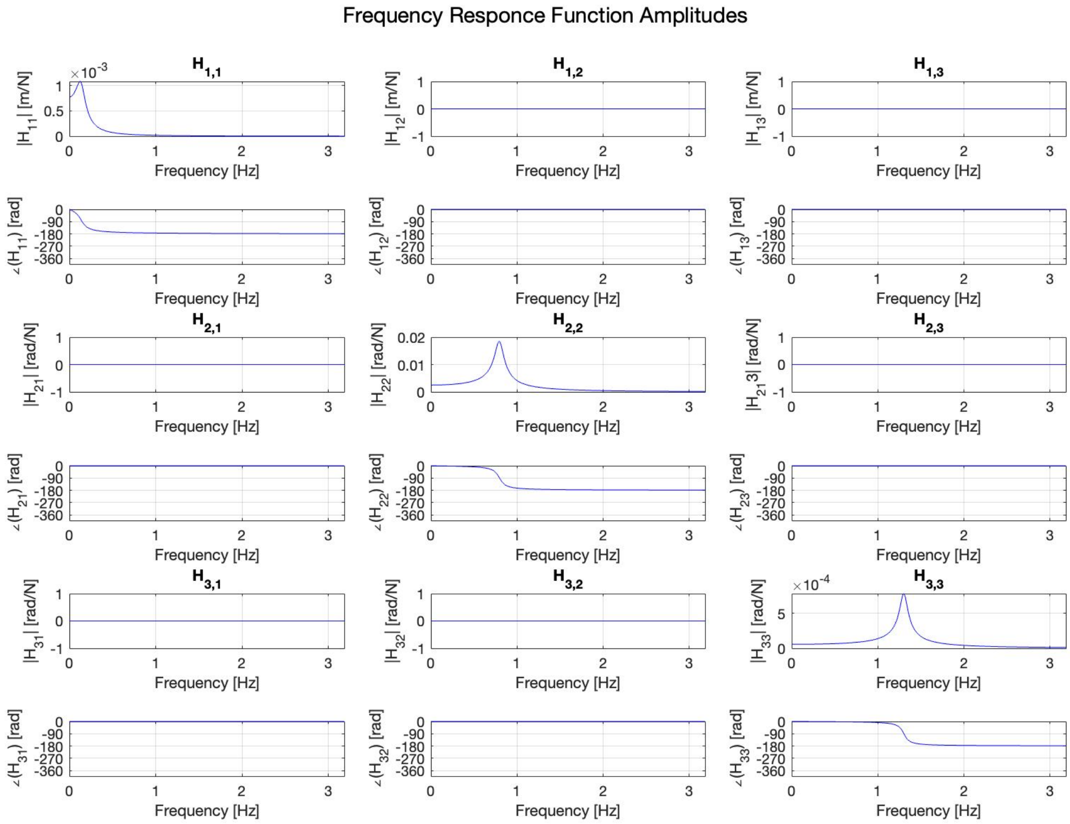

Figure 9: Modal Frequency Response Matrix

4.2 Co-located FRM in modal approach

The modal matrix $[H_q(\Omega)]$ defines the steady state behaviour of the modal coordinates under the effect of an external harmonic force of frequency Ω. Furthermore, by means of an appropriate matrix transformation, it is possible to deduce, starting from the modal coordinates, the steady state behaviour of new independent coordinates under the effect of the same external harmonic force.

The aim is to rewrite the FRM $[H'(\Omega)]$ but related to the new set of independent coordinates $\underline{x}'$. By calling $[\Phi']$ the correspondent transformation matrix such that

$$\underline{x}' = [\Phi']\underline{x}$$

Hence,

$$\begin{cases} \underline{x} = [\Phi]\underline{q} \\ \underline{x}' = [\Phi']\underline{x} \end{cases} \Rightarrow \underline{x}' = [\Phi'] \cdot [\Phi]\underline{q} \;\Rightarrow \underline{x}'_0 = ([\Phi'] \cdot [\Phi][H_q(\Omega)][\Phi]^T \cdot [\Phi']^T)[\Lambda_f]^T F_0$$

from which the co-located FRM is obtained:

$$[H'(\Omega)] = [\Phi'] \cdot [\Phi][H_q(\Omega)][\Phi]^T \cdot [\Phi']^T$$

The FRF related to the displacement of point A is obtained by substituting $[\Phi']$ with $[\Lambda_f]$ in the previous formula as follows:

$$H'_{3,3}(\Omega) = [\Lambda_f] \cdot [\Phi][H_q(\Omega)][\Phi]^T \cdot [\Lambda_f]^T$$

4.2 Co-located FRF of point A at the centre of M_3 disk (modal approach)

We now want to reconstruct, using the modal approach, the co-located Frequency Response Function (FRF) related to the displacement of the point A when the force is seen in action on A itself

$$\underline{x}' = \begin{Bmatrix} x_1 \\ \theta_2 \\ x_3 \end{Bmatrix}$$

x_3 is the vertical absolute motion in the positive direction of the centre of the disk of mass M_3 (point A) about its static equilibrium position.
For this particular case the transformation matrix $[\Phi']$ is such that

$$[\Phi'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & r \end{bmatrix}$$

It is now possible to compute the FRM $[H'(\Omega)]$ related to this particular case. Since it is required to reconstruct the co-located FRF of point A at the centre of the disk 3, the element is then extracted and displayed $H'_{3,3}(\Omega)$.
It is now possible to compute $[H'(\Omega)]$ for this particular case and display the co-located FRF of point A at the centre of the disk 3, which corresponds to the element $H'_{3,3}(\Omega)$ (both displacement and force seen and applied in A).

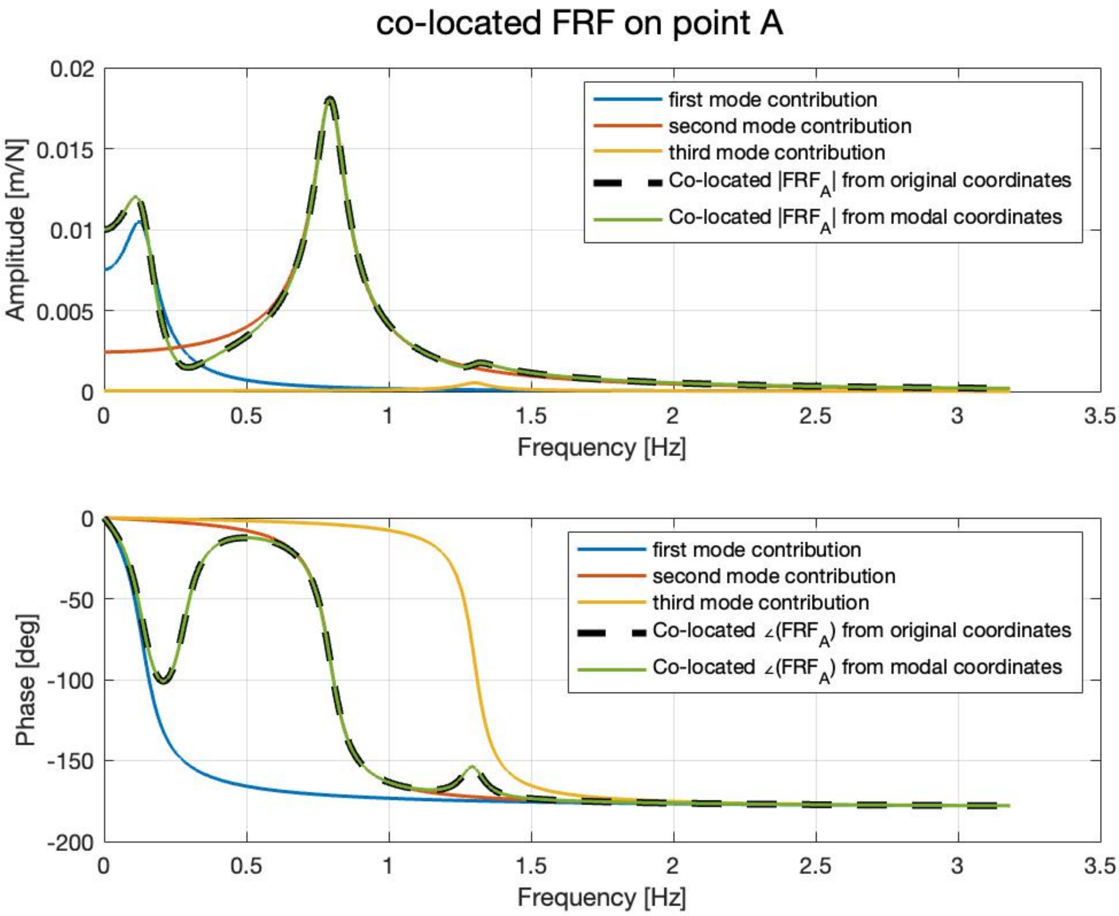

Figure 10: Co-Located FRF on point A

4.3 Co-located FRF between the rotation of the M_3 disk and the torque applied onto the disk itself (modal approach).

This time it is required to re-compute the co-located FRF between the rotation of the disk of mass M_3 and the torque applied onto the disk itself. To do this the matrix $[H(\Omega)] = [\Phi][H_q(\Omega)][\Phi]^T$ is evaluated and considered only the element $H'_{3,3}(\Omega)$.

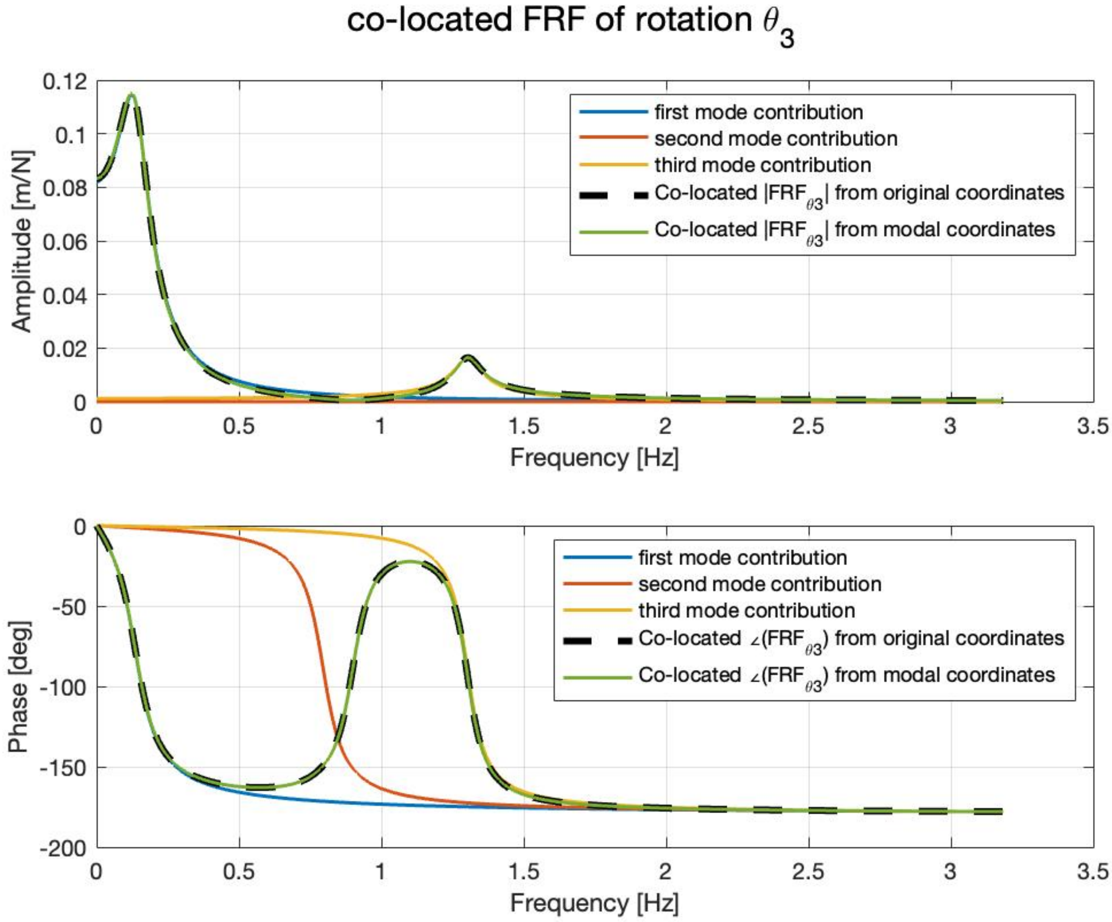

Figure 11: Co-Located FRF of rotation θ_3

4.4 Complete system steady state response (modal approach)

It is now required to study the complete system steady state response $\underline{x}_p(t)$, assuming Rayleigh damping and using modal approach,for two different cases:

$$F_1(t) = A_1 \cos(2\pi f_1 t)$$

$$F_2(t) = A_2 \cos(2\pi f_2 t)$$

The complete steady state response $\underline{x}_p(t)$ can be easily computed by means of $[H_q(\Omega)]$ and $[\Phi]$. In fact, assuming in general that $F(t) = A_k e^{i\Omega_k t}$ (where $\Omega_k = 2\pi f_k$), then

$$\begin{aligned}
\underline{x}_p(t) &= \Re e\{\underline{\tilde{x}}_{p,k,0} e^{i\Omega_k t}\} \\
&= \Re e\{[\Phi]\underline{\tilde{q}}_{p,k,0} e^{i\Omega_k t}\} \\
&= \Re e\{[\Phi][H_q(\Omega_k)]\underline{Q}_k e^{i\Omega_k t}\} \\
&= \Re e\{[\Phi][H_q(\Omega_k)][\Phi]^T[\Lambda_f]^T F_0 e^{i\Omega_k t}\}
\end{aligned}$$

Furthermore, it is now required to compute and display the steady state response of the system considering only one mode of vibration we need to modify Equation. So it is needed to keep different from zero just the mode we want to know the response of. Hence, it is needed to compute the solution with the following relation

$$\underline{x}_p(t) = \Re e\{[\Phi]_{\text{SM}}[H_q(\Omega_k)] \cdot [\Phi]_{\text{SM}}^T[\Lambda_f]^T F_0 e^{i\Omega_k t}\}$$

where

$$[\Phi]_{\text{SM}} = [\underline{X}_{I,appr} \quad \underline{0} \quad \underline{0}]$$

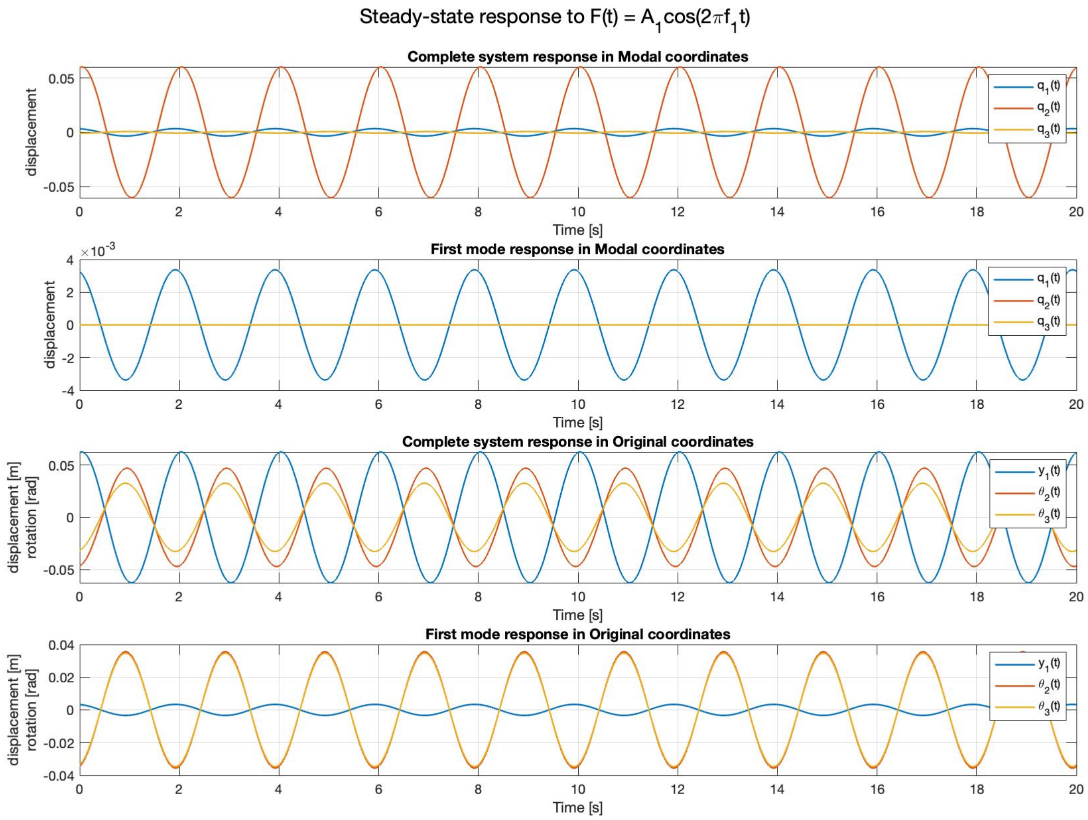

Figure 12: Steady state response to $F_1(t)$

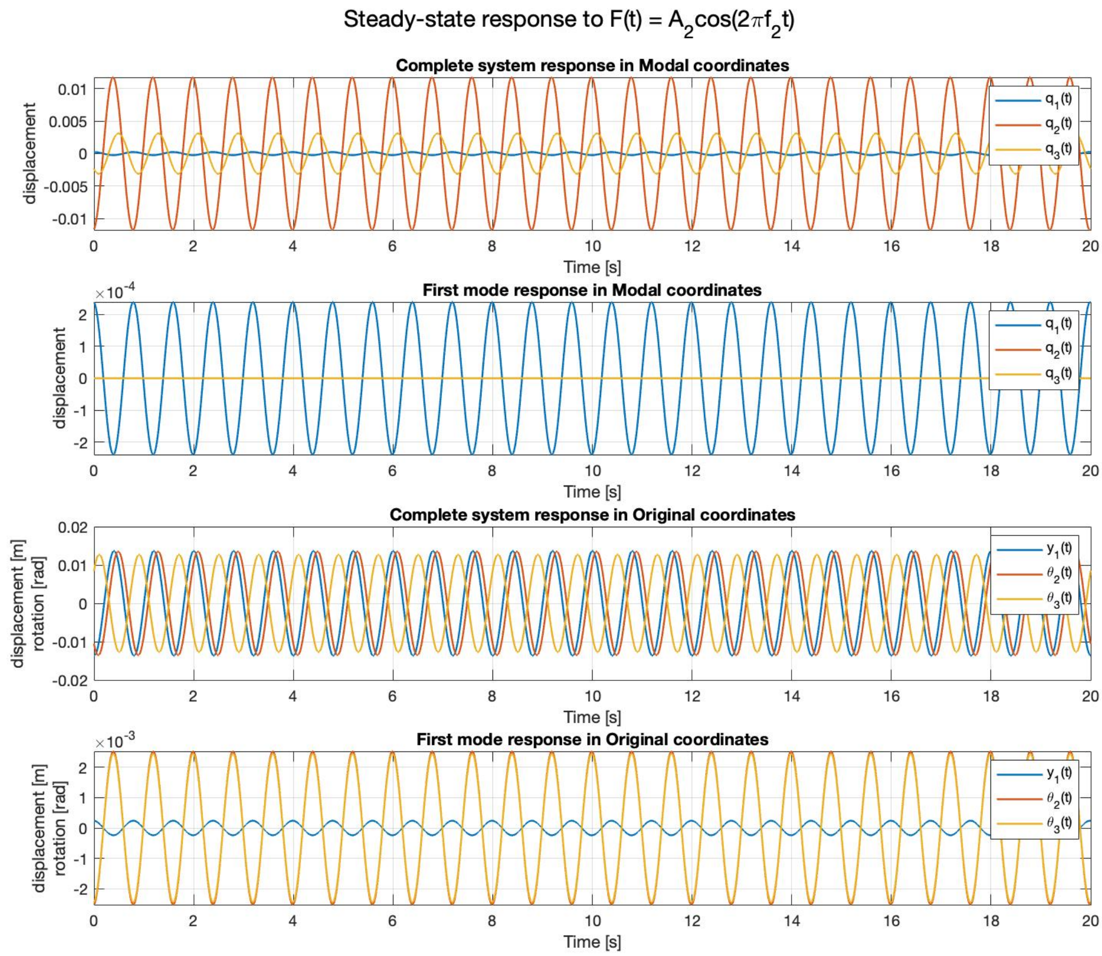

Figure 13: Steady state response to $F_2(t)$

Vibration Analysis and Vibroacoustic

Assignment 3: Modal parameter identification

Homework 3 Report

Students

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Experimental Frequency Response Functions

It is useful to first visualise the data collected in the "Data.mat" file

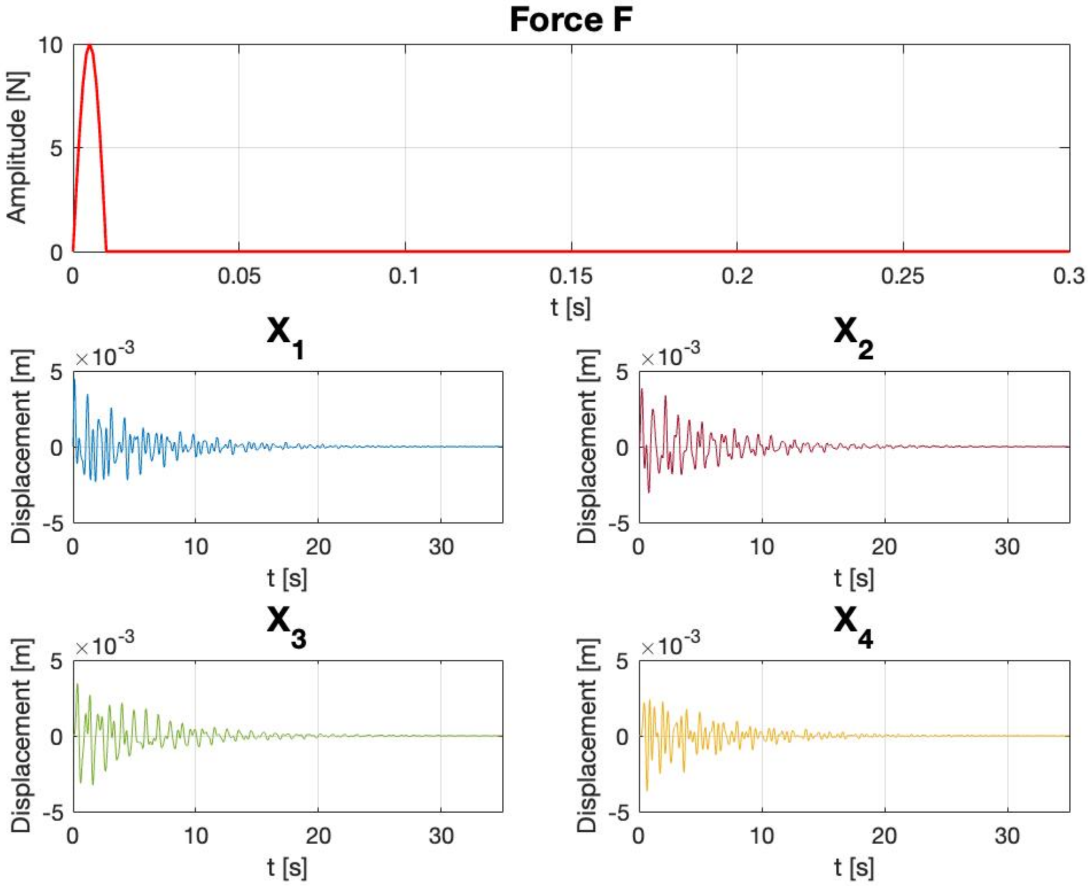

Figure 1: Force F and displacements in four distinct points

It is clear that the force is an impulsive one and so all the responses are different impulse responses at different points of the structure. The frequency response function is now computed for every point. First of all it is necessary to compute the Fourier transform of the force and the four time responses, easily achieved by the function *fft()* of MATLAB. They will be called $F_1(\Omega)$ and $X_i(\Omega)$ where i refers to the point of measurement. Then the ratio between the Fourier Transforms of the displacements and of the force will be computed

$$H_{11}(\Omega) = \frac{X_1(\Omega)}{F_1(\Omega)} \quad H_{21}(\Omega) = \frac{X_2(\Omega)}{F_1(\Omega)}$$

$$H_{31}(\Omega) = \frac{X_3(\Omega)}{F_1(\Omega)} \quad H_{41}(\Omega) = \frac{X_4(\Omega)}{F_1(\Omega)}$$

Remembering that the force is applied in the same location of the first point of measurement, the first FRF $H_{11}(\Omega)$ is the co-located FRF of point 1.
The results are shown below. It is clear that all the four FRFs have four peaks at the same frequencies, which means that we are studying a four degrees of freedom system.

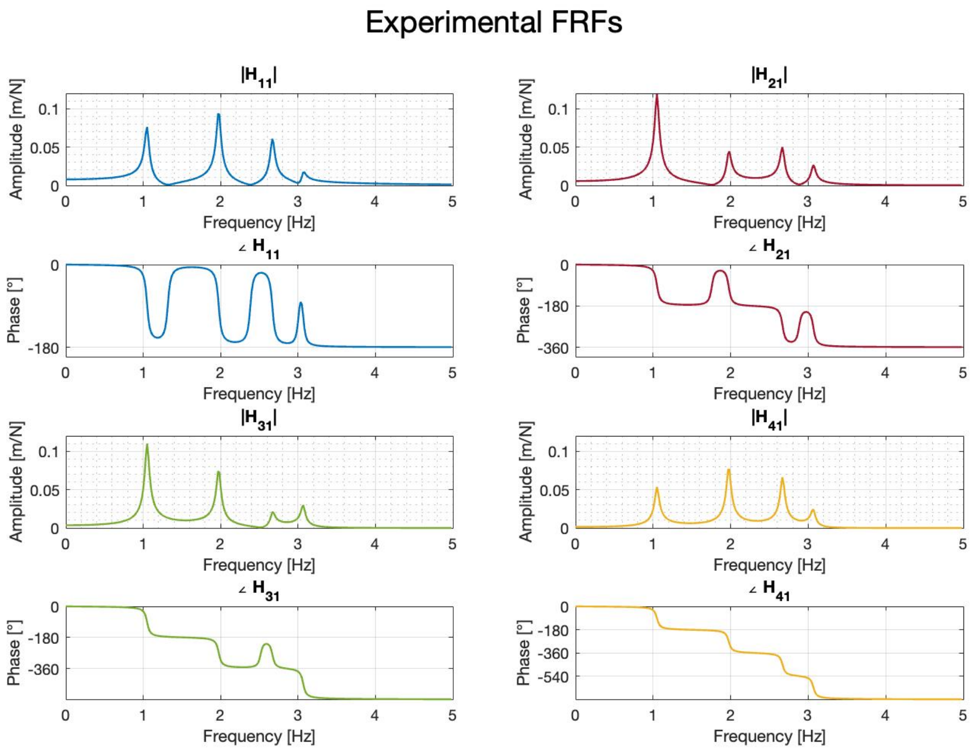

Figure 2: Experimental FRFs of the structure

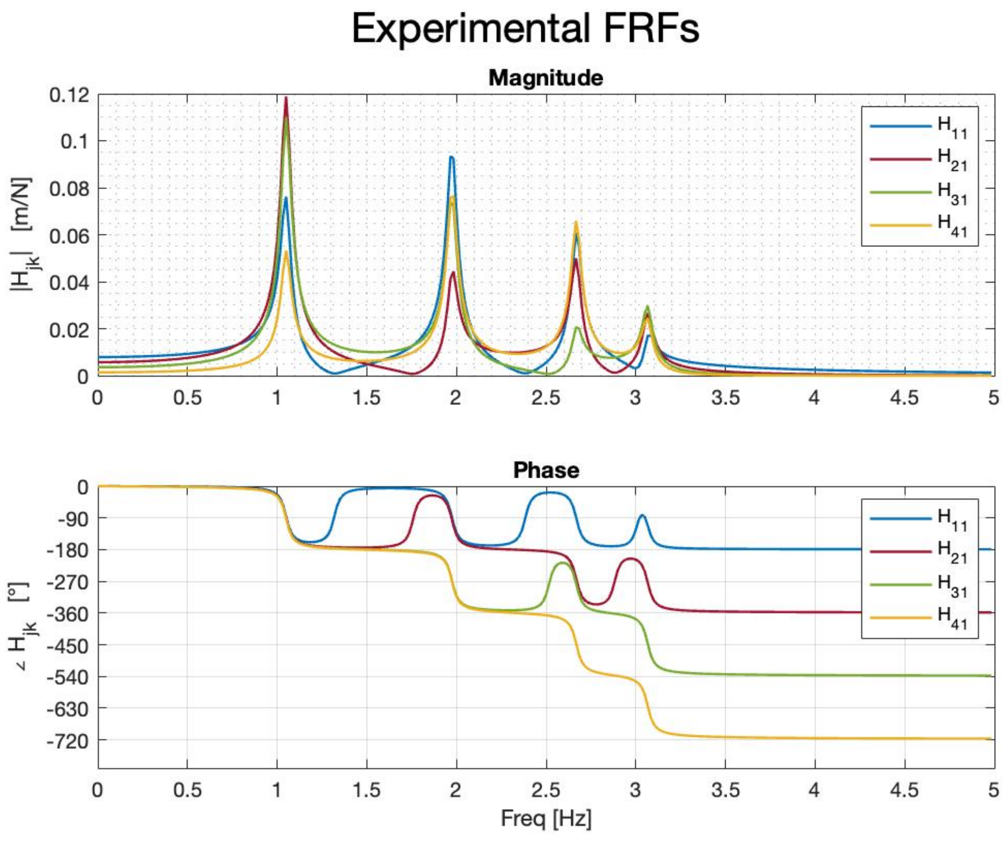

Figure 3: Superimposition of the Experimental FRFs

Estimation of the Parameters through Simplified Methods

To evaluate the parameters of the system through the following simplified method it is first considered, for every FRF, one peak at a time, as if the peaks at lower and higher frequencies have no effect on it. So every FRF is divided in ranges of frequencies centered in the peaks. In other words it is possible to say that it is assumed that the point of measurements are located in the modal coordinates.
It is also assumed that the system is lightly damped, so that the positions of the peaks correspond to the natural frequencies of the system.
Under this assumptions, the FRF of the k-th peak of the j-th point of measurement can be expressed as:

$$H_{j1}^{(k)}(\Omega) = \frac{X_j^{(k)}\ X_1^{(k)}}{-\Omega^2 m_{qkk} + i\Omega c_{qkk} + k_{qkk}}$$

To evaluate the natural frequencies of the system finding the location of the peaks is needed. This is possible by using the MATLAB function *findpeaks()*.
The procedure is done for every FRF, as it will be done for the following parameters evaluations.

Resonance frequencies [Hz]

x_1	x_2	x_3	x_4
1.0500	1.0500	1.0500	1.0500
1.9666	1.9833	1.9666	1.9833
2.6666	2.6666	2.6666	2.6666
3.0833	3.0666	3.0666	3.0666

The adimensional damping ratio through the phase derivation method consists to evaluate the adimensional damping ratio h_k using the derivative of the phase at $\Omega = \omega_{0k}$, by inverting the following equation:

$$\frac{\partial H_{j1}^{(k)}(\omega_{0k})}{\partial \Omega} = -\frac{1}{h_k\ \omega_{0k}}$$

The results are shown below:

Csi computed through the Phase Derivation Method

x_1	x_2	x_3	x_4
0.0286	0.0283	0.0283	0.0282
0.0161	0.0160	0.0160	0.0158
0.0113	0.0113	0.0115	0.0111
0.0144	0.0099	0.0097	0.0096

To compute the adimensional damping ratio through the half power point method, we have to find the frequencies ω_{1k} and ω_{2k} corresponding to the points in which the value of the FRF is

$\frac{1}{\sqrt{2}}$ the value of the considered peak.
Then we apply the following formula:

$$h_k = \frac{\omega_{2k}^2 - \omega_{1k}^2}{4\omega_{0k}^2}$$

The results are shown below:

Csi computed through the Half Power Method

x_1	x_2	x_3	x_4
0.0236	0.0317	0.0317	0.0317
0.0169	0.0168	0.0169	0.0168
0.0094	0.0125	0.0126	0.0125
0.0163	0.0109	0.0109	0.0108

Mode Shapes

Now it is possible to derive the following:

$$H_{j1}^{(k)}(\omega_{0k}) = \frac{X_j^{(k)} \, X_1^{(k)}}{i\omega_{0k} c_{qkk}}$$

It is possible to invert this equation and by posing $X_1^{(k)} = 1$ and $m_{qkk} = 1$ we can evaluate the value of $X_j^{(k)}$.
The results are shown below:

Mode Shapes

x_1	x_2	x_3	x_4
0.1563	0.4564	0.3205	0.1370
0.3284	0.2130	-0.3449	-0.2141
0.3037	-0.3590	-0.1449	0.2389
0.1471	-0.3804	0.4605	-0.1948

Results

The tables in the last paragraphs show the results computed with MATLAB using the previous methods. For natural frequencies and adimensional damping ratios the rows corresponds to different natural frequencies and the columns to different point of measurements.
As we can see the columns are quite similar, which means that for every FRFs the results are almost the same.
The mode shapes are represented by the columns of the last table. We can tell that the results are quite good by looking at the ratio between the same peak in different point of measurements.

Residual Minimization Technique

In this technique it is no longer assumed that the effect of the other peaks is negligible, but it is also to be considered the effect of the lower frequencies peaks as a constant, while considering the effect of the higher ones as inversely proportional to Ω^2. So the expression of the FRF of the k-th peak of the j-th point of measurement becomes:

$$H_{j1}^{NUM(k)}(\Omega) = \frac{A_j + iB_j}{-\Omega^2 m_{qkk} + i\Omega c_{qkk} + k_{qkk}} + C_j + iD_j + \frac{E_j + iF_j}{\Omega^2}$$

It is now needed to find the following parameters (m_{qkk} is assumed equal to 1):
$[c_{qkk}, k_{qkk}, A_j, B_j, C_j, D_j, E_j, F_j]$
To do so, the error will be minimized ϵ between $H_{j1}^{NUM(k)}(\Omega)$ and the experimental FRF, in the range of frequencies considered, that is going to be called $H_{j1}^{EXP(k)}(\Omega)$.

$$\epsilon = \Re e \left(H_{j1}^{NUM(k)}(\Omega) - H_{j1}^{EXP(k)}(\Omega) \right)^2 + \Im m \left(H_{j1}^{NUM(k)}(\Omega) - H_{j1}^{EXP(k)}(\Omega) \right)^2$$

By putting as initial values of the unknowns the one found in the previous section $\left[c_{qkk}, k_{qkk}, X_j^{(k)}, 0, 0, 0, 0, 0\right]$ and by using the function of MATLAB *fminsearch()* to find them. These are the results.

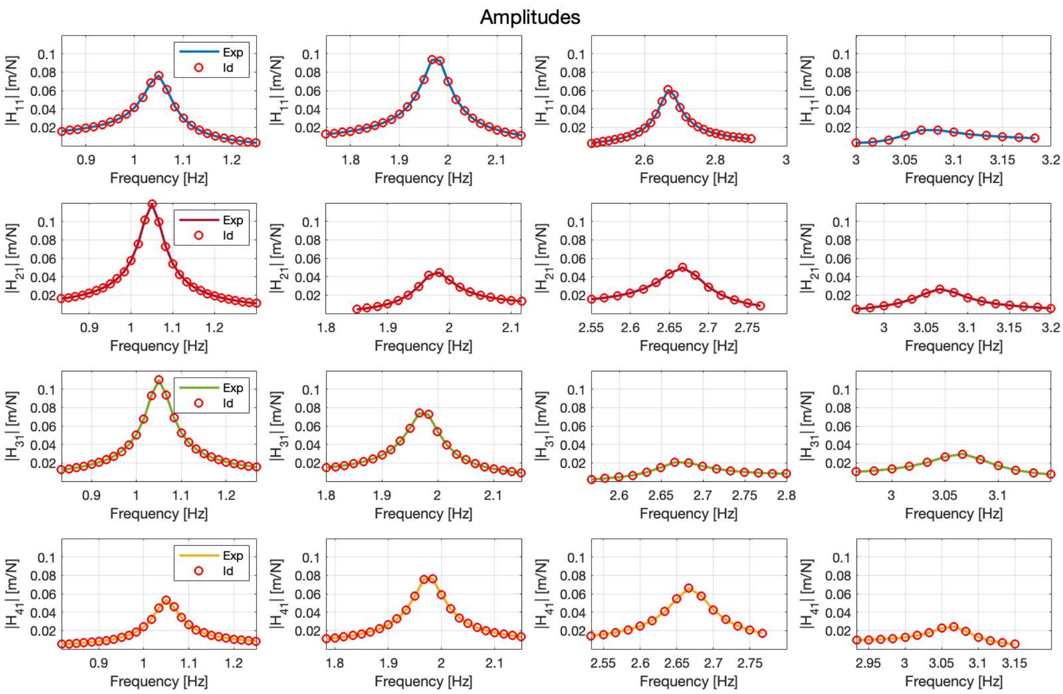

Figure 4: Amplitude of the FRFs

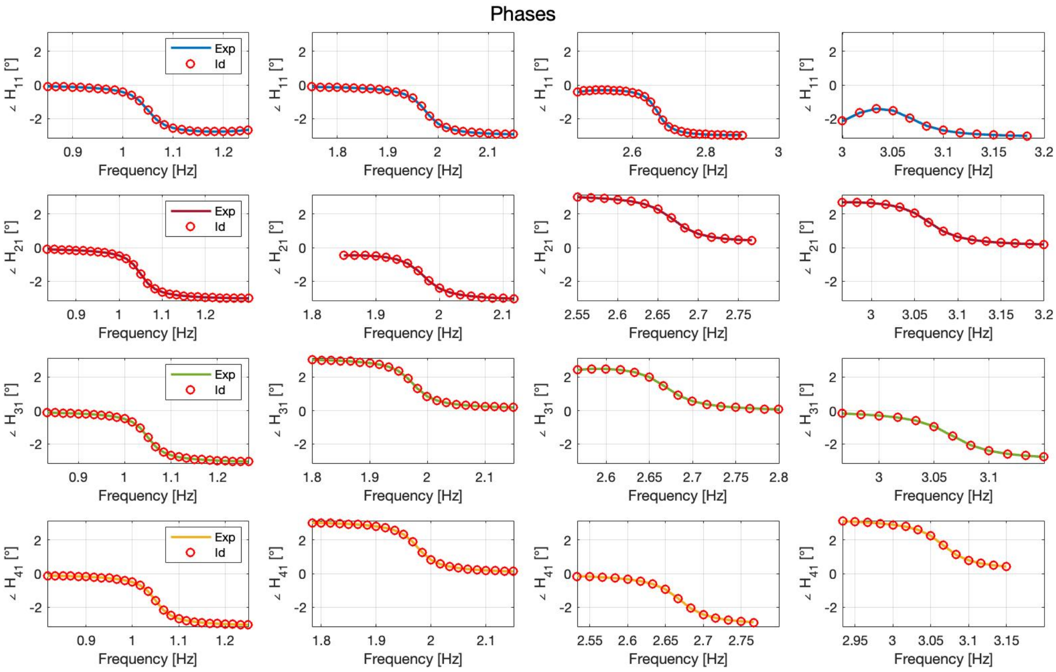

Figure 5: Phase of the FRFs

We can see that the results are almost equal to the experimental FRFs, which means that in this case the residual minimization technique gives us a good approximation of the studied system.

Comparison between Methods

The following tables show the parameters of the system found by using the residual minimization technique.
For the natural frequencies and the adimensional damping ratios we can see a better convergence of the results, compared to the ones obtained with the simplified methods. This is due to the fact that in this case we have considered more than one peak at a time.

Identified frequencies [Hz]

x_1	x_2	x_3	x_4
1.0501	1.0501	1.0500	1.0500
1.9754	1.9753	1.9754	1.9753
2.6691	2.6691	2.6687	2.6689
3.0661	3.0661	3.0664	3.0666

Identified Csi

x_1	x_2	x_3	x_4
0.0252	0.0253	0.0253	0.0253
0.0134	0.0134	0.0134	0.0134
0.0099	0.0099	0.0099	0.0101
0.0086	0.0086	0.0086	0.0086

Identified Modes

x_1	x_2	x_3	x_4
0.1668	0.4013	0.3405	0.0976
0.2613	0.1847	-0.2768	-0.1685
0.2420	-0.3169	-0.1154	0.1908
0.1174	-0.3306	0.3766	-0.1561

Reconstructed FRFs through Modal Approach

First of all it is possible to build the mass, damping and stiffness matrices in modal coordinates from the parameters just evaluated through the residual minimization technique, remembering that it was imposed $m_{qkk} = 1 \ \forall k$.

$$[M_q] = \begin{bmatrix} m_{q11} & 0 & 0 & 0 \\ 0 & m_{q22} & 0 & 0 \\ 0 & 0 & m_{q33} & 0 \\ 0 & 0 & 0 & m_{q44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[C_q] = \begin{bmatrix} c_{q11} & 0 & 0 & 0 \\ 0 & c_{q22} & 0 & 0 \\ 0 & 0 & c_{q33} & 0 \\ 0 & 0 & 0 & c_{q44} \end{bmatrix}$$

$$[K_q] = \begin{bmatrix} k_{q11} & 0 & 0 & 0 \\ 0 & k_{q22} & 0 & 0 \\ 0 & 0 & k_{q33} & 0 \\ 0 & 0 & 0 & k_{q44} \end{bmatrix}$$

The modal frequency response matrix is evaluated as:

$$[H_q] = \left[-\Omega^2 [M_q] + i\Omega [C_q] + [K_q] \right]^{-1}$$

Now we define the modal matrix as the combination of the mode shapes we found:

$$[\phi] = \left[\underline{X}^{(1)} \ \underline{X}^{(2)} \ \underline{X}^{(3)} \ \underline{X}^{(4)} \right]$$

Finally the frequency response function with respect to the measurement points is:

$$[H] = [\phi] [H_q] [\phi]^T$$

The elements of the first row of the matrix are the reconstructed FRFs. Below are the individual modal FRFs and a comparison with the experimental ones.

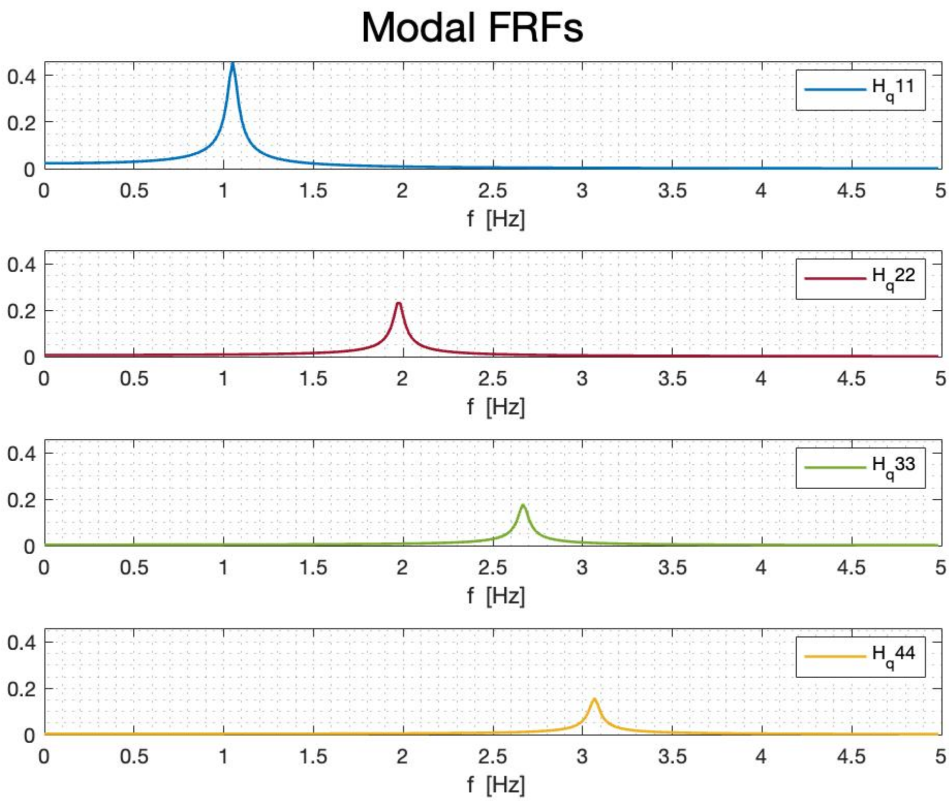

Figure 6: Individual Modal FRFs

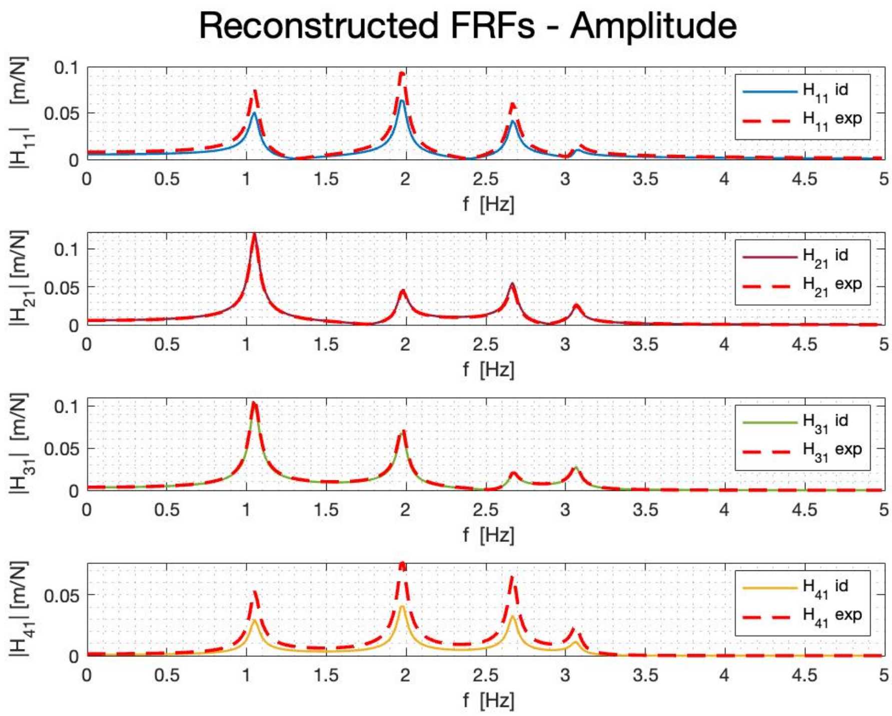

Figure 7: Amplitude comparison between original and reconstructed FRFs

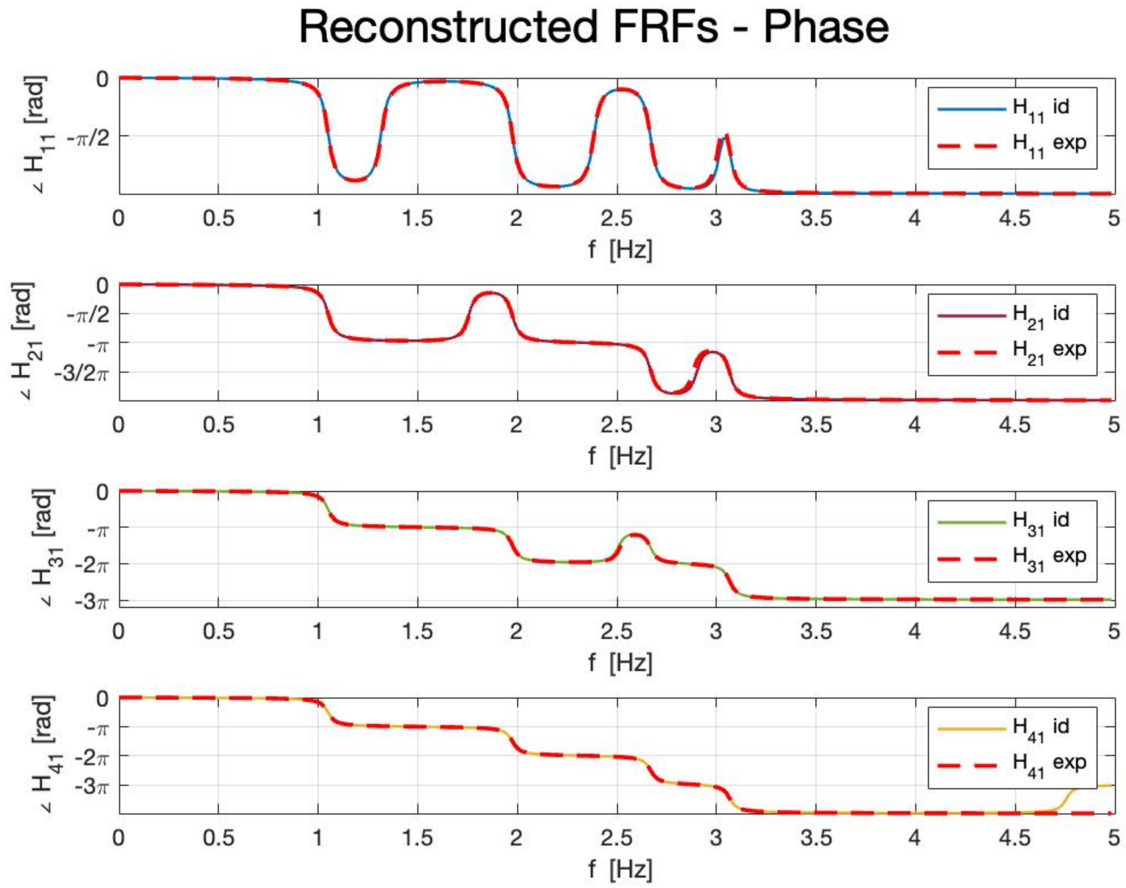

Figure 8: Phase comparison between original and reconstructed FRFs

Co-Located FRF of the Second Point of Measurement

The co-located FRF of x_2 is simply the element of the frequency response matrix $[H]$ at the second row and second column: H_{22}.

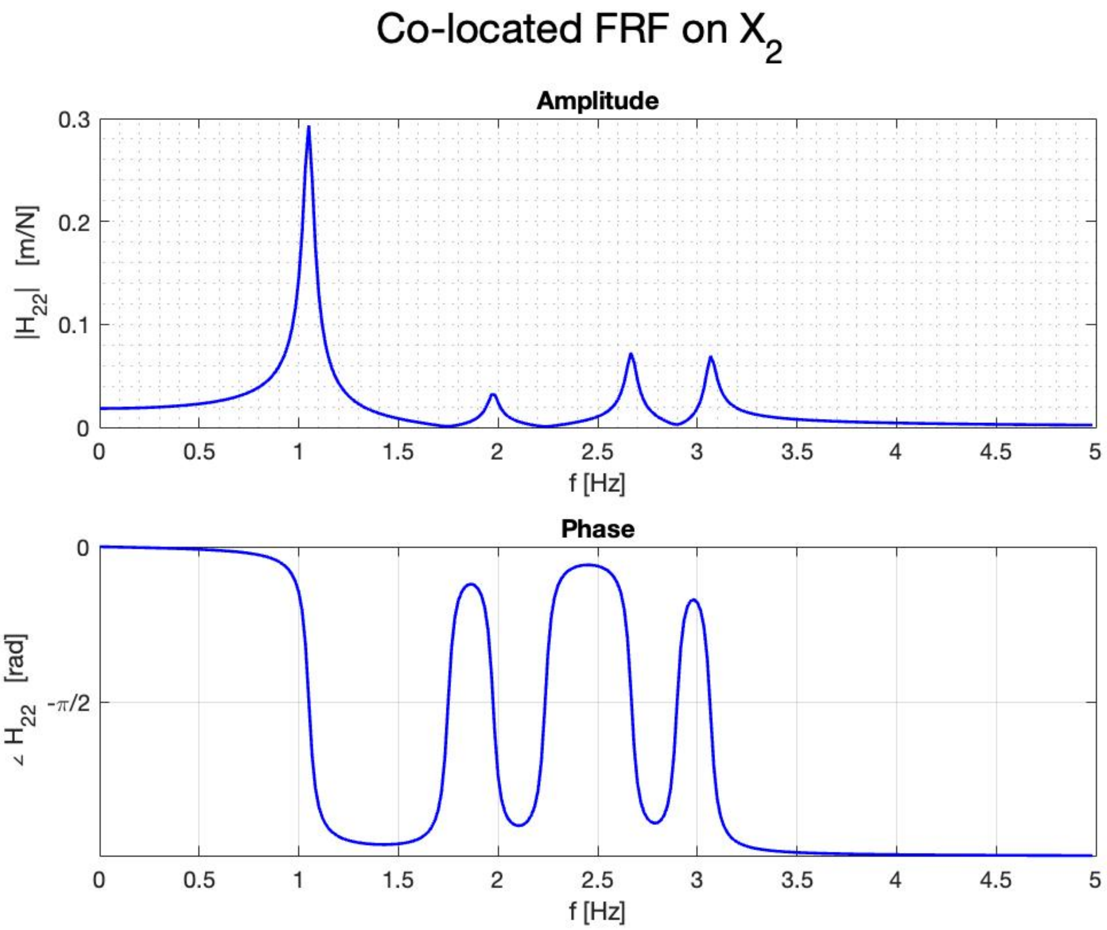

Figure 9: Co-Located FRF

Mass, Damping an Stiffness matrices

The mass, damping and stiffness matrices with respect to the point of measurements can be evaluated through the modal matrices as follows:

$$[M] = \left[[\phi]^T\right]^{-1} [M_q] \, [\phi]^{-1}$$

$$[C] = \left[[\phi]^T\right]^{-1} [C_q] \, [\phi]^{-1}$$

$$[K] = \left[[\phi]^T\right]^{-1} [K_q] \, [\phi]^{-1}$$

The following figure shows the obtained results.

Mass matrix

x_1	x_2	x_3	x_4
3.9584	-0.5267	2.2389	-0.6591
-0.5267	6.2021	-1.1392	2.5579
2.2638	-1.1392	6.4744	-1.7666
-0.6591	2.5579	-1.7666	4.7802

Damping matrix

x_1	x_2	x_3	x_4
1.3222	-0.1744	0.7419	-0.2123
-0.1744	2.0657	-0.3712	0.8446
0.7419	-0.3712	2.1528	-0.5838
-0.2123	0.8446	-0.5838	1.5936

Stiffness matrix

1.0e+03*

x_1	x_2	x_3	x_4
0.8455	-0.7256	0.5660	-0.3873
-0.7256	1.4064	-1.1035	0.7532
0.5660	-1.1035	1.5863	-1.0866
-0.3873	0.7532	-1.0866	1.3709