Vibration Analysis and Vibroacoustic

Assignment 1: Analysis of a 1-dof System

Homework 1 Report

Students

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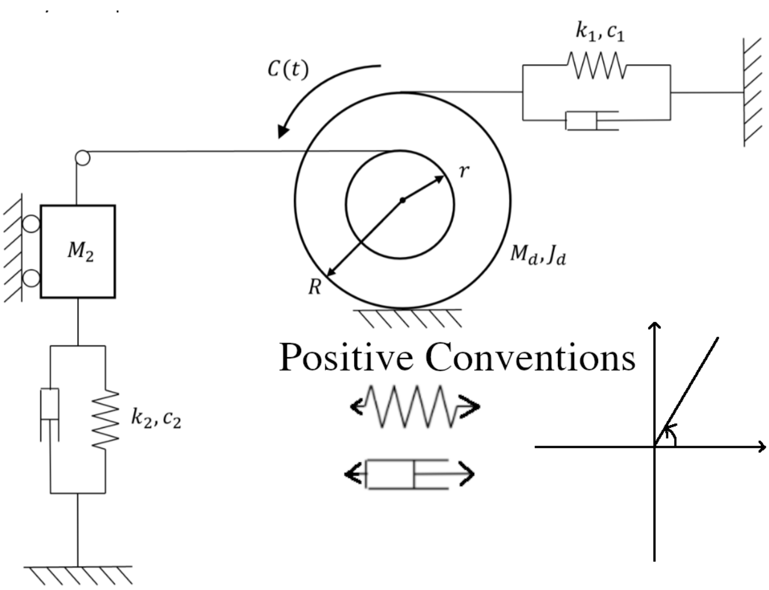

Figure 1: Physical System and Positive Conventions

1. Equation of Motion

1.1 Preliminary passages

The aim of this section is to define the equations that describe movements around the equilibrium position of the system described in figure 1. To achieve this goal we need to define some conventions which will be necessary in order to correctly define the equations of motion. As represented in the picture we define as positive the elongation of a spring or of a damper or an angle taken anti-clockwise from the equilibrium state.
In order to correctly analyse the system it is needed to first determine the total number of Degrees of Freedom (DOF). To achieve this result two steps are required:

- Determine the Degrees of Constraint (DOC) of the system
- Subtract the DOC from the total possible amount of DOF to get the effective number of DOF of the system

1.2.1 Determination of the Degrees of Constraint

To correctly characterize the system it is needed to determine how the constraints act in a mechanical system. This system in particular has three types of constraint: inextensible strings, contact points and trolleys.

- A string does not allow free motion to the object in the same direction of the string itself thus adding 1 DOC to the system
- A contact point (given that the object in question cannot slip on the surface) prevents the object from translating along both axis and only allows rotation around the point itself. Two DOC are introduced
- Trolleys only allow motion along the surface they roll against. This results in 2 DOC being added to the system

Constraint	DOC
String	1
Trolley	2
Contact Point	2

Table 1: DOC for each element in the system

These results are summarised in Table 1
Having gathered all the necessary information it is now possible to determine the total amount of DOC introduced in the system

$$n_{DOC} = 1 * 1_{string} + 1 * 2_{trolley} + 1 * 2_{\text{contact point}} = 5$$

1.2.2 Determination of the Degrees of Freedom

In a 2D system like the one taken into consideration in this report an object is free to move along the x and y axis and to rotate. This means that an object without any constraint introduces 3 DOF into the system. By taking this into consideration, along with the results above it is possible to determine the effective amount of DOF of this system

$$n_{DOF} = 3 * 2_{objects} - n_{DOC} = 1$$

1.3.1 Choice of Independent Variable

There are different possible choices in this regard but in this case the angle θ of counter-clockwise rotation of the disc from its equilibrium position was chosen.

1.3.2 Derivation of Parameters

All other parameters of the system need to be expressed as a function of the independent variable

$\Delta\ell_1$	$\Delta\ell_2$	x_2	$\delta\theta$	$\dot{\Delta\ell_1}$	$\dot{\Delta\ell_2}$	$\dot{x_2}$
$2\theta R$	$-(R+r)\theta$	$-(R+r)\theta$	θ	$2\dot{\theta}R$	$-(R+r)\dot{\theta}$	$-(R+r)\dot{\theta}$

Now that the system is fully characterized only by functions of the independent variable it is possible to proceed with the computation of the Equation of Motion

1.4.1 Equation of Motion

The derivation of the Equation of Motion (EOM) will be done through an energy approach using the Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\partial E_k}{\partial \dot{\theta}}\right) - \frac{\partial E_k}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} + \frac{\partial V}{\partial \theta} = Q_\theta$$

Computing the EOM requires first to determine the functions for the kinetic and potential energy, the energy dissipation function and the virtual work function.

1.4.2 Kinetic Energy

Starting from the kinetic energy it is needed to consider both translation and rotation of the bodies. In this system the disc will both translate and rotate at the same time (a motion which is called rototranslation) and the mass M_2 is only able to translate along the y axis. These considerations will be used to determine the Kinetic Energy Function.

$$E_k = \frac{1}{2}J_d\dot{\theta}^2 + \frac{1}{2}M_dR^2\dot{\theta}^2 + \frac{1}{2}M_2(R+r)^2\dot{\theta}^2 = \frac{1}{2}\left(J_d + M_dR^2 + M_2(R+r)^2\right)\dot{\theta}^2$$

1.4.3 Potential Energy

The potential energy is made of two main components: gravitational and elastic energy. Being that the system is assumed in its equilibrium position we assume the gravitational potential energy of the mass M_2 to be zero and since the disc is in contact on a plane we get that the potential energy function will be only comprised of the elastic part

$$V = \frac{1}{2}k_1\Delta\ell_1^2 + \frac{1}{2}k_2\Delta\ell_2^2 = \frac{1}{2}(4k_1R^2 + k_2(R+r^2))\theta^2$$

1.4.4 Dissipative Energy

In this system the dissipative function only has contributions from the dampers c_1 and c_2. This results in the following expression

$$D = \frac{1}{2}c_1\dot{\Delta\ell}_1^2 + \frac{1}{2}c_2\dot{\Delta\ell_2}^2 = \frac{1}{2}(4c_1R^2 + c_2(R+r)^2))\dot{\theta}^2$$

1.4.5 Virtual Work

By applying a torque $C(t)$ the work applied is evaluated in the following expression.

$$\delta w = c(t)\delta\theta$$

1.4.6 Derivation of the Equation of Motion

It is now necessary to individually deriive all the elements of the Lagrange equation:

$$\frac{1}{\partial t}\frac{\partial E_k}{\partial \dot{\theta}} = (J_d + M_dR^2 + M_2(r+R)^2)\ddot{\theta}$$

$$\frac{\partial E_k}{\partial \theta} = 0$$

$$\frac{\partial V}{\partial \theta} = (4k_1R^2 + k_2(R+r)^2)\theta$$

$$\frac{\partial D}{\partial \dot{\theta}} = (4c_1R^2 - c_2(R+r)^2\dot{\theta}$$

by substituting into the main equation the result is:

$$(J_d + M_dR^2 + M_2(R+r)^2)\ddot{\theta} + (4c_1R^2 + c_2(R+r)^2)\dot{\theta} + (4k_1R^2 + k_2(R+r)^2)\theta = C(t)$$

with the data of the system the resulting equation is:

$$3.05\ddot{\theta} + 2.48\dot{\theta} + 179.5 = C(t)$$

1.5.1 Natural Frequency of the System

Now that the EOM has been found it is possible to extrapolate a number of different kind of information about the system. The first is the natural frequency or resonant frequency of the system. To achieve this result it is necessary to resolve the EOM as a differential equation and in particular the free motion solution is of interest.

$$3.05\lambda^2 + 2.48\lambda + 179.5 = 0$$

which gives the result $f_0 = 1,24hz, \omega_0 = 7.67rad/s$
Also, by rewriting the EOM in the form:

$$\lambda^2 + 2\alpha\lambda + \omega_0^2$$

it is possible to achieve the following result

$$\alpha = \frac{C_{eq}}{2M_{eq}} = 0.4$$

whith α being the damping coefficient

1.5.2 Adimensional Damping Ratio and Damped Frequency

At this point it is desirable to compute the effect of the damping on the natural frequency of the system and then adjusting it accordingly to obtain a better representation overall. To achieve this the adimensional damping ratio of the system is computed first

$$h = \frac{c}{2M_{eq}\omega_0} = 0.053$$

Being h very small we do not expect the damped natural frequency to differ a lot from the original natural frequency found at point 1.5.1.

$$f_{damped} = f_0\sqrt{1-h^2} = 1.238hz = 7.66s^-1$$

It is clearly seen, as expected that the two results only differ by $0.01hz$

2.0 Free Motion of the System

By considering the system as linear and time-invariant and being it a 1 DOF system with damping it was possible, as in chapter 1, to determine the EOM. The solution of the equation is the following:

$$x(t) = C_1 e^{\lambda 1t} + C_2 e^{\lambda_2 t}$$

where C_1 and C_2 are constants to be determined by applying the opportune boundary conditions for the system. When the boundary conditions are set as follows this is the result achieved.

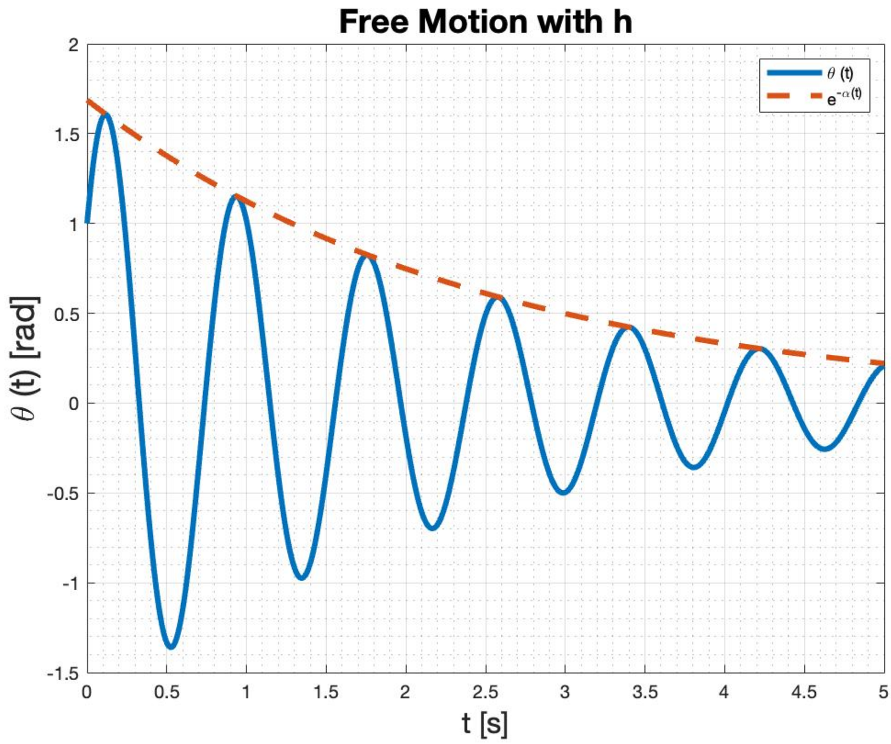

Figure 2: Plot of Free motion with arbitrary initial conditions

This plot perfectly represents the expectations for the system. Since the system in question is a lightly damped system the free motion of the system is supposed to be a sinusoid at the natural frequency of the system which will slowly fade out in time following the envelope of a decaying exponential function. This is because the losses of the system slowly dissipate the energy of the system since no restoring force is applied and it tends to a steady state.

2.1 Free Motion of the System (with $h_1 = 4h_0$)

By quadrupling the adimensional damping ratio, the response will decay faster than the before example at point 2.0. This is due to the fact that λ shrinks for higher values of h so the result is a faster-decaying exponential which multiplies the sinusoid. These considerations are perfectly represented in the following plot

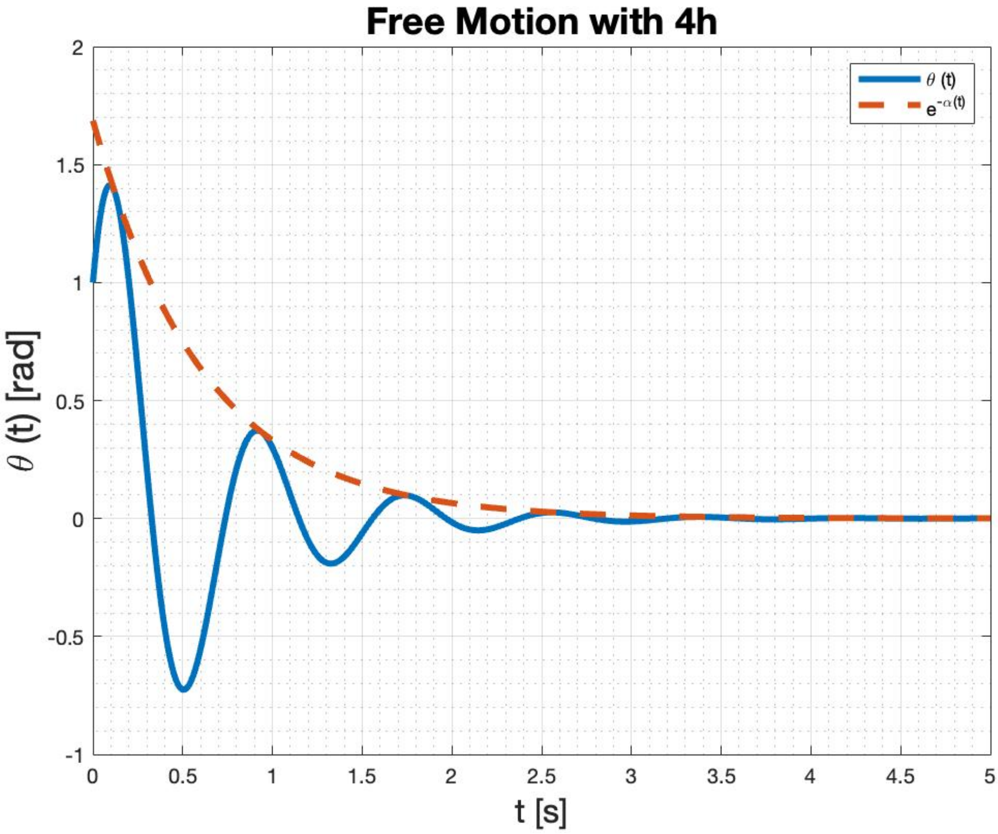

Figure 3: Plot of Free motion with four times the adimensional damping ratio

2.2 Free Motion of the System (with $h_1 = 25h_0$)

The considerations made for the previous point still apply here. The result is even clearer as the damping ratio is now way bigger.

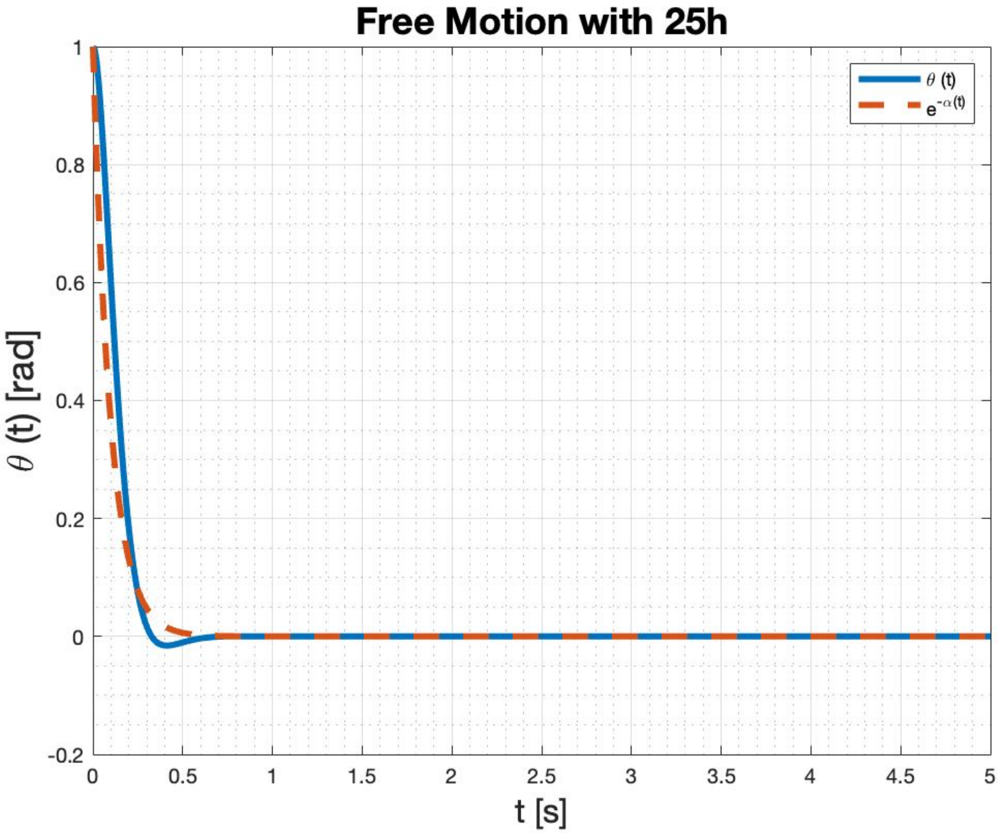

Figure 4: Plot of Free motion with twenty-five times the adimensional damping ratio

3.1 Forced Motion of the System: Frequency Response Function

The frequency response function (FRF) is a common way to define a linear time-invariant system since it describes how the system reacts to an harmonic input in terms of amplification (or attenuation) of certain frequencies and in terms of phase shift of those same frequencies. This is a useful way to describe a system and it can give different information in different scenarios. A FRF is computes as follows

$$H(\Omega) = \frac{x_0}{F_0}$$

where x_0 represents the movement of the system with respect to the applied harmonic force F_0. From this representation it is possible to extract the magnitude (amplitude response) of the system by taking the absolute value of the function and the phase response by taking the angle of function. In the specific case of this system the FRF is the following:

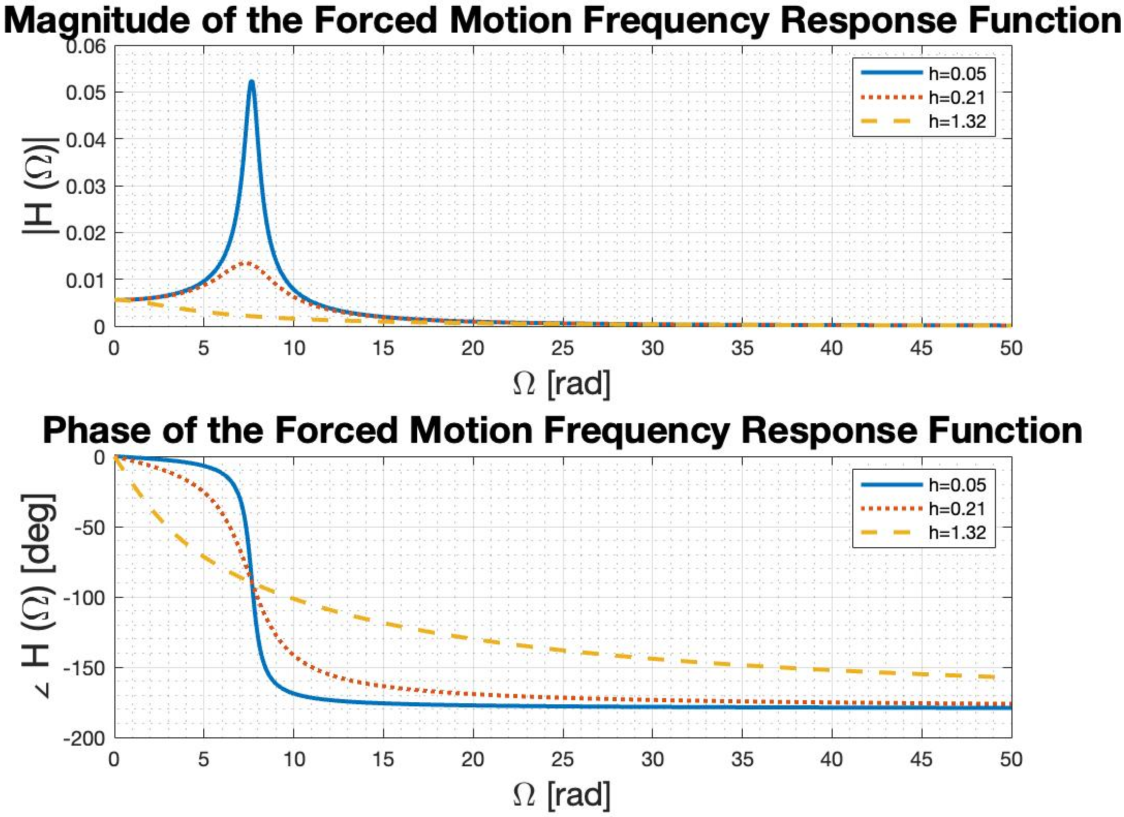

Figure 5: Plot of the Frequency Response Function with different values of h

3.2 Complete Time Response to a Harmonic Torque

It is now required to evaluate the response of the system to an harmonic torque

$$C(t) = Acos(2\pi f_1 + \varphi)$$

with $A = 2.5N$, $f_1 = 0.35hz$ and $\varphi = \frac{\pi}{3}$. The following plot shows the amplitude and phase response of the system

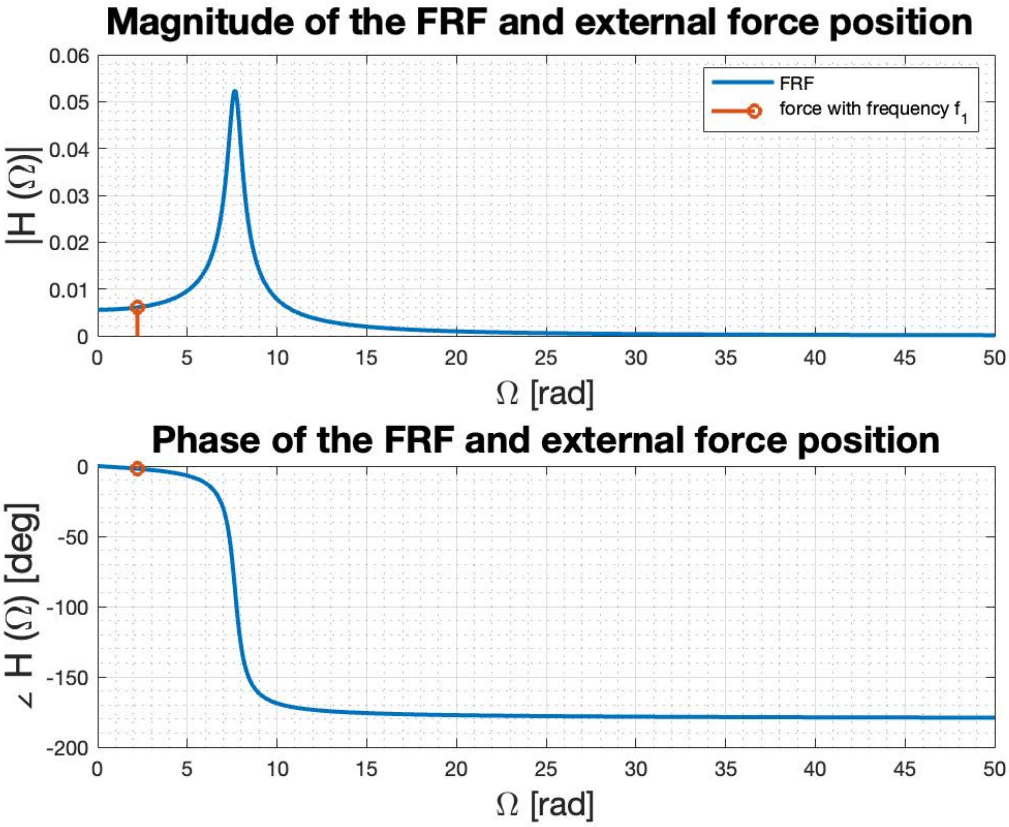

Figure 6: Representation of the torque C(t) in the FRF plot

It is now possible to compute the complete response of the system. As it is easily foreseeable the input force will not have much effect of the system as the FRF at that frequency has less than a 0.01 amplification factor. To achieve better visual results plots for both individual responses (free and forced) are given and also a comparison plot to visually show how small the forced response really is.

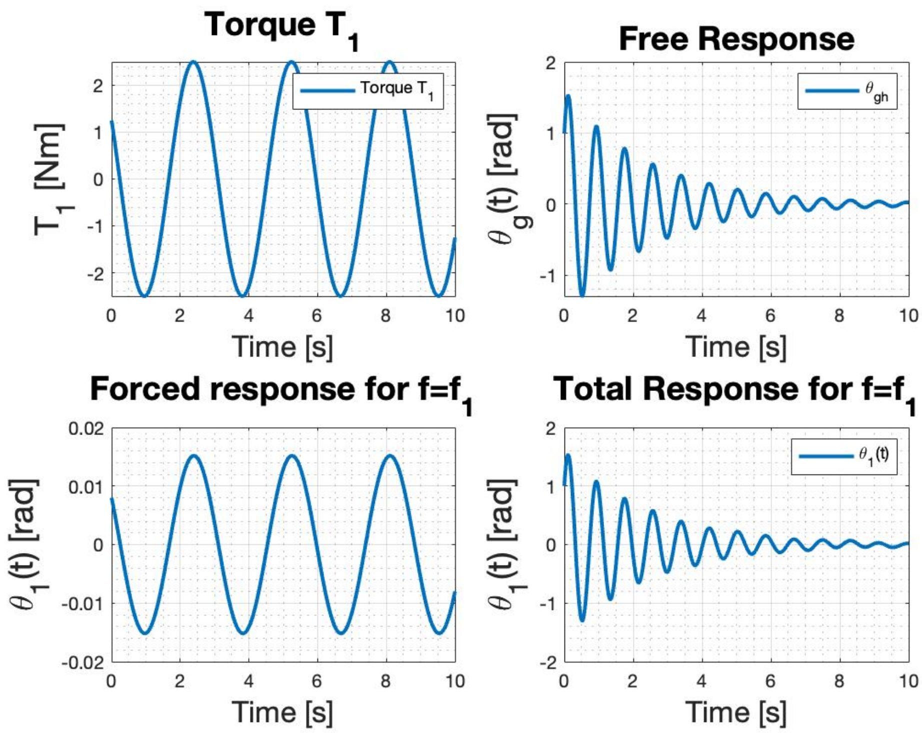

Figure 7: From top-left to bottom right: torque applied, forced response, free response, total response

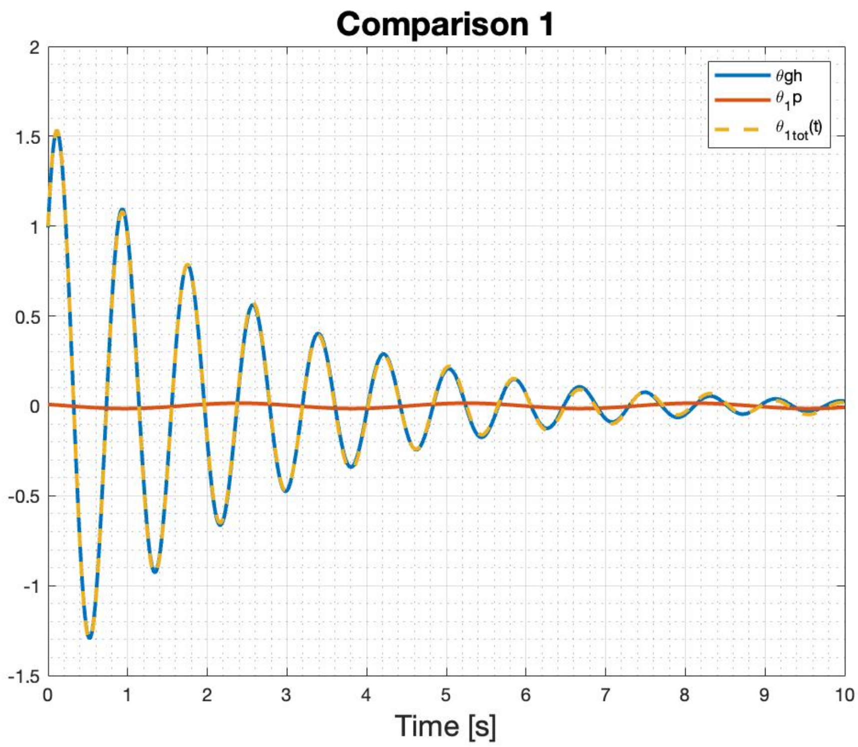

Figure 8: Visual comparison of the free and forced responses

3.3 Different cases of response of the system

First the same calculations as the previous point are done with a torque which has a frequency of $10hz$. The same considerations apply as also this frequency lies way beyond the peak of the FRF

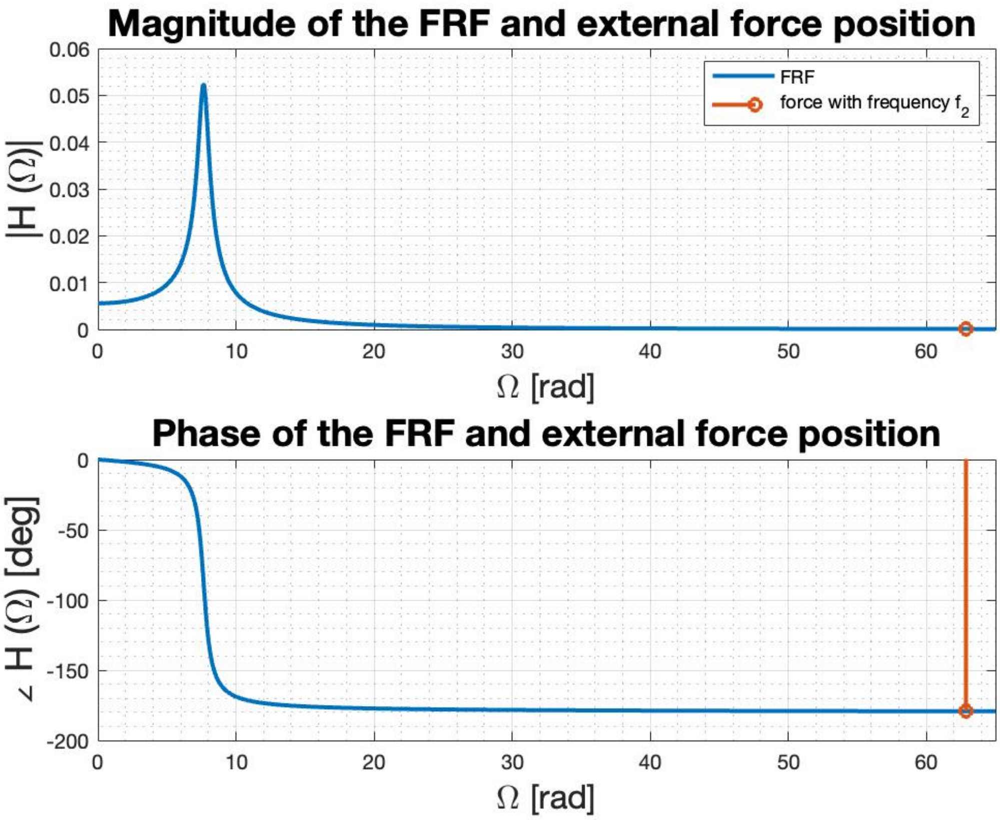

Figure 9: Representation of the torque C(t) in the FRF plot

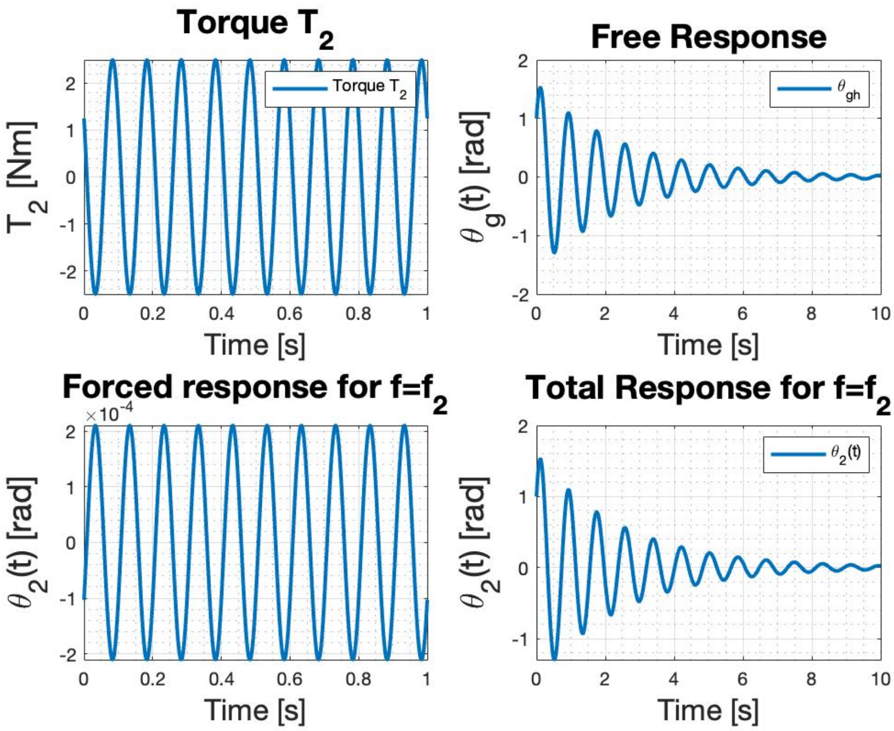

Figure 10: From top-left to bottom right: torque applied, forced response, free response, total response

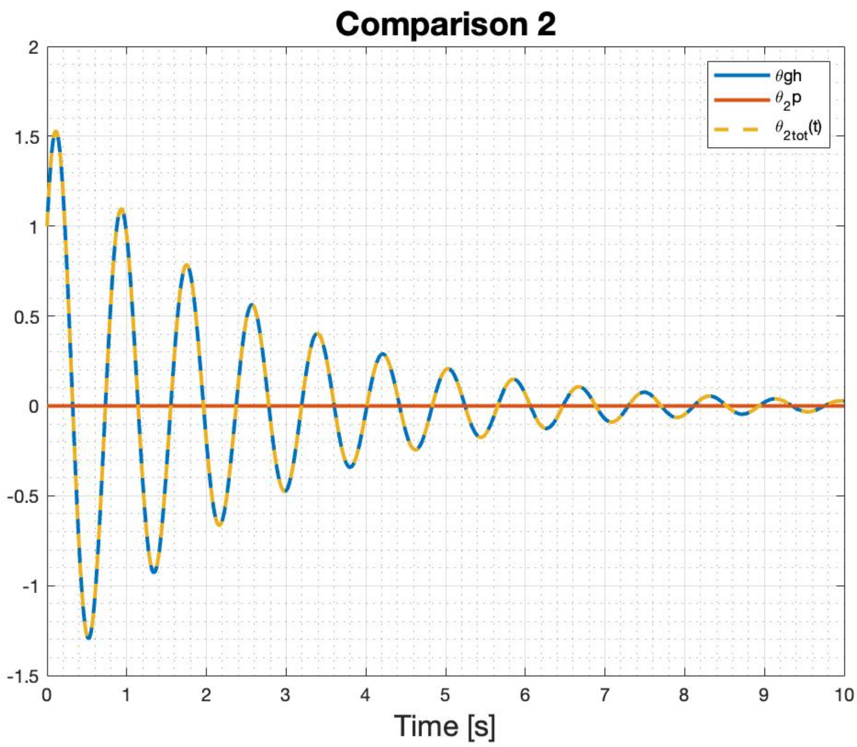

Figure 11: Visual comparison of the free and forced responses

Statically-applied torque

It is now considered the case of a statically-applied torque such as the frequency of the force is $0hz$ with the same amplitude as the previous examples. The following plots represent the results achieved

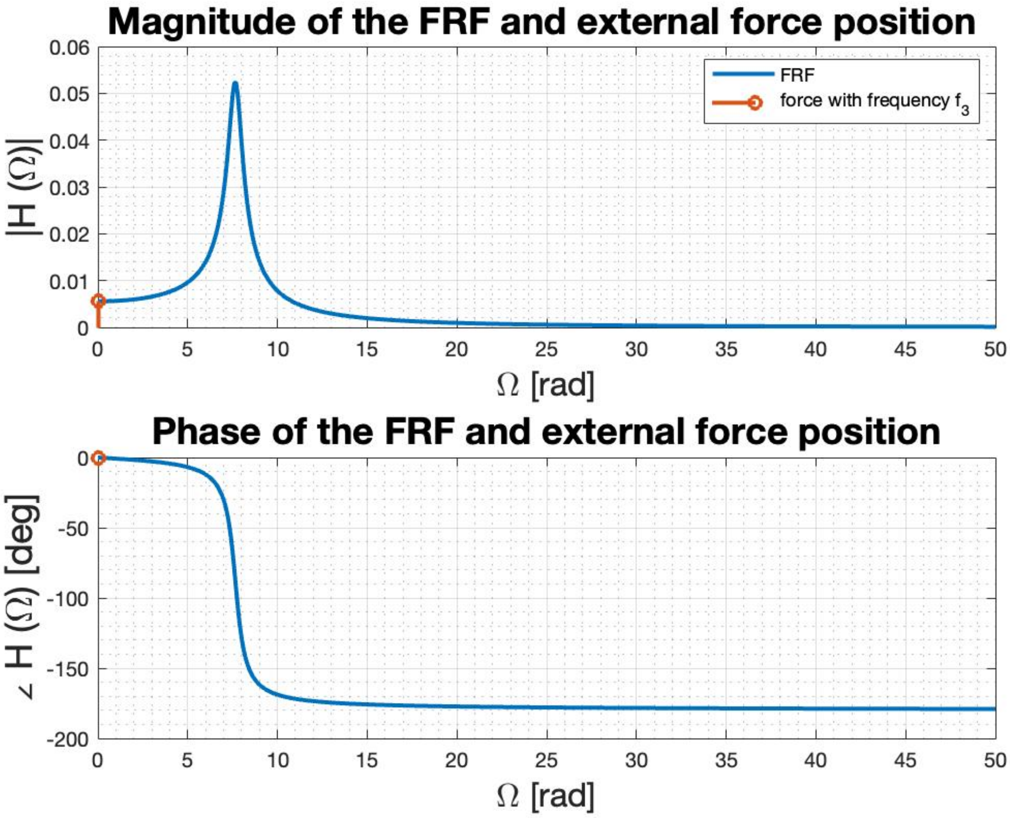

Figure 12: Representation of the constant torque in the FRF plot

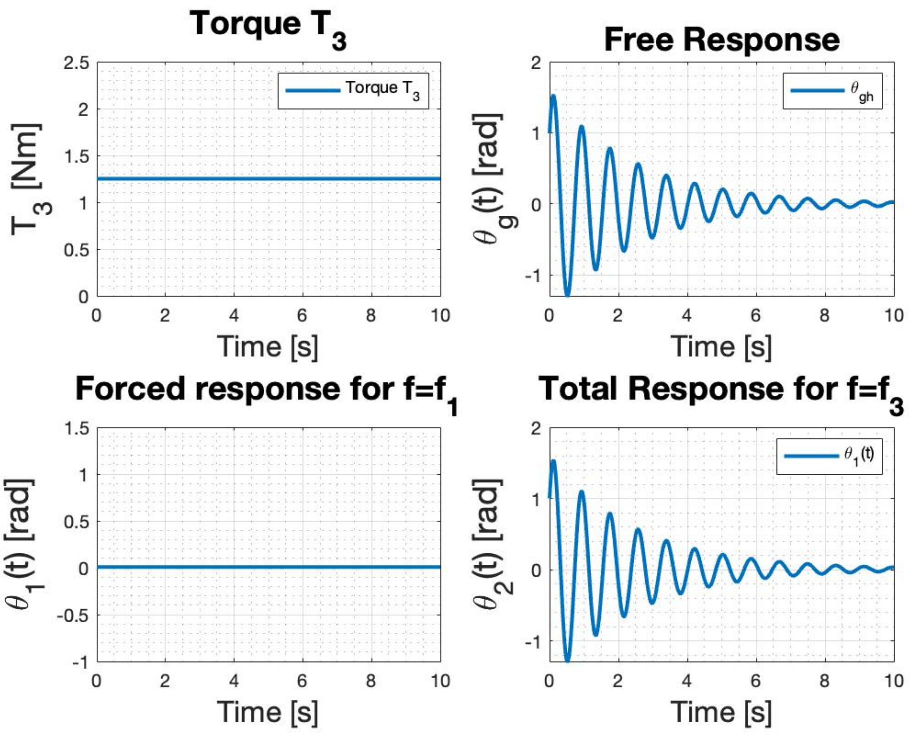

Figure 13: From top-left to bottom right: torque applied, free response, forced response, total response

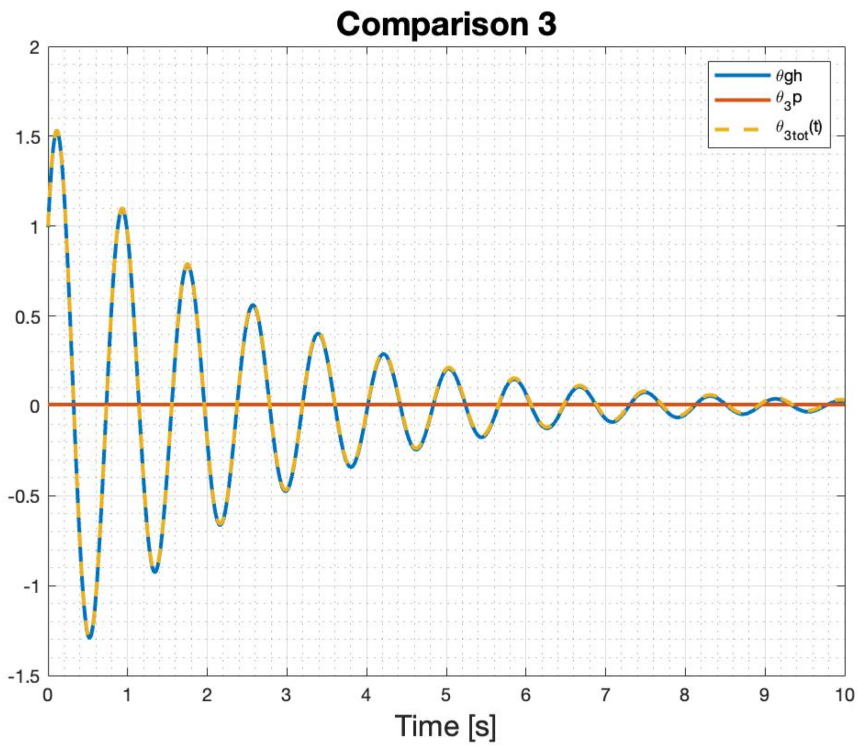

Figure 14: Visual comparison of the free and forced responses

3.4 Response to a multi-harmonic torque

This time the superposition of three different harmonic torques is applied to the system. The total torque has the following form:

$$C(t) = \sum_{k=1}^{3} B_k \cos(2\pi f_k t + \beta_k)$$

with B_k, f_k and β_k having the following values: $B_1 = 1.2$, $B_2 = 0.5$, $B_3 = 5$, $f_1 = 0.35hz$, $f_2 = 2.5hz$, $f_3 = 10hz$, $\beta_1 = \frac{\pi}{4}$, $\beta_2 = \frac{\pi}{5}$ and $\beta_3 = \frac{\pi}{6}$ The total torque is represented in the following plot:

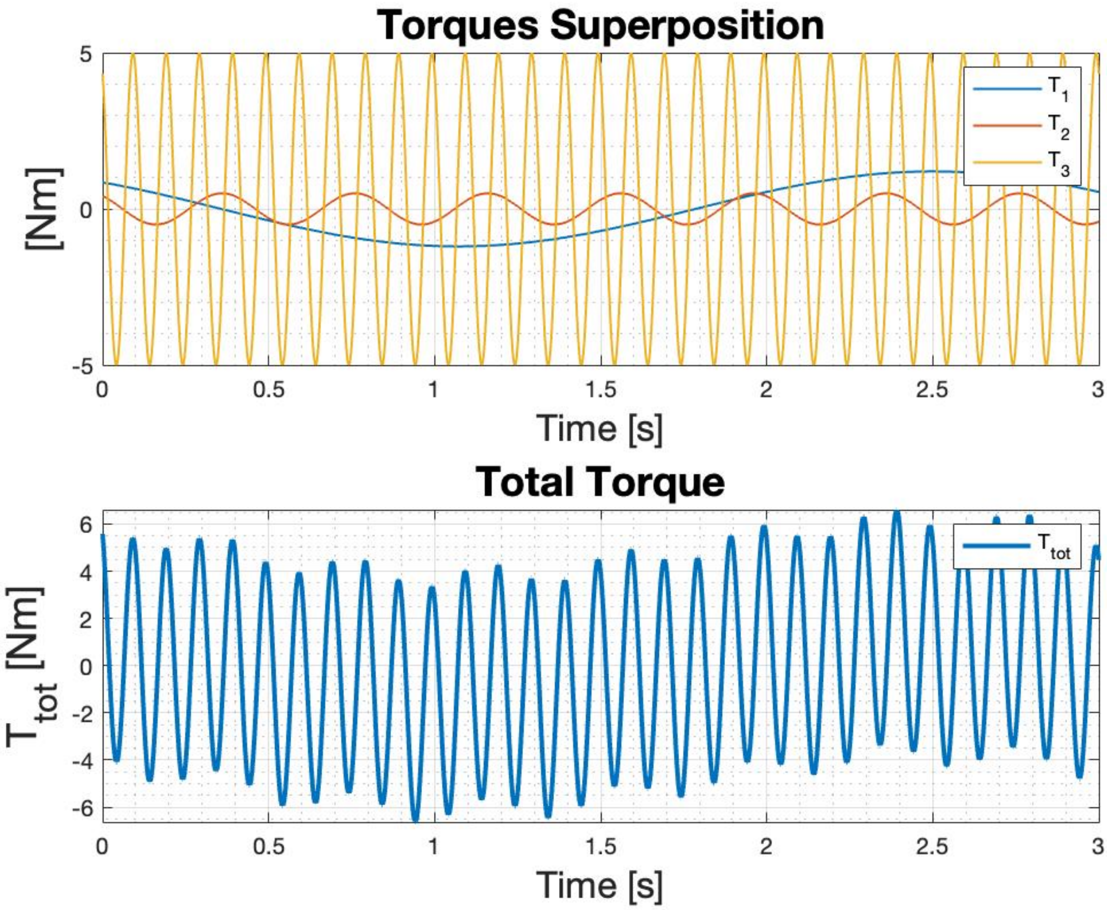

Figure 15: Total torque applied (superposition of 3 harmonic torques)

Since the system in study is linear it is possible to apply the superposition of effects to compute the total output of the system. This means that the motion will be the sum of the motion for each of the three harmonic torques that make the total torque $C(t)$. The following plot will show the result of these computations

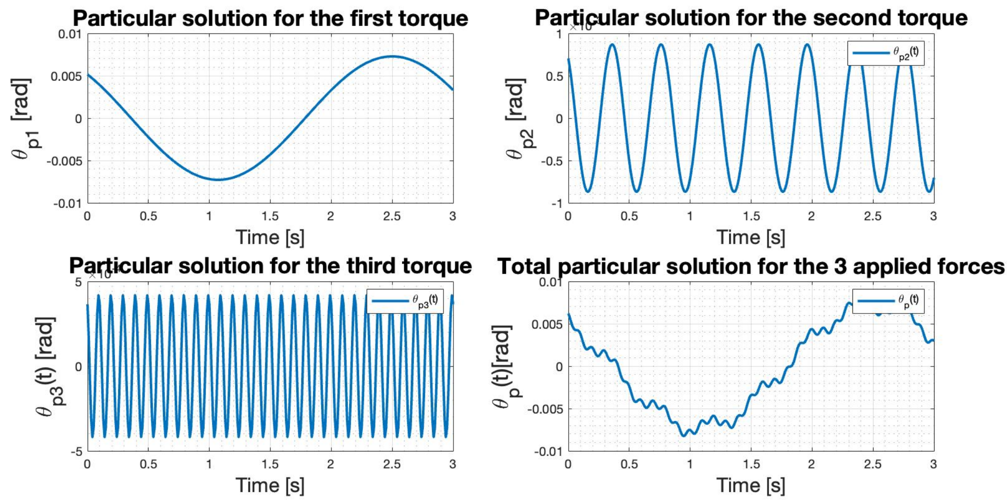

Figure 16: Forced response of the system to the multi-harmonic torque

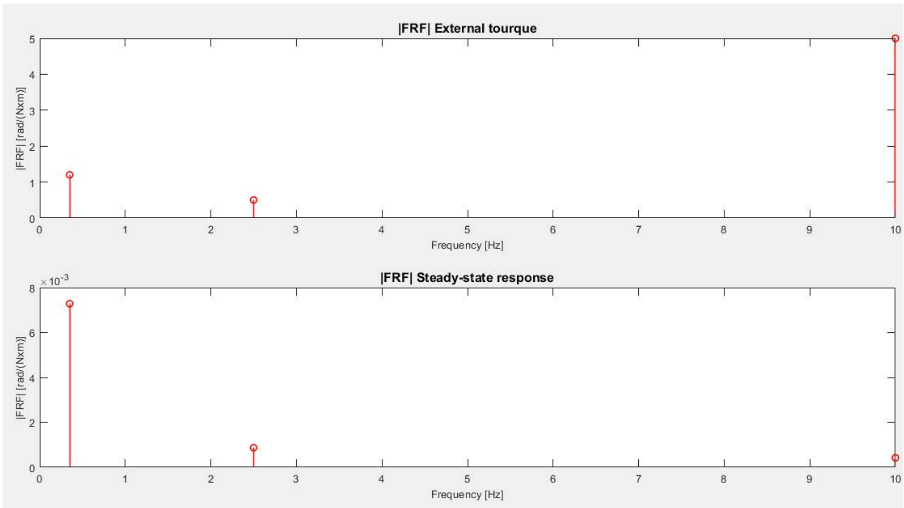

Figure 17: |FRF| of C(t) and of the motion of the system

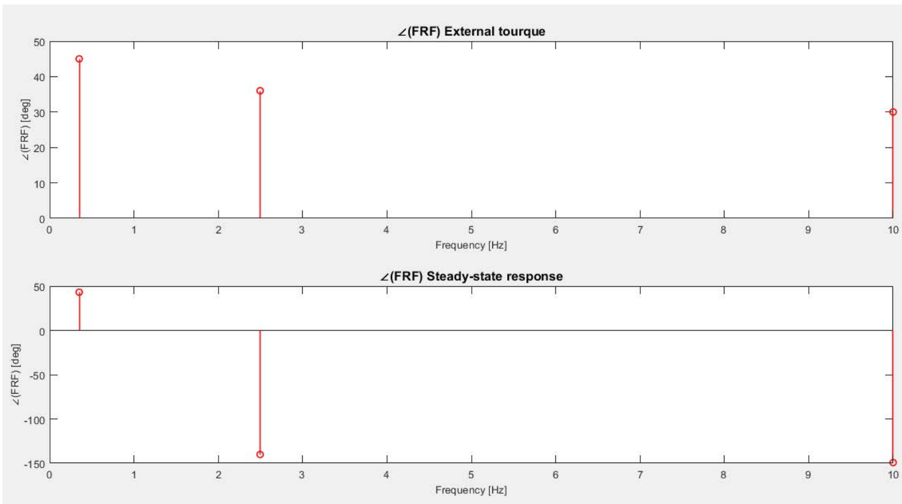

Figure 18: Phse of the FRF of C(t) and of the motion of the system