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1 Introduction to market risk measurement

The focus of this homework will be on the methods for measuring market risk, understood as the risk incurred by the investor following general market variations. For this reason, attention has been paid to financial companies because their problems can be transferred to investors/savers and the entire financial system. The starting point is given by the need to regulate banks and financial institutions with the aim of inducing prudent management (with limited risks) of the resources collected by savers and investors, while at the same time avoiding that the default of a company translates into significant losses in others (or in other defaults). For this need, in 1974 the Basel Committee for banking supervision was created by the ten most industrialized countries (G10). The first resolution, Basel I, was issued in 1988 and concerned only the capital requirements to deal with credit risk. In this agreement there was for the first time a universally recognized definition of minimum banking capital. The basic reasoning is very simple: each opening of credit must correspond to a share of capital set aside as a precaution, the percentage of which is directly proportional to the riskiness of the loan granted. As time went by, however, this agreement showed notable limitations and therefore in 1996 it was established that the minimum capital had to consider both the credit risk, from the insolvency of the debtor, and the market risk, which became increasingly relevant with the growing globalization in the financial sector. In this context, *Value at Risk* was introduced, which means the maximum potential loss that can be incurred with a given level of confidence and a given time horizon.

Subsequently, over the years, further revisions of the Basel I agreement were proposed, especially following the great financial crisis of 2008, which showed some of its limits.

The evaluation of Value-at-Risk, as has just been observed, represents an extremely important aspect both for investors and savers and for the banks themselves. For this reason, as previously stated, this homework will deal with the Value-at-Risk aspect and specifically the paper will be divided into two parts:

- in the first part the VaR one step forward of a basket of securities belonging to the STOXX600 will be calculated;
- while in the second part, different VaR estimates for the "Naturgy Energy" stock will be carried out and compared.

2 Methods and Models for Value at Risk (VaR)

2.1 Estimation of VaR with a portfolio of financial instruments

To create the portfolio, given the importance that the energy market is acquiring in recent years, I considered the historical series of the last 10 years of shares belonging to companies in the sectors: electricity, gas, water, oil, transport and automotive. In particular, 51 shares were selected[1]. We assume that we are working with a relatively simple portfolio, made up of $K = 51$ financial instruments of the same class, all shares, whose returns are assumed i.i.d. and normals $r_t \sim N(\mu, \Sigma)$ and with weights equal to $\mathbf{w}$ (vector of dimension K whose sum is equal to 1).
Of interest is the VaR of the portfolio, i.e. the VaR obtained as a linear combination of the returns of the individual instruments, with weightings equal to $\mathbf{w}$. For the normal property:

$$\mu_p = \mathbf{w}'\mu, \quad \sigma_p^2 = \mathbf{w}'\Sigma\mathbf{w}', \quad r_t \sim N(\mu_p, \sigma_p^2) \tag{1}$$

$$\begin{aligned} VaR(t+h, \alpha) &= h\mu_p + \sqrt{h}\sigma_p\Phi(\alpha)^{-1} \\ &= h\mathbf{w}'\mu + \sqrt{h}(\mathbf{w}'\Sigma\mathbf{w}')^{\frac{1}{2}}\Phi(\alpha)^{-1} \end{aligned} \tag{2}$$

The dataset initially available included all the share prices of companies belonging to the STOXX600 starting from 29 December 1989. From this set, a basket of 51 companies was selected starting from 31 December 2010. The selection of these values was carried out before reading the dataset using R. Below is the code used for reading, transforming prices into returns and cleaning the dataset.

```
data=read_excel('STOXX600-2.xlsx','TotalReturnsIndexes',
                'B7:AZ3202',col_names=FALSE,na="NA")
T=dim(data)[1]
data1=date
i=is.na(data1)
idr=rowSums(i)
idc=colSums(i)
data1=data1[,idc==0]
i=is.na(data1)
idr=rowSums(i)
idc=colSums(i)
T=dim(data1)[1]
```

[1] In Appendix A it is possible to acquire more information on the companies belonging to the chosen basket

```
N=dim(data1)[2]
data1 = data.frame(data1)

r=100*(log(data1[2:T,])-log(data1[1:T-1,]))
r = as.matrix(r)
i0= r==0
i0r=rowSums(i0)
```

The portfolio in question represents a special case in which the return is expressed as a function of a set of risk factors that reflect the impact of changes in the underlying financial instruments on the portfolio. It is important to highlight that in this context we are considering a linear portfolio, i.e. the portfolio return is a linear function of the risk factors, even though the relationship between the risk factors and the underlying financial instruments may not be linear. In particular, the model taken into consideration can be expressed as follows:

$$r_t = \alpha + \beta F_t + \epsilon_t \tag{3}$$

where the K equity instruments are a function of $M << K$ factors of risk, the model parameters are estimated with OLS and the returns are approximated with:

$$\hat{r_t} = \hat{\alpha} + \hat{\beta} F_t \tag{4}$$

Depending on the type of instrument taken into consideration, different risk factors can be identified. For example, in the case of an option, you can use the log-prices of the underlying asset or the short-term interest rate. In the case of shares, however, it is possible to consider various indicators, including macroeconomic indicators such as GDP, the industrial production index, inflation, statistical indicators such as Principal Component Analysis (PCA) and indicators derived from the characteristics of the companies, known as Fama-French factors. Among the latter, the most commonly used are:

- Small-minus-Big (SMB): This factor is based on the size of the companies. Small companies tend to generate higher returns than large companies. The SMB factor is calculated by taking the weighted average stock returns of companies with the lowest market capitalization and subtracting the weighted average returns of companies with the highest market capitalization.

- High-minus-Low (HML): This factor is based on companies' book values. Companies with a high book value (High) tend to generate higher returns than companies with a low book value (Low). The HML factor is calculated by taking the weighted average stock returns of companies with a high book value and subtracting the weighted average returns of companies with a low book value.

- Winners-minus-Loosers (WML): This factor is based on the historical performance of companies. Companies that have achieved high returns in the past (Winners) tend to continue to generate high returns in the future, while companies that have achieved low returns in the past (Loosers) tend to continue to generate low returns in the future. The WML factor is calculated by taking the weighted average stock returns of companies that have achieved high returns in the past and subtracting the weighted average returns of companies that have achieved low returns in the past.

- Robust-minus-Weak (RMW): This factor is based on the profitability of companies. Companies that have high profitability (Robust) tend to generate higher returns than companies with low profitability (Weak). The RMW factor is calculated by taking the weighted average stock returns of companies with high profitability and subtracting the weighted average returns of companies with low profitability.

- Conservative-minus-Aggressive (CMA): This factor is based on companies' investment. Companies that invest conservatively (Conservative) tend to generate higher returns than companies that invest aggressively (Aggressive). The CMA factor is calculated by taking the weighted average stock returns of companies that invest conservatively and subtracting the weighted average returns of companies that invest aggressively.

Where, in addition to these, a market index is usually also considered.
For the analysis, in addition to considering the factors proposed by Fama-French, I took into consideration 4 additional factors:

- Endex-TTF-gas base load, i.e. a standardized futures contract for the supply of natural gas on the TTF (Title Transfer Facility) market, the natural gas trading point in north-west Europe. The Endex-TTF-gas base load contract specifies the amount of natural gas that will be delivered in a given delivery month, at a pre-agreed price. This contract is used to hedge price risks by producers, suppliers, distributors and end users of natural gas.

- TRPC-SRMC-Coal-API2 is a price index for thermal coal used for electricity generation, published by Thomson Reuters/Platts. The price is calculated based on coal production costs from the main producers and suppliers of the API2 market (North-west Europe, South Africa, Colombia), and is used as a reference for trading thermal coal futures contracts.

- EEX Phelix Base is a price index used to evaluate the electricity market in continental Europe and is used as a benchmark for electricity purchase and sales contracts in continental Europe.

- Crude Oil WTI Spot is an indicator that represents the price per barrel of WTI (West Texas Intermediate) crude oil traded on the New York Mercantile Exchange[2] (NYMEX), with delivery to Cushing, Oklahoma (USA). The price of this contract is used as a reference for the price of crude oil in North America and other parts of the world.

I also took these factors into consideration because I believe they can be excellent indicators of the basket of securities taken into consideration. As stated previously, I believe that companies operating with activities closely linked to the energy sector are having a great impact on the markets, just think of the effect on the European economy caused by the war in Ukraine; but not only that, individuals, and therefore investors, are becoming aware of the impact that the energy sector, and in particular the renewable one, is having on all of us.
The model with the factors considered will have the following form:

$$\begin{aligned} r_{i,t} = \alpha_i + \beta_{i,Mkt} r_t^{Mkt} + \beta_{i,SMB} r_t ... CMA + \\ + \beta_{i,TTF-gas} r_t^{TTF-gas} + \beta_{i,Coal} r_t^{Coal} + \beta_{i,EEX} r_t^{EEX} + \beta_{i,Oil} r_t^{Oil} + \epsilon_{i,t} \end{aligned} \tag{5}$$

The model is estimated equation by equation, so as to save the parameters and residuals in a matrix. Below, Figure 1 shows a graph containing all the estimated coefficients for the 9 different factors considered.

[2] It is underlined that this value has been converted into euros in order not to be influenced from exchange rate issues.

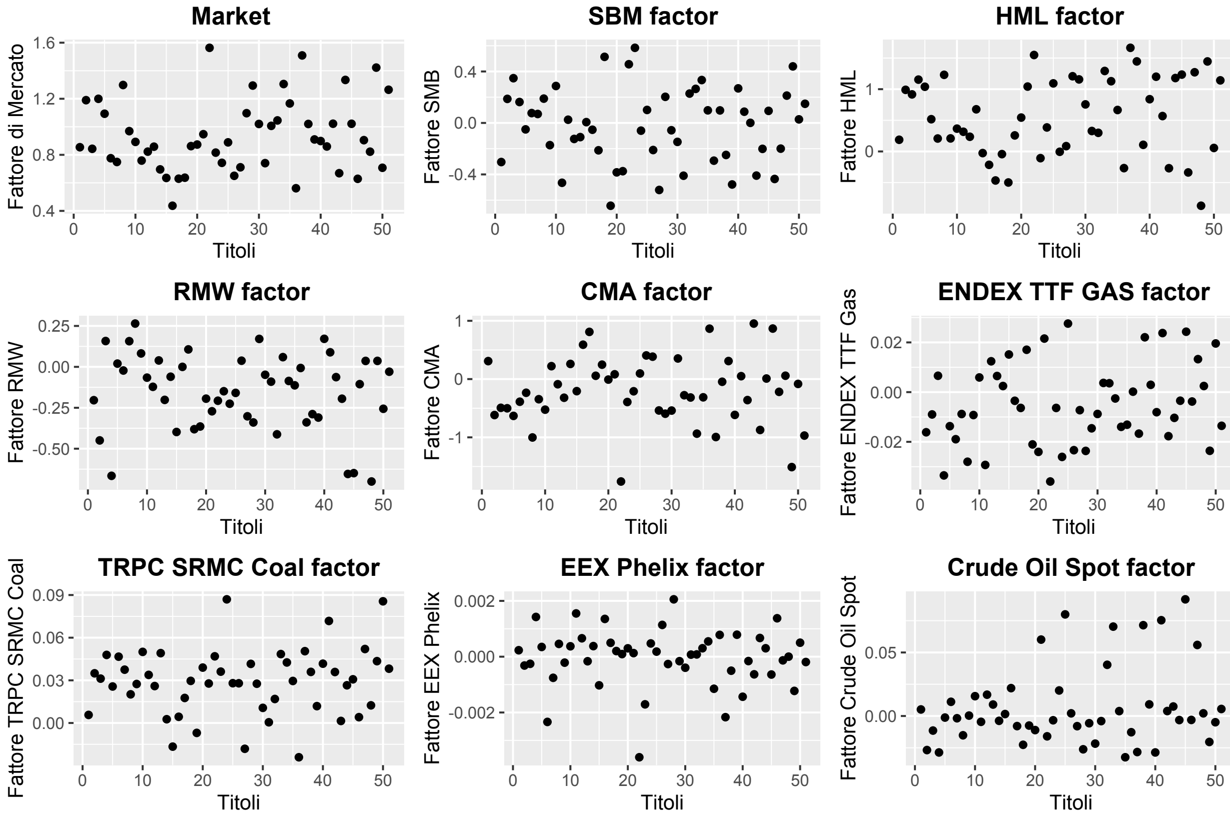

Figure 1: Estimated coefficients for each factor

Furthermore, to observe another measure of fit of the model to the data the following boxplots are shown. Two graphic representations are made in order to better observe the boxplots of the factors I inserted, this is because they represent values that vary little around zero.

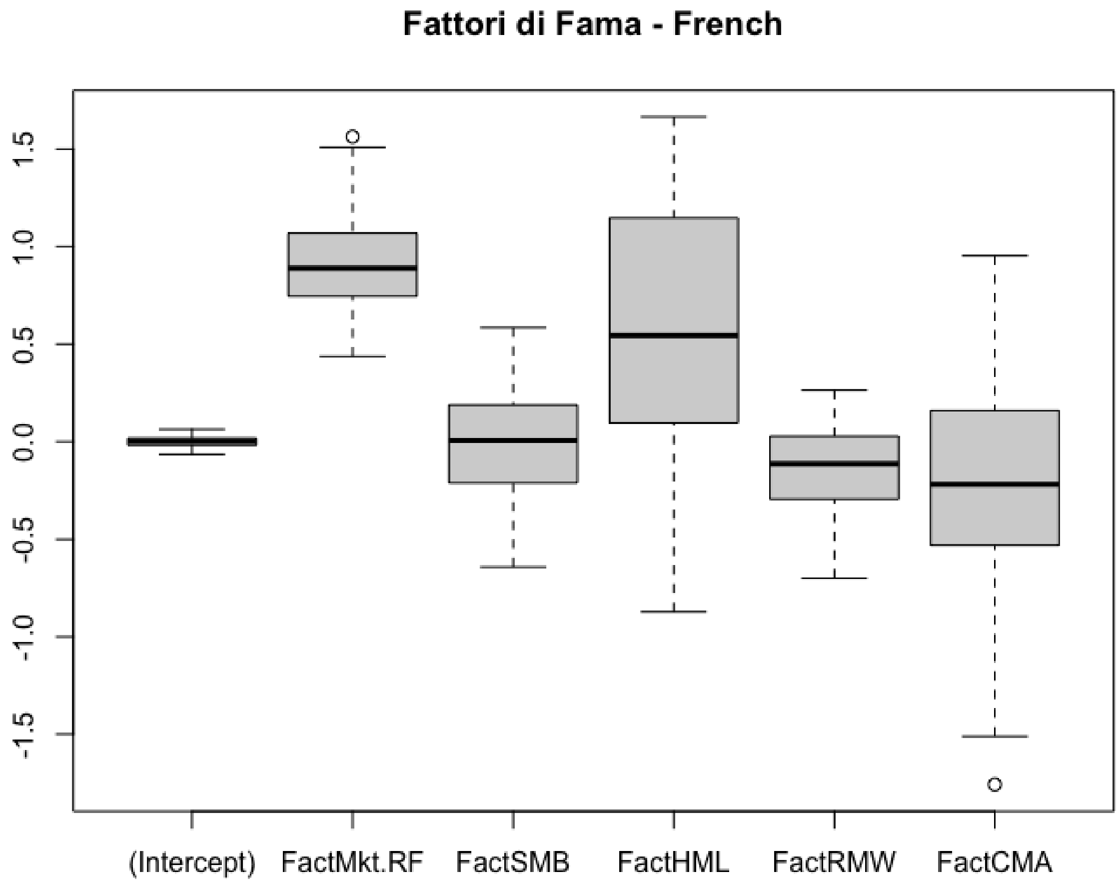

Figure 2: Boxplot of the estimated coefficients for the Fama factors - French

It is interesting to note that the market factor (represented in the second boxplot: FactMkt.RF) has positive values, which indicates a good fit. In fact, if we had observed negative values for this factor, we would have encountered a data alignment problem, probably caused by the difference in length of the data matrices, resulting from the fact that a holiday had been considered in one dataset but not in the other. The other coefficients, however, vary both positively and negatively (there is no aggregate standard). However, if you analyze the individual groups of securities, it is possible to identify patterns: the SMB factor shows positive coefficients for small companies and negative coefficients for large ones, while the HML factor presents positive coefficients for "High" securities. " and negative coefficients for "Low" stocks.

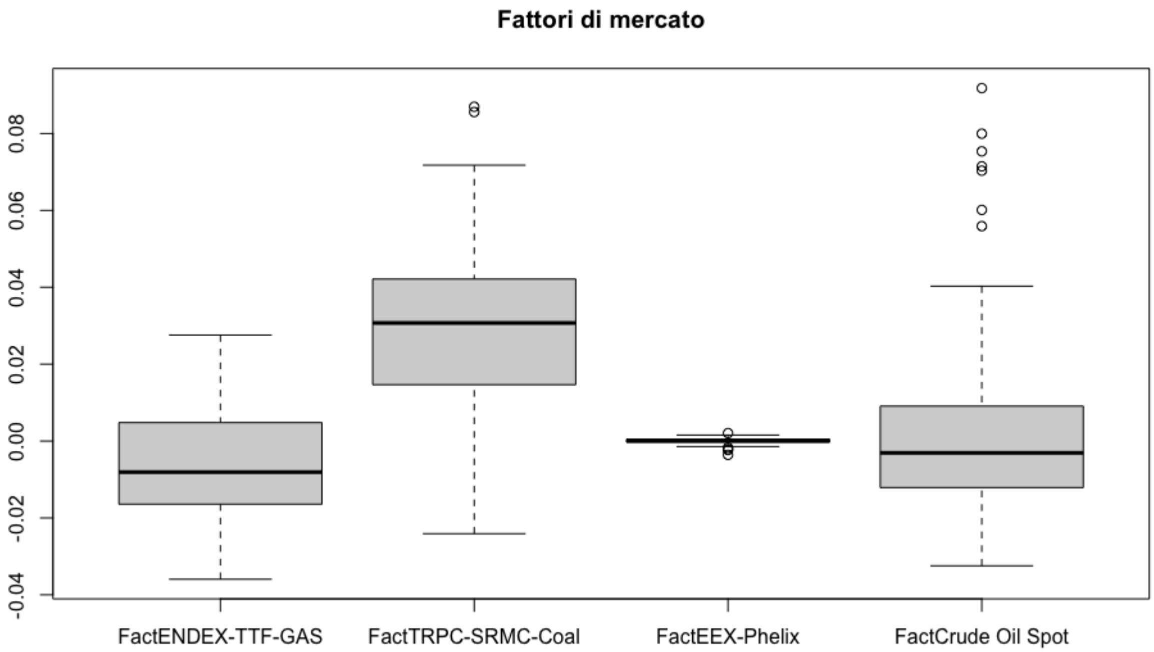

Figure 3: Boxplot of the estimated coefficients for the factors I entered

As regards the factors I inserted, as already observed previously and as highlighted in this boxplot, the coefficients show a distribution concentrated around zero. In order to delve deeper into this aspect, it is interesting to observe the price and return graphs relating to the four factors.

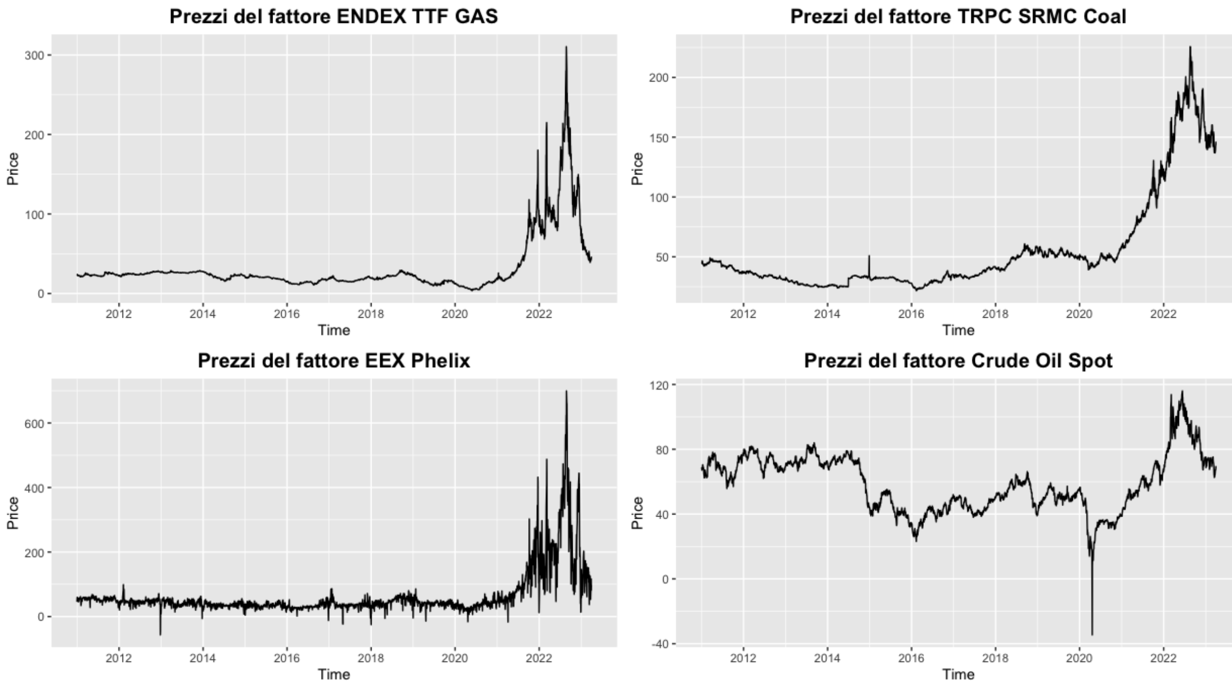

Figure 4: Trends of market factor price

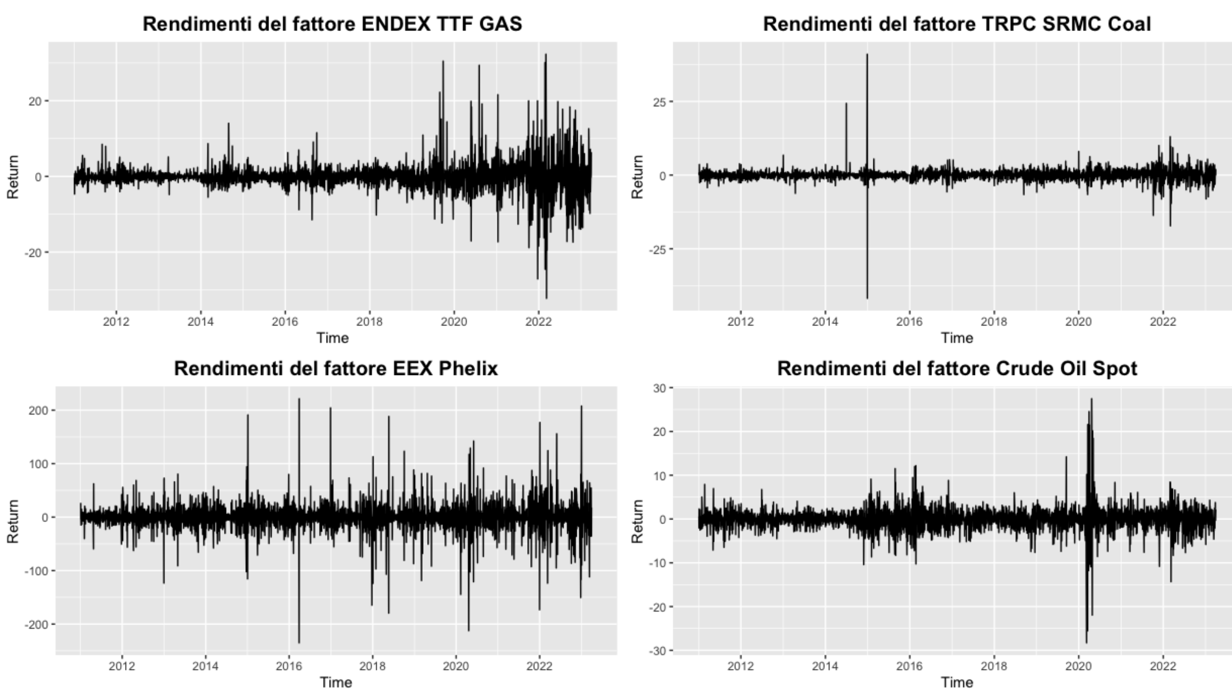

Figure 5: Trends of market factor returns

Below you can see the code used for this section, where, in the first part, a simple reading of the data is carried out, share prices are transformed into returns and the data sets relating to the Fama - French factors are combined with those relating to energy sector indicators; subsequently, using a *for* loop, I estimated the model equation by equation; finally I represented the results obtained graphically.

```
# Reading data sets
     dataFF = read.csv("Factors_Fama.csv", sep = ";")
     F=as.matrix(dataFF[,-c(1,7)])

     dataFF2 = read_excel('Factors_Futures.xlsx','Sheet1',
                         'B2:E3197',col_names=FALSE,na="NA")
     T=dim(dataFF2)[1]
     N=dim(dataFF2)[2]
     dataFF2 = data.frame(dataFF2)

     rFF2=100*(log(abs(dataFF2[2:T,]))-log(abs(dataFF2[1:T-1,])))

     Fact = cbind(F[2:3196,],F2)
     colnames(Fact) = c("Mkt.RF","SMB","HML","RMW","CMA", "ENDEX-TTF-GAS",
                       "TRPC-SRMC-Coal", "EEX-Phelix", "Crude Oil Spot")
# Estimation of the factorial model
     T=dim(r)[1]
     N=dim(r)[2]
     K=dim(Fact)[2]
     pars=NULL
     resLM=NULL
     for(i in 1:N){
         out=lm(r[,i]~Fact)
         pars=rbind(pars,out$coefficients)
         resLM=cbind(resLM,out$residuals)
     }
# Graphical representations
     ggplot(pars) +
         aes(x = seq(1:51), y = X.Intercept.) +
         geom_point(shape = "circle", size = 1.5, color = "black") +
         labs(title = "Intercept ") +
         theme_gray() +
         theme(plot.title = element_text(face = "bold", hjust = 0.5))
```

```
boxplot(pars)

hist(resLM, nclass = 40, xlim = c(-10,10),
    main = "Histogram of frequency of residuals",
    xlab = "Residuals", ylab = "Frequency")
```

In the case of linear models for returns, the use of a linear factor model undoubtedly generates an approximation because we leave out the idiosyncratic or asset-specific component, i.e. the variance linked to ϵ_t. As a first estimate, it is assumed that the risk factors follow a Normal distribution:

$$F_t \sim N(\mu_F, \Sigma_F) \quad \text{then} \quad \hat{r}_t \sim N(\hat{\alpha} + \hat{\beta}\mu_F, \hat{\beta}\Sigma_F\hat{\beta}') \tag{6}$$

The VaR of the portfolio is then approximated as

$$\begin{aligned} VaR(t+h,\alpha) &= h\mu_p + \sqrt{h}\sigma_p \Phi(\alpha)^{-1} \\ &\approx h\mathbf{w}'(\hat{\alpha} + \hat{\beta}\mu_F) + \sqrt{h}(\mathbf{w}'\hat{\beta}\ Sigma_F\hat{\beta}'\mathbf{w}')^{\frac{1}{2}}\Phi(\alpha)^{-1} \end{aligned} \tag{7}$$

From this model we obtain a VaR at 5% of **1.53%**, i.e. in one day the maximum acceptable loss is -1.53%, where the higher this value the higher the risk of this portfolio will be. For its calculation, the Formula **??** was used, and therefore the code used is:

```
muF=as.matrix(colMeans(Fact)) # Mean of the factors
sigmaF=cov(Fact) # Variance of factors
w=matrix(1/N,N,1) # <-- define the weights, in this case equi - weight
a=0.05 # <-- the quantile
muP=t(w) %*% (pars[,1] + pars[,2:10]%*%muF[1:9]) # portfolio average
SigmaP= t(w) %*% (pars[,2:10] %*% sigmaF[1:9,1:9] %*% t(pars[,2:10])) %*% w
                            # portfolio variance
VaRC1=muP + sqrt(SigmaP)*qnorm(a)
```

If we assume that the correlation between returns r_t is completely captured by the factors r_t then the innovations are uncorrelated and Σ_ϵ is diagonal. In this case the VaR becomes:

$$\begin{aligned} VaR(t+h,\alpha) &= h\mu_p + \sqrt{h}\sigma_p \Phi(\alpha)^{-1} \\ &\approx h\mathbf{w}'(\hat{\alpha} + \hat{\beta}\mu_F) + \sqrt{h}(\mathbf{w}'\hat{\beta}\ Sigma_F\hat{\beta}'\mathbf{w} + \mathbf{w}'\hat{\Sigma}_\epsilon\mathbf{w})^{\frac{1}{2}}\Phi(\alpha)^{-1} \end{aligned} \tag{8}$$

The VaR is therefore influenced by a systematic component ($\mathbf{w}'\hat{\beta}\Sigma_F\hat{\beta}'\mathbf{w}$) linked to the risk factors and by an idiosyncratic component ($\mathbf{w}'\hat{\Sigma}_\epsilon\mathbf{w}$) relating to the individual financial instruments. From this model we obtain an estimate of the VaR at 5% equal to **1.95%**.
In this case the code is:

```
sigmaE = cov(resLM)
idios = t(w) %*% sigmaE %*% w
VaR_dis=muP + sqrt(SigmaP+idios)*qnorm(a)
```

The two results obtained with the two different estimation methods present an important difference. To describe what the causes of this difference may be, it is necessary to make some notes on the matter to specify how the systematic and idiosyncratic components are distinguished.

The idiosyncratic component refers to the specific or unique part of a single financial instrument that is not related to or explained by systemic or market factors. It represents the individual characteristics or specific events that influence the performance or behavior of a particular financial instrument. The idiosyncratic component is considered to be the “noise” or non-systemic variability associated with a financial instrument. It can be influenced by factors such as specific company news, unique events or changes within the company. On the other hand, the systemic component refers to the part of the performance or behavior of an individual financial instrument that is related to market or systemic factors. These are the factors that affect the entire market or a large segment of the market, such as interest rates, global economic fluctuations, or changes in market policies. Of these two quantities, it is important to note how the idiosyncratic component, in the context of portfolios, can be reduced through diversification, since the specific effects of a single financial instrument tend to balance each other; while the systemic component can be managed through allocation strategies taking into account market factors.

From what has just been stated, I believe that the difference in the two VaR values obtained is due to the fact that the introduction of the factors used has had a negative influence[3] this value. The systemic component is related to market factors and therefore can be explained by macroeconomic or market variables. In addition to the first factor, introduced as a market proxy, the remaining ones represent more specific indicators which, in my opinion, had the greatest influence on the discrepancy between the results obtained.

[3] I define "negatively", since it means that there is a greater probability of suffering losses and for banks this translates into greater exposure to risk and increased capital requirements

2.2 Estimation of VaR with a single financial instrument

In this second part, we will consider the entire historical series available for the "Naturgy Energy" stock. The company focuses mainly on the production and marketing of energy from renewable sources, such as solar, wind, hydroelectric or biomass energy. Its main objective is to contribute to the transition to a low-carbon economy and to provide sustainable energy solutions to address climate change and the reduction of fossil resources.
For the security considered here, the price and yield graphs for the entire historical series are shown.

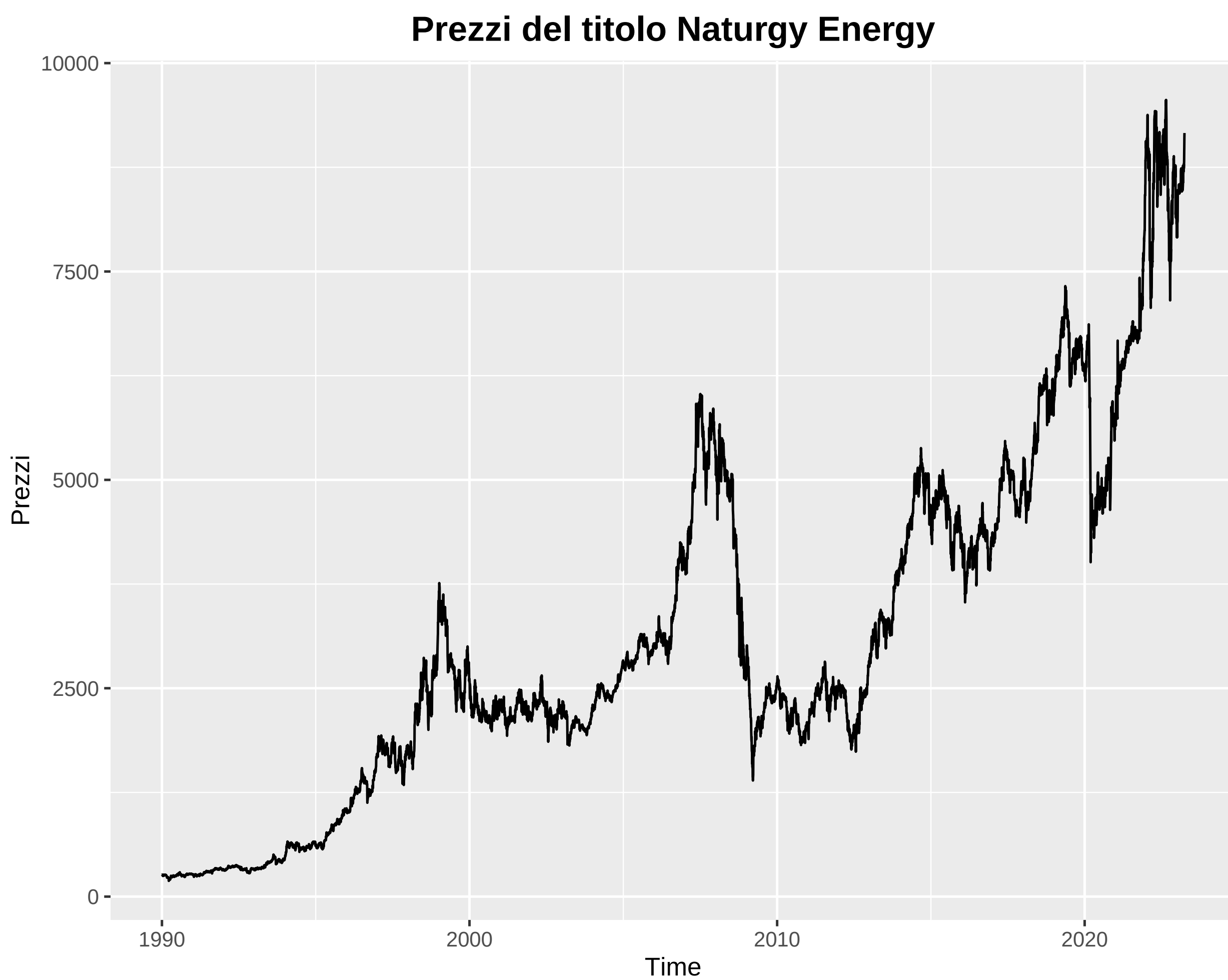

Figure 6: Prezzi titolo Naturgy Energy

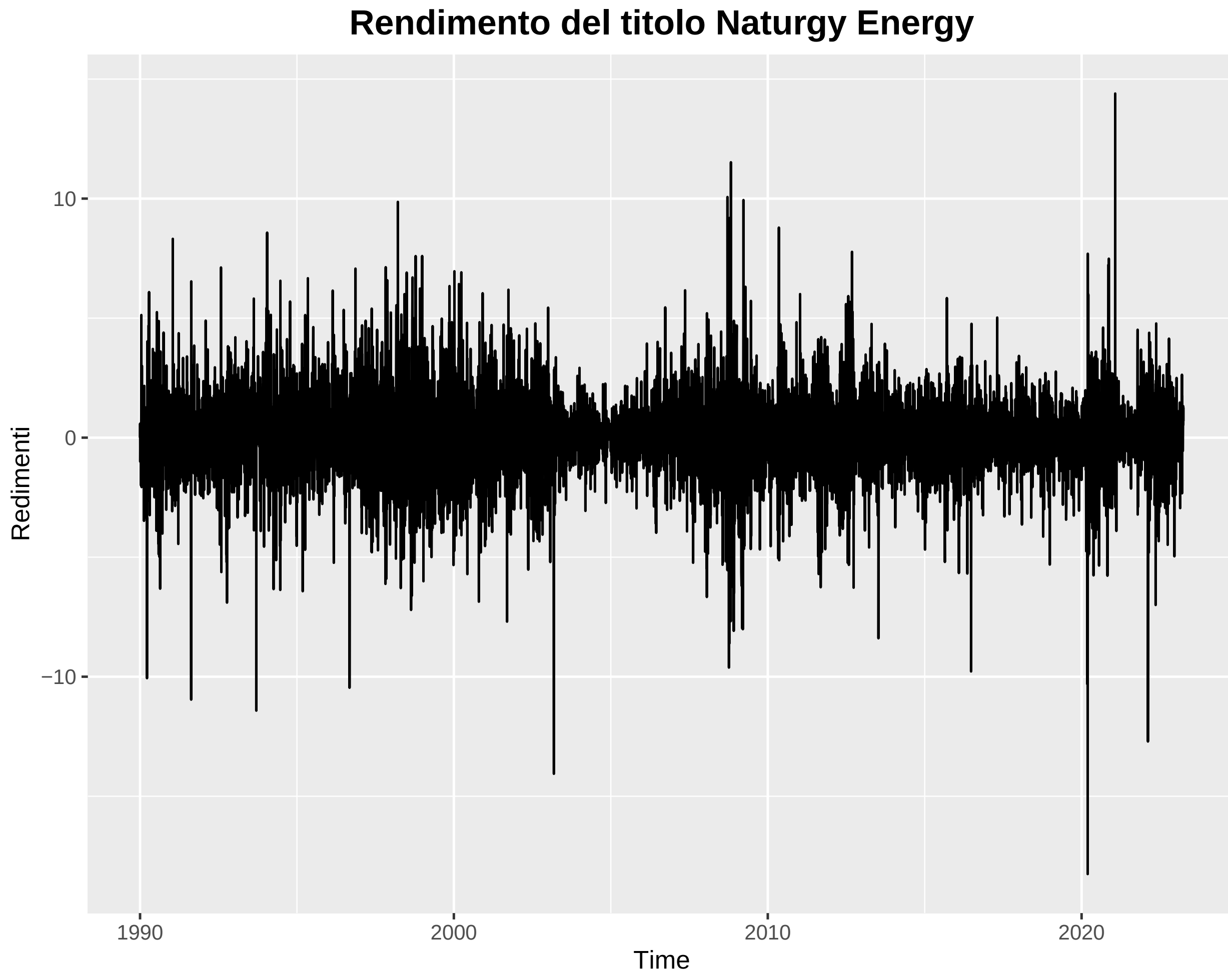

Figure 7: Rendimento titolo Energy

In the two series represented, the shocks dictated by the crisis of 2008 and 2020 with the Covid - 19 pandemic are easily noticeable.
As mentioned in the introduction of this paper, different VaR estimates will be carried out and compared for this security, in particular the VaR estimate will be carried out using: GEV, GPD, quantile regression and GARCH approach.

2.2.1 Estimation of VaR using GEV

Extreme value theory has been proposed as a method for directly estimating VaR by characterizing the distribution of the maximum of a sequence of losses.
In this context, researchers Fischer-Tippet and Gnedenko demonstrated how the distribution function of the maxima converges to a non-degenerate random variable that follows a GEV

distribution (Generalized[4] Extreme Value)

$$F_z(z) = \begin{cases} \exp(-(1+\zeta z)^{-\frac{1}{\zeta}}) & \zeta \neq 0 \\ \exp(-e^{-z}) & \zeta = 0 \end{cases} \tag{9}$$

where $1 + \zeta z > 0$ w therefore if $\zeta < 0$ then $z < 1/\zeta$; while if $\zeta > 0$ we have that $z > 1/\zeta$. Note how the parameter ζ determines the shape of the distribution. Consequently it is also possible to determine the density function:

$$f_z(z) = \begin{cases} (1+\zeta z)^{-\frac{1}{\zeta}-1}\exp(-(1+\zeta z)^{-\frac{1}{\zeta}}) & \zeta \neq 0 \\ \exp(-z - e^{-z}) & \zeta = 0 \end{cases} \tag{10}$$

For different values of ζ it is possible to obtain different known distributions. Specifically, if $\zeta < 0$ we obtain a Weibull distribution; for $\zeta = 0$ we obtain a Gumbel distribution; finally for $\zeta > 0$ we obtain a Frèchet distribution. The latter case is of greater interest in the case under analysis, since it is characterized by the presence of tails that decay into similar to a power function with a decay rate equal to $\alpha = 1/\zeta$, where α is also called tail index, it indicates the speed with which the tail goes towards 0.

In a time series of losses there is a single maximum, so to characterize the distribution of the maximum, it is possible to use the *block maximum* method. The method is applicable in the case under consideration here since independent observations are considered, therefore there is the same distribution of the maxima in the different blocks. Specifically, the time series of losses $x_1, x_2, ..., x_T$ is divided into $n = T/m$ blocks of size equal to m

$$\{x_1, x_2, ..., x_m | x_{m+1}, x_{m+2}, ..., x_{2m} | ... | x_{T-m+1}, ..., x_T\} \tag{11}$$

where for each $i-th$ block the maximum $x_{M,i}$ is identified, obtaining n observations.

If the size of the individual blocks is sufficiently large, we can invoke the Fisher-Tippet and Gnedenko results and hypothesize that the distribution of the maxima $x_{M,i}$ belongs to the class of GEV distributions. At this point we have a sample and can use maximum likelihood. However, it should be remembered that the results seen previously are related to the distribution of normalized maximums. For this reason it is necessary to normalize them in the following way:

$$z_{M,i} = \frac{x_{M,i} - \mu_M}{\sigma_M} \tag{12}$$

In the case under examination, a number of m blocks equal to 23 was considered. The evaluation of this value is linked to greater stability of the estimates and above all to the

[4]It is defined as generalization because it includes as special cases three other distributions known in literature and which are obtained depending on the values assumed by ζ.

good fit observed in the QQPlot of the residuals, observable in Figure **??**. The frequency histogram of the block maxima is also shown below.

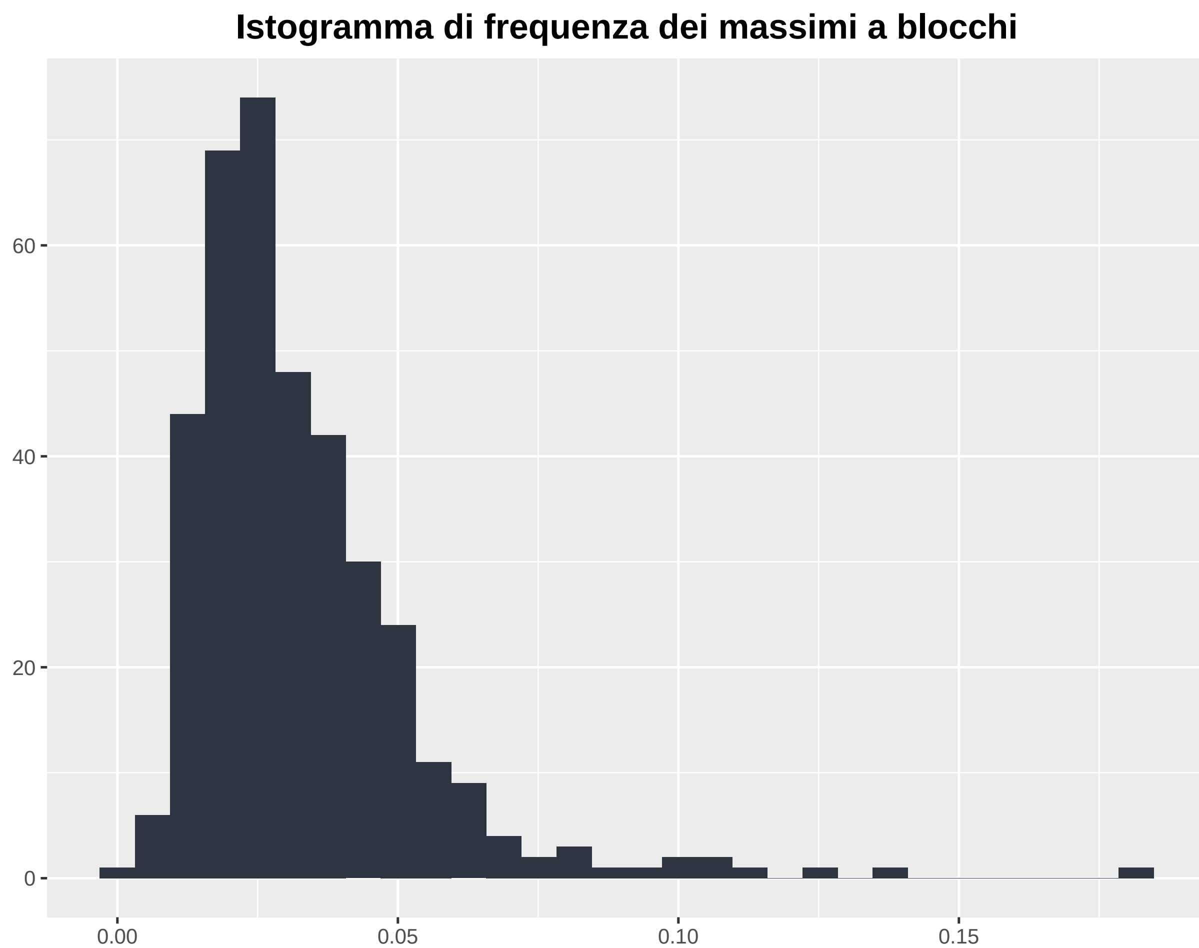

Figure 8: Frequency histogram of maxima

Furthermore, to give greater evidence on the number of blocks to be used, it is possible to observe the graph of the estimate of the scale parameter as the number of blocks varies, where the best m will be that value above which the curve of the parameter ζ will tend to stabilize. When just stated, it can be observed in the graph in the figure **??**, where the red dotted lines represent the Wald confidence intervals.

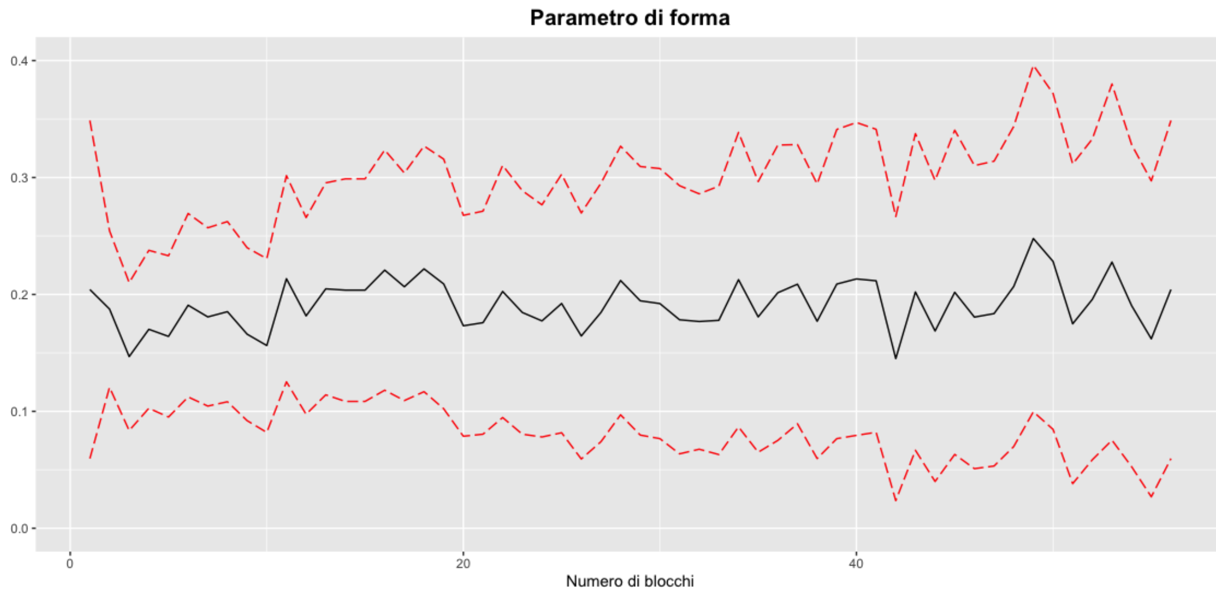

Figure 9: Estimation of the shape parameter as the number of blocks varies

Therefore, in relation to the density function defined in equation 10, it is possible to define the density of the maxima:

$$f_x(x) = \begin{cases} \frac{1}{\sigma}(1+\zeta\frac{x-\mu}{\sigma})^{-\frac{1}{\zeta}-1}\exp(-(1+\zeta\frac{x-\mu}{\sigma})^{-\frac{1}{\zeta}}) & \zeta \neq 0 \\ \frac{1}{\sigma}\exp(-\frac{x-\mu}{\sigma} - e^{-\frac{x-\mu}{\sigma}}) \quad quad\zeta = 0 \end{cases} \tag{13}$$

The **maximization of the likelihood** requires the use of numerical optimization procedures. The ML estimators, under the usual hypotheses, are consistent, asymptotically normal and the covaziness matrix is equal to the inverse of the information matrix evaluated at the optimal point. Below is the code used for the maximum likelihood estimate of the GEV.

```
library("stats4")

  llgev <- function(muM,lnsigM,xiM) {
   sigM=exp(lnsigM)
   z=(bmax-muM)/sigM
  u=1+(1/xiM)
  if(min(1+xiM*z)>0){
    llt=-log(sigM)-(u)*log(1+xiM*z)-((1+xiM*z)^(-1/xiM))
   } else{
    llt=NA
    }
```

```
-sum(llt)
}
est=stats4::mle(minuslogl=llgev,start=list(muM=0.01,lnsigM=log(0.1),xiM=1))
```

To verify the **adaptability of the GEV distribution** it is possible to observe that under correct specification of the density the residuals are distributed as an exponential random variable with unit parameter, therefore it is possible to graphically evaluate the residuals using a QQplot.

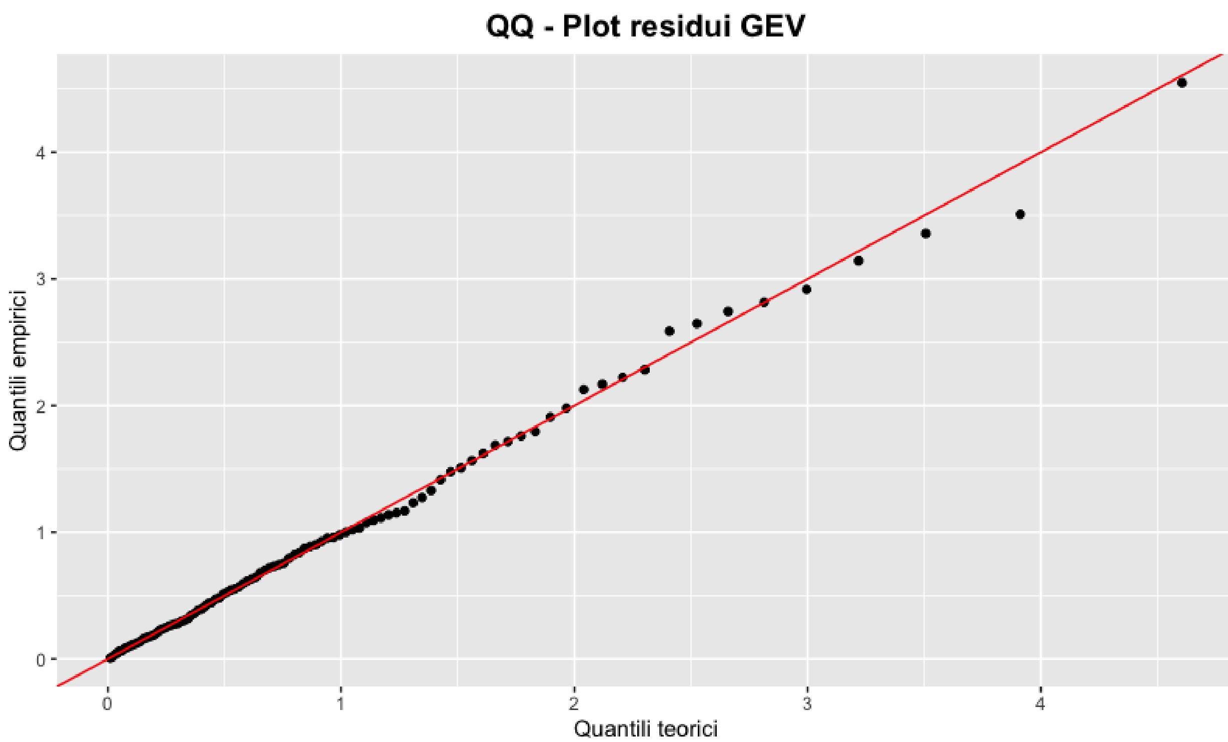

Figure 10: QQplot residuals

```
# residual calculation
c=coef(est)
e=(1+c[3]*((bmax-c[1])/exp(c[2])))^(-1/c[3])

# qqplot
p=0.01*(1:99)
qemp=quantile(e,probs=p) # quant. empirical
qteor=qexp(p,rate=1) # quant. theorists
qqplot(qteor,qemp)
qqline(qemp, distribution = function(p) qexp(p,rate=1))
```

Once the GEV parameters have been estimated, it is possible to **calculate the VaR** with a two-step procedure, first by determining the quantile of the GEV and then using the relationship that exists between the quantile of the GEV and the quantile of the losses (or of returns). Consequently, the Value-at-Risk at 1% for losses is equal to **2.14%**, calculated as:

$$q_{1-\alpha,x} = \begin{cases} \mu - \frac{\sigma}{\zeta}(1 - (-m\log(1-\alpha))^{-\zeta}) & \zeta \neq 0 \\ \mu - \sigma\log(-m\log(1-\alpha)) & \zeta = 0 \end{cases} \tag{14}$$

```
a=0.01 # VaR quantile (for losses) --> tail above 99%
mu=coef(est)[1]
sig=exp(coef(est)[2])
xi=coef(est)[3]
VaR=mu-(sig/xi)*(1-((-m*log(1-a))^(-xi)))
```

So far we have considered the limiting distribution of the maxima if the data are independent. However, in reality it is observed that returns (and consequently losses) present time dependence and heteroskedasticity. Despite this peculiarity, it is possible to demonstrate that the limiting distribution of the maxima remains a distribution of extreme generalized type (GEV), even if the data are dependent, provided that the dependence is not excessively strong and that the process generating the data is strictly stationary .
In this context, we consider a sequence of dependent random variables x_t, which are strictly stationary and follow a marginal distribution $F_X(x)$. We also consider a sequence of random variables i.i.d. $\tilde{x}_t$, which have the same marginal distribution as x_t. It is possible to prove that

$$\lim_{T\to\infty} P\left(\frac{x_{M_T} - a_T}{bt} < z\right) = F_Z^{\theta}(z) = \exp\left(-\theta\left(1 + \zeta\frac{z-\mu}{\sigma}\right)^{-\frac{1}{\zeta}}\right) \tag{15}$$

where θ[5] it is called an extremal index, in order to impact the distribution of the maximums in order to grasp the dependence. This value is estimated as:

$$\hat{\theta} = \frac{1}{k}\frac{\log(1 - G(u))}{\log(1 - N(u))} \tag{16}$$

where $N(u)$[6] and $G(u)$[7] correspond to the fraction of observations above the threshold and the fraction of blocks whose maximum is above the threshold. Given the extremal index,

[5] If $\theta = 1$ there are no effects related to the presence of temporal dependence in the data.

[6] $N(u) = \frac{1}{T}\sum_{t=1}^{T} I(x_t > u)$ with T size of the sample, x_t the losses and u the threshold defined as a quantile of the losses.

[7] $G(u) = \frac{1}{[T/K]}\sum_{t=1}^{[T/k]} I(x_{M,i} > u)$ with T sample size, x_t losses, u the threshold defined as a quantile of the losses and k the block size.

the Value-at-Risk estimate is obtained combining the previous results relating to the limit distribution of the maximums for stationary series and the calculation of the VaR through the GEV, therefore

$$q_{1-\alpha,x} = \begin{cases} \mu - \frac{\sigma}{\zeta}(1 - (-m\theta \log(1-\alpha))^{-\zeta}) & \zeta \neq 0 \\ \mu - \sigma \log(-m\theta \log(1-\alpha)) & \zeta = 0 \end{cases} \tag{17}$$

In this case the VaR is equal to which means that there is a 1% probability of having a loss of **2.73%**

```
# calculation of the extremal index
u=quantile(l,1-a) # threshold
Nu=sum(l>u)/T # percentage of observations above threshold
Gu=sum(bmax>u)/length(bmax) # percentage of maximums above threshold
th=(1/m)*(log(1-Gu)/log(1-Nu)) # extremal index

# VaR calculation for non-i.i.d. series
VaRe=mu-(sig/xi)*(1-((-m*th*log(1-a))^(-xi)))
```

It is possible to observe some problems related to the use of extreme value theory (EVT) for calculating Value-at-Risk:

- Increased sensitivity to the quantile selected for the GEV
- Parameter estimation depends on block size
- The dependence of the returns, certainly present in the second moment, is considered only through the introduction of the extremal index whose estimate depends on the size of the blocks and the choice of the threshold
- In fact it is based on a limited information set, it only considers the maxima of the blocks, without exploiting much of the sample information

2.2.2 Estimation of VaR using GPD

To overcome the limits of EVT it is possible to use the information contained on the excess losses compared to a pre-established threshold. This shifts attention from the maximum $x_t = -r_t$ to the losses in excess of a preferred value, i.e. to the quantities $y_t = x_t - u$ which are observed only in presence of losses exceeding the u threshold. The available data is characterized by two elements, the time at which the event occurs and the magnitude of the event. In this context we will focus only on the analysis of the magnitude of losses in excess of the threshold. In this context, a threshold u is considered by taking an empirical quantile at 99%. Below you can see the excess loss graph.

```
u=quantile(l,probs=0.99)
el=l[l>u]-u
```

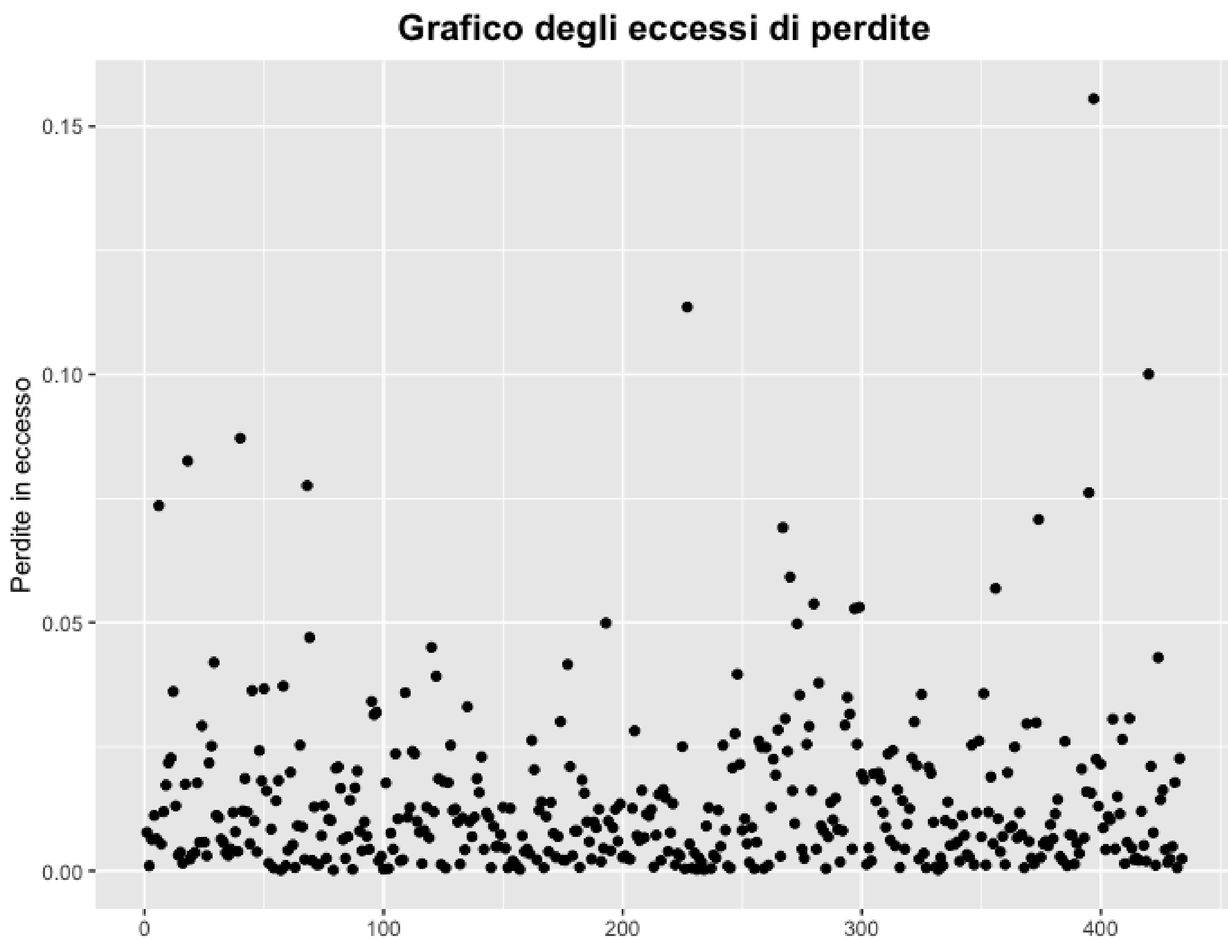

Figure 11: Excess Losses Chart

For this reason, F_X is defined as the density of the losses and $x_t = y_t + u$ is considered when the losses exceed the threshold, therefore the distribution of the excesses is obtained by considering:

$$P\left(x_t \leq y_t + u | x_t > u\right) = \frac{P\left(u \leq x_t \leq y_t + u\right)}{P\left(x_t > u\right)} \\ = \frac{F_X(y_t + u) - F_X(u)}{1 - F_X(u)} \\ = F_{Y,u}(y_t) \quad y_t > 0 \tag{18}$$

It is shown that if the distribution of normalized maxima converges towards a GEV then for sufficiently high thresholds the distribution of excesses with respect to the threshold converges towards a Generalized Pareto distribution (GPD) with distribution function:

$$G_{\zeta,\mu,\sigma,u}(y_t) = \begin{cases} 1 - \left(1 + \frac{\zeta y_t}{\sigma + \zeta(u-\mu)}\right)^{-\frac{1}{\zeta}} & \zeta\ ne 0 \\ 1 - exp\left(\frac{y_t}{\sigma + \zeta(u-\mu)}\right) & \zeta = 0 \end{cases} \tag{19}$$

It can be seen that the shape parameter of the generalized extreme type distribution (GEV) and the generalized Pareto distribution (GPD) is the same, and that the parameters of the GEV can be used to determine the parameters of the GPD.
The GPD parameters can be estimated using the maximum likelihood method. However, it is important to be careful when choosing the threshold u, since the maximum likelihood estimate depends on this selection.

```
llgpd <- function(muM,lnsigM,xiM) {
sigM=exp(lnsigM)
 rhou=sigM+xiM*(u-muM)
llt=-log(rhou)-(1+(1/xiM))*log(1+(xiM*el)/rhou)
 -sum(llt)
}

est=stats4::mle(minuslogl=llgpd,start=list(muM=0.001,lnsigM=log(0.2),xiM=0.01))
summary(est)
```

For its estimate there can be different approaches

- Economic-financial evaluation: a threshold is chosen in relation to the perception of what the significant losses are, an element that depends on risk aversion;
- Evaluation of the stability of the parameter estimates as the thresholds vary, looking for thresholds for which the parameters estimated via ML are stable;
- A third approach is based on a graphical tool, the mean excess function

In the latter case it is possible to demonstrate that the average excess losses y_t for a given threshold is equal to

$$\mathbb{E}[x_t - u|x_t > u] = \mathbb{E}[y_t|x_t > u] = \frac{\rho(u)}{1-\zeta} \tag{20}$$

Furthermore, for any threshold $u' > u$ we have the mean excess function

$$\epsilon(u') = \mathbb{E}[x_t - u'|x_t > u'] = \frac{\rho(u') + \zeta(u' - u)}{1-\zeta} \tag{21}$$

Therefore the mean excess function is linear in $u' - u$ for a fixed value of ζ.

```
ustep=seq(0.01,0.06,0.001) #fixed values for the threshold
mef=NULL
for (i in ustep){
 eloc=l[l>i]-i
 mef=rbind(mef,mean(eloc))}
```

The mean excess function is represented graphically as the threshold u varies and the value u is chosen above which the graph becomes linear.

Figure 12: Mean excess plot

Once the threshold has been selected and the GPD parameters have been estimated, an evaluation of the goodness of fit to the data can be based, once again, on the residuals which should be distributed as an exponential with a unit parameter, thus allowing to compare the empirical distribution with the theoretical one through a QQ-plot.

```
c=coef(est)
rhou=exp(c[2])+c[3]*(u-c[1])
z=(1/c[3])*log(1+c[3]*el/rhou)
p=0.01*(1:99)
qemp=quantile(z,probs=p)
qteor=qexp(p,rate=1)
qqplot(qteor,qemp)
qqline(qemp, distribution = function(p) qexp(p,rate=1))
```

Figure 13: QQ-Plot of residuals

By explaining the link between GPD and Value-at-Risk it is possible to define the quantile:

$$q_{1-\alpha,x} = \mu - \frac{\sigma + \zeta(u-\mu)}{\zeta}\left[1 - \left(\frac{T}{T_u}\alpha\right)^{-\zeta}\right] \tag{22}$$

```
a=0.01
VaRGPD=u-(rhou/c[3])*(1-((1-a)*length(el)/length(l))^(-c[3]))
```

The estimate of the Value-at-Risk at 1%, in the case under consideration here, turns out to be equal to **7.20**%.

This value, compared to that obtained with the GEV in the previous point, is greater. One reason for this result may be caused by the heavy tail effect: in fact, if the data has heavy tails, i.e. a higher frequency of extreme events than what would be expected from the light tail distribution, the GPD could overestimate the VaR compared to to GEV, which, being designed to model the tail of the distribution, could provide more accurate estimates.

2.3 Variance-covariance approach

2.3.1 GARCH methods

This series of approaches aims to determine the Value-at-Risk using a parametric approach in order to remove the assumption of independence of observations, moving from estimation methods that are based on the marginal distribution towards methods that consider conditional distributions. The most famous approach is that of RiskMetrics proposed between the end of the 80s and the beginning of the 90s within JP-Morgan. This method uses a Gaussian conditional distribution for returns in which the variances are dynamic, based on the exponential smoothing of past squared returns. Specifically, the following form is proposed for the variance of a security:

$$\begin{aligned}
\sigma_t^2 &= (1-\lambda)\sum_{s=1}^{t-1}\lambda^{s-1}r_{t-s}^2 \\
&= (1-\lambda)r_{t-1}^2 + (1-\lambda)\sum_{s=2}^{t-1}\lambda^{s-1}r_{t-s}^2 \\
&= (1-\lambda)r_{t-1}^2 + \lambda\sigma_{t-1}^2
\end{aligned} \tag{23}$$

You can see a relationship between the RiskMetrics approach and GARCH models. Although the RiskMetrics method can be considered a special case of a GARCH model in which $\omega = 0$ and $\alpha + \beta = 1$, the literature has shown that $\sigma_t^2 = 0$ if $\omega = 0$. Therefore it is more appropriate to consider this method a filter.

In this case the calculation of the Value-at-Risk is based on the estimate of the quantile of the hypothesized conditional distribution for the returns (which to obtain a general representation we assume have a non-zero mean and equal to μ):

$$q_{\alpha,t} = \mu + \sigma_t\Phi^{-1}(\alpha) \tag{24}$$

If it is necessary to calculate them, let's assume we have the weight vector $\mathbf{w}$ available, the VaR corresponds to

$$q_{\alpha,t} = \mu + (\mathbf{w}'\Sigma_t\mathbf{w})^{\frac{1}{2}})\Phi^{-1}(\ alpha) \tag{25}$$

From an operational point of view, the hypothesis of returns conditionally distributed as a Normal with heteroscedasticity can be modified by varying the density, thus moving, for example, to distributions with thick tails or with asymmetry. The parameters of the distribution can be estimated with the appropriate methods using standardized returns, i.e. $z_t = \sigma_t^{-1} r_t$.

It is possible to estimate the Value-at-Risk for an individual security based on a conditional model

$$r_t | I_{t-1} \sim \mathcal{D}(\mu, \sigma_t^2, \Theta) \tag{26}$$

where it is assumed that μ is the conditional mean of the returns, σ_t^2 is the conditional variance obtained from any GARCH model and Θ are the possible parameters of the density $\mathcal{D}$. In particular, two of the most frequently used models are APARCH and EGARCH. APARCH is an extension of the GARCH model that uses a power-based skewness function to model skewness in conditional volatility, where this function allows for greater flexibility in capturing skewness in the data. The conditional variance is described by the following function:

$$\sigma_t^2 = \omega + \alpha(|\epsilon_{t-1} - \gamma\epsilon_{t-1}|)^2 + \beta\sigma_{t-1}^2 \tag{27}$$

EGARCH is a model that captures skewness via the conditional volatility function which is modeled using the logarithmic scale. This function allows you to capture the effect of positive or negative shocks on financial returns, so that volatility increases differently depending on the sign of the return. The conditional variance is described by the following function:

$$\log\left(\sigma_t^2\right) = \omega + \alpha\frac{|\epsilon_{t-1}|}{\sigma_{t-1}} + \gamma\frac{\epsilon_{t-1}}{\sigma_{t-1}} + \beta\log\left(\sigma_{t-1}^2\right) \tag{28}$$

Both of these models, as noted, are intended to model volatility asymmetry more flexibly. Between the two different modeling I chose the EGARCH with T-Student distribution. In the evaluation to define which model was most appropriate, I considered the T-Student distribution, since it weights the tails more and the QQ-plot of the residuals of the estimated models fit better than the normal distribution; furthermore, I considered how the conditional variance is defined in the two different models, considering the EGARCH more flexible and suitable for the Naturgy Energy stock.

For greater clarity in the choice of the model used, below, I report the graphs relating to the QQ-plots of the residuals (on the left) and the ACF of the standardized residuals (on the right), for the EGARCH models with normal distribution and for the APARCH with both normal distribution and T-Student.

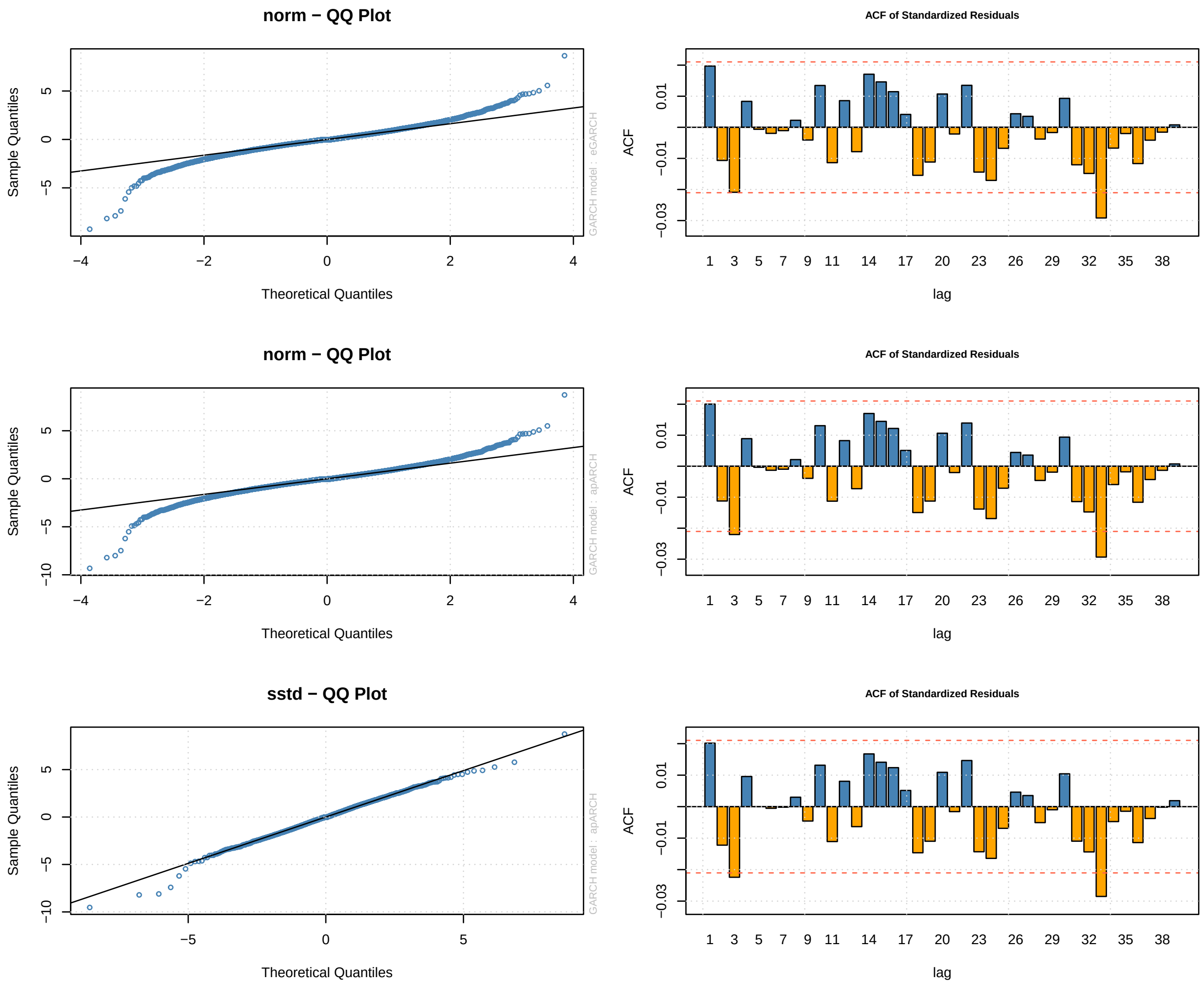

Figure 14: Comparison of EGACH and APARCH models for different distributions

As can be seen from the Figure 14, the models with normal distribution, as expected, do not present a good fit on the tails, compared to the T-Student distribution, despite all the models capture the heteroscedasticity component. In particular, observing Figure 15, in which I have reported the graphs relating to the QQ-plot of the residuals (on the left), the ACF of the standardized residuals (in the center) and the impact of the news curve (on the right) of the EGARCH model with T-Student distribution, no differences are noted with the APARCH model with T-Student distribution.

```
spec2 <- ugarchspec(variance.model = list(model="eGARCH", garchOrder = c(1, 1)),
                mean.model = list(armaOrder = c(0, 0), include.mean = TRUE),
                distribution.model="sstd")
fit2 <- ugarchfit(spec2,Y)
```

```
c2=fit2@fit$coef
z2=fit2@fit$z
for2=ugarchforecast(fit2,n.ahead=1)

sigma2for=(t(for2@forecast$sigmaFor))
q2=qdist(distribution="sstd",p=0.01,mu=0,sigma=1,skew=c2[6],shape=c2[7])
VaR2=c2[1]+q2*sigma2for
```

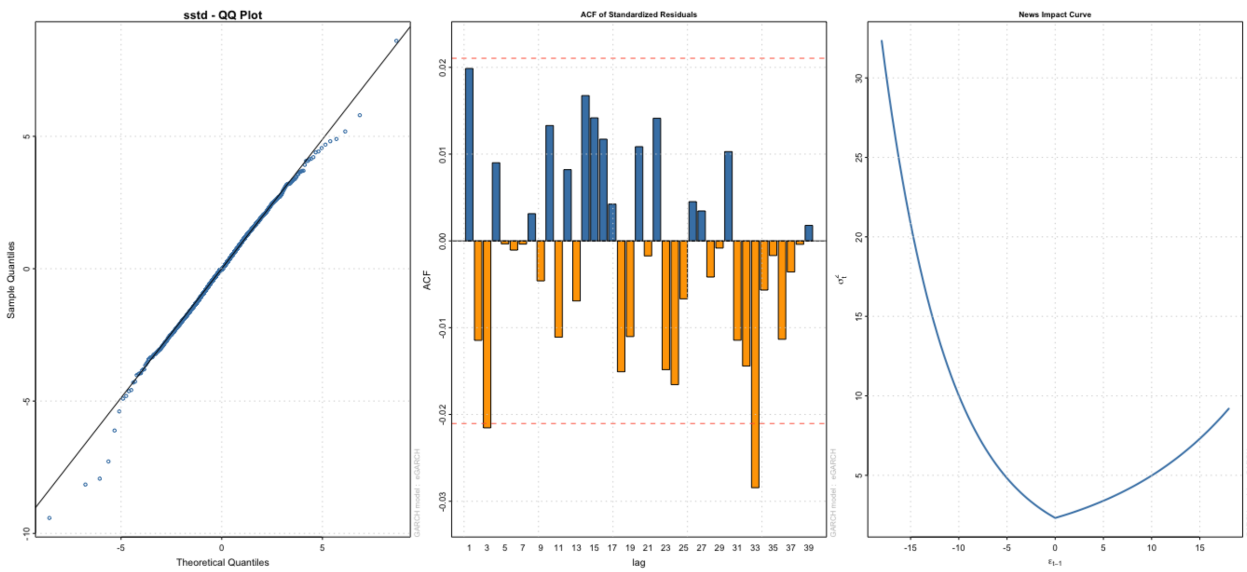

Figure 15: EGARCH

As can be seen, the model presents a good fit, visible from the QQ-plot and the ACF. Furthermore, it is interesting to observe the news impact curve, which highlights how volatility increases more in correspondence with negative shocks than in correspondence with positive ones and a greater sensitivity to the arrival of new news, due to a high openness of the curve. The estimate of the VaR at 1% occurs using methods equivalent to those adopted in the RiskMetriks

$$q_{\alpha,t} = \mu + \sigma_t F^{-1}(\alpha; \Theta) \tag{29}$$

where F is the cumulative density distribution function $\mathcal{D}$. In the case under consideration here the VaR estimate is equal to **2.88%**.

2.3.2 Quantile regression

Quantile regression was introduced by Koenker and Basset (1978) with the aim of generalizing and overcoming the linear regression model. Let X be a random variable, $F(x) =$

$P(X \leq x)$ be its cumulative density function, and $F^{-1}(\tau)$ be the quantile of order τ of X. We introduce the asymmetric loss function check function:

$$\rho_\tau(u) = u \times (\tau - I(u <)) \tag{30}$$

At this point we want to find the value of $\hat{x}$ that minimizes the loss for the random variable X, i.e. solve the problem:

$$\min_{\hat{x}} \mathbb{E}[\rho_\tau(X - \hat{x}] \tag{31}$$

To do this you need to minimize

$$(\tau - 1) \int_{-\infty}^{\hat{x}} (x - \hat{x}) dF(x) + \tau \int_{\hat{x}}^{+\ infty} (x - \hat{x}) dF(x) \tag{32}$$

Differentiating with respect to $\hat{x}$ we obtain

$$(\tau - 1) \int_{-\infty}^{\hat{x}} dF(x) + \tau \int_{\hat{x}}^{+\infty} dF(x) \tag{33}$$

Consequently, equating the differential to zero gives $F(\hat{x}) - \tau = 0$, and this implies that the value of X that minimizes the asymmetric loss function for a given $\tau\ in(0, 1)$ is nothing other than the quantile of order τ for X. So given a sample, if we minimize the loss function we obtain the empirical quantile:

$$\min_{\hat{x}} \frac{1}{n} \sum_{i=1}^{n} \rho_\tau(x_i - \hat{x}) \tag{34}$$

As you can see there are some similarities with the linear regression model; in fact if I try to estimate the average I get

$$\min_{\hat{\mu}} \sum_{i=1}^{n} (x_i - \hat{\mu})^2 \tag{35}$$

In particular, in the context of regression we are mainly interested in the conditional mean $(\min_{\hat{\beta}} \sum_{i=1}^{n} (x_i - \hat{\beta}' z_i)^2)$, where the role of a set of covariates or control variables is taken into account. In quantile regression it can be assumed that the quantile of order τ is conditional on a set of covariates. By indicating the quantile as $Q_x(\tau, z) = \beta(\tau)' z_i$, highlighting that the coefficients depend on the quantile, to estimate $\beta(\tau)$ one minimizes

$$\min_{\beta(\tau)} \sum_{i=1}^{n} \rho_\tau(x_i - \beta\tau' z_i)^2 \tag{36}$$

From the estimated coefficients, it is easily possible to determine the conditional quantile as:

$$Q_x(\hat{\tau}, z) = \beta(\tau)' z_i \tag{37}$$

It has been observed that the intercept absorbs the quantile of the error term in most cases and is therefore always dependent on τ. The other parameters of the conditional quantile can depend on τ. If the parameters do not depend on τ, quantile regression is of little use (although it can be useful in the presence of deviations from normality because it is more efficient than OLS); in these cases least squares approaches may be sufficient. Interesting cases are observed in those contexts in which the coefficients associated with the covariates change with τ.
The nature of the QR allows the quantile of interest, i.e. the Value-at-Risk, to be estimated, also conditioning this estimate on the presence of covariates. Different approaches can be used, and 3 of them will be considered here: Quantile Auto-Regression, inclusion of a risk factor and considering the dependence on the level of returns and their square.
I start by considering Quantile Auto-Regression, where the quantile depends on lags of the modeled variable, specifically:

$$Q_{r_t}(\tau) = \alpha_\tau + \phi_\tau r_{t-1} \tag{38}$$

As previously observed the relationship between the linear model and the quantile regression, in the following graph I report the comparison between the linear model and the quantile regression for various tau values $(0.01, 0.05, 0.10, 0.90, 0.95, 0.99)$, where the blue line represents the linear model, while the red lines represent the QR for the different values of τ. Below I represent the table with the relative estimate of the coefficients for the different values of τ.

	Intercept and Coefficient	
$\tau = 0.01$	-4.82	0.16
$\tau = 0.05$	-2.70	0.078
$\tau = 0.1$	-1.91	0.056
$\tau = 0.9$	2.01	-0.027
$\tau = 0.95$	2.85	-0.023
$\tau = 0.99$	4.82	-0.036
Linear model	0.040	0.0066

Table 1: Parameter estimation for QR and LM

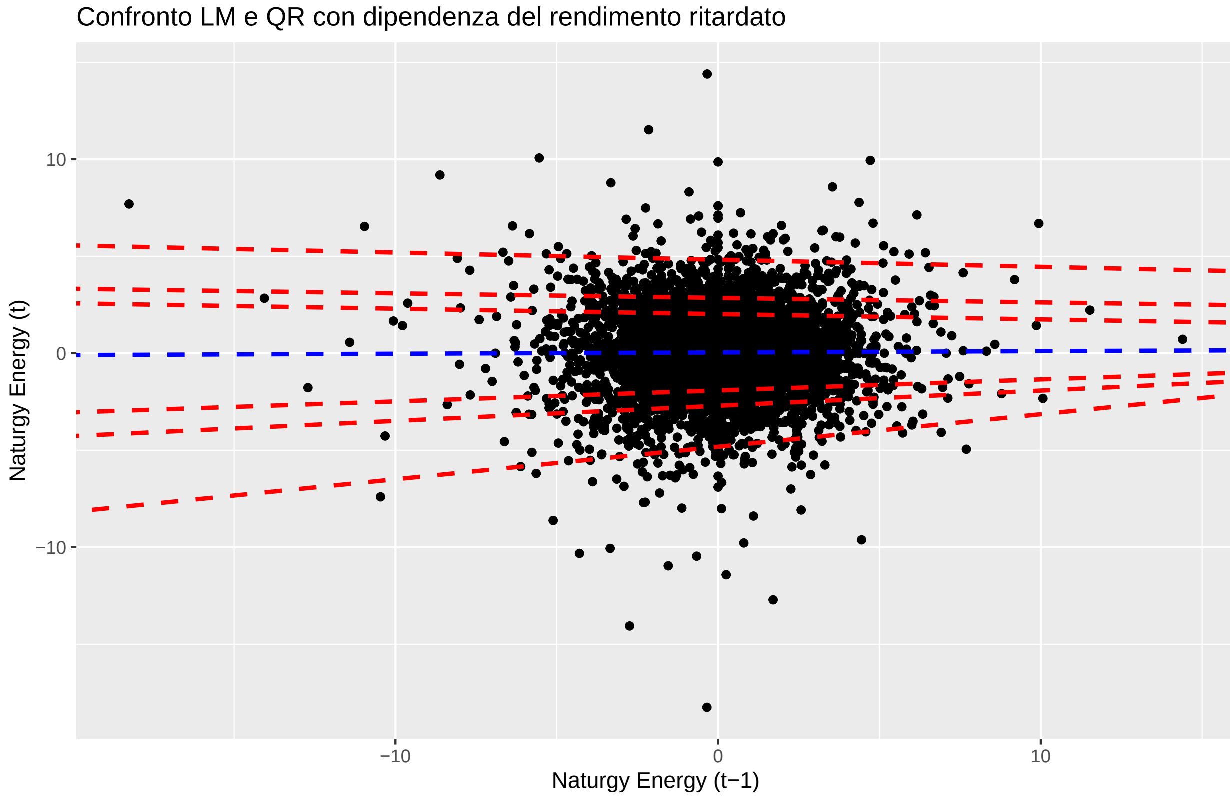

Figure 16: Comparison of LM and QR lagged performance

From the results observed so far we can see how the value of the intercept changes due to the presence of the quantile of the error term. The aspect to pay attention to is the estimate of the term relating to the slope of the line, therefore ϕ. As can be seen from Table 9 and from the Graph **??**, there is little variability, especially for small values of τ. In this context, it is important to highlight a special case, if the lines are parallel. In fact, in this situation, lines associated with a slope that does not change are not very informative. The importance deriving from the QR is that it provides additional information when the parameters associated with the covariates change as τ varies, establishing that the relationship between the variables considered does not only impact the mean of the conditional distribution. It allows us to observe a broader effect, highlighting how the quantiles of the conditional distribution vary as a function of the covariates.

We also observe the comparison between coefficients estimated via quantile regression for a sequence of τ values ranging from 0.01 to 0.99.

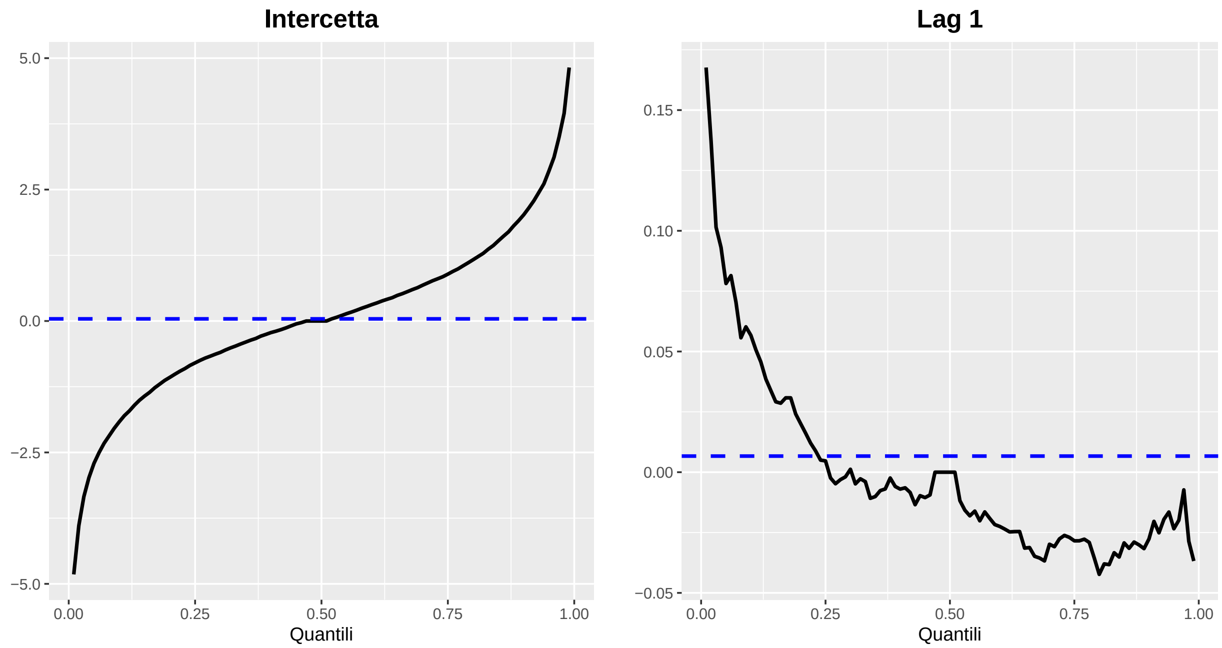

Figure 17: Parameters - QR delayed yield

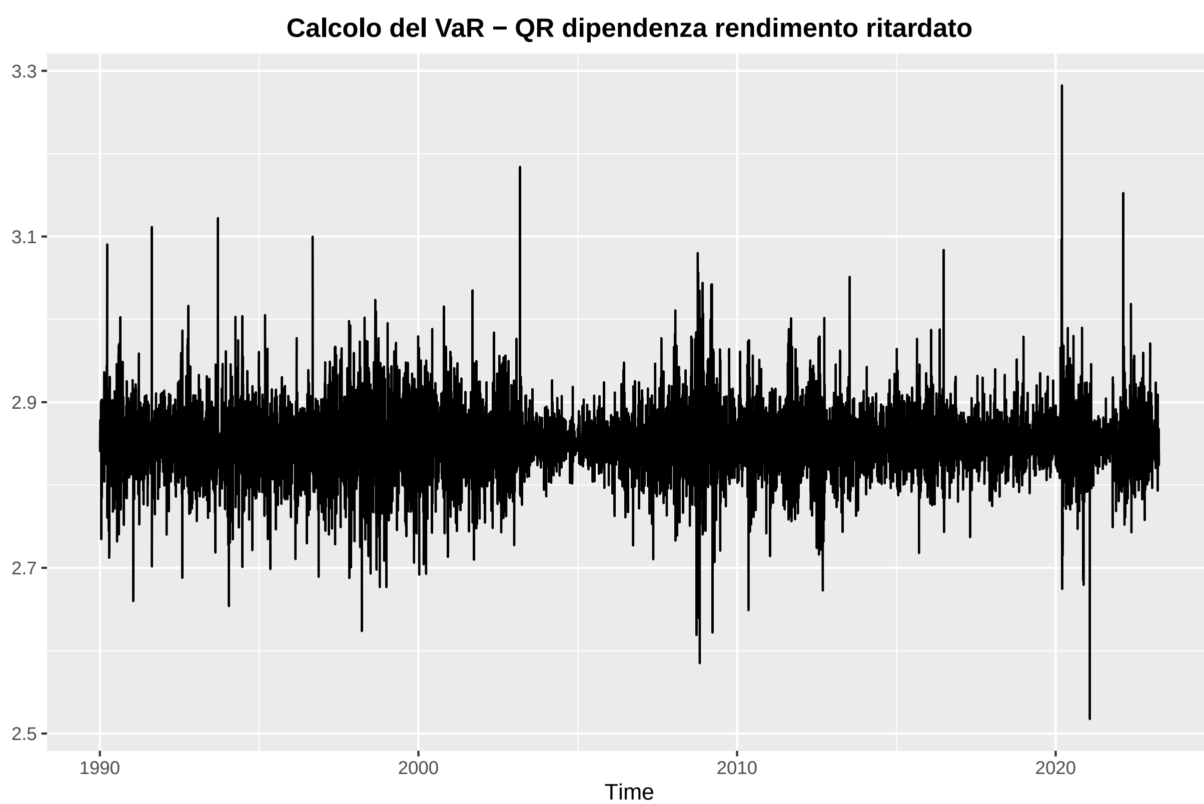

Figure 18: Estimation of VaR - QR with lagged return dependence

The proposed VaR graph, by construction ($Q_{r_t}(\tau) = \alpha_\tau + \phi_\tau r_{t-1}$) and as previously observed in relation to the importance of the parameter ϕ, it presents this flattened shape very similar to that of the returns of the Naturgy Energy stock. In fact, the VaR is calculated by considering the lagged returns multiplied by ϕ_τ and translated by the parameter α_τ. Below is the code for calculating VaR:

```
rqfit1 <- rq(Y[2:(T)] ~ Y[1:(T-1)], tau = 0.99)
VaR_qr1 = coef(rqfit1)[1]+coef(rqfit1)[2]*Y[1:(T-1)]
```

In this context, the VaR estimate at 1% one step forward presents a value equal to **2.60%**.

Subsequently I considered Quantile Regression considering the dependence on the level of returns and their square, where the quantile depends on lags of the modeled variable, specifically:

$$Q_{r_t}(\tau) = \alpha_\tau + \gamma_{1,\tau} r_{t-1} + \gamma_{2,\tau} r_{t-1}^2 \tag{39}$$

Also in this case I evaluated both the comparison of the estimated coefficients and the graphic analysis of the regression parabolas. As noted previously, also in this case the quantile regression was estimated for several values of τ $(0.01, 0.05, 0.10, 0.90, 0.95, 0.99)$. Below I represent the table with the relative estimate of the coefficients for the different values of τ.

	Intercept	Coefficient r_{t-1}	Coefficient r_{t-1}^2
$\tau = 0.01$	-4.38	0.24	-0.11
$\tau = 0.05$	-2.44	0.11	-0.07
$\tau = 0.1$	-1.71	0.083	-0.06
$\tau = 0.9$	1.79	-0.035	0.078
$\tau = 0.95$	2.52	0.017	0.096
$\tau = 0.99$	4.27	0.088	0.096
Linear model	0.010	0.009	0.009

Table 2: Parameter estimation for QR and LM

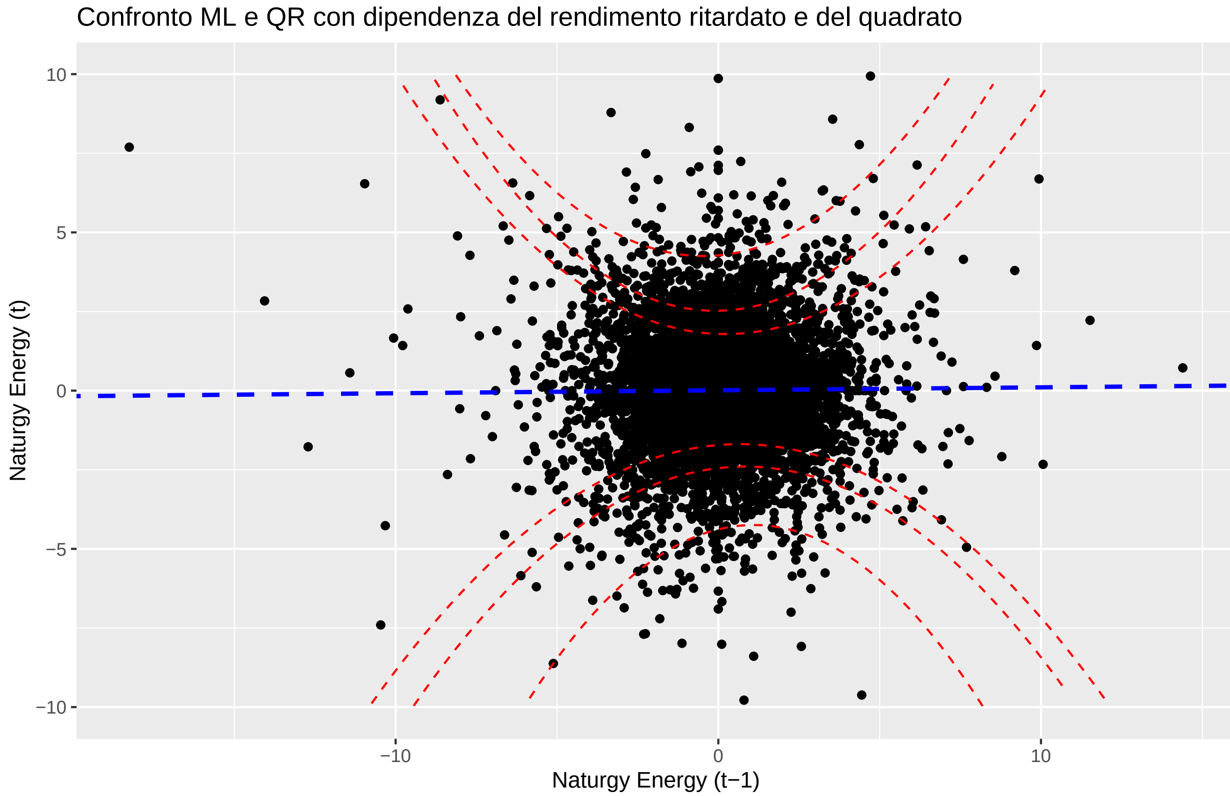

Figure 19: Comparison of LM and QR squared yield

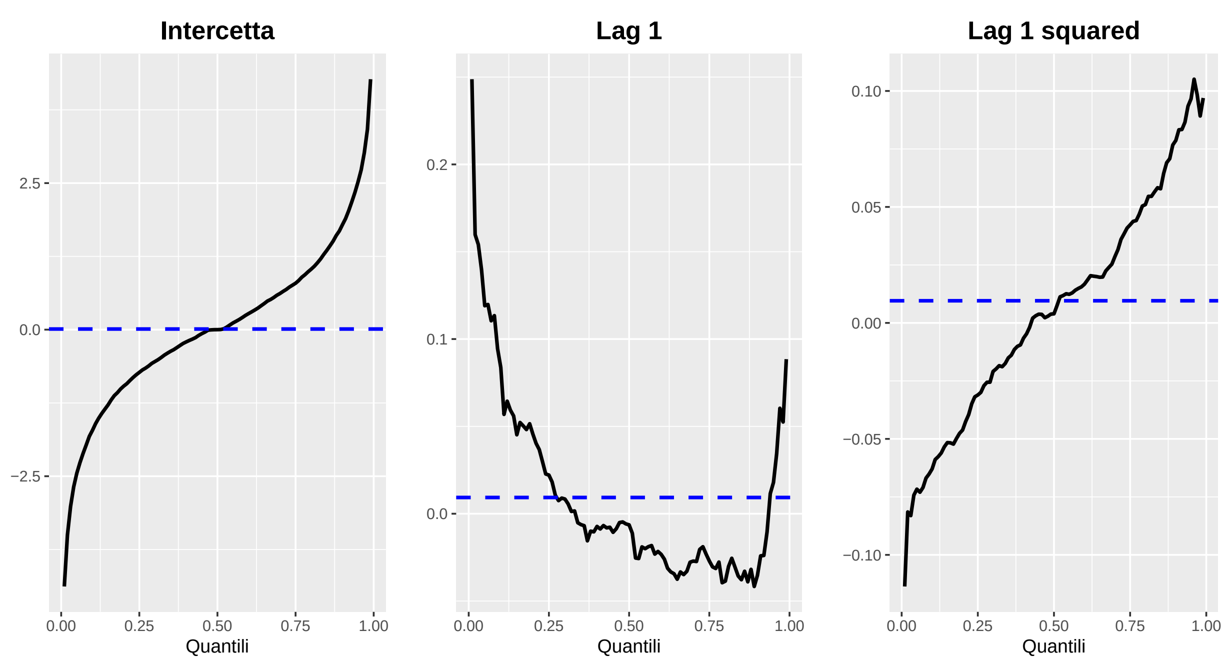

Figure 20: Parameters - QR return squared

The considerations relating to the results obtained here are similar to those made previously. An aspect to underline in this case is related to the parameter $\gamma_{2,\tau}$ which determines the degree of concavity of the parabola. As the value increases, its concavity decreases. Therefore, to have greater relevance, it is more appropriate to consider estimates for smaller values of τ, where the estimates provide more information.

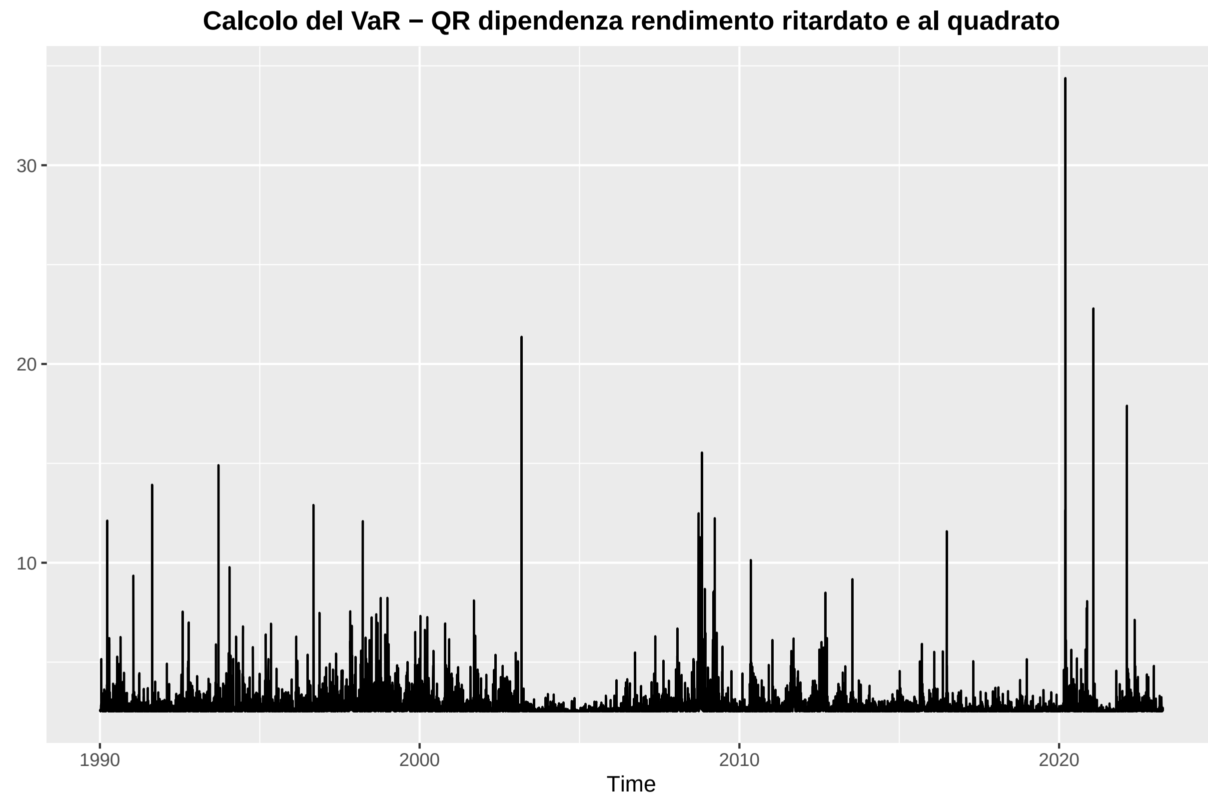

Figure 21: Estimation of VaR - QR with squared dependence

The proposed VaR graph, by construction ($Q_{r_t}(\tau) = \alpha_\tau + \gamma_{1,\tau} r_{t-1} + \gamma_{2,\ tau} r_{t-1}^2$) and as previously observed in relation to the importance of the estimated parameters, presents this flattened shape with many peaks. In fact, the VaR is calculated by considering the lagged squared returns multiplied by $\gamma_{2,\tau}$, the lagged returns multiplied by $\gamma_{1,\tau}$ and shifted by the parameter α_τ. Below is the code for calculating VaR:

```
r_2 =(Y[1:(T-1)])^2
rqfit <- rq(Y[2:(T)] ~ Y[1:(T-1)] + r_2, tau = 0.99)
VaR_qr = coef(rqfit)[1]+coef(rqfit)[2]*Y[1:(T-1)]+coef(rqfit)[3]*r_2
```

In this context, the VaR estimate at 1% one step forward presents a value equal to **2.58%**. Finally, I considered Quantile Regression considering the dependence of a contemporary risk factor, of interest since we want to observe whether the relationship that exists between the two variables changes depending on the quantile. In particular:

$$Q_{r_t}(\tau) = \alpha_\tau + \beta_\tau' F_t \tag{40}$$

Also in this case I evaluated both the comparison of the estimated coefficients and the graphical analysis of the regression lines. As noted previously, also in this case the quantile regression was estimated for several values of tau $(0.01, 0.05, 0.10, 0.90, 0.95, 0.99)$. Below I represent the table with the relative estimate of the coefficients for the different values of τ.

	Intercept	Coefficient
$\tau = 0.01$	-4.21	0.89
$\tau = 0.05$	-2.43	0.87
$\tau = 0.1$	-1.76	0.84
$\tau = 0.9$	1.87	0.77
$\tau = 0.95$	2.63	0.78
$\tau = 0.99$	4.47	0.88
Linear model	0.02408	0.77255

Table 3: Parameter estimation for QR and LM

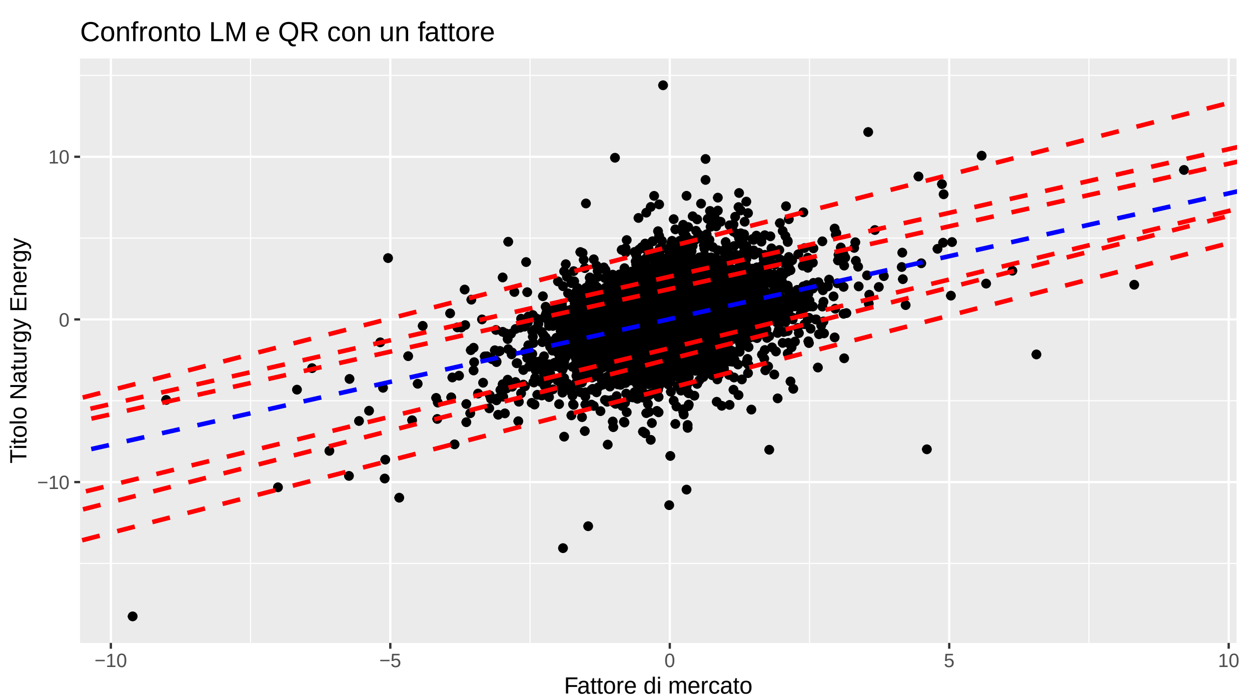

Figure 22: Comparison of LM and QR with one factor

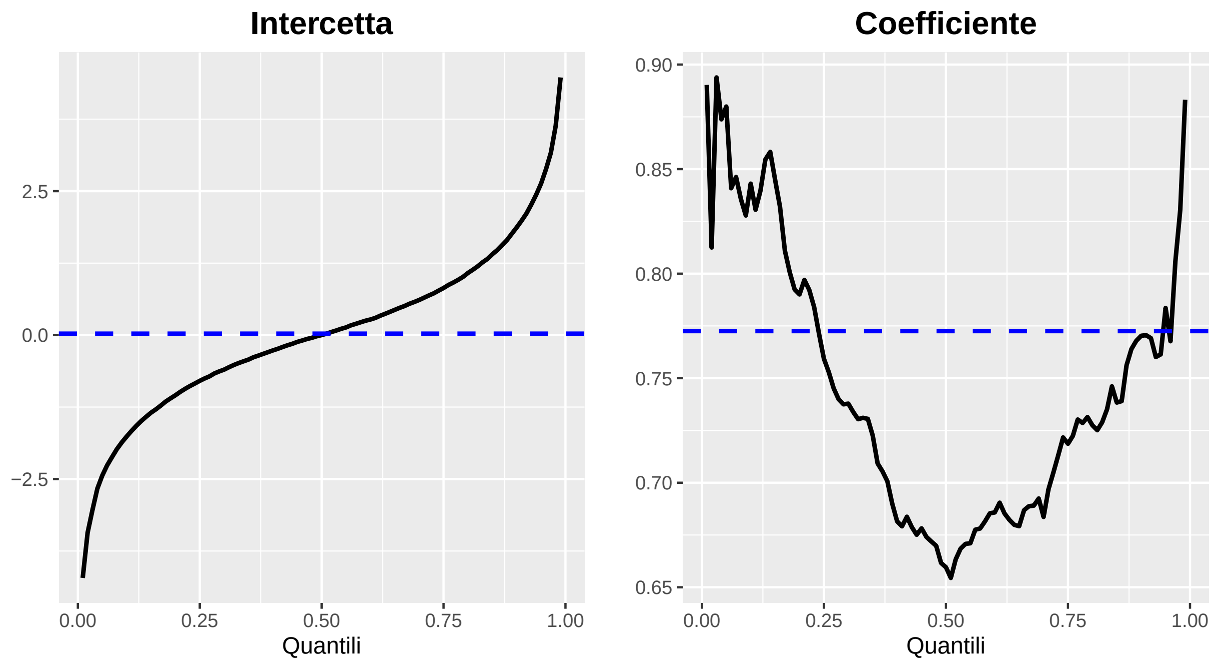

Figure 23: Parameters - QR with factor

The considerations relating to the results obtained here are similar to those made previously. As might be expected, by construction, the estimate of the intercept is different in the two different models. It is observed that the estimate of the parameter β with least squares is statistically greater than the estimate of the QR around the median. The importance deriving from the QR is that it provides additional information when the parameters associated with the covariates change as τ varies, establishing that the relationship between the variables considered does not only impact the mean of the conditional distribution. From Table 3 and Figures 22 and 23, it can be seen, as in the previous cases, that we are in the presence of an extreme case, in which the QR lines are almost parallel to the linear regression line.

The VaR graph proposed in Figure 24, by construction ($Q_{r_t}(\tau) = \alpha_\tau + \beta'_\tau F_t$) and as previously observed in relation to the importance of the parameter β, presents this flattened shape very similar to that of the returns of the Naturgy Energy stock. In fact, the VaR is calculated by considering the lagged returns multiplied by β'_τ and translated by the parameter α_τ. Below is the code for calculating VaR:

```
rqfit_F <- rq(Y_qr ~ Fact, tau = 0.99)
VaR_qr_F = coef(rqfit_F)[1]+coef(rqfit_F)[2]*Fact
```

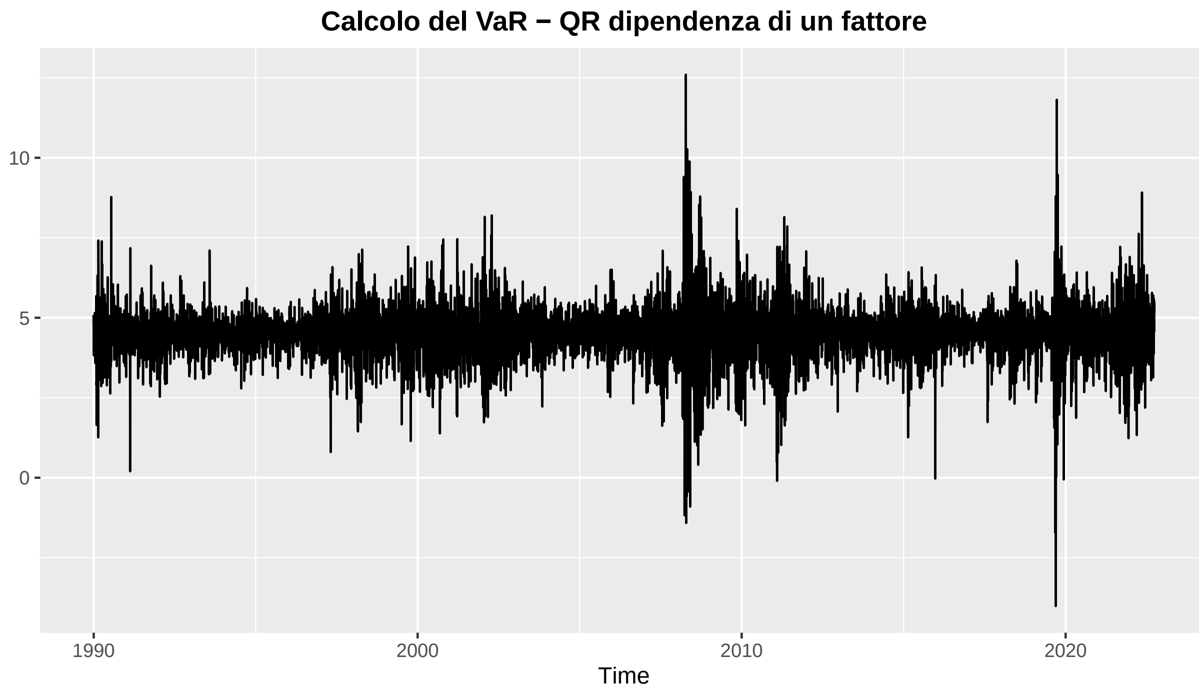

Figure 24: Estimation of VaR - QR with one factor

In this context, the VaR estimate at 1% one step forward presents a value equal to **3.45%**. Unlike the VaR values previously obtained, in this case a higher value is obtained, to indicate how the value of the factor has the greatest influence on the estimate.

3 Backtesting

Following the analyzes conducted so far, it is of interest to understand which of the approaches proposed for predicting Value-at-Risk is best, i.e. whether the chosen model provides adequate predictions. The evaluation of the quality of the model is based on the use of a set of statistical procedures that analyze the performance of the model in an out-of-sample evaluation. In particular, here, two of the methods considered previously will be used to estimate the VaR for the years 2021 and 2022 and, by comparing the different models, define which model is the best in the two different years and whether this is the same.

In the out-of-sample procedure I will use the rolling estimation method. Specifically for this procedure, we start with a data set $[1, T]$, where T represents the latest available data point. This range is used to specify the model, that is, to establish its characteristics. Next, to obtain forecasts for the period $[T + 1, T + h]$, where h represents the desired forecast horizon, a new data set is used that goes from point 2 to point $T+1$. In this case, the model remains the same as the initial specification, but subsequent data is used to estimate predicted values for the next period. This process is repeated using sliding windows. Thus, to obtain forecasts for the period $[T+2, T+1+h]$, the data set from point 3 to point $T+2$ is used. In practice, you move the data window forward one point at a time, keeping the model specification unchanged.

To obtain out-of-sample Value-at-Risk predictions, several models are used and compared with the realizations of the variable of interest, the information of which is known in the period $[1 : T + M]$. Since the available data do not concern the conditional (or unconditional) quantile of the variable of interest, but contain its realizations, the approach used does not consist in comparing point forecasts, but rather in analyzing approaches related to the evaluation of interval forecasts or density. However, the observations allow us to evaluate whether the predicted quantiles are adequate. This means checking whether the number of observations of the variable of interest, which fall on the tail, is consistent with the probability corresponding to the coverage level specified for the Value-at-Risk. In this context, a first fundamental aspect to consider concerns the VaR coverage level, which requires consistency between the specified nominal coverage level and the observed one. This is known as unconditional coverage, and overshoots, i.e., observations that fall into the tails, play a key role in this context.

Furthermore, it is important to evaluate the independence of the overshoots, which represents the second relevant characteristic. The goal is to determine whether the extreme

observations are correlated with each other or whether they are independent. It is essential to keep in mind that overruns, i.e. events in which losses in excess of the VaR level occur, are of particular relevance since VaR itself represents the maximum acceptable level of loss for which financial institutions must protect themselves. VaR overruns represent unexpected losses for a bank, for which no coverage is provided. Therefore, there is a strong interest on the part of the bank to reduce the number of overruns and to prevent an overrun today from increasing the probability of observing one tomorrow. An appropriate VaR model must ensure that the number of breaches observed is in line with the expected ones. However, it is not correct to choose a model just because it does not produce overshoots, especially considering that overshoots represent only a small percentage of the data, generally between 1% and 5%. In fact, acceptable losses associated with VaR require banks to immobilize financial resources based on the level of VaR. If a model does not produce overruns, it could be associated with very high VaR levels, which would lead to significant immobilization of resources that would penalize the bank's operations.
The backtesting procedure that will be used is often evaluated considering a time horizon of one year, corresponding to 250 working days. This approach aims to avoid evaluating the model only in periods of low volatility, ensuring a more complete evaluation of its capabilities. It is important to underline how the use of long historical series allows us to mitigate the risk of undersampling of extreme events, which represent the greatest risk for financial institutions. By having a wide range of historical data available, you can capture a greater variety of scenarios and market conditions, including those extreme events that can significantly influence model results.

There are two models considered: EGARCH and Quantile - Regression models. In particular, however, three different estimates of the VaR will be carried out using the EGARCH model, specifically: estimate of the VaR using EGARCH, estimate of the VaR using EGARCH with empirical quantile of innovations and estimate of the VaR using EGARCH with the approximation of Cornish - Fisher; furthermore, two for the QR method: calculation of the VaR by estimating the QR dependent on the lagged returns and calculating the VaR by estimating the QR dependent on the lagged returns and the absolute value of the lagged returns. Below, for the models considered, the graphic representations of the estimate of the VaR at 1% in the 5 different methods for the two years considered are proposed (the relative comments on these representations will be postponed later in order to provide a better description of the results).

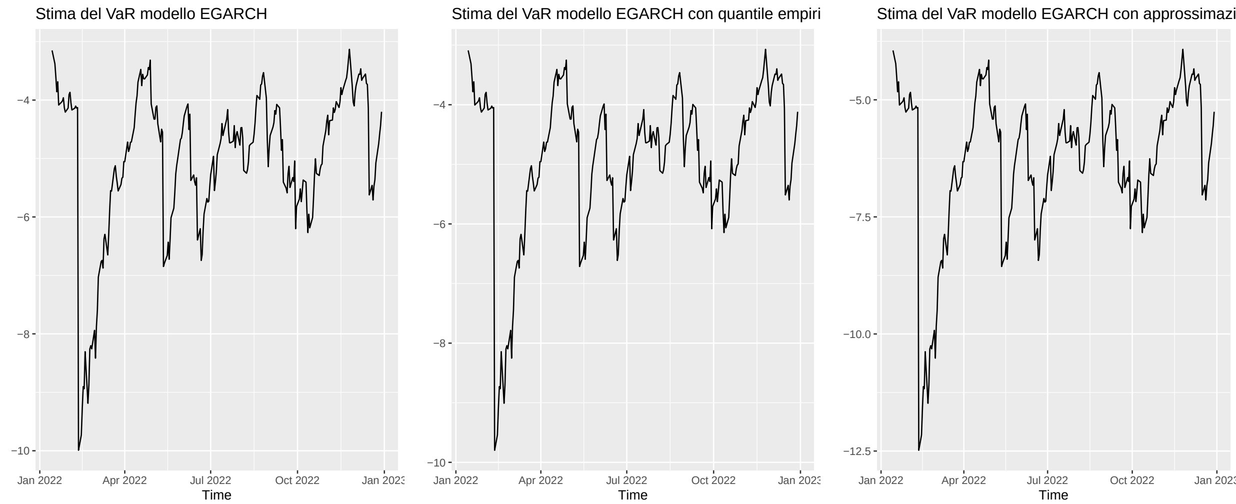

Figure 25: VaR estimates using EGARCH - 2022

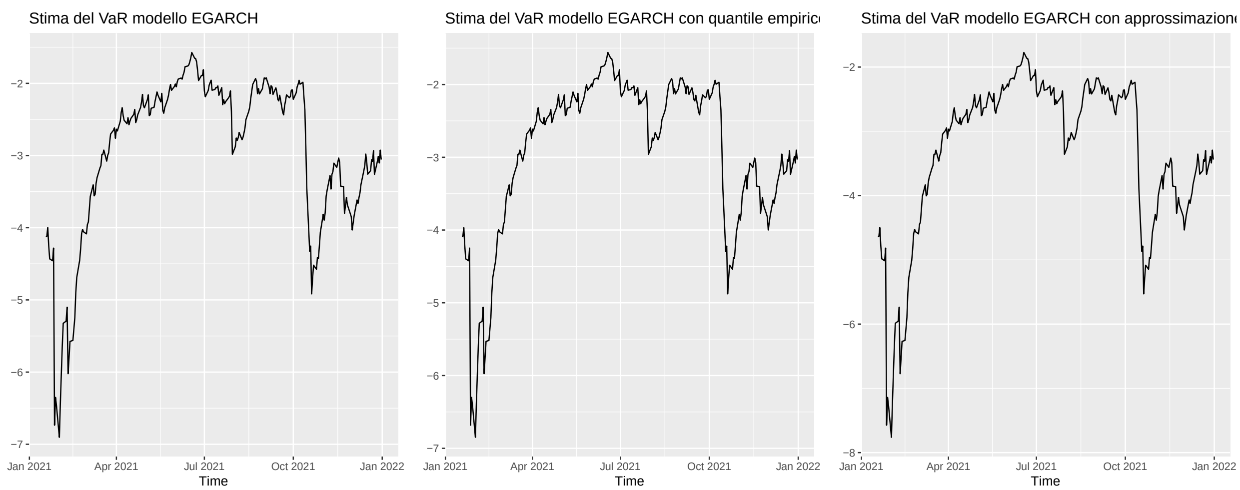

Figure 26: VaR estimates using EGARCH - 2021

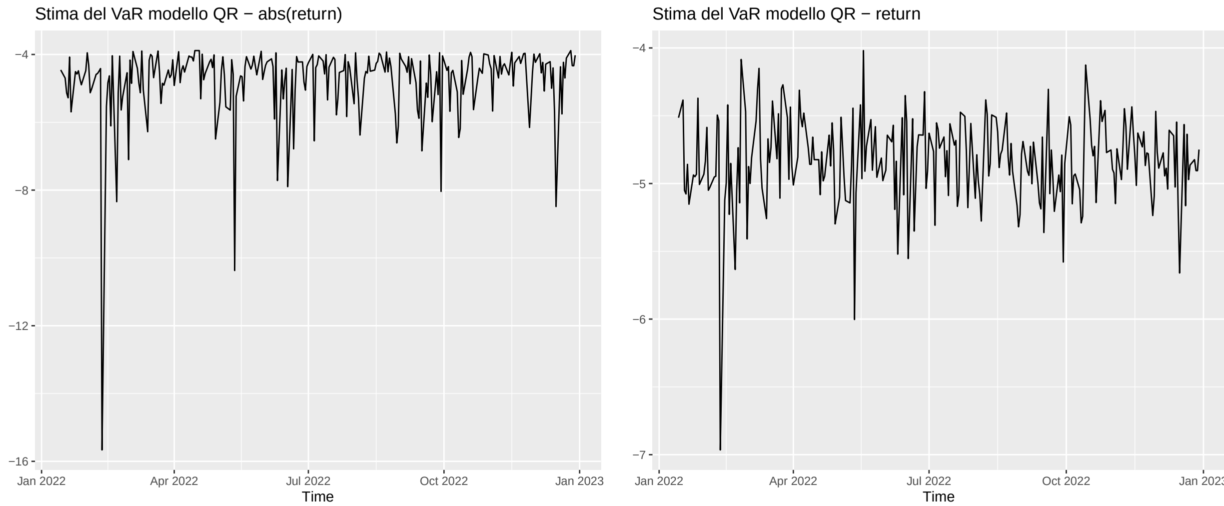

Figure 27: VaR estimate using QR - 2022

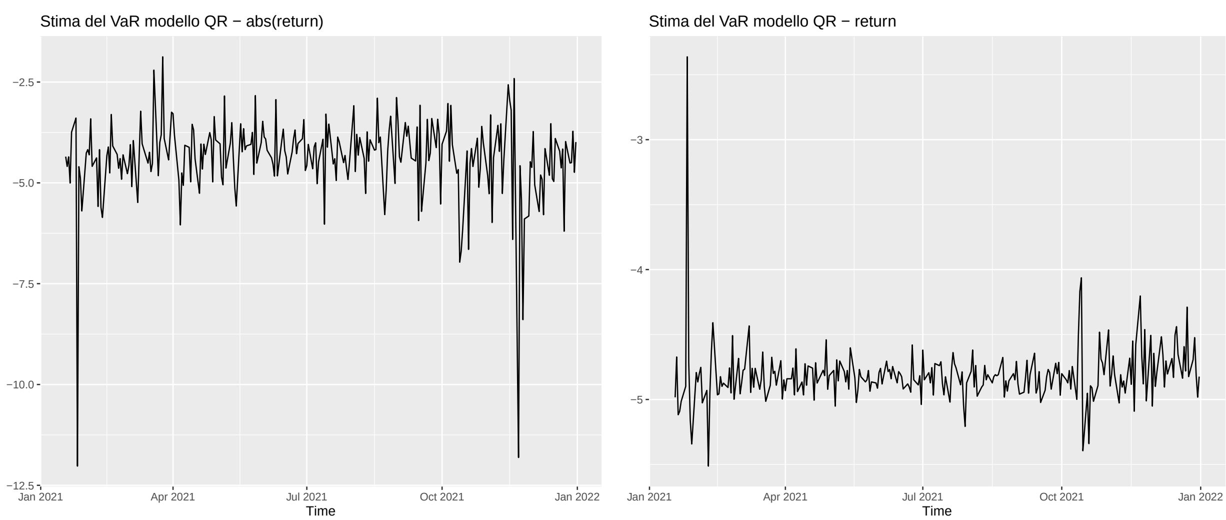

Figure 28: VaR estimate using QR - 2021

Below is the code used to estimate the out-of-sample models.

```
# EGARCH
spec2 <- ugarchspec(variance.model = list(model="eGARCH", garchOrder = c(1, 1)),
                mean.model = list(armaOrder = c(0, 0), include.mean = TRUE),
                distribution.model="sstd")
fit2 <- ugarchfit(spec2,Y,out.sample = 250)

c2=fit2@fit$coef
```

```
z2=fit2@fit$z
for2=ugarchforecast(fit2,n.ahead=1,n.roll=249)

sigma2for=(t(for2@forecast$sigmaFor))
q2=qdist(distribution="sstd",p=0.01,mu=0,sigma=1,skew=c2[6],shape=c2[7])
VaR1=c2[1]+q2*sigma2for

# EGARCH + empirical quantile of innovations (first sample)
VaR2=c2[1]+quantile(z2,0.01)*sigma2for

# EGARCH + Cornish-Fisher on innovations
sk2=skewness(z2)
kt2=kurtosis(z2)
q1=qdist(distribution="norm",p=0.01,mu=0,sigma=1)
q2CF=q1+(sk2[1]/6)*((q1^2)-1)+((kt2[1]-3)/24)*
        ((q1^3)-3)-((sk2[1]^2)/36)*q1*(2*q1*q1-5)
VaR3=c2[1]+q2CF*sigma2for

#QR - abs(return)
T = length(Y)
VaR4 = NULL
for (i in 0:250) {
    out=rq(Y[(2+i):(T-250+i)]~Y[(1+i):(T-251+i)]+
        abs(Y[(1+i):(T-251+i)]),tau=0.01)
    c=coefficients(out)
    VaR4[i]=c[1]+c[2]*Y[(T-250-i)]+c[3]*abs(Y[(T-250+i)])
}

# QR - return
VaR5 = NULL
for (i in 0:250) {
     out1=rq(Y[(2+i):(T-250+i)]~Y[(1+i):(T-251+i)],tau=0.01)
     c1=coefficients(out1)
    VaR5[i]=c1[1]+c1[2]*Y[(T-250+i)]
}
```

In the first part of the code, an estimation of the EGARH models is performed. Using the ”out.sample” command allows you to estimate outside the data sample. Regarding Quantile-

Regression, I have taken an approach that involves using a for loop. During this cycle, the VaR estimate is performed by advancing the time window by one step, i.e. by shifting the range of observations.
To test and compare the different methods, one of the possible tests that can be used to evaluate the best approach for predicting Value-at-Risk is the Kupiec test for evaluating unconditional coverage. This approach is based on the sequence of overruns. Specifically, we consider the Value-at-Risk prediction $VaR_{t+h}(r_t, \alpha)$ made at time t with horizon h steps forward for the random variable r_t and with coverage level α. The Kupiec test considers the random variable associated with the presence of overruns, i.e. the quantity

$$\begin{cases} 1 & r_{t+h} < VaR_{t+h}(r_t, \alpha) \\ 0 & r_{t+h} \geq VaR_{t+h}(r_t, \alpha) \end{cases} \tag{41}$$

The quantity H_t is also called *Hit* and is assumed to be evaluated over M periods (the out of sample has size M). If the VaR is adequate, the underlying random variable H_t is distributed as a Bernoulli with a probability of success equal to α. The test statistic that Kupiec suggests is based on the likelihood ratio

$$LR_I = 2\log\big((1-\hat{\rho})^{M-k}\hat{\rho}^k\big) - 2\log\big((1-\hat{\alpha})^{M-k}\hat{\alpha}^k\big) \sim \chi^2_1 \tag{42}$$

In the tables below it is possible to observe the values obtained from the text for the two years considered.

EGARCH	EGARCH + quantile	EGARCH + CF	QR - abs(return)	QR return
0.7579883	0.7579883	0.7419327	0.0249815	0.7419327

Table 4: Kupiec test estimate - year 2022

EGARCH	EGARCH + quantile	EGARCH + CF	QR - abs(return)	QR return
0.0249815	0.0249815	0.0249815	0.0249815	0.0249815

Table 5: Kupiec test estimate - year 2021

Below is the test calculation code, where in the first part we calculate for which values the VaR estimate exceeds the returns:

```
YF=Y[(T-249):T]

AllVaR=cbind(VaR1,VaR2,VaR3,VaR4)
```

```
e1=YF<VaR1
e2=YF<VaR2
e3=YF<VaR3
e4=YF<VaR4
e5=YF<VaR5
E=cbind(e1,e2,e3,e4, e5)
colSums(E)

phat=colSums(E)/250;
k=colSums(E)  # eccezioni al VaR
LRK=2*log(((1-phat)^(250-k))*(phat^k))-2*log(((1-0.01)^(250-k))*(0.01^k))
1-pchisq(LRK,1)
```

The test previously considered allows you to evaluate hypotheses also by referring to a single model for calculating the Value-at-Risk, however, there is often a reference model (benchmark) compared to which an improvement is aimed, or it is necessary to identify the model to use within a set of alternatives. In this case it is possible to use loss functions to have a comparison, where the best model will be the one with the lowest loss.
The first loss function that will be considered is the one proposed by Lopez who proposes to combine the overshoots with their amplitude:

$$Loss_{2.t} = \begin{cases} 1 + (r_t - VaR(t,\alpha))^2 & r_t < VaR(t,\alpha) \\ 0 & r_t \geq VaR(t,\alpha) \end{cases} \tag{43}$$

Lopez's proposal therefore introduces a quadratic loss associated only with overruns, thus leaving out of the evaluation the deviation between the Value-at-Risk sequence during the backtesting period when the VaR is not associated with an overrun. Below are the results obtained with respect to the 5 methods for calculating VaR in the years 2022 and 2021:

EGARCH	EGARCH + quantile	EGARCH + CF	QR - abs(return)	QR return
83.35385	85.33337	60.65083	2.547411	36.01313

Table 6: Loss function test estimate 1 - year 2022

EGARCH	EGARCH + quantile	EGARCH + CF	QR - abs(return)	QR return
0	0	0	0	0

Table 7: Loss function test estimate 1 - year 2022

Here is the code used, which as you can easily see, reflects the system in Formala 44:

```
LFLopez1=colSums((YF<VaR1)*(1+(YF-VaR1)^2))
LFLopez2=colSums((YF<VaR2)*(1+(YF-VaR2)^2))
LFLopez3=colSums((YF<VaR3)*(1+(YF-VaR3)^2))
LFLopez4=colSums((YF<VaR4)*(1+(YF-VaR4)^2))
LFLopez5=colSums((YF<VaR5)*(1+(YF-VaR5)^2))
```

A second method, proposed by Caporin and McAleer, consists in applying a more flexible approach that considers a generic loss function

$$Loss_{2.t} = \begin{cases} f(r_t - VaR(t,\alpha))^2 & r_t < VaR(t,\alpha) \\ g(r_t - VaR(t,\alpha))^2 & r_t < VaR(t,\alpha) \qquad r_t \geq VaR(t,\ alpha) \end{cases} \tag{44}$$

where the two functions $f()$ and $g()$ can be chosen within a set of possible specifications and can even be identical. In the case under consideration here I considered the following specification for the $f()$ function:

$$f = (|r_t| - |VaR(t,\alpha|)|h(r_t) \tag{45}$$

where $h(r_t)$ is a selection function that allows evaluating the loss function over the entire support r_t.

EGARCH	EGARCH + quantile	EGARCH + CF	QR - abs(return)	QR return
454.3836	442.8571	602.6287	414.698	457.8007

Table 8: Estimation of loss function tests 2 - year 2022

EGARCH	EGARCH + quantile	EGARCH + CF	QR - abs(return)	QR return
284.3089	281.4259	329.4193	432.0802	513.4513

Table 9: Estimation of loss function tests 2 - year 2021

Here is the code used for the estimate:

```
LFAbs1=colSums((YF<0)*(abs(abs(YF)-abs(VaR1))))
LFAbs2=colSums((YF<0)*(abs(abs(YF)-abs(VaR2))))
LFAbs3=colSums((YF<0)*(abs(abs(YF)-abs(VaR3))))
LFAbs4=colSums((YF<0)*(abs(abs(YF)-abs(VaR4))))
LFAbs5=colSums((YF<0)*(abs(abs(YF)-abs(VaR5))))
```

In conclusion, following the results obtained and commented, for greater interpretation and clarity, the graphs of the performance of the Naturgy Energy stock are shown with the related estimates of the different VaR approaches.

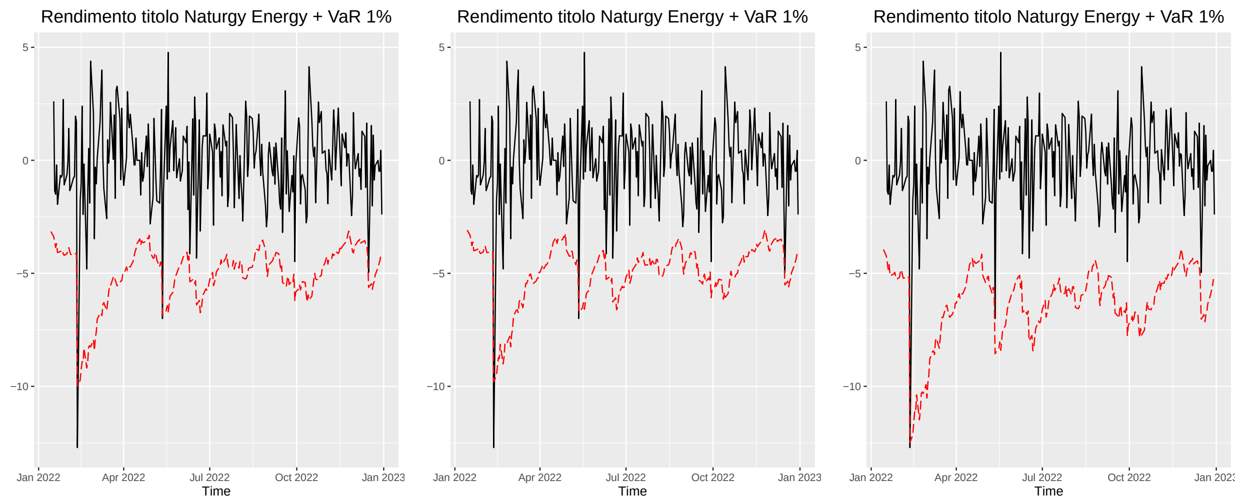

Figure 29: Comparison of Naturgy Energy stock performance and VaR estimated with EGARCH - 2022

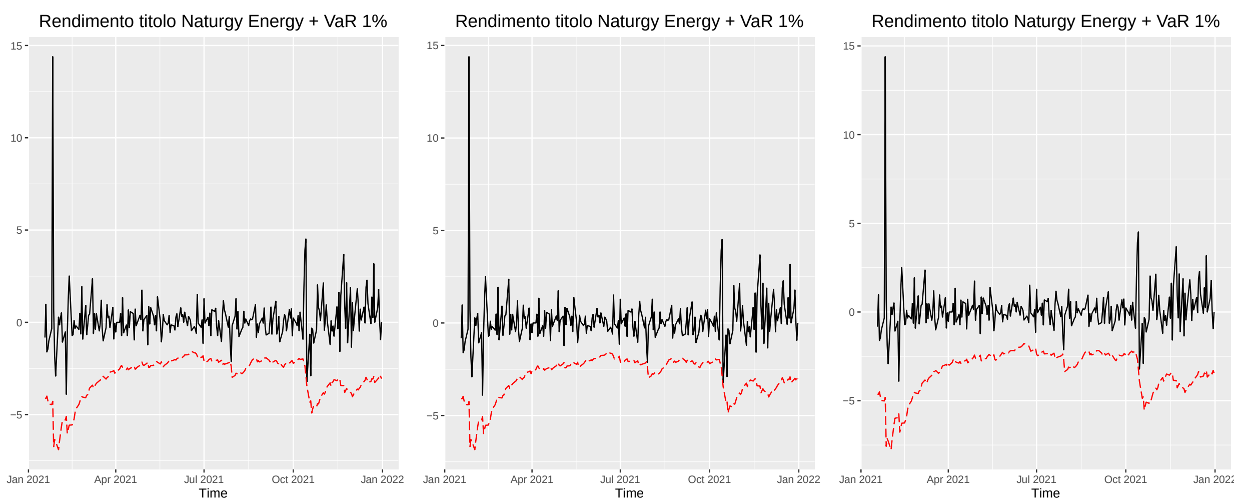

Figure 30: Comparison of Naturgy Energy stock performance and VaR estimated with EGARCH - 2021

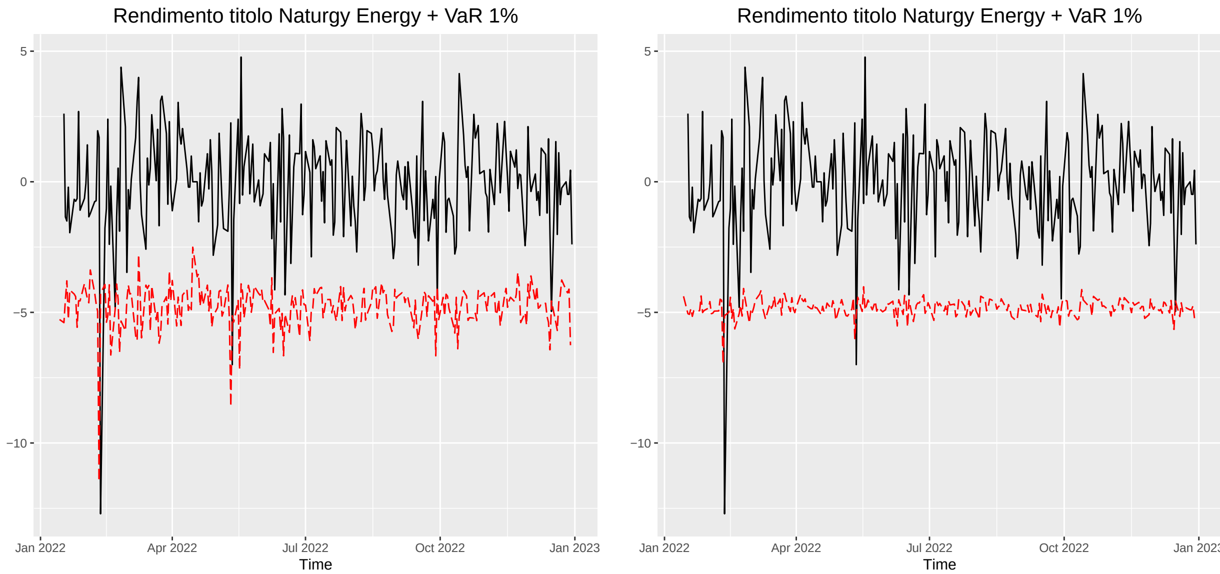

Figure 31: Comparison of Naturgy Energy stock performance and estimated VaR with QR - 2022

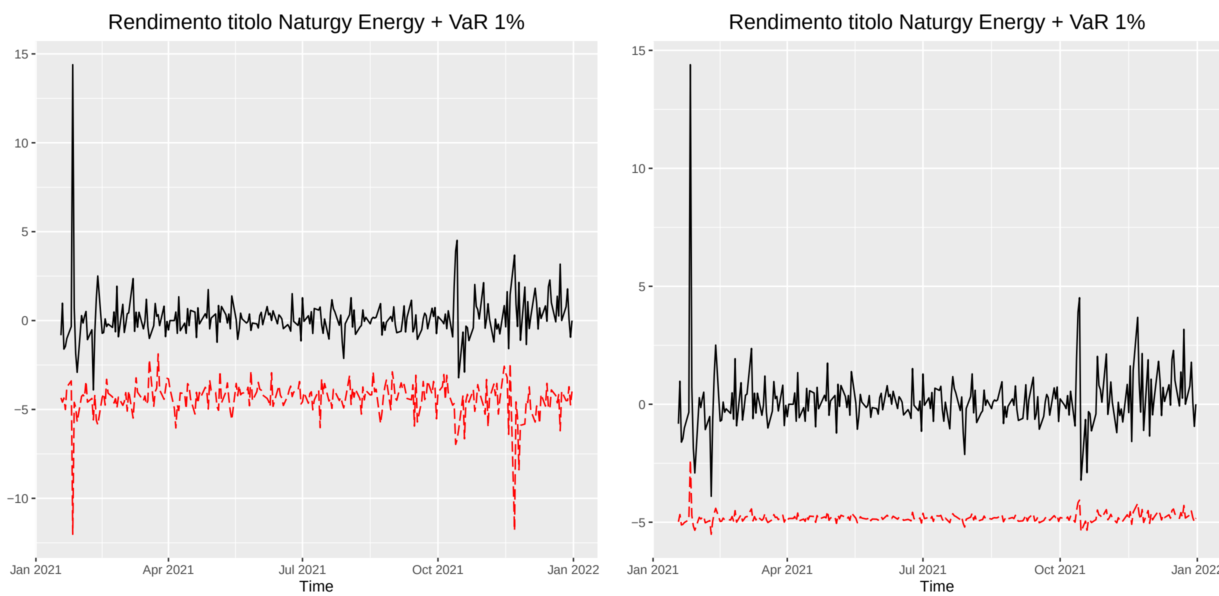

Figure 32: Comparison of Naturgy Energy stock performance and estimated VaR with QR - 2021

The objective of this paragraph is to determine which of the different approaches for forecasting Value-at-Risk is best for the years 2022 and 2021 and whether this approach is the same in both years. From the results and graphs obtained, the particularity of the VaR for the year 2021 clearly emerges. As highlighted in the Figures 30 and **??**, the VaR at 1% (red dotted line) appears not to outperform returns. As already highlighted, it is of fundamental importance to observe overruns, which represent events in which losses greater than the VaR level occur and for which financial institutions must protect themselves. However, in this context, the particular VaR estimate does not represent an exceptional event. What has been achieved is attributable to the peculiarities of the historical series of prices and returns in 2021 compared to 2022, as can be seen in Figures **??** and **??**.

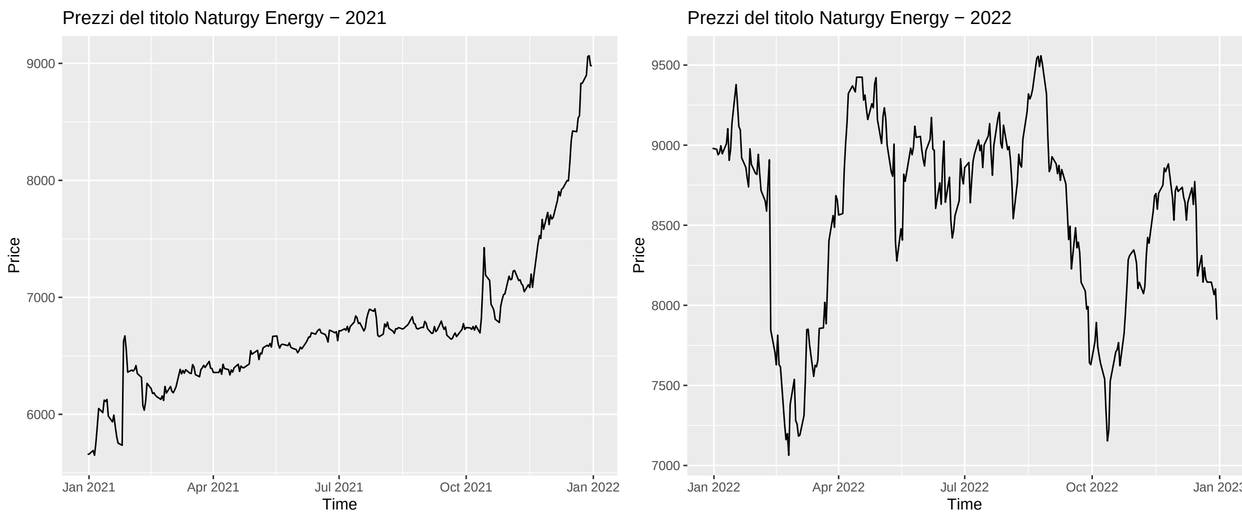

Figure 33: Naturgy Energy stock price comparison 2022 - 2021

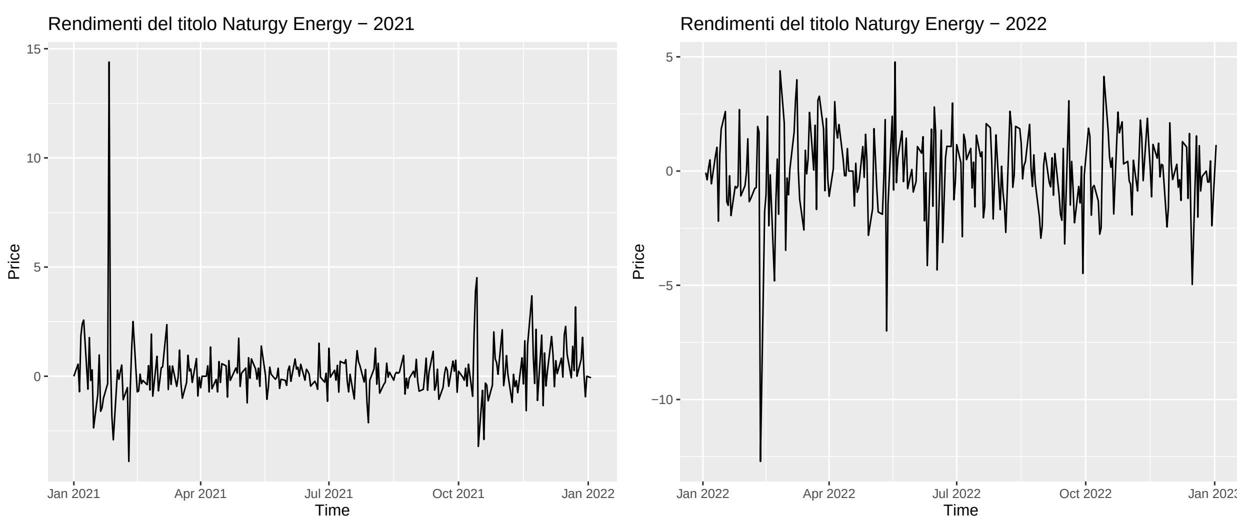

Figure 34: Comparison of Naturgy Energy stock returns 2022 - 2021

The year 2021 stands out from 2022 for a lower volatility of returns, which remained close to zero, and for a constant growth in prices. The increase in gas prices began in the summer of 2021, following the recovery of the economy and consumption following the COVID-19 pandemic. During the autumn period, there was a decline in gas deliveries from Russia and Norway to Europe, along with an increase in deliveries to the more profitable Asian markets. This situation has led to an increase in gas prices in Europe. In 2022, the war in Ukraine was the major event characterizing the year, evident in the data through a downward spike in prices and yields at the beginning of the series. In the price data, two negative peaks can be observed in 2022, the first related to the outbreak of war in Ukraine and the second to the increase in gas prices during the winter period, due to the increase in demand for this commodity. Naturgy Energy, operating both in the distribution of gas in Spain and Latin America and in the conversion of gas into electricity, has suffered significantly from increases in the prices of this resource.

Despite the peculiarities of 2021, thanks to the more flexible approach proposed by Caporin and McAller, it was possible to determine which of the proposed models was better, unlike other evaluation methods that did not provide useful results. The EGARCH model with empirical quantile of innovations proved to be the best. However, this does not apply to the following year. In fact, all three valuation methods agree that the Value at Risk (VaR) estimation method using lagged returns and the absolute value of lagged returns as determining factors is the best model.

4 APPENDIX A

Nome azienda	Settore - level 2	Settore - level 5
A2A	Utilities	Conv. Electricity
ACCOR	Consumer Discretion	Hotels & Motels
ADP	Industrials	Transport Services
AIR FRANCE	Consumer Discretion	Airlines
BMW	Consumer Discretion	Automobiles
BOLLORE	Industrials	Transport Services
BUREAU VERITAS	Industrials	Prof.Business Support
CONTINENTAL	Consumer Discretion	Auto Parts
DEUTSCHE POST	Industrials	Delivery Services
D IETEREN GROUP	Consumer Discretion	Auto Services
E ON N	Utilities	Multi-Utilities
EDF	Utilities	Conv. Electricity
EDP ENERGIAS DE PORTUGAL	Utilities	Alt. Electricity
EDP RENOVAVEIS	Utilities	Alt. Electricity
ELIA GROUP	Utilities	Conv. Electricity
ENAGAS	Energy	Pipelines
ENCAVIS	Utilities	Alt. Electricity
ENEL	Utilities	Conv. Electricity
ENGIE	Utilities	Multi-Utilities
ENI	Energy	Integrated Oil & Gas
FAURECIA	Consumer Discretion	Auto Parts
FLUTTER (DUB) ENTERTAINMENT	Consumer Discretion	Casinos & Gambling
FORTUM	Utilities	Conv. Electricity
GALP ENERGIA SGPS	Energy Integrated	Oil & Gas
HERA	Utilities	Multi-Utilities
IBERDROLA	Utilities	Conv. Electricity
DEUTSCHE LUFTHANSA	Consumer Discretion	Airlines
MERCEDES-BENZ GROUP	Consumer Discretion	Automobiles
CMPG.DES ETS.MICH.	Consumer Discretion	Tires
NATURGY ENERGY	Utilities	Gas Distribution
NESTE	Energy Oil	Refining & Mkting
OMV	Energy	Integrated Oil & Gas
PORSCHE AML.HLDG.	Consumer Discretion	Automobiles
RANDSTAD	Industrials	Training, Emp. Agency
RED ELECTRICA	Utilities	Conv. Electricity
RENAULT	Consumer Discretion	Automobiles

Table 10: Companies included in the composition of the constructed portfolio

Company Name and Sector - level 2	Sector - level 5	
REPSOL YPF	Energy	Integrated Oil & Gas
RWE	Utilities	Multi-Utilities
RYANAIR HOLDINGS	Consumer Discretion	Airlines
SHELL	Energy	Integrated Oil & Gas
SIEMENS	Energy	Renewable Energy Eq.
SNAM	Energy	Pipelines
STELLANTIS	Consumer Discretion	Automobiles
TENARIS	Energy	Oil Equipment & Svs
TERNA	Utilities	Conv. Electricity
TOTALENERGIES	Energy	Integrated Oil & Gas
UBISOFT ENTERTAINMENT	Consumer Discretion	Elec. Entertainment
VALEO	Consumer Discretion	Auto Parts
VERBUND	Utilities	Alt. Electricity
VOLKSWAGEN	Consumer Discretion	Automobiles

Table 11: Companies included in the composition of the constructed portfolio