VIBRATION ANALYSIS AND VIBROACOUSTIC

ASSIGNMENT 2: ANALYSIS OF A N-DOF SYSTEM

Homework 2 Report

Students

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1. Equations of Motion

1.0.1 Determination of the Degrees of Constraint

As stated in report 1 it is paramount to correctly analyze a system to determine its Degrees Of Freedom. To get to that result it is first needed to determine the total amount of DOC of the system. By following the same path the result is the following:

$$n_{DOC} = 2 * 2_{cart} + 2 * 1_{hinge} + 1 * 1_{roller} + 2 * 1_{string} = 9$$

1.0.2 Degrees of Freedom

Having now concluded the number of DOC of the system, and by remembering that in a 2D system each body has a maximum of 3 DOF it is easy to conclude that the system has the following number of DOF:

$$n_{DOF} = 3 * 4_{bodies} - n_{DOC} = 3$$

The system has 3 DOF

1.0.3 Choice of Independent Variable

Since the system has 3 DOF it is first needed to find 3 independent variables that describe the system in a way that all other variables can be expressed as a linear combination of the independent ones. For this purpose the displacement of mass M1 and the angles θ_2 and θ_3 respectively the angular displacement of the discs M2 and M3 with respect to their equilibrium position.

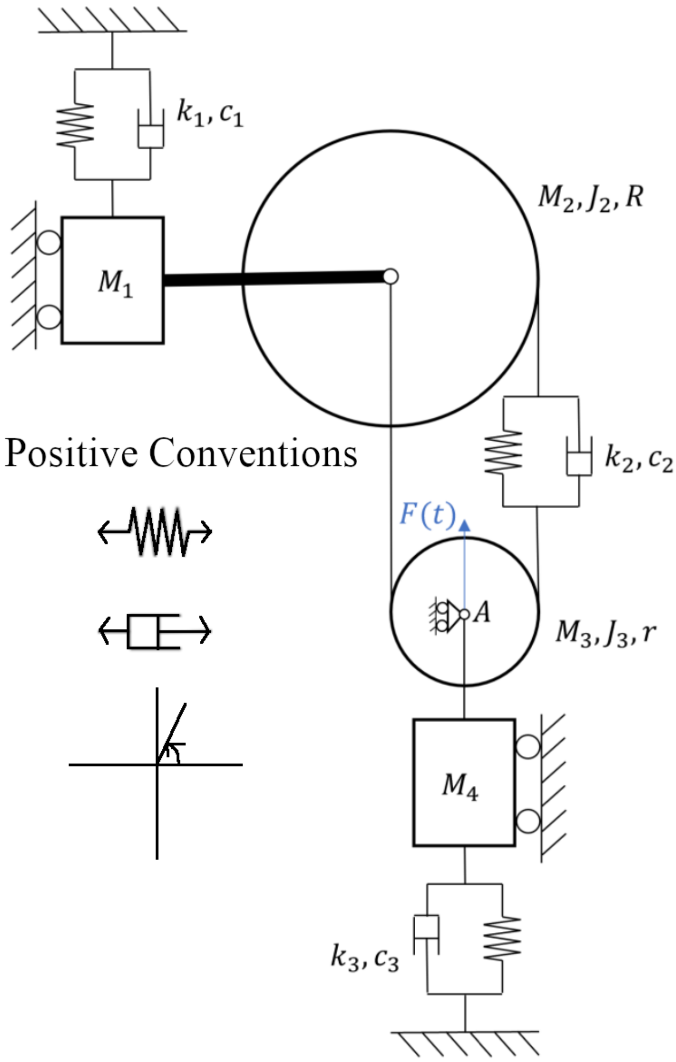

Figure 1: Representation of the system and positive conventions

1.0.4 Derivation of Parameters

All other parameters of the system need to be expressed as a function of the independent variables

$\Delta\ell_1$	$-y_1$
$\Delta\ell_2$	$y_1 - r\theta_3 + R\theta_2$
$\Delta\ell_3$	$y_1 + r\theta_3$
y_4	$\Delta\ell_3$
y_4	y_3
$\dot{\Delta\ell}_1$	$-\dot{y}_1$
$\dot{\Delta\ell}_2$	$\dot{y}_1 - r\dot{\theta}_3 + R\dot{\theta}_2$
$\dot{\Delta\ell}_3$	$\dot{y}_1 + r\dot{\theta}_3$
y_2	y_1

1.1.0 Derivation of the Equations of Motion

The general formulation of the equations of motion can be computed by means of the Lagrange's equation in its vectorial form:

$$\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \underline{\dot{x}}}\right)\right\}^T - \left\{\frac{\partial T}{\partial \underline{x}}\right\}^T + \left\{\frac{\partial D}{\partial \underline{\dot{x}}}\right\}^T + \left\{\frac{\partial V}{\partial \underline{x}}\right\}^T = \underline{Q}$$

where T is the total kinetic energy, D is the total dissipation function, V is the total potential energy and $\underline{Q}$ is the lagrangian component of the external forces applied to the system. Every element will now be individually computed. For the sake of simplicity the three independent variables will be grouped in a column vector as follows:

$$\underline{x} = \begin{Bmatrix} y_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

1.1.1 Kinetic Energy

The total kinetic energy is a sum of individual translation and rotation components of the different bodies present in the system

$$T = \frac{1}{2}M_1 v_1^2 + \frac{1}{2}M_2 v_2^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}M_3 v_3^2 + \frac{1}{2}J_3\omega_3^2 + \frac{1}{2}M_4 v_4^2$$

where v_i is the vertical velocity of the mass M_i, while ω_i is the angular velocity of the mass M_i with moment of inertia J_i.

To write the kinetic energy as a function of the chosen independent variables the vector $\underline{y}$ is introduced such that each of the variables that characterize the motion of the bodies of the system can be considered.

$$\underline{y} = \begin{Bmatrix} y_1 \\ y_2 \\ \theta_2 \\ y_3 \\ \theta_3 \\ y_4 \end{Bmatrix}$$

From this it is possible to derive $\underline{\dot{y}}$

$$\underline{\dot{y}} = \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_2 \\ \dot{y}_3 \\ \dot{\theta}_3 \\ \dot{y}_4 \end{Bmatrix} \equiv \begin{Bmatrix} v_1 \\ v_2 \\ \omega_2 \\ v_3 \\ \omega_3 \\ v_4 \end{Bmatrix}$$

Now the previous equation can be rewritten as $T = \frac{1}{2}\underline{\dot{y}}^T[M]\underline{\dot{y}}$, where

$$[M] = \begin{bmatrix} M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_4 \end{bmatrix}$$

The vector $\underline{\dot{y}}$ can be reformulated considering the dependencies of its coordinates on the independent variables:

$$\underline{\dot{y}} = \left(\frac{\partial \underline{y}}{\partial \underline{x}}\right)\underline{\dot{x}} = [\Lambda_m]\underline{\dot{x}}$$

where the Jacobian matrix $[\Lambda_m]$ contains the partial derivatives of the vector $\underline{y}$ with respect to the vector $\underline{x}$ of the independent variables. Since

	$\dot{y}_1$	$\dot{\theta}_2$	$\dot{\theta}_3$
v_1	1	0	0
v_2	1	0	0
ω_2	0	1	0
v_3	1	0	r
ω_3	0	0	1
v_4	1	0	r

Which results in the following Jacobian matrix

$$[\Lambda_m] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & r \\ 0 & 0 & 1 \\ 1 & 0 & r \end{bmatrix}$$

Now the total kinetic energy can be written as a function of the independent variables:

$$T = \frac{1}{2}\underline{\dot{y}}^T[M]\underline{\dot{y}} \equiv \frac{1}{2}\underline{\dot{x}}^T[\Lambda_m]^T[M][\Lambda_m]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{x}}^T[M^*]\underline{\dot{x}}$$

In the what has been obtained is the following:

$$[M^*] = [\Lambda_m]^T[M][\Lambda_m] \equiv \begin{bmatrix} M_1 + M_2 + M_3 + M_4 & 0 & r(M_3 + M_4) \\ 0 & J_2 & 0 \\ r(M_3 + M_4) & 0 & r^2(M_3 + M_4) + J_3 \end{bmatrix}$$

1.1.2 Potential Energy

Just as in the previous paragraph the potential energy is computed as a sum of the individual potential energies of the single bodies

$$V_{el} = \frac{1}{2}k_1\Delta l_1^2 + \frac{1}{2}k_2\Delta l_2^2 + \frac{1}{2}k_3\Delta l_3^2$$

As the EOM is computed for small vibrations around the point of equilibrium of the system it is safe to assume that the gravitational potential energy will not have an impact on our computations so it is assumed 0 at all times.
As done previously a vector is introduced to help in the computations

$$\underline{\Delta l} = \begin{Bmatrix} \Delta l_1 \\ \Delta l_2 \\ \Delta l_3 \end{Bmatrix}$$

And a matrix to keep track of all the springs is created

$$[K] = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}$$

Just as before the jacobian matrix is computed for this case:

	y_1	θ_2	θ_3
Δl_1	-1	0	0
Δl_2	0	R	$-R$
Δl_3	1	0	r

Which results in the following matrix

$$[\Lambda_k] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & R & -R \\ 1 & 0 & r \end{bmatrix}$$

By substitution it is obtained:

$$V_{el} = \frac{1}{2}\underline{\Delta l}^T[K]\underline{\Delta l} \equiv \frac{1}{2}\underline{x}^T[\Lambda_k]^T[K][\Lambda_k]\underline{x} = \frac{1}{2}\underline{x}^T[K^*]\underline{x}$$

where, in conclusion,

$$[K^*] = [\Lambda_k]^T[K][\Lambda_k] \equiv \begin{bmatrix} k_1 + k_2 & 0 & rk_3 \\ 0 & R^2k_2 & -R^2k_2 \\ rk_3 & -R^2k_2 & R^2k_2 + r^2k_3 \end{bmatrix}$$

1.1.3 Dissipation Function

Again, the total dissipation function D is given by

$$D = \frac{1}{2}c_1\dot{\Delta l_1}^2 + \frac{1}{2}c_2\dot{\Delta l_2}^2 + \frac{1}{2}c_3\dot{\Delta l_3}^2$$

As done for the elastic potential energy, the vector $\dot{\underline{\Delta l}}$ is introduced so that the dissipation function can be first rewritten in a vectorial form by means of the diagonal damping matrix $[C]$, and then can be written as a function of the vector of the independent coordinates thanks to a jacobian matrix $[\Lambda_c]$.
Since the springs and the dampers in the system are all in parallel, then the Jacobian matrix $[\Lambda_c]$ is equal to the jacobian matrix $[\Lambda_k]$. Hence, for

$$[C] = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

$$\dot{\underline{\Delta l}} = [\Lambda_c]\dot{\underline{x}} \equiv [\Lambda_k]\dot{\underline{x}}$$

$$D = \frac{1}{2}\underline{\dot{\Delta l}}^T[C]\underline{\dot{\Delta l}} \equiv \frac{1}{2}\underline{\dot{x}}^T[\Lambda_c]^T[C][\Lambda_c]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{x}}^T[C^*]\underline{\dot{x}}$$

$$[C^*] = [\Lambda_c]^T[C][\Lambda_c] \equiv \begin{bmatrix} c_1 + c_2 & 0 & rc_3 \\ 0 & R^2c_2 & -R^2c_2 \\ rc_3 & -R^2c_2 & R^2c_2 + r^2c_3 \end{bmatrix}$$

1.1.4 Lagrangian Component

The Lagrangian component

$$\underline{Q} = \left\{\frac{\partial^* W}{\partial \underline{x}}\right\}^T$$

is referred to the virtual work $\partial^* W$ of the external force applied to the system in the point A. Since $\partial^* W = F(t)\delta^*$ and the virtual displacement δ^* can be written as a function of the independent variables by means of a proper jacobian matrix $[\Lambda_f]$ as follows $\delta^* = [\Lambda_f]\delta\underline{x}$, then $\partial^* W \equiv F(t)[\Lambda_f]\delta\underline{x}$ and so

$$\underline{Q} = \left\{\frac{\partial^* W}{\partial \underline{x}}\right\}^T \equiv F(t)[\Lambda_f]^T$$

Since

	y_1	θ_2	θ_3
δ	1	0	r

$$\Rightarrow [\Lambda_f] = \begin{bmatrix} 1 & 0 & r \end{bmatrix}$$

then, in conclusion,

$$\underline{Q} = F(t)[\Lambda_f]^T = \begin{Bmatrix} F(t) \\ 0 \\ rF(t) \end{Bmatrix}$$

1.1.5 Derivation of the equations of motion

Now that all preliminary passages have been made it is possible to compute the EOM of the system

$$\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \underline{\dot{x}}}\right)\right\}^T = \left\{\frac{d}{dt}(\underline{\dot{x}}^T[M^*])\right\}^T = [M^*]^T\underline{\ddot{x}} \equiv [M^*]\underline{\ddot{x}}$$

$$\left\{\frac{\partial T}{\partial \underline{x}}\right\}^T \equiv \underline{0} \text{ (The system is linear)}$$

$$\left\{\frac{\partial D}{\partial \underline{\dot{x}}}\right\}^T = \{\underline{\dot{x}}^T[C^*]\}^T = [C^*]^T\underline{\dot{x}} \equiv [C^*]\underline{\dot{x}}$$

$$\left\{\frac{\partial V_{el}}{\partial \underline{x}}\right\}^T = \{\underline{x}^T[K^*]\}^T = [K^*]^T\underline{x} \equiv [K^*]\underline{x}$$

then the equation becomes

$$[M^*]\underline{\ddot{x}} + [C^*]\underline{\dot{x}} + [K^*]\underline{x} = \underline{Q}$$

which is a system of 3 linear differential equations.

1.2 Eigenvalues and Eigenvectors

It is now possible to further analyse the system by computing its eigenvalues and eigenvectors

1.2.1 Free Undamped Case

The free undamped system is characterized by $\underline{Q} = 0$ and $[C^*] = 0$. In this case the equations of motion (EOM) are simply the following:

$$[M^*]\underline{\ddot{x}} + [K^*]\underline{x} = \underline{0}$$

The solution of the EOM is of the form $\underline{x} = \underline{X}e^{\lambda t}$. Since $\underline{\ddot{x}} = \lambda^2 \underline{X}e^{\lambda t}$, then it is possible to rewrite the equation as

$$(\lambda^2[M^*] + [K^*])\underline{X}e^{\lambda t} = 0$$

Since $e^{\lambda t} \neq 0 \; \forall t \in R$ and the trivial solution is not of interest $\underline{X} = 0$, then the previous equation is satisfied if and only if

$$\lambda^2[M^*] + [K^*] = 0$$

Since $\det([M^*]) \neq 0$ then $\exists \; [M^*]^{-1}$ and so it is possible to rearrange the latter equation in the following way:

$$\lambda^2[I] + [M^*]^{-1}[K^*] = 0$$

where $[I]$ is the 3×3 identity matrix.

The different λ_i are the roots of the determinant of $\lambda^2[I] + [M^*]^{-1}[K^*]$. Hence, by imposing

$$\det(\lambda^2[I] + [M^*]^{-1}[K^*]) = 0$$

We can easily solve the problem through a MATLAB code, that gives us the following results:

$$\lambda^2_{1,2} = -0.8623, \qquad \lambda^2_{3,4} = -25.0586, \qquad \lambda^2_{5,6} = -67.0143$$

$$\lambda_{1,2} = \pm\; i\;\omega_{01}, \qquad \lambda_{3,4} = \pm\; i\;\omega_{02}, \qquad \lambda_{5,6} = \pm\; i\;\omega_{03}$$

$$\omega_{01} = 0.9286\; rad/s, \qquad \omega_{02} = 5.0059\; rad/s, \qquad \omega_{03} = 8.1862\; rad/s$$

$$\underline{X}^{(I)} = \begin{Bmatrix} -0.0677 \text{ m} \\ 0.7151 \text{ rad} \\ 0.6958 \text{ rad} \end{Bmatrix}, \; \underline{X}^{(II)} = \begin{Bmatrix} 0.9894 \text{ m} \\ -0.1417 \text{ rad} \\ -0.0307 \text{ rad} \end{Bmatrix}, \; \underline{X}^{(III)} = \begin{Bmatrix} 0.1600 \text{ m} \\ 0.6659 \text{ rad} \\ -0.7287 \text{ rad} \end{Bmatrix}$$

which can be normalized with respect to one of their coordinates. For simplicity, for each of them, the first coordinate was chosen which represents the amplitude of the vertical motion of M_1.

$$\underline{X}^{(I)} = \begin{Bmatrix} 1.00 \text{ m} \\ -10.5598 \text{ rad} \\ -10.2753 \text{ rad} \end{Bmatrix}, \; \underline{X}^{(II)} = \begin{Bmatrix} 1.00 \text{ m} \\ -0.1432 \text{ rad} \\ -0.0311 \text{ rad} \end{Bmatrix}, \; \underline{X}^{(III)} = \begin{Bmatrix} 1.00 \text{ m} \\ 4.1612 \text{ rad} \\ -4.5532 \text{ rad} \end{Bmatrix}$$

1.2.2 Free Damped Case

The free damped system is characterized by $\underline{Q} = 0$ and $[C^*] \neq 0$. In this case the equations of motion (EOM) are the following:

$$[M^*]\underline{\ddot{x}} + [C^*]\underline{\dot{x}} + [K^*]\underline{x} = \underline{0}$$

To compute the general solution it is noted that

$$\begin{cases} [M^*]\underline{\ddot{x}} + [C^*]\underline{\dot{x}} + [K^*]\underline{x} = \underline{0} \\ [M^*]\underline{\dot{x}} - [M^*]\underline{\dot{x}} = \underline{0} \end{cases} \Leftrightarrow \begin{bmatrix} [M^*] & [0] \\ [0] & [M^*] \end{bmatrix} \begin{Bmatrix} \underline{\ddot{x}} \\ \underline{\dot{x}} \end{Bmatrix} + \begin{bmatrix} [C^*] & [K^*] \\ -[M^*] & [0] \end{bmatrix} \begin{Bmatrix} \underline{\dot{x}} \\ \underline{x} \end{Bmatrix} = \underline{0} \Leftrightarrow [A]\underline{\dot{z}} + [B]\underline{z} = \underline{0}$$

for

$$\underline{z} = \begin{Bmatrix} \underline{\dot{x}} \\ \underline{x} \end{Bmatrix}, \qquad [A] = \begin{bmatrix} [M^*] & [0] \\ [0] & [M^*] \end{bmatrix}, \qquad [B] = \begin{bmatrix} [C^*] & [K^*] \\ -[M^*] & [0] \end{bmatrix}$$

Since $[M^*]$ is invertible, so is $[A]$ and then the EOM is simply

$$\underline{\dot{z}} - [\Lambda_z]\underline{z} = \underline{0} \text{ for } [\Lambda_z] = -[A]^{-1}[B]$$

which solution is of the form $\underline{z} = \underline{Z}e^{\lambda t}$.

To compute the different eigenvalues λ_i the trivial solution is excluded $\underline{z}_0 = 0$ and so it is desirable to look for the roots of the determinant of $\lambda[I]_{3\times 6} - [\Lambda_z]$. These solutions happen to be of the form $\lambda_i = -\alpha_i + \omega_i$, where ω_i are quite the same eigenvalues found previously for the undamped system

$$\underline{\lambda} = \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{Bmatrix} = \begin{Bmatrix} (-0.0335 + 0.9286i) \text{ rad/s} \\ (-0.0335 - 0.9286i) \text{ rad/s} \\ (-0.3995 + 4.9870i) \text{ rad/s} \\ (-0.3995 - 4.9870i) \text{ rad/s} \\ (-0.1124 + 8.1846i) \text{ rad/s} \\ (-0.1124 - 8.1846i) \text{ rad/s} \end{Bmatrix} = \begin{Bmatrix} \lambda_I \\ \lambda_I^* \\ \lambda_{II} \\ \lambda_{II}^* \\ \lambda_{III} \\ \lambda_{III}^* \end{Bmatrix}$$

The corresponding eigenvectors $\underline{Z}^{(i)}$ are the one that solve

$$(\lambda_i[I]_{6\times 6} - [\Lambda_z])\underline{Z}^{(i)} = 0, \ \forall\, i = 1, \dots, 6$$

Discarding the 3 first elements of each vector (which are the ones related to velocity),the following eigenvectors are obtained

$$\underline{X}_I^{(1,2)} = \begin{Bmatrix} (-0.0496 \mp 0.0032i) \text{ m} \\ (0.5238 \pm 0.0000i) \text{ rad} \\ (0.5097 \mp 0.0010i) \text{ rad} \end{Bmatrix}$$

$$\underline{X}_{II}^{(3,4)} = \begin{Bmatrix} (0.0155 \pm 0.1930i) \text{ m} \\ (-0.0121 \mp 0.0267i) \text{ rad} \\ (-0.0057 \mp 0.0046i) \text{ rad} \end{Bmatrix}$$

$$\underline{X}_{III}^{(5,6)} = \begin{Bmatrix} (0.0017 \pm 0.0196i) \text{ m} \\ (0.0017 \pm 0.0807i) \text{ rad} \\ (-0.0012 \mp 0.0883i) \text{ rad} \end{Bmatrix}$$

which can be normalized with respect to one of their coordinates. As done before, for each of them, the first coordinate are chosen

$$\underline{X}_{I}^{(1,2)} = \begin{Bmatrix} (1.0000 \pm 0.0000i)\ \mathrm{m} \\ (-10.5210 \pm 0.6702i)\ \mathrm{rad} \\ (-10.2366 \pm 0.6718i)\ \mathrm{rad} \end{Bmatrix}$$

$$\underline{X}_{II}^{(3,4)} = \begin{Bmatrix} (1.0000 \mp 0.0000i)\ \mathrm{m} \\ (-0.1421 \pm 0.0511i)\ \mathrm{rad} \\ (-0.0262 \pm 0.0277i)\ \mathrm{rad} \end{Bmatrix}$$

$$\underline{X}_{III}^{(5,6)} = \begin{Bmatrix} (1.0000 \pm 0.0000i)\ \mathrm{m} \\ (4.1062 \pm 0.2576i)\ \mathrm{rad} \\ (-4.4893 \mp 0.3168i)\ \mathrm{rad} \end{Bmatrix}$$

1.3 Rayleigh Damping

It is now required to compute the two constants α and β that allow to approximate the generalized damping matrix $[C^*]$ through the Rayleigh proportional damping formula:

$$[C^*] \approx \alpha[M^*] + \beta[K^*]$$

In this particular case, the system of equations related to the Rayleigh proportional damping formula is analytically unsolvable: we have a system of nine equations and 2 unknowns to determine. Hence, to compute α and β we decide to apply the least square method (LSM), which is based on the computation of the values α and β that minimise the sum of the residuals squared. In particular, assuming to re-shape the 3×3 matrices $[M^*]$, $[C^*]$, $[K^*]$ of the system in such a way that their coefficients can be stored in a 9×1 array, the LSM assumes that the best values for α and β are the ones that minimize the following function

$$S = \sum_{i,j=1}^{3} (C_{i,j}^* - \alpha M_{i,j}^* - \beta K_{i,j}^*)^2 = \sum_{k=1}^{9} (C_k^* - \alpha M_k^* - \beta K_k^*)^2$$

where the $C_k^* - \alpha M_k^* - \beta K_k^*$ are the so called residuals. Hence, by imposing

$$\begin{cases} \dfrac{\partial S}{\partial \alpha} = 0 \\ \dfrac{\partial S}{\partial \beta} = 0 \end{cases}$$

it is found that the two equations that allow to compute α and β. In conclusion:

$$\alpha = 0.725\ 1/\mathrm{s},\ \beta = -0.002\ \mathrm{s}$$

Lastly, assuming that $[C^*] = \alpha[M^*] + \beta[K^*]$, we compute again the eigenvalues and the eigenvectors for the damped system.

Repeating the same steps presented earlier, we find the following eigenvalues and eigenvectors:

$$\underline{\lambda}_{appr} = \begin{Bmatrix} \lambda_{I,appr}^{(1,2)} \\ \lambda_{II,appr}^{(3,4)} \\ \lambda_{III,appr}^{(5,6)} \end{Bmatrix} = \begin{Bmatrix} (-0.3617 \pm 0.8553i)\ \mathrm{rad/s} \\ (-0.3392 \pm 4.9943i)\ \mathrm{rad/s} \\ (-0.3002 \pm 8.1807ii)\ \mathrm{rad/s} \end{Bmatrix}$$

$$\underline{X}^{(1,2)}_{I,appr} = \begin{Bmatrix} 1.00 \text{ m} \\ -10.5598 \text{ rad} \\ -10.2753 \text{ rad} \end{Bmatrix}, \ \underline{X}^{(3,4)}_{II,appr} = \begin{Bmatrix} 1.0000 \text{ m} \\ -0.1432 \text{ rad} \\ -0.0311 \text{ rad} \end{Bmatrix}, \ \underline{X}^{(5,6)}_{III,appr} = \begin{Bmatrix} 1.0000 \text{ m} \\ 4.1612 \text{ rad} \\ -4.5532 \text{ rad} \end{Bmatrix}$$

As expected the eigenvectors are real and have a null imaginary part. This property is a consequence of the Rayleigh proportional damping formula.

2. Free Motion of the System

2.1 Free Motion from given Initial Conditions

We want to solve the EOM in the damped case considering Rayleigh damping for a particular set of initial conditions. In other words, we want to solve the following *Cauchy problem*:

$$\begin{cases} [M^*]\ddot{\underline{x}} + [C^*_{appr}]\dot{\underline{x}} + [K^*]\underline{x} = \underline{0} \\ [C^*_{appr}] = \alpha[M^*] + \beta[K^*] \\ \underline{x}(t=0) = \underline{x}_0 \\ \dot{\underline{x}}(t=0) = \dot{\underline{x}}_0 \end{cases}$$

The solution is given by the following relation:

$$\underline{x}(t) = \sum_{i=1}^{6} a_i \underline{X}^{(i)}_{appr} e^{\lambda^{(i)}_{appr} t}$$

where $\lambda^{(i)}_{appr}$ and $\underline{X}^{(i)}_{appr}$ are the estimated eigenvalues and eigenvectors presented earlier, while the a_i coefficients are the ones specifically related to the chosen initial conditions. In particular:

$$\underline{a} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{Bmatrix} = \begin{bmatrix} \underline{X}^{(1)}_{appr} & \underline{X}^{(2)}_{appr} & \dots & \underline{X}^{(6)}_{appr} \\ \lambda^{(1)}_{appr}\underline{X}^{(1)}_{appr} & \lambda^{(2)}_{appr}\underline{X}^{(2)}_{appr} & \dots & \lambda^{(6)}_{appr}\underline{X}^{(6)}_{appr} \end{bmatrix}^{-1} \begin{Bmatrix} \underline{x}_0 \\ \dot{\underline{x}}_0 \end{Bmatrix}$$

Assuming that

$$\underline{x}_0 = \begin{Bmatrix} 0.1 \text{ m} \\ \pi/12 \text{ rad} \\ -\pi/12 \text{ rad} \end{Bmatrix}, \ \dot{\underline{x}}_0 = \begin{Bmatrix} 1.0 \text{ m/s} \\ 0.5 \text{ rad/s} \\ 2.0 \text{ rad/s} \end{Bmatrix}$$

we are able to determine and plot a particular solution

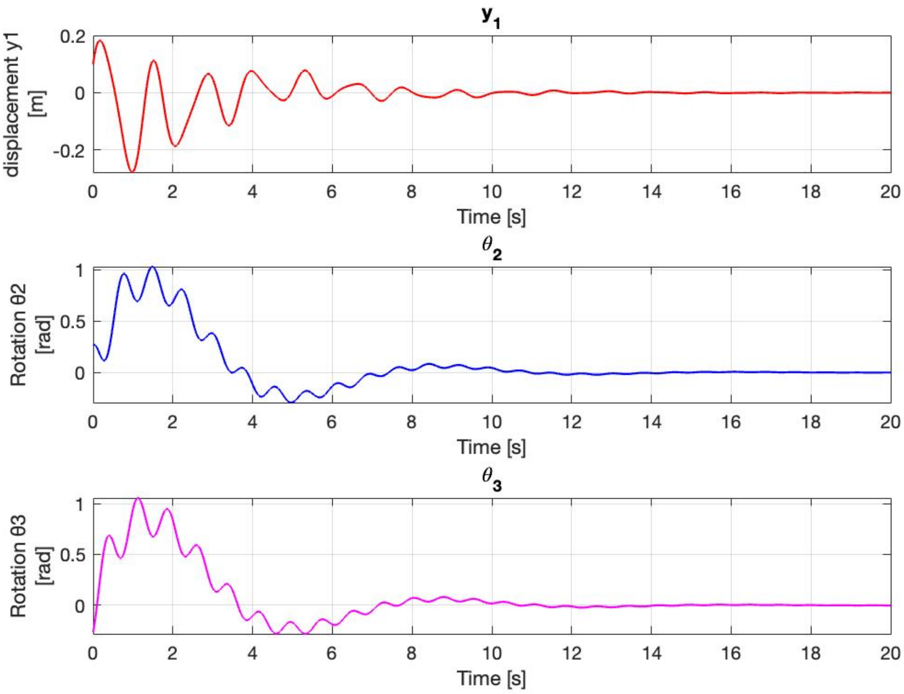

Figure 2: Free Motion from given initial conditions

2.2 Single mode initial conditions

In order to have a set of initial conditions that ensure that only one mode contributes to the free motion of the system, it is necessary to impose that, for a certain i, $\underline{x}_0 \equiv \underline{X}^{(i)}_{appr}$ and $\underline{\dot{x}}_0 = \underline{0}$. Under these conditions it is assured that there are no contributions, due to any initial energy, which could enhance different modes from the selected one. An equivalent way to reach the same results is to put as initial velocity a vector given by $\underline{\dot{x}} = \lambda^{(i)}_{appr} X^{(i)}_{appr}$.

It was decided to compute and display the motion correspondent to all the normalized modes taken one by one as given in Equation

$$\underline{x}_0 = \underline{X}^{(i)}_{appr},\ \underline{\dot{x}}_0 = \underline{0}\ where\ i = 1, 2, 3, ...$$

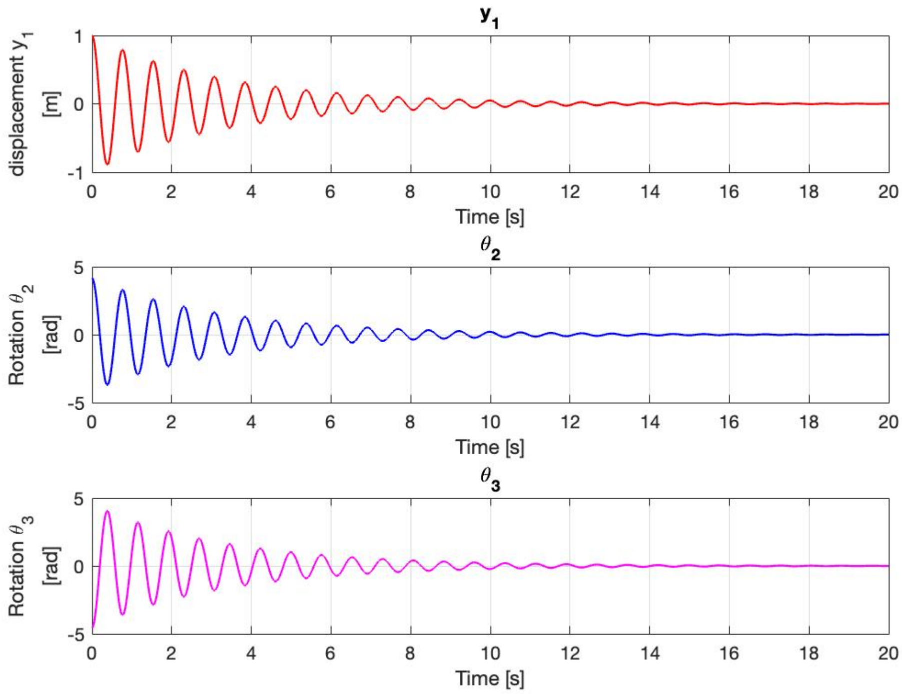

Figure 3: Free Motion with single mode

Forced Motion of the System

3.1 Frequency Response Matrix

It is now required to solve the EOM considering Rayleigh damping for a particular set of initial conditions and for a given external force $F(t)$. In other words, it is desirable to solve the following *Cauchy problem*:

$$\begin{cases} [M^*]\underline{\ddot{x}} + [C^*_{appr}]\underline{\dot{x}} + [K^*]\underline{x} = F(t)[\Lambda_f]^T \\ [C^*_{appr}] = \alpha[M^*] + \beta[K^*] \\ \underline{x}(t=0) = \underline{x}_0 \\ \underline{\dot{x}}(t=0) = \underline{\dot{x}}_0 \end{cases}$$

The complete time response (or general integral) of the linear system is given by the superposition of two solutions:

$$\underline{x}(t) = \underline{x}_g(t) + \underline{x}_p(t)$$

where $\underline{x}_g(t)$ is the general solution related to the free homogeneous problem, while $\underline{x}_p(t)$ is the so called steady state response or particular solution (since it is associated to the external force). The former solution has already been computed. Hence, we now only the general expression of $\underline{x}_p(t)$ will be computed.
The steady state response is the one that becomes dominant after the initial transient given by the general solution and that directly reflects the presence of an external force. Hence, assuming

to excite the system with an harmonic force $F(t) = F_0 e^{i\Omega t}$, the steady state response is given by the following expression:

$$\underline{x}_p = \underline{X}_{p,0} e^{i\Omega t}$$

$$\dot{\underline{x}}_p = i\Omega \underline{X}_{p,0} e^{i\Omega t}, \ \ddot{\underline{x}}_p = -\Omega^2 \underline{X}_{p,0} e^{i\Omega t}$$

and then EOM can be rewritten as follows:

$$(-\Omega^2[M^*] + i\Omega[C^*_{appr}] + [K^*])\underline{X}_{p,0} e^{i\Omega t} = \underline{F}_0 e^{i\Omega t}$$

where $\underline{F}_0 = F_0[\Lambda_f]^T$.

If we introduce in the equation the matrix $[D^*(\Omega)] = -\Omega^2[M^*] + i\Omega[C^*_{appr}] + [K^*]$ and simplify all the time dependencies, then we get the following expression

$$[D^*(\Omega)]\underline{X}_{p,0} = \underline{F}_0$$

Since $[D^*(\Omega)]$ is a matrix with complex coefficients then its determinant is never null and then we are able to compute the *Frequency Response Matrix* (FRM) of the system: $[H^*(\Omega)] = [D^*(\Omega)]^{-1}$. Thanks to $[H^*(\Omega)]$, it is possible to derive the amplitude of the steady state response:

$$\underline{X}_{p,0} = [D^*]^{-1}\underline{F}_0 = [H^*(\Omega)]\underline{F}_0$$

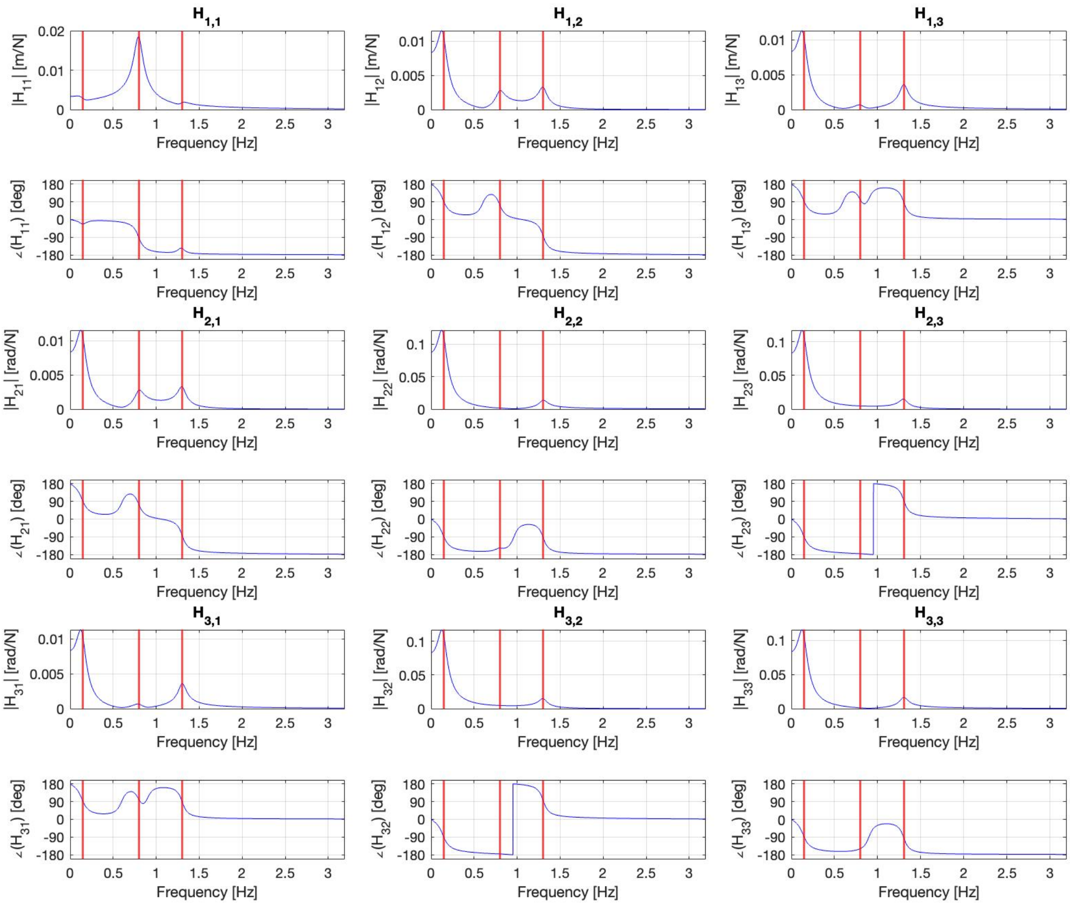

Figure 4: Frequency Response Matrix plot

The elements of the matrix $[H^*(\Omega)]$ tell us what is the steady state behaviour of the independent coordinates under the effect of an external harmonic force with frequency Ω. In particular, the element $[H^*(\Omega)]_{j,k}$ is the *Frequency Response Function* that describes how the j-th independent coordinate of the system responses if the force acts on the k-th independent coordinate.

Furthermore, each element $[H^*(\Omega)]_{j,k}$ shows which is the behaviour of the coordinates when the external force has a frequency similar to the natural frequencies ω_I, ω_{II}, ω_{III} of the system. In fact, as can be seen in the previous figures, all the FRM elements show a certain behaviour in correspondence of all the three computed natural frequencies. We can generally say that in the amplitude's graphs the presence or absence of peaks in correspondence of the natural frequencies is correlated to the modeshapes.

3.2 Co-Located FRF at point A

Thanks to an appropriate matrix transformation, it is possible to understand, by starting from the original independent coordinates, the steady state behaviour of new independent coordinates under the effect of the same external harmonic force.

Supposing the FRM $[H'(\Omega)]$ related to the new set of independent coordinates $\underline{x}'$ and calling $[\Phi']$ the transformation matrix. In this way

$$\underline{x}' = [\Phi']\underline{x} \text{ and } \underline{x} = [\Phi']^{-1}\underline{x}'$$

Since $\dot{\underline{x}} = [\Phi']^{-1}\dot{\underline{x}}'$, $\ddot{\underline{x}} = [\Phi']^{-1}\ddot{\underline{x}}'$ and $\delta\underline{x} = [\Phi']^{-1}\delta\underline{x}'$, then, without loss of generality, it is possible to compute the system's total kinetic energy T, dissipation function D, potential energy V and lagrangian component $\underline{Q}'$ by means of $\underline{x}'$, hence obtaining the following new set of equations of motion:

$$[M']\ddot{\underline{x}}' + [C']\dot{\underline{x}}' + [K']\underline{x}' = \underline{Q}'$$

where

$$\begin{cases} [M'] = ([\Phi']^{-1})^T[M^*][\Phi']^{-1} \\ [C'] = ([\Phi']^{-1})^T[C^*_{appr}][\Phi']^{-1} \\ [K'] = ([\Phi']^{-1})^T[K^*][\Phi']^{-1} \\ \underline{Q}' = F(t)([\Phi']^{-1})^T[\Lambda_f]^T \end{cases}$$

From this it is possible to study the steady state response of the system and hence compute the *co-located* Frequency Response Matrix $[H'(\Omega)]$ with the same previously shown relation:

$$[H'(\Omega)] = [-\Omega^2[M'] + i\Omega[C'] + [K']]^{-1}$$

It is now required to compute the co-located Frequency Response Function (FRF) related to the displacement of the point A when the force acts on A itself. To this end, we introduce the new set of independent coordinates

$$\underline{x}' = \begin{Bmatrix} x_1 \\ \theta_2 \\ x_3 \end{Bmatrix}$$

where x_1 and θ_2 are still the same independent coordinates, while x_3 is the vertical absolute motion in the positive direction of the centre of the disk of mass M_3 (point A) about its static equilibrium position. For this particular case the transformation matrix $[\Phi']$ is such that

$$[\Phi'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & r \end{bmatrix}$$

Now it is possible to compute $[H'(\Omega)]$ for this particular case and display the co-located FRF of point A at the centre of the disk 3, which corresponds to the element $[H'(\Omega)]_{3,3}$ (both displacement and force seen and applied in A).

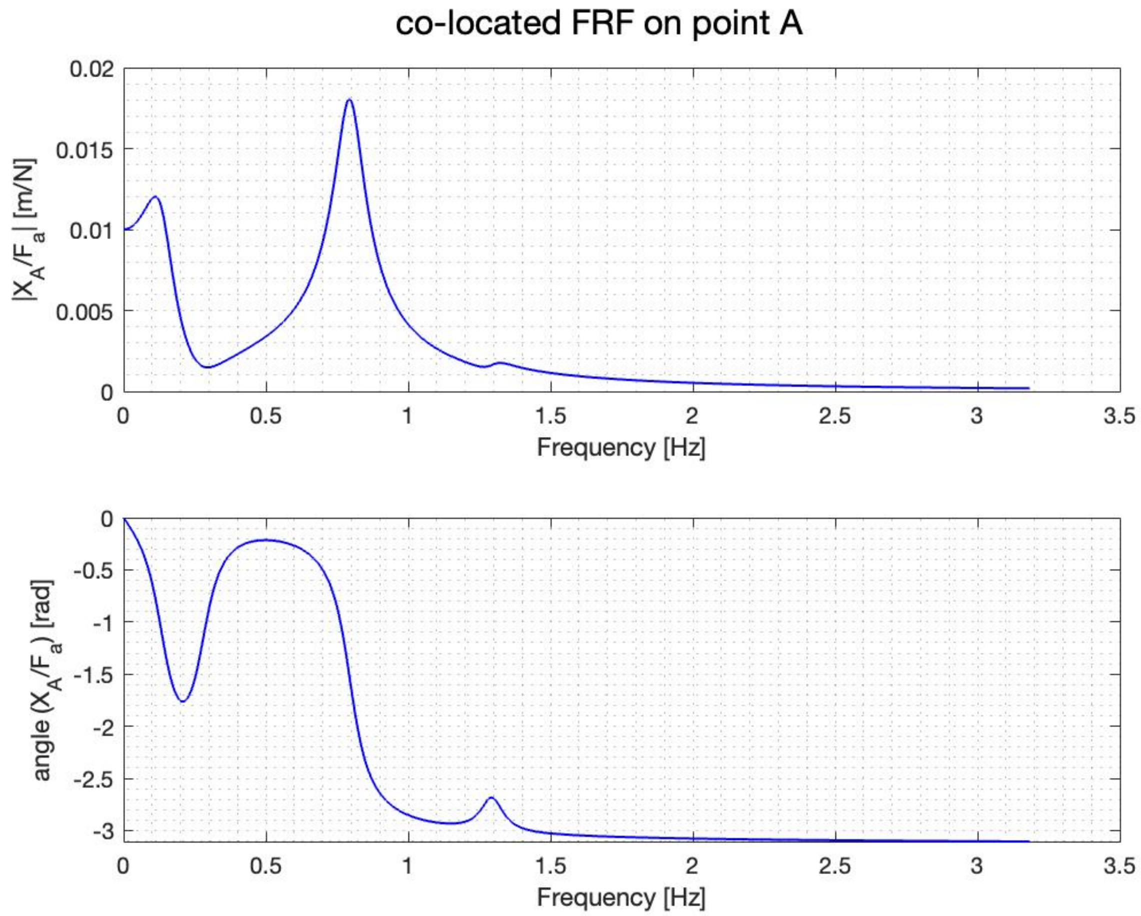

Figure 5: Co-Located FRF on point A

3.3 Co-Located FRF between the rotation of $M3$ and torque applied to the same object

Now it is required to compute the co-located FRF between the rotation of the disk of mass M_3 and the torque applied onto the disk itself. Since the rotation of the disk of mass M_3 is described by the independent coordinate θ_3 and the torque applied onto the disk is exactly given by the correspondent component of $F(t)[\Lambda_f]^T$, then it is possible to avoid defining a new set of coordinates and directly computing and displaying the co-located FRF, which corresponds to the element $[H(\Omega)]_{3,3}$ of the original FRM.

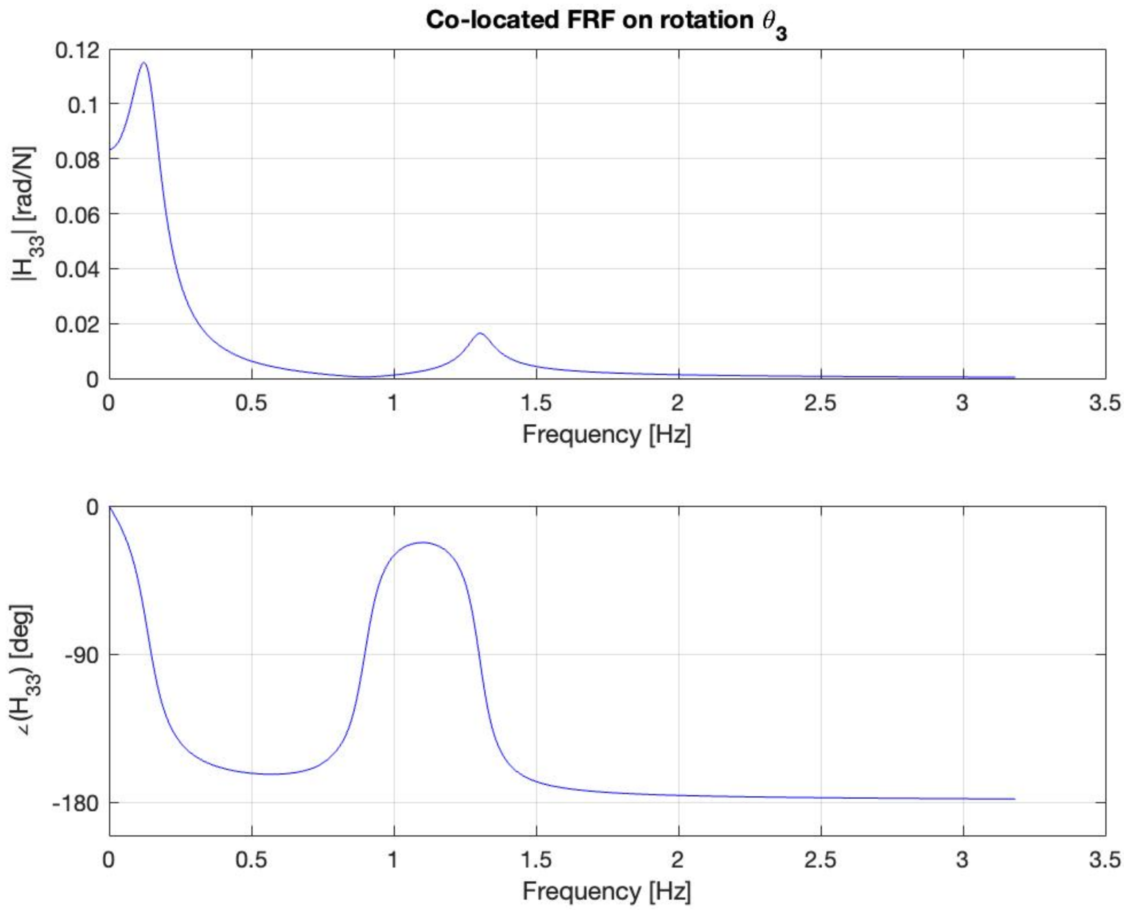

Figure 6: Co-Located FRF on rotation θ_3

3.4 Complete response to a bi-harmonic force

It is requested to compute, assuming Rayleigh damping the complete time solution of the EOM assuming that on the system acts the following bi-harmonic force:

$$F(t) = A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t)$$

The complete time solution of the system is given by the superposition of two different contributions: the general solution and the steady state response, both of which were previously discussed. Now it is possible to write as follows:

$$\underline{x}_p(t) = \Re e\left\{ \sum_{k=1,2} \tilde{\underline{x}}_{p,k,0} e^{i\Omega_k t} \right\} \equiv \Re e\left\{ \sum_{k=1,2} A_k [H^*(\Omega_k)][\Lambda_f]^T e^{i\Omega_k t} \right\}$$

$$\underline{x}(t) = \underline{x}_g(t) + \underline{x}_p(t)$$

where $\Omega_k \equiv 2\pi f_k$.

Using again the relations, we are able to compute the a_i coefficients.

$$\underline{a} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{Bmatrix} = \begin{bmatrix} \underline{X}^{(1)}_{appr} & \underline{X}^{(2)}_{appr} & \dots & \underline{X}^{(6)}_{appr} \\ \lambda^{(1)}_{appr}\underline{X}^{(1)}_{appr} & \lambda^{(2)}_{appr}\underline{X}^{(2)}_{appr} & \dots & \lambda^{(6)}_{appr}\underline{X}^{(6)}_{appr} \end{bmatrix}^{-1} \begin{Bmatrix} \underline{x}_0 - \sum\limits_{k=1,2} A_k[H^*(\Omega_k)][\Lambda_f]^T \\ \underline{\dot{x}}_0 - i \sum\limits_{k=1,2} \Omega_k A_k[H^*(\Omega_k)][\Lambda_f]^T \end{Bmatrix}$$

Hence, setting in conclusion the same initial conditions used earlier

$$\underline{x}_0 = \begin{Bmatrix} 0.1 \text{ m} \\ \pi/12 \text{ rad} \\ -\pi/12 \text{ rad} \end{Bmatrix}, \; \underline{\dot{x}}_0 = \begin{Bmatrix} 1.0 \text{ m/s} \\ 0.5 \text{ rad/s} \\ 2.0 \text{ rad/s} \end{Bmatrix}$$

it is now possible to compute the plots for this case

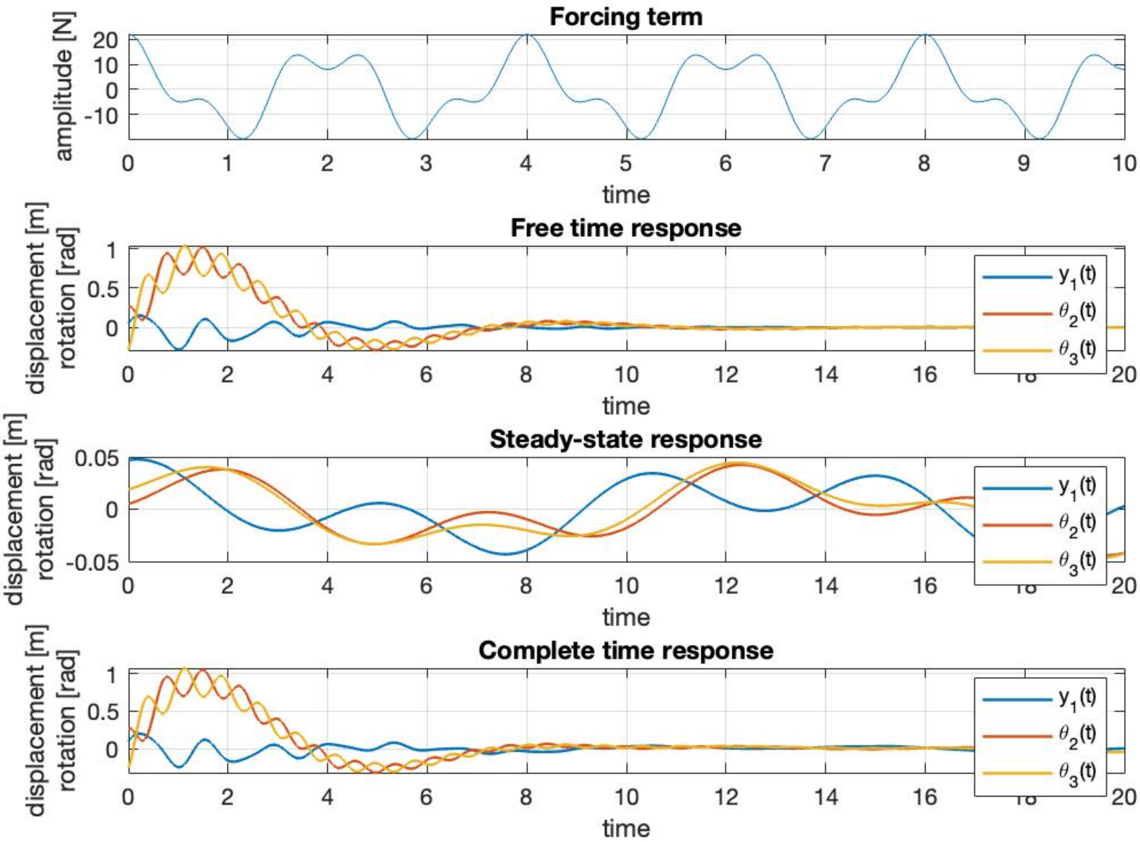

Figure 7: Response to a bi-harmonic force

3.5 Response to a force described by a triangular wave

Lastly, it is required to evaluate only the steady-state response of the vertical displacement of point A with respect to the vertical force applied in A considering that the force is a periodic triangular wave, with fundamental frequency f_0, of the form

$$F(t) = \frac{8}{\pi^2} \sum_{k=0}^{4} (-1)^k \frac{\sin\left((2k+1)2\pi f_0 t\right)}{(2k+1)^2}$$

Since it is required to compute the steady-state response of the vertical displacement of point A, it is necessary to compute the solution for the vector $\underline{x}$ and then apply the same coordinate transformation used to deduce the co-located FRF of point A at the centre of M_3 disk, so that we are able to correctly plot the asked solution.

First of all, the expression for the triangular wave is rewritten with a complex notation:

$$F(t) = \frac{8}{\pi^2}\sum_{k=0}^{4}(-1)^k\frac{\cos\left((2k+1)2\pi f_0 t - \pi/2\right)}{(2k+1)^2} = \Re e\left\{\sum_{k=0}^{4}\left[\frac{8}{\pi^2}\frac{(-1)^k e^{-i\pi/2}}{(2k+1)^2}\right]e^{i(2k+1)2\pi f_0 t}\right\} =$$
$$= \Re e\left\{\sum_{k=0}^{4}\tilde{F}_{k,0}e^{i\Omega_k t}\right\}$$

where the amplitudes $\tilde{F}_{k,0} = \frac{8}{\pi^2}\frac{(-1)^k}{(2k+1)^2}e^{-i\pi/2}$ and the set of frequencies $\Omega_k = (2k+1)2\pi f_0$ are introduced. In this way, the steady state response is simply given by

$$\underline{x}_p(t) = \Re e\left\{\sum_{k=0,4}\tilde{F}_{k,0}[H^*(\Omega_k)][\Lambda_f]^T e^{i\Omega_k t}\right\}$$

Then the solution is applied to the same transformation matrix $[\Phi']$ used earlier for the problem of the co-located FRF. Since $\underline{x}' = [\Phi']\underline{x}$ and $[\Phi']$ is real

$$\underline{x}'_p(t) = \Re e\left\{\sum_{k=0,4}\tilde{F}_{k,0}[\Phi'][H^*(\Omega_k)][\Lambda_f]^T e^{i\Omega_k t}\right\}$$

from which it is easy to extract and display the third component, which is exactly the steady state response of the vertical displacement of point A to the vertical force applied in A. Furthermore another way to find directly the displacement of the point A is to substitute $[\Phi']$ with $[\Lambda_f]$ in the previous formula as follows

$$y_{A_p(t)} = \Re e\left\{\sum_{k=0,4}\tilde{F}_{k,0}[\Lambda_f][H^*(\Omega_k)][\Lambda_f]^T e^{i\Omega_k t}\right\}$$

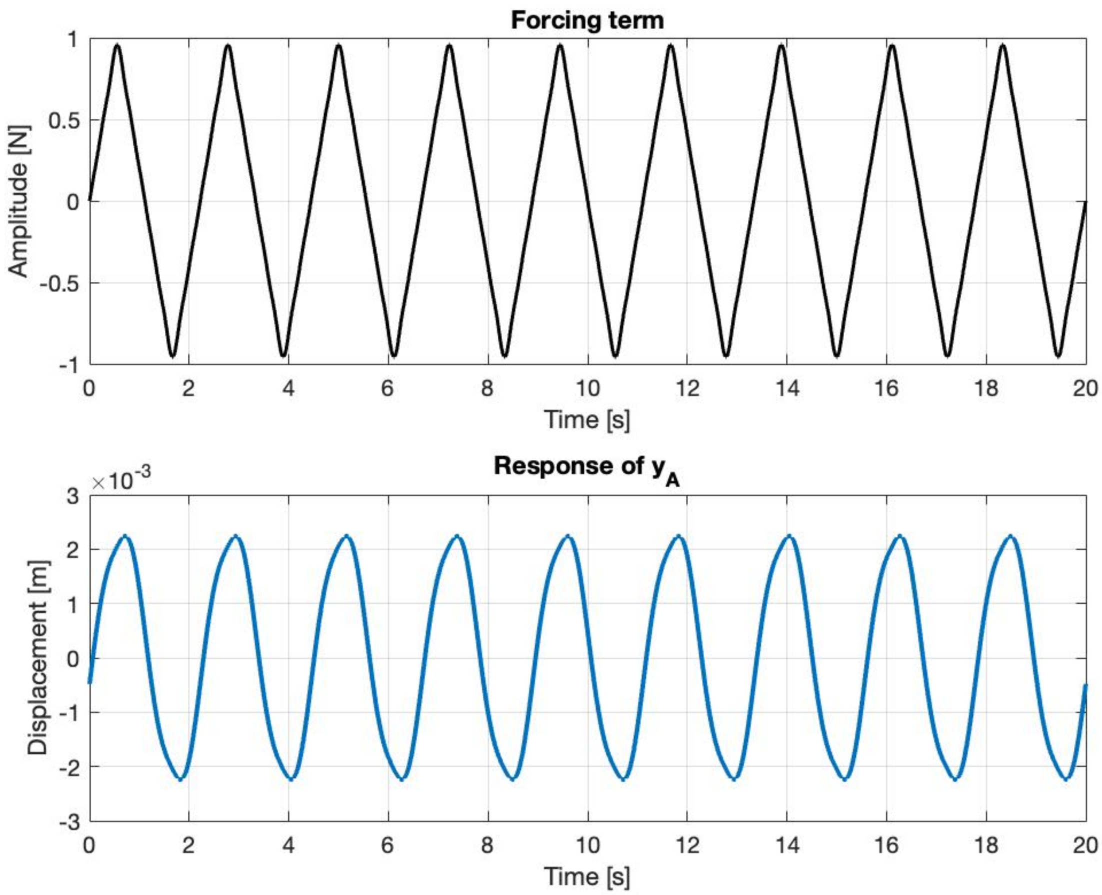

Figure 8: Response to a triangular force

Modal approach considering Rayleigh damping

4.1 Derivation of the EOM in modal coordinates

The modal approach is based on the following transformation of coordinates:

$$\underline{x} = [\Phi]\underline{q}$$

where $\underline{q}$ is the vector that contains the so called modal coordinates and $[\Phi]$ is the matrix containing the mode shapes computed in the first section considering the Rayleigh damping

$$[\Phi] = [\underline{X}_{I,appr} \quad \underline{X}_{II,appr} \quad \underline{X}_{III,appr}]$$

Since $\dot{\underline{x}} = [\Phi]\dot{\underline{q}}$, $\ddot{\underline{x}} = [\Phi]\ddot{\underline{q}}$ and $\delta\underline{x} = [\Phi]\delta\underline{q}$, then, without loss of generality, the system's total kinetic energy T, dissipation function D, potential energy V and lagrangian component $\underline{Q}_q$ can be computed by means of $\underline{q}$, hence obtaining the following new set of equations of motion:

$$[M_q]\ddot{\underline{q}} + [C_q]\dot{\underline{q}} + [K_q]\underline{q} = \underline{Q}_q$$

where

$$\begin{cases} [M_q] = [\Phi]^T[M^*][\Phi] \\ [C_q] = [\Phi]^T[C^*_{appr}][\Phi] \\ [K_q] = [\Phi]^T[K^*][\Phi] \\ \underline{Q}_q = F(t)[\Phi]^T[\Lambda_f]^T \end{cases}$$

This can be shown starting from the Lagrange's equation written in modal coordinates:

$$\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \underline{\dot{q}}}\right)\right\}^T - \left\{\frac{\partial T}{\partial \underline{q}}\right\}^T + \left\{\frac{\partial D}{\partial \underline{\dot{q}}}\right\}^T + \left\{\frac{\partial V}{\partial \underline{q}}\right\}^T = \underline{Q}_q$$

Using the relations $\underline{\dot{x}} = [\Phi]\underline{\dot{q}}$, $\underline{\ddot{x}} = [\Phi]\underline{\ddot{q}}$ and $\delta\underline{x} = [\Phi]\delta\underline{q}$ it is possible to deduce

$$T = \frac{1}{2}\underline{\dot{x}}^T[M^*]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{q}}^T[\Phi]^T[M][\Phi]\underline{\dot{q}} = \frac{1}{2}\underline{\dot{q}}^T[M_q]\underline{\dot{q}}$$

$$V = \frac{1}{2}\underline{x}^T[K*]\underline{x} = \frac{1}{2}\underline{q}^T[\Phi]^T[K][\Phi]\underline{q} = \frac{1}{2}\underline{q}^T[K_q]\underline{q}$$

$$D \approx \frac{1}{2}\underline{\dot{x}}^T[C^*_{appr}]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{q}}^T[\Phi]^T[C^*_{appr}][\Phi]\underline{\dot{q}} = \frac{1}{2}\underline{\dot{q}}^T[C_q]\underline{\dot{q}}$$

Taking into account that $\partial^* W = F(t)[\Lambda_f]\delta\underline{x} = F(t)[\Lambda_f][\Phi]\delta\underline{q}$, lastly the result is

$$\underline{Q}_q = \left\{\frac{\partial^* W}{\partial \underline{q}}\right\}^T \equiv F(t)[\Phi]^T[\Lambda_f]^T$$

from which it is possible to compute all the different contribution of the Lagrange's equation in modal coordinates

$$\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \underline{\dot{q}}}\right)\right\}^T = \left\{\frac{d}{dt}(\underline{\dot{q}}^T[M_q])\right\}^T = [M_q]^T\underline{\ddot{q}} \equiv [M_q]\underline{\ddot{q}}, \quad \left\{\frac{\partial T}{\partial \underline{q}}\right\}^T \equiv \underline{0}$$

$$\left\{\frac{\partial D}{\partial \underline{\dot{q}}}\right\}^T = \{\underline{\dot{q}}^T[C_q]\}^T = [C_q]^T\underline{\dot{q}} \equiv [C_q]\underline{\dot{q}}, \quad \left\{\frac{\partial V}{\partial \underline{q}}\right\}^T = \{\underline{q}^T[K_q]\}^T = [K_q]^T\underline{q} \equiv [K_q]\underline{q}$$

and finally getting the set of equations of motion as previously presented
Starting from the modal EOM we are able to compute the Frequency Response Modal Matrix $[H_q(\Omega)]$ with the relation:

$$[H_q(\Omega)] = [-\Omega^2[M_q] + i\Omega[C_q] + [K_q]]^{-1}$$

Since $[M_q]$, $[K_q]$ and, thanks to Rayleigh damping, also $[C_q]$ are all diagonal matrices, then, also $[H_q(\Omega)]$ is a diagonal matrix.

It is now possible to write the following expression:

$$\begin{aligned} \underline{x}_0 =& [\Phi]\underline{q}_0 = [\Phi]\cdot[H_q(\Omega)]\underline{F}_{q,0} = [\Phi][H_q(\Omega)]([\Phi]^T[\Lambda_f]^T F_0) \\ =& ([\Phi][H_q(\Omega)][\Phi]^T)[\Lambda_f]^T F_0 \equiv [H^*(\Omega)][\Lambda_f]^T F_0 \end{aligned}$$

from which it is clear that $[H^*(\Omega)] = [\Phi][H_q(\Omega)][\Phi]^T$. This equation tells that the FRF $[H^*(\Omega)]$ is a linear superposition of the elements of $[H^*(\Omega)]$.

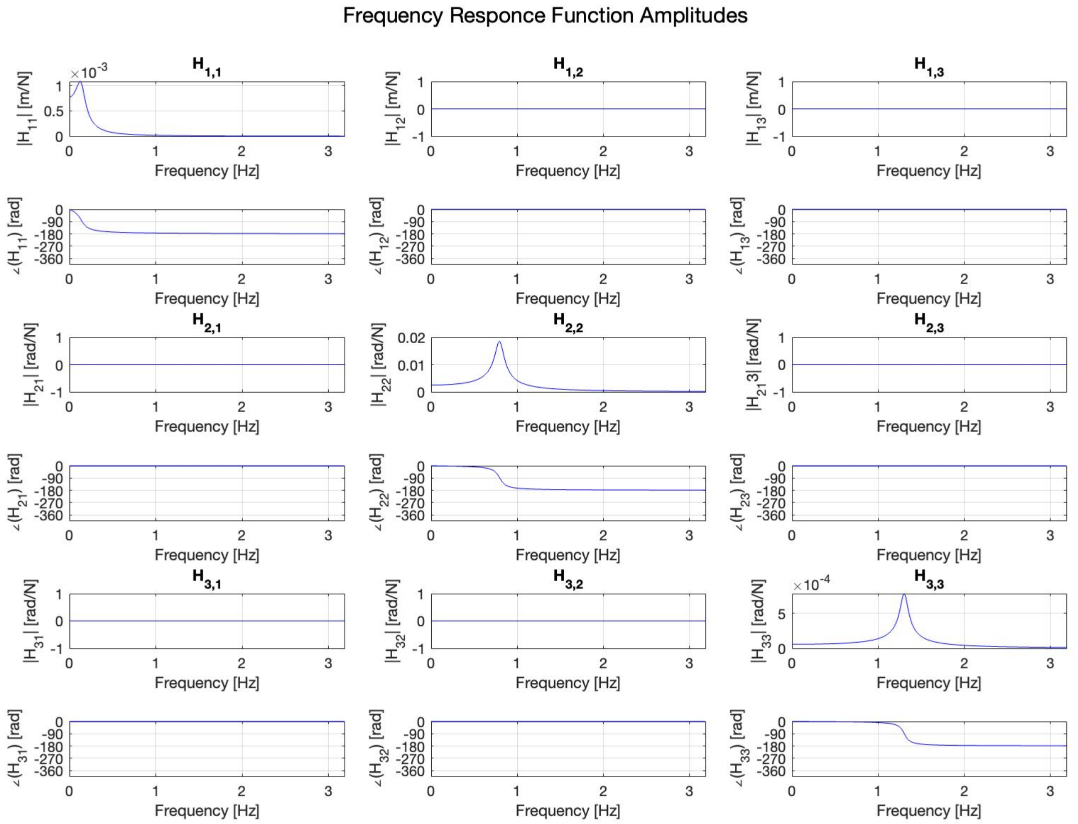

Figure 9: Modal Frequency Response Matrix

4.2 Co-located FRM in modal approach

The modal matrix $[H_q(\Omega)]$ defines the steady state behaviour of the modal coordinates under the effect of an external harmonic force of frequency Ω. Furthermore, by means of an appropriate matrix transformation, it is possible to deduce, starting from the modal coordinates, the steady state behaviour of new independent coordinates under the effect of the same external harmonic force.

The aim is to rewrite the FRM $[H'(\Omega)]$ but related to the new set of independent coordinates $\underline{x}'$. By calling $[\Phi']$ the correspondent transformation matrix such that

$$\underline{x}' = [\Phi']\underline{x}$$

Hence,

$$\begin{cases} \underline{x} = [\Phi]\underline{q} \\ \underline{x}' = [\Phi']\underline{x} \end{cases} \Rightarrow \underline{x}' = [\Phi'] \cdot [\Phi]\underline{q} \ \Rightarrow \underline{x}'_0 = ([\Phi'] \cdot [\Phi][H_q(\Omega)][\Phi]^T \cdot [\Phi']^T)[\Lambda_f]^T F_0$$

from which the co-located FRM is obtained:

$$[H'(\Omega)] = [\Phi'] \cdot [\Phi][H_q(\Omega)][\Phi]^T \cdot [\Phi']^T$$

The FRF related to the displacement of point A is obtained by substituting $[\Phi']$ with $[\Lambda_f]$ in the previous formula as follows:

$$H'_{3,3}(\Omega) = [\Lambda_f] \cdot [\Phi][H_q(\Omega)][\Phi]^T \cdot [\Lambda_f]^T$$

4.2 Co-located FRF of point A at the centre of M_3 disk (modal approach)

We now want to reconstruct, using the modal approach, the co-located Frequency Response Function (FRF) related to the displacement of the point A when the force is seen in action on A itself

$$\underline{x}' = \begin{Bmatrix} x_1 \\ \theta_2 \\ x_3 \end{Bmatrix}$$

x_3 is the vertical absolute motion in the positive direction of the centre of the disk of mass M_3 (point A) about its static equilibrium position.
For this particular case the transformation matrix $[\Phi']$ is such that

$$[\Phi'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & r \end{bmatrix}$$

It is now possible to compute the FRM $[H'(\Omega)]$ related to this particular case. Since it is required to reconstruct the co-located FRF of point A at the centre of the disk 3, the element is then extracted and displayed $H'_{3,3}(\Omega)$.
It is now possible to compute $[H'(\Omega)]$ for this particular case and display the co-located FRF of point A at the centre of the disk 3, which corresponds to the element $H'_{3,3}(\Omega)$ (both displacement and force seen and applied in A).

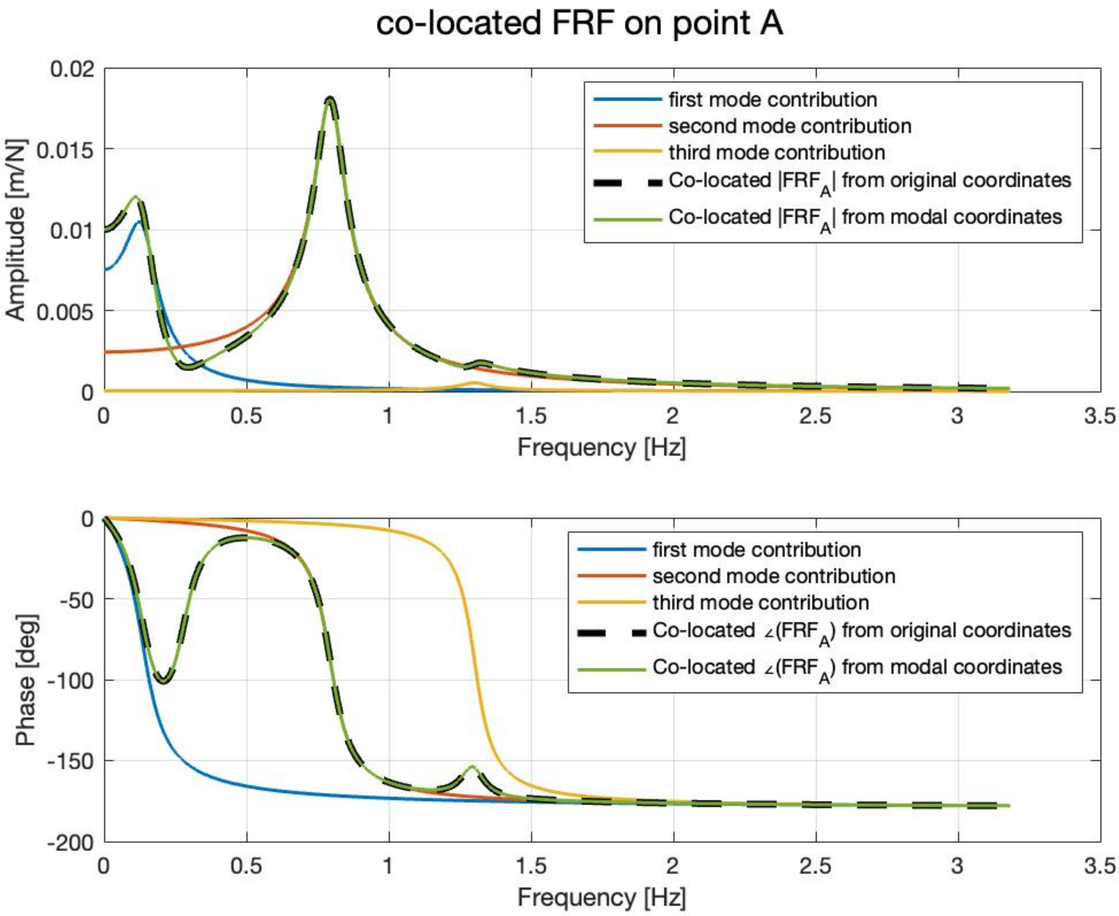

Figure 10: Co-Located FRF on point A

4.3 Co-located FRF between the rotation of the M_3 disk and the torque applied onto the disk itself (modal approach).

This time it is required to re-compute the co-located FRF between the rotation of the disk of mass M_3 and the torque applied onto the disk itself. To do this the matrix $[H(\Omega)] = [\Phi][H_q(\Omega)][\Phi]^T$ is evaluated and considered only the element $H'_{3,3}(\Omega)$.

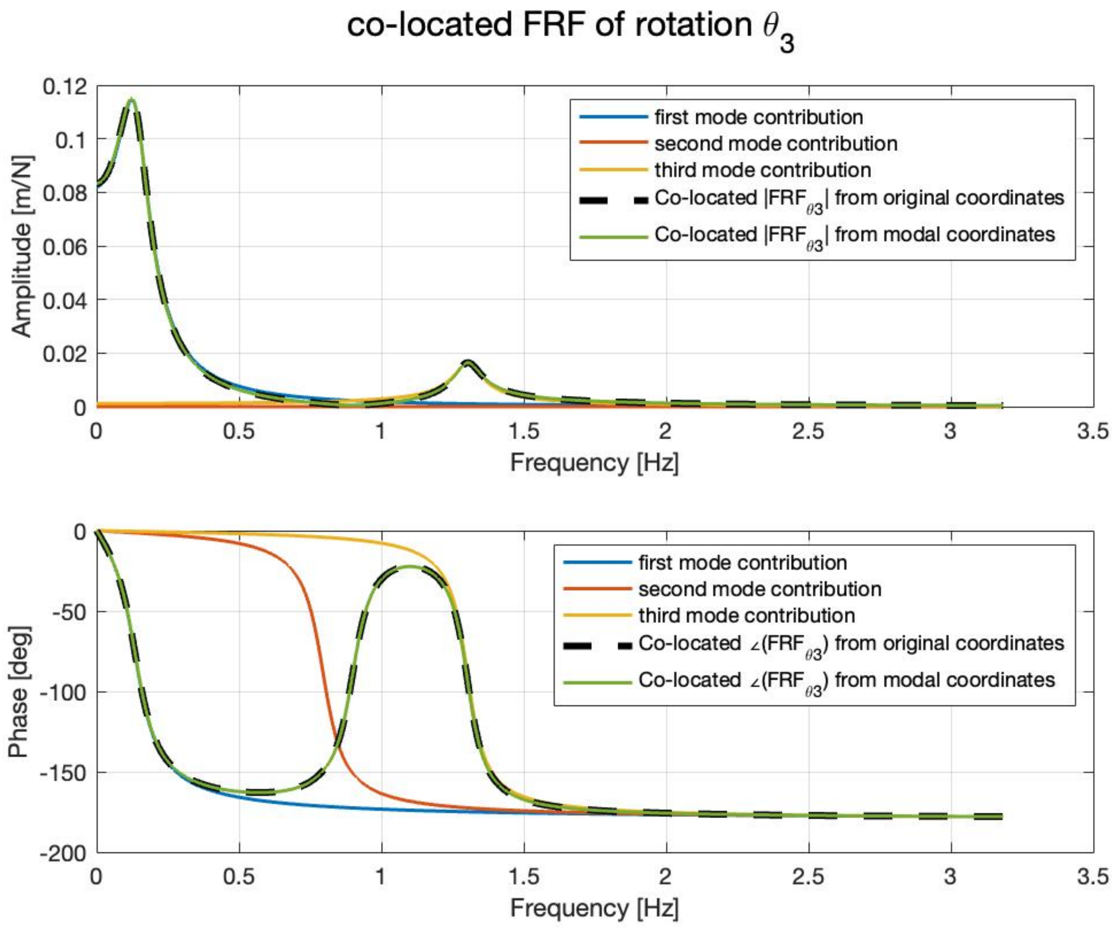

Figure 11: Co-Located FRF of rotation θ_3

4.4 Complete system steady state response (modal approach)

It is now required to study the complete system steady state response $\underline{x}_p(t)$, assuming Rayleigh damping and using modal approach,for two different cases:

$$F_1(t) = A_1 \cos(2\pi f_1 t)$$

$$F_2(t) = A_2 \cos(2\pi f_2 t)$$

The complete steady state response $\underline{x}_p(t)$ can be easily computed by means of $[H_q(\Omega)]$ and $[\Phi]$. In fact, assuming in general that $F(t) = A_k e^{i\Omega_k t}$ (where $\Omega_k = 2\pi f_k$), then

$$\begin{aligned}\underline{x}_p(t) &= \Re e\{\underline{\tilde{x}}_{p,k,0} e^{i\Omega_k t}\} \\ &= \Re e\{[\Phi]\underline{\tilde{q}}_{p,k,0} e^{i\Omega_k t}\} \\ &= \Re e\{[\Phi][H_q(\Omega_k)]\underline{Q}_k e^{i\Omega_k t}\} \\ &= \Re e\{[\Phi][H_q(\Omega_k)][\Phi]^T[\Lambda_f]^T F_0 e^{i\Omega_k t}\}\end{aligned}$$

Furthermore, it is now required to compute and display the steady state response of the system considering only one mode of vibration we need to modify Equation. So it is needed to keep different from zero just the mode we want to know the response of. Hence, it is needed to compute the solution with the following relation

$$\underline{x}_p(t) = \Re e\{[\Phi]_{\text{SM}}[H_q(\Omega_k)] \cdot [\Phi]_{\text{SM}}^T[\Lambda_f]^T F_0 e^{i\Omega_k t}\}$$

where

$$[\Phi]_{\text{SM}} = [\underline{X}_{I,appr} \quad \underline{0} \quad \underline{0}]$$

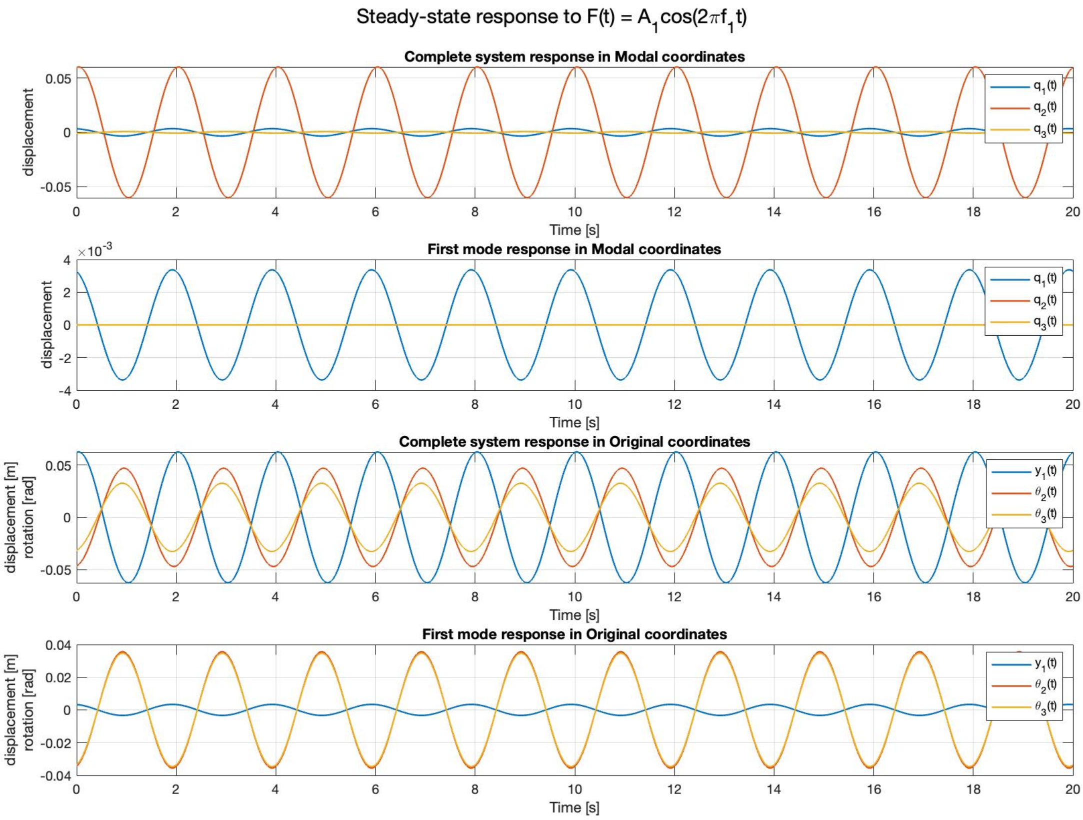

Figure 12: Steady state response to $F_1(t)$

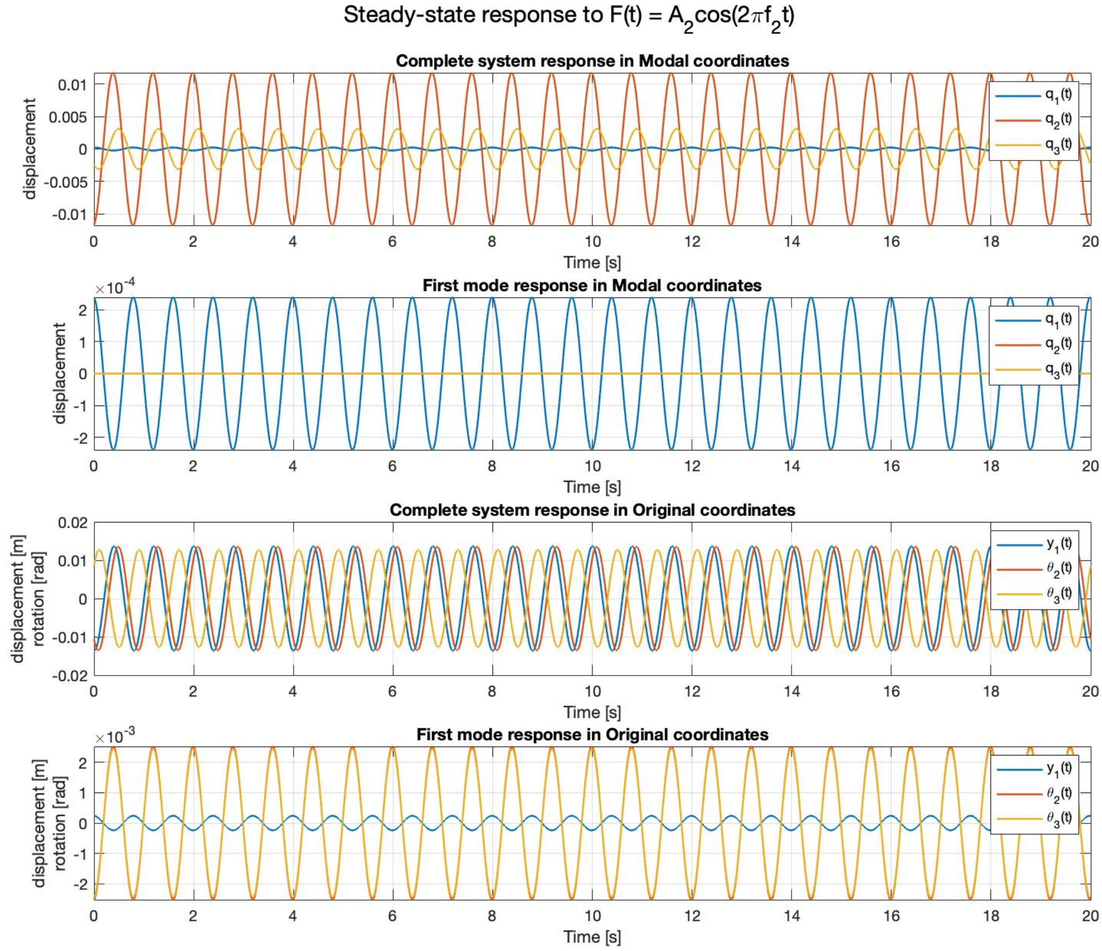

Figure 13: Steady state response to $F_2(t)$