COMP3224

Coursework II

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1 Introduction

A clear explanation of the solutions for Exercise 1 and 2 is provided in the sections that follow. This report also contains a list of Appendices that delve deeper into the causal reasoning concepts encountered in the exercises. It is highly recommended to read the Appendices to appreciate the quality of this report and the rationale under which the tasks of this exercise have been carried out. Furthermore, the Appendices prove a deep understanding of the material covered in the lectures. Of particular interest is Appendix A. It goes through the development of Causal Reasoning and the crucial problem it tries to solve: *the missing data problem*. This journey inside the Causal Reasoning development clearly shows the origin of ATE, which is fundamental to solve Exercise 2.

2 Exercise 1

This exercise requires us to verify the correctness of the claims (a)-(j) using the rules of d-separation. The table below summarises the truth values of the claims (a)-(j). In the subsequent subsections, the truth values in the table are proved by providing a table or multiple tables if necessary. Each table lists all possible paths between two variables, stating whether they are blocked or not and why. For claim (j), a proof by counterexample and a proof by contradiction is provided showing that $\{X, S, Z\}$ cannot be d-separated from W by conditioning on any set Q in the DAG.

Claim	Truth value
(a)	True
(b)	True
(c)	False
(d)	True
(e)	True
(f)	False
(g)	False
(h)	True
(i)	False
(j)	False

(a) $W \perp\!\!\!\perp S \mid \{Y, X\}$

$W \perp\!\!\!\perp S \mid \{Y, X\}$ is a true claim if and only if all paths between W and S are blocked given that we condition on the set $\{Y, X\}$. The table below shows that, conditioning on $\{Y, X\}$, all paths between W and S are blocked. Consequently, $W \perp\!\!\!\perp S \mid \{Y, X\}$ is a true claim.

$W \perp\!\!\!\perp S \mid \{Y, X\}$			
Paths	Blocked	Kind	Why
$W \to Y \leftarrow X \to Z \to S$	Yes	Fork	$Y \leftarrow X \to Z$
$W \to Y \leftarrow X \to Z \to R \to S$	Yes	Fork	$Y \leftarrow X \to Z$
$W \to Y \to R \leftarrow Z \to S$	Yes	Chain	$W \to Y \to R$
$W \to Y \to R \to S$	Yes	Chain	$W \to Y \to R$

(b) $W \perp\!\!\!\perp S \mid \{Y, Z\}$

$W \perp\!\!\!\perp S \mid \{Y, Z\}$ is a true claim if and only if all paths between W and S are blocked given that we condition on the set $\{Y, Z\}$. The table below shows that, conditioning on $\{Y, Z\}$, all paths between W and S are blocked. Consequently, $W \perp\!\!\!\perp S \mid \{Y, Z\}$ is a true claim.

$W \perp\!\!\!\perp S \mid \{Y, Z\}$			
Paths	Blocked	Kind	Why
$W \to Y \leftarrow X \to Z \to S$	Yes	Chain	$X \to Z \to S$
$W \to Y \leftarrow X \to Z \to R \to S$	Yes	Chain	$X \to Z \to R$
$W \to Y \to R \leftarrow Z \to S$	Yes	Chain	$W \to Y \to R$
$W \to Y \to R \to S$	Yes	Chain	$W \to Y \to R$

(c) $W \perp\!\!\!\perp S \mid \{R, X\}$

$W \perp\!\!\!\perp S \mid \{R, X\}$ is a true claim if and only if all paths between W and S are blocked given that we condition on the set $\{R, X\}$. The table below shows that, conditioning on $\{R, X\}$, there exists one paths between W and S which is not blocked. Consequently, $W \perp\!\!\!\perp S \mid \{R, X\}$ is a false claim.

$W \perp\!\!\!\perp S \mid \{R, X\}$			
Paths	Blocked	Kind	Why
$W \to Y \leftarrow X \to Z \to S$	Yes	Fork	$Y \leftarrow X \to Z$
$W \to Y \leftarrow X \to Z \to R \to S$	Yes	Fork	$Y \leftarrow X \to Z$
$W \to Y \to R \leftarrow Z \to S$	No	Collider	$Y \to R \leftarrow Z$
$W \to Y \to R \to S$	Yes	Chain	$Y \to R \to S$

(d) $\{W, X\} \perp\!\!\!\perp \{S, T\} \mid \{R, Z\}$

By definition, $\{W, X\} \perp\!\!\!\perp \{S, T\} \mid \{R, Z\}$ is a true claim if and only if $W \perp\!\!\!\perp S \mid \{R, Z\} \ \wedge \ W \perp\!\!\!\perp T \mid \{R, Z\} \ \wedge \ X \perp\!\!\!\perp S \mid \{R, Z\} \ \wedge \ X \perp\!\!\!\perp T \mid \{R, Z\}$ is a true statement as well. Given that the latter statement is a concatenation of AND statements, we need to verify whether each of them is a true claim. The tables below show that all the paths for each of the AND statements are blocked. As a consequence, $\{W, X\} \perp\!\!\!\perp \{S, T\} \mid \{R, Z\}$ is a true claim.

$W \perp\!\!\!\perp S \mid \{R, Z\}$			
Paths	Blocked	Kind	Why
$W \to Y \leftarrow X \to Z \to S$	Yes	Chain	$X \to Z \to S$
$W \to Y \leftarrow X \to Z \to R \to S$	Yes	Chain	$X \to Z \to R$
$W \to Y \to R \leftarrow Z \to S$	Yes	Fork	$R \leftarrow Z \to S$
$W \to Y \to R \to S$	Yes	Chain	$Y \to R \to S$

$W \perp\!\!\!\perp T \mid \{R, Z\}$			
Paths	Blocked	Kind	Why
$W \to Y \to R \to T$	Yes	Chain	$Y \to R \to T$
$W \to Y \leftarrow X \to Z \to R \to T$	Yes	Chain	$X \to Z \to R$
$W \to Y \leftarrow X \to Z \to S \leftarrow R \to T$	Yes	Chain	$X \to Z \to S$

$X \perp\!\!\!\perp S \mid \{R, Z\}$			
Paths	Blocked	Kind	Why
$X \to Z \to S$	Yes	Chain	$X \to Z \to S$
$X \to Z \to R \to S$	Yes	Chain	$X \to Z \to R$
$X \to Y \to R \to S$	Yes	Chain	$Y \to R \to S$
$X \to Y \to R \leftarrow Z \to S$	Yes	Fork	$R \leftarrow Z \to S$

$X \perp\!\!\!\perp T \mid \{R, Z\}$			
Paths	Blocked	Kind	Why
$X \to Y \to R \to T$	Yes	Chain	$Y \to R \to T$
$X \to Z \to R \to T$	Yes	Chain	$X \to Z \to R$
$X \to Z \to S \leftarrow R \to T$	Yes	Chain	$X \to Z \to S$

(e) $\{Y, Z\} \perp\!\!\!\perp T \mid \{R, S\}$

By definition, $\{Y, Z\} \perp\!\!\!\perp T \mid \{R, S\}$ is a true claim if and only if $Y \perp\!\!\!\perp T \mid \{R, S\} \ \wedge \ Z \perp\!\!\!\perp T \mid \{R, S\}$ is a true statement as well. Given that the latter statement is a concatenation of AND statements, we need to verify whether each of them is a true claim. The tables below show that all the paths for each of the AND statements are blocked. As a consequence, $\{Y, Z\} \perp\!\!\!\perp T \mid \{R, S\}$ is a true claim.

$Y \perp\!\!\!\perp T \mid \{R, S\}$			
Paths	Blocked	Kind	Why
$Y \to R \to T$	Yes	Chain	$Y \to R \to T$
$Y \leftarrow X \to Z \to R \to T$	Yes	Chain	$Z \to R \to T$
$Y \leftarrow X \to Z \to S \leftarrow R \to T$	Yes	Fork	$S \leftarrow R \to T$

$Z \perp\!\!\!\perp T \mid \{R, S\}$			
Paths	Blocked	Kind	Why
$Z \rightarrow R \rightarrow T$	Yes	Chain	$Z \rightarrow R \rightarrow T$
$Z \rightarrow S \leftarrow R \rightarrow T$	Yes	Fork	$S \leftarrow R \rightarrow T$
$Z \leftarrow X \rightarrow Y \rightarrow R \rightarrow T$	Yes	Chain	$Y \rightarrow R \rightarrow T$

(f) $\{X, S\} \perp\!\!\!\perp \{W, T\} \mid \{R, Z\}$

By definition, $\{X, S\} \perp\!\!\!\perp \{W, T\} \mid \{R, Z\}$ is a true claim if and only if $X \perp\!\!\!\perp W \mid \{R, Z\} \;\wedge\; X \perp\!\!\!\perp T \mid \{R, Z\} \;\wedge\; S \perp\!\!\!\perp W \mid \{R, Z\} \;\wedge\; S \perp\!\!\!\perp T \mid \{R, Z\}$ is a true statement as well. Given that the latter statement is a concatenation of AND statements, we need to verify whether each of them is a true claim. The tables below show that not all the paths for each of the AND statements are blocked. Specifically, $X \perp\!\!\!\perp W \mid \{R, Z\}$ is a false claim. As a consequence, $\{X, S\} \perp\!\!\!\perp \{W, T\} \mid \{R, Z\}$ is a false claim too.

$X \perp\!\!\!\perp W \mid \{R, Z\}$			
Paths	Blocked	Kind	Why
$X \rightarrow Y \leftarrow W$	No	Collider	$X \rightarrow Y \leftarrow W$
$X \rightarrow Z \rightarrow R \leftarrow Y \leftarrow W$	Yes	Chain	$X \rightarrow Z \rightarrow R$
$X \rightarrow Z \rightarrow S \leftarrow R \leftarrow Y \leftarrow W$	Yes	Chain	$X \rightarrow Z \rightarrow S$

$X \perp\!\!\!\perp T \mid \{R, Z\}$			
Paths	Blocked	Kind	Why
$X \rightarrow Y \rightarrow R \rightarrow T$	Yes	Chain	$Y \rightarrow R \rightarrow T$
$X \rightarrow Z \rightarrow R \rightarrow T$	Yes	Chain	$X \rightarrow Z \rightarrow R$
$X \rightarrow Z \rightarrow S \leftarrow R \rightarrow T$	Yes	Chain	$X \rightarrow Z \rightarrow S$

$S \perp\!\!\!\perp W \mid \{R, Z\}$			
Paths	Blocked	Kind	Why
$S \leftarrow R \leftarrow Y \leftarrow W$	Yes	Chain	$S \leftarrow R \leftarrow Y$
$S \leftarrow Z \leftarrow X \rightarrow Y \leftarrow W$	Yes	Chain	$S \leftarrow Z \leftarrow X$
$S \leftarrow Z \rightarrow R \leftarrow Y \leftarrow W$	Yes	Fork	$S \leftarrow Z \rightarrow R$
$S \leftarrow R \leftarrow Z \leftarrow X \rightarrow Y \leftarrow W$	Yes	Chain	$S \leftarrow R \leftarrow Z$

$S \perp\!\!\!\perp T \mid \{R, Z\}$			
Paths	Blocked	Kind	Why
$S \leftarrow R \rightarrow T$	Yes	Fork	$S \leftarrow R \rightarrow T$
$S \leftarrow Z \rightarrow R \rightarrow T$	Yes	Fork	$S \leftarrow Z \rightarrow R$
$S \leftarrow Z \leftarrow X \rightarrow Y \rightarrow R \rightarrow T$	Yes	Chain	$S \leftarrow Z \leftarrow X$

(g) $\{X, S, Z\} \perp\!\!\!\perp \{W, T\} \mid R$

By definition, $\{X, S, Z\} \perp\!\!\!\perp \{W, T\} \mid R$ is a true claim if and only if $X \perp\!\!\!\perp W \mid R \;\wedge\; X \perp\!\!\!\perp T \mid R \;\wedge\; S \perp\!\!\!\perp W \mid R \;\wedge\; S \perp\!\!\!\perp T \mid R \;\wedge\; Z \perp\!\!\!\perp W \mid R \;\wedge\; Z \perp\!\!\!\perp T \mid R$ is a true statement as well. Given that the latter statement is a concatenation of AND statements, we need to verify whether each of them is a true claim. The tables below show that not all the paths for each of the AND statements are blocked. Specifically, $X \perp\!\!\!\perp W \mid R$, $Z \perp\!\!\!\perp W \mid R$ and $S \perp\!\!\!\perp W \mid R$ are false claims. As a consequence, $\{X, S, Z\} \perp\!\!\!\perp \{W, T\} \mid R$ is a false claim too.

$X \perp\!\!\!\perp W \mid R$			
Paths	Blocked	Kind	Why
$X \rightarrow Y \leftarrow W$	No	Collider	$X \rightarrow Y \leftarrow W$
$X \rightarrow Z \rightarrow R \leftarrow Y \leftarrow W$	No	Collider	$Z \rightarrow R \leftarrow Y$
$X \rightarrow Z \rightarrow S \leftarrow R \leftarrow Y \leftarrow W$	Yes	Collider	$Z \rightarrow S \leftarrow R$

$Z \perp\!\!\!\perp W \mid R$			
Paths	Blocked	Kind	Why
$Z \rightarrow R \leftarrow Y \leftarrow W$	No	Collider	$Z \rightarrow R \leftarrow Y$
$Z \rightarrow S \leftarrow R \leftarrow Y \leftarrow W$	Yes	Collider	$Z \rightarrow S \leftarrow R$
$Z \leftarrow X \rightarrow Y \leftarrow W$	No	Collider	$X \rightarrow Y \leftarrow W$

$S \perp\!\!\!\perp W \mid R$			
Paths	Blocked	Kind	Why
$S \leftarrow R \leftarrow Y \leftarrow W$	Yes	Chain	$S \leftarrow R \leftarrow Y$
$S \leftarrow Z \leftarrow X \rightarrow Y \leftarrow W$	No	Collider	$X \rightarrow Y \leftarrow W$
$S \leftarrow R \leftarrow Z \leftarrow X \rightarrow Y \leftarrow W$	Yes	Chain	$S \leftarrow R \leftarrow Z$

$X \perp\!\!\!\perp T \mid R$			
Paths	Blocked	Kind	Why
$X \rightarrow Y \rightarrow R \rightarrow T$	Yes	Chain	$Y \rightarrow R \rightarrow T$
$X \rightarrow Z \rightarrow R \rightarrow T$	Yes	Chain	$Z \rightarrow R \rightarrow T$
$X \rightarrow Z \rightarrow S \leftarrow R \rightarrow T$	Yes	Collider	$Z \rightarrow S \leftarrow R$

$S \perp\!\!\!\perp T \mid R$			
Paths	Blocked	Kind	Why
$S \leftarrow R \rightarrow T$	Yes	Fork	$S \leftarrow R \rightarrow T$
$S \leftarrow Z \rightarrow R \rightarrow T$	Yes	Chain	$Z \rightarrow R \rightarrow T$
$S \leftarrow Z \leftarrow X \rightarrow Y \rightarrow R \rightarrow T$	Yes	Chain	$Y \rightarrow R \rightarrow T$

$Z \perp\!\!\!\perp T \mid R$			
Paths	Blocked	Kind	Why
$Z \rightarrow R \rightarrow T$	Yes	Chain	$Z \rightarrow R \rightarrow T$
$Z \rightarrow S \leftarrow R \rightarrow T$	Yes	Collider	$Z \rightarrow S \leftarrow R$
$Z \leftarrow X \rightarrow Y \rightarrow R \rightarrow T$	Yes	Chain	$Y \rightarrow R \rightarrow T$

(h) $\{X, Z\} \perp\!\!\!\perp W$

By definition, $\{X, Z\} \perp\!\!\!\perp W$ is a true claim if and only if $X \perp\!\!\!\perp W \wedge Z \perp\!\!\!\perp W$ is a true statement as well. Given that the latter statement is a concatenation of AND statements, we need to verify whether each of them is a true claim. The tables below show that all the paths for each of the AND statements are blocked. As a consequence, $\{X, Z\} \perp\!\!\!\perp W$ is a true claim.

$X \perp\!\!\!\perp W$			
Paths	Blocked	Kind	Why
$X \rightarrow Y \leftarrow W$	Yes	Collider	$X \rightarrow Y \leftarrow W$
$X \rightarrow Z \rightarrow R \leftarrow Y \leftarrow W$	Yes	Collider	$Z \rightarrow R \leftarrow Y$
$X \rightarrow Z \rightarrow S \leftarrow R \leftarrow Y \leftarrow W$	Yes	Collider	$Z \rightarrow S \leftarrow R$

$Z \perp\!\!\!\perp W$			
Paths	Blocked	Kind	Why
$Z \rightarrow R \leftarrow Y \leftarrow W$	Yes	Collider	$Z \rightarrow R \leftarrow Y$
$Z \rightarrow S \leftarrow R \leftarrow Y \leftarrow W$	Yes	Collider	$Z \rightarrow S \leftarrow R$
$Z \leftarrow X \rightarrow Y \leftarrow W$	Yes	Collider	$X \rightarrow Y \leftarrow W$

(i) $\{X, S, Z\} \perp\!\!\!\perp W$

By definition, $\{X, S, Z\} \perp\!\!\!\perp W$ is a true claim if and only if $X \perp\!\!\!\perp W \wedge S \perp\!\!\!\perp W \wedge Z \perp\!\!\!\perp W$ is a true statement as well. Given that the latter statement is a concatenation of AND statements, we need to verify whether each of them is a true claim. The tables below show that not all the paths for each of the AND statements are blocked. Specifically, $S \perp\!\!\!\perp W$ is a false claim. As a consequence, $\{X, S, Z\} \perp\!\!\!\perp W$ is a false claim.

$X \perp\!\!\!\perp W$			
Paths	Blocked	Kind	Why
$X \rightarrow Y \leftarrow W$	Yes	Collider	$X \rightarrow Y \leftarrow W$
$X \rightarrow Z \rightarrow R \leftarrow Y \leftarrow W$	Yes	Collider	$Z \rightarrow R \leftarrow Y$
$X \rightarrow Z \rightarrow S \leftarrow R \leftarrow Y \leftarrow W$	Yes	Collider	$Z \rightarrow S \leftarrow R$

$Z \perp\!\!\!\perp W$			
Paths	Blocked	Kind	Why
$Z \rightarrow R \leftarrow Y \leftarrow W$	Yes	Collider	$Z \rightarrow R \leftarrow Y$
$Z \rightarrow S \leftarrow R \leftarrow Y \leftarrow W$	Yes	Collider	$Z \rightarrow S \leftarrow R$
$Z \leftarrow X \rightarrow Y \leftarrow W$	Yes	Collider	$X \rightarrow Y \leftarrow W$

$S \perp\!\!\!\perp W$			
Paths	Blocked	Kind	Why
$S \leftarrow R \leftarrow Y \leftarrow W$	No		
$S \leftarrow Z \leftarrow X \rightarrow Y \leftarrow W$	Yes	Collider	$X \rightarrow Y \leftarrow W$
$S \leftarrow R \leftarrow Z \leftarrow X \rightarrow Y \leftarrow W$	Yes	Collider	$X \rightarrow Y \leftarrow W$

(j)

Let us denote the set of nodes in the whole DAG with D. In what follows, two proofs are provided which demonstrate that $\{X, S, Z\}$ cannot be d-separated from W by conditioning on any set $Q \in 2^D$. The first proof is a simple proof by counterexample, which shows that there exists a set $T \in 2^D$ such that $\{X, S, Z\}$ are not d-separated from W conditioning on T. The second proof demonstrates that actually there exists no set Q such that $\{X, S, Z\}$ are d-separated from W conditioning on Q.

(j).1 Proof by counterexample

$\{X, S, Z\}$ are d-separated from W by conditioning on any set $Q \in 2^D$ in the DAG, if for all such sets Q, all paths between $\{X, S, Z\}$ and W are blocked. Hence, if there exists a set $T \in 2^D$ such that not all paths between $\{X, S, Z\}$ and W are blocked, then $\{X, S, Z\}$ are not d-separated from W by conditioning on any set Q. It turns out that such a set T exists. Indeed, if $T = \{\}$, we know that $\{X, S, Z\}$ are not d-separated from W as shown in claim (i).

However, since this solution was trivial, let us rephrase the question into: can $\{X, S, Z\}$ be d-separated from W by conditioning on any set $Q \in 2^D$ containing at least one node? We can easily provide a counterexample of why also for this new question $\{X, S, Z\}$ are not d-separated from W by conditioning on any set $Q \in 2^D$ such that $|Q| \geq 1$. Indeed, if $T = \{Y\}$, (X, W), (S, W) and (Z, W) become dependent conditioned on T. This holds because the following paths are open:

- $X \rightarrow Y \leftarrow W$
- $S \leftarrow Z \leftarrow X \rightarrow Y \leftarrow W$
- $S \leftarrow R \leftarrow Z \leftarrow X \rightarrow Y \leftarrow W$
- $Z \leftarrow X \rightarrow Y \leftarrow W$

Notice that it possible to provide other counterexamples as well. For example, if $T = \{R\}$ or $T = \{S\}$, $\{X, S, Z\}$ are not d-separated from W by conditioning on T as well.

(j).2 Proof by contradiction

Let us suppose that there exists a set $Q \in 2^D$ in the DAG such that $\{X, S, Z\}$ are d-separated from W by conditioning on the set Q. Hence, all paths between $\{X, S, Z\}$ and W must be blocked given that we condition on Q. As a consequence, the paths $X \rightarrow Y \leftarrow W$ and $S \leftarrow R \leftarrow Y \leftarrow W$ must be blocked conditioning on Q.

The first path is blocked if and only if Q does not contain Y and its descendants R, T and S. Then, Q must necessarily be one of the subsets in the superset of $D - \{Y, R, T, S\}$, $Q \in 2^{D - \{Y,R,T,S\}}$.

The second path is blocked if and only if Q contains R or Y or both. Thus, Q must necessarily be one of the sets in the set A, $Q \in A$, where $A = \{B \in 2^D \mid R \in B \textit{ or } Y \in B\}$.

We have reached a contradiction as $2^{D - \{Y,R,T,S\}} \cap A = \{\}$. In other words, there exists no such $Q \in 2^D$ so that $\{X, S, Z\}$ are d-separated from W by conditioning on Q.

3 Exercise 2

Exercise 2 provides us with a decision-making problem to be solved. Given the observational data gathered in the kidney stone table, should we give treatment $T = 1$ (open surgery) or treatment $T = 0$ (needle) to a new patient P? Naturally, the objective of this decision-making problem comes down to maximising the chances of a successful recovery for patient P by giving him the best treatment.

In order to make this critical decision, we need to find out the magnitude of the treatment's causal effect on the recovery outcome. In other words, we would like to intervene on the treatment variable and see how the two different surgeries affect the recovery outcome. This amounts to computing the Average Treatment Effect(ATE), also known as Average Causal Effect(ACE) or simply Causal Effect:

$$\begin{aligned} ATE &= \mathbb{E}[Y^1] - \mathbb{E}[Y^0] \\ &= P(Y = 1|do(T = 1)) - P(Y = 1|do(T = 0)) \\ &= P_{do(T=1)}(Y = 1) - P_{do(T=0)}(Y = 1) \end{aligned}$$

$\mathbb{E}[Y^1]$ is the expected successful outcome given an intervention that sets $T = 1$. This is equivalent to $P(Y = 1|do(T = 1))$. Similarly, $\mathbb{E}[Y^0]$ is the expected succesful outcome given an intervention that sets $T = 0$. Thus, $\mathbb{E}[Y^0] = P(Y = 1|do(T = 0))$. Appendix B shows that if we could observe data generated by the interventional graph $G_{do(T)}$ then we would be able to extimate the quantity $P(Y = 1|do(T))$. Indeed, given that $P_{do(T)}(Y = 1|T)$ is the probability of successfull outcome on the interventional graph given a treatment T, it follows that $P(Y = 1|do(T)) = P_{do(T)}(Y = 1|T)$. Therefore, $P(Y = 1|do(T = 1)) = P_{do(T=1)}(Y = 1)$ and $P(Y = 1|do(T = 0)) = P_{do(T=0)}(Y = 1)$.

At first, it may seem impossible to estimate the ATE as we do not have access to the post-intervention probabilities. However, if we could rewrite the post-intervention probabilities in terms of the observed data's distribution, the problem would be overcome. This is called the Identification stage in Causal Reasoning(Appendix B). Appendix A claims that no causal question can be answered directly from the observed data without some causal assumptions. Therefore, we need to construct a Graph Causal Model for the data in the kidney stone table, and the one proposed in the slide lectures seems reasonable for the data at hand. Since the stone size Z is a valid adjustment set as it satisfies the Backdoor Criterion(Appendix D), we can use the Adjustment formula(Appendix D) to identify $\mathbb{E}[Y^1]$ and $\mathbb{E}[Y^0]$. Appendix C provides two alternative derivations for $\mathbb{E}[Y^1]$ and $\mathbb{E}[Y^0]$. Given that $\mathbb{E}[Y^1]$ and $\mathbb{E}[Y^0]$ are identified, we can proceed to the Estimation stage(Appendix A):

$$\begin{aligned} \mathbb{E}[Y^1] &= \sum_{z=0,1} P(Y = 1|Z = z, T = 1)\ P(Z = z) \\ &= P(Y = 1|Z = 0, T = 1)\ P(Z = 0) + P(Y = 1|Z = 1, T = 1)\ P(Z = 1) \\ &= 0.93 \times \frac{87 + 270}{87 + 270 + 263 + 80} + 0.73 \times \frac{263 + 80}{87 + 270 + 263 + 80} \\ &= 0.93 \times \frac{357}{700} + 0.73 \times \frac{343}{700} \\ &= 0.4743 + 0.3577 \\ &= 0.832 \end{aligned}$$

$$\begin{aligned} \mathbb{E}[Y^0] &= \sum_{z=0,1} P(Y = 1|Z = z, T = 0)\ P(Z = z) \\ &= P(Y = 1|Z = 0, T = 0)\ P(Z = 0) + P(Y = 1|Z = 1, T = 0)\ P(Z = 1) \\ &= 0.87 \times \frac{357}{700} + 0.69 \times \frac{343}{700} \\ &= 0.4437 + 0.3381 \\ &= 0.7818 \end{aligned}$$

$$\begin{aligned} ATE &= \mathbb{E}[Y^1] - \mathbb{E}[Y^0] \\ &= 0.832 - 0.7818 \\ &= 0.0502 \end{aligned}$$

The result of the ATE solves the kidney stone decision problem. Since its value is positive, a new patient P should be treated with treatment $T = 1$ (open surgery) to maximise his chances of a successful recovery.

The exercise also requires us to compute the difference $\mathbb{E}[Y=1|T=1]-\mathbb{E}[Y=1|T=0]$:

$$\begin{aligned}
\mathbb{E}[Y=1|T=1] &= P(Y=1|T=1) \\
&= \sum_{z=0,1} P(Y=1, Z=z|T=1) \\
&= \sum_{z=0,1} P(Y=1|T=1, Z=z)\ P(Z=z|T=1) \\
&= P(Y=1|T=1, Z=0)\ P(Z=0|T=1) + P(Y=1|T=1, Z=1)\ P(Z=1|T=1) \\
&= 0.93 \times \frac{87}{263+87} + 0.73 \times \frac{263}{263+87} \\
&= 0.93 \times \frac{87}{350} + 0.73 \times \frac{263}{350} \\
&= 0.2311 + 0.5485 \\
&= 0.7796
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[Y=1|T=0] &= P(Y=1|T=0) \\
&= \sum_{z=0,1} P(Y=1, Z=z|T=0) \\
&= \sum_{z=0,1} P(Y=1|T=0, Z=z)\ P(Z=z|T=0) \\
&= P(Y=1|T=0, Z=0)\ P(Z=0|T=0)\ + P(Y=1|T=0, Z=1)\ P(Z=1|T=0) \\
&= 0.87 \times \frac{270}{80+270} + 0.69 \times \frac{80}{80+270} \\
&= 0.87 \times \frac{270}{350} + 0.69 \times \frac{80}{350} \\
&= 0.6711 + 0.1577 \\
&= 0.8288
\end{aligned}$$

$$\mathbb{E}[Y=1|T=1]-\mathbb{E}[Y=1|T=0] = 0.7796 - 0.8288 = -0.0492$$

Unlike the ATE, the above difference favours needle surgery to open surgery. However, such a result is erroneous. The whole point of this exercise is to find the causal effect of the treatment T on the recovery outcome Y. Therefore, we need to block any "backdoor" path between T and Y as they do not transmit causal influences. If the "backdoor" paths are not blocked, they will confound the effect of T on Y. While in the ATE, the "backdoor" path $T \leftarrow Z \rightarrow Y$ is inactive as $\mathbb{E}[Y^1]$ and $\mathbb{E}[Y^0]$ are identified by adjusting for the stone size Z, the same spurious path is open when computing $\mathbb{E}[Y=1|T=0]$ and $\mathbb{E}[Y=1|T=0]$. Thus, the stone size Z confounds the effect of T on Y, creating bias and yielding a wrong answer. The "backdoor" path $T \leftarrow Z \rightarrow Y$ introduces spurious answers because patients are assigned to treatment according to their kidney stone size. Indeed, the kidney stone table clearly shows that while roughly 75% of the patients assigned to open surgery have big stones, only 22% of the patients receiving needle surgery have the same stone size. Clearly, doctors are not performing a randomised controlled experiment, which would be unethical. Rather, they are giving the best treatment (open surgery) to the most severe clinical conditions.

Appendices

A A journey through the development of Causal Reasoning

Many scientific fields rely on Causal Reasoning to assess theories and answer substantive questions about the phenomena occurring in the world that we inhabit. However, despite its paramount importance in scientific enquiries, most of the Causal Reasoning's statistical tools have only been developed in recent years. The reason behind this late development is a lack of consensus on the definition of causality. Indeed, starting from the ancient Greek philosophers up to current philosophy's exponents, several definitions have been proposed, but none of them has been largely accepted.

Given that causality is such a nebulous concept, Causal Reasoning researchers have developed a flexible reasoning framework that works irrespective of any philosophical abstraction. Nonetheless, it is evident that much of this development has been inspired by the interventionist definition of causality. According to interventionists, an event B is caused by another event A, if changing A and keeping everything else constant leads to a difference in B's value. In the Causal Reasoning's jargon, A is known as the *treatment* while B is the *outcome*.

The interventionist definition of causality highlights two critical concepts in Causal Reasoning: *interventions* and *counterfactuals*. An intervention is an action that actively changes the value of a given treatment variable. From a probabilistic perspective, an intervention changes the distribution of a variable. The difference between intervening on a variable and conditioning on it may be confusing at this point. The key insight is that when we intervene in a variable, we change the causal mechanism behind a certain phenomenon; in contrast, conditioning on a variable merely changes our perception of the phenomenon.

However, as the interventionist definition of causality states, intervening on a treatment variable is not sufficient to evaluate its effect on the outcome. We need to isolate the effect of the intervention, which translates to keeping all other variables constant. Essentially, we need to ensure that actively changing a certain variable does not create side effects that directly alter other variables. Ideally, this can be achieved by observing two worlds that are completely identical up to one point where a certain treatment occurs in one world but not in the other. Logically, any following difference is a consequence of the treatment. Unfortunately, we still are not able to observe the multiverse or create real-life simulations. As a consequence, the first world is called the *factual world*, while the second world is known as the *counterfactual world* because it cannot be observed. By combining the concepts of interventions and counterfactuals, the following definition of causal effect can be derived: **the causal effect of an intervention is the difference between its observed outcome and its counterfactual outcome without the intervention**. Causal effect is also known as *Causal effect difference* or *Average causal effect* (ACE).

Given the above definition of Causal effect, we can drive our decision-making on whether to give a *treatment* or not. The fundamental challenge is that this calculation requires taking the difference between an observed outcome and a counterfactual that we cannot observe. This is an essential problem of Causal Reasoning known as the *missing data problem*.

Randomised controlled experiments are one possible solution to the missing data problem. They were developed by Ronald Fisher in the early twentieth century and are considered the golden standard in Causal Reasoning. The concept of randomised controlled experiments is straightforward: split a large population randomly into two groups and assign a treatment to one group but not to the other. Splitting a large population in a completely random manner ensures that, on average, the two subpopulations do not differ. Therefore, randomised controlled experiments effectively allow us to create two worlds, a factual world and a counterfactual world, from which we can estimate the causal effect of a given intervention. Despite their effectiveness, randomised controlled experiments cannot always be used because of monetary, ethical or feasibility issues. For instance, it is not even possible to split a large population in a completely random way in some scenarios.

Thus, how are we going to learn causal effects when we cannot perform randomised controlled experiments?. Equivalently, how are we going to untangle the causal effect from the mere correlations in observational studies?. This is where the recent statistical tools developed in Causal Reasoning come into place as they allow us to answer causal questions directly from the observed data. It is no surprise that this is a big leap forward because observing data is way more straightforward than controlling it.

Causal Reasoning allows us to learn causal questions from observational studies by means of a four-step process:

- **Causal assumptions**: the first step consists of making assumptions on the observational data's causal story. This is a critical step because causal effects cannot be learned directly from the data itself but rather from the modelling assumptions. The causal mechanism can be represented in the form of Structural Causal Models or Graph causal models.
- **Identification**: given the observed data and the causal model assumptions, the identification stage allows us to decide whether we have enough information to answer a specific causal inference question.
- **Estimation**: once we know how to identify the causal effect, several statistical tools can be employed to estimate its magnitude from the data.

- **Refute**: are we sure that the modelling assumptions are correct? How can we validate them? It turns out that we cannot answer those questions directly from the data. Like scientific theories, data only allows us to refute a set of assumptions, but it cannot prove the validity of a theory.

B Identification stage

The identification stage consists of analysing the causal model assumptions to answer a specific causal inference question. It is critical to understand that causal inference questions cannot be answered directly from the mere observational data, and the Simpson's paradox is a concrete example of this claim. Thus, given the observed data, we need to inspect the modelling assumptions to compute the magnitude of a given causal relationship.

Suppose that we need to answer a causal inference question, such as, how does the value of B changes given that we intervene on the treatment variable A? Statistics does not give us the tools to describe interventions formally as $P(B|A)$ merely denotes statistical correlation. On the other hand, Causal Reasoning introduces a notation that addresses the difference between correlations and causal relationships. The intervention on a treatment variable is denoted as $do(A)$; consequently, the causal relationship between A and B is the quantity $P(B|do(A))$.

When we intervene on a treatment variable A, we create a new causal model where all edges directed to A are removed. Essentially, we actively change the value of A, so its parents do not affect its value directly. In the previous section, we have defined the *Average Causal Effect*(ACE) of an intervention as the difference between its observed outcome and its counterfactual outcome without the intervention. By means of the do-operator, given an intervention on a binary treatment A, its Average Causal Effect on the outcome B is equivalent to $P(B|do(A = 1)) - P(B|do(A = 0))$.

By denoting the original causal model with G, the post-intervention graph, also known as interventional graph, is denoted as $G_{do(A)}$. If we could observe data generated by the interventional graph $G_{do(A)}$, we would be able to estimate the quantity $P(B|do(A))$. Indeed, given that $P_{do(A)}(B|A)$ is the probability of B given A in the post-intervention graph, then it holds that $P(B|do(A)) = P_{do(A)}(B|A)$.

However, how could we identify the quantity $P(B|do(A))$ if we cannot observe data generated by the interventional graph $G_{do(A)}$? We need to resort to a strategy that allows us to identify an expression for $P(B|do(A))$ in terms of the distribution $P_G(\cdot)$. In other words, given some observed data generated by a graph causal model G, we would like to transform an expression involving the do-operator into an equivalent expression that is only based on the observed probability distribution $P_G(\cdot)$.

Do-calculus is a set of rules that allows us to mechanically rewrite an expression involving the do-operator into an expression solely based on the observed probability distributions. A nice property of Do-calculus is that it has been proven to be complete. That is, if repeated applications of the Do-calculus rules cannot remove all references to the do-operator, then we cannot identify the causal target without any further assumption.

C Alternative derivations for $\mathbb{E}[Y^1]$ and $\mathbb{E}[Y^0]$

$\mathbb{E}[Y^1]$ derivation using the interventional graph probabilities:

$$\begin{aligned}
\mathbb{E}[Y^1] &= P_{do(T=1)}(Y = 1) \\
&= \sum_{z=0,1} P_{do(T=1)}(T = 1, Y = 1, Z = z) \\
&= \sum_{z=0,1} P_{do(T=1)}(Y = 1|T = 1, Z = z)\ P_{do(T=1)}(T = 1, Z = z) \\
&= \sum_{z=0,1} P_{do(T=1)}(Y = 1|T = 1, Z = z)\ P_{do(T=1)}(Z = z|T = 1) P_{do(T=1)}(T = 1) \\
&= \sum_{z=0,1} P_{do(T=1)}(Y = 1|T = 1, Z = z)\ P_{do(T=1)}(Z = z)\ P_{do(T=1)}(T = 1) \\
&= \sum_{z=0,1} P_{do(T=1)}(Y = 1|T = 1, Z = z)\ P_{do(T=1)}(Z = z) \\
&= \sum_{z=0,1} P(Y = 1|T = 1, Z = z)\ P(Z = z)
\end{aligned}$$

$\mathbb{E}[Y^1]$ derivation using DO-Calculus:

$$\begin{aligned}
\mathbb{E}[Y^1] &= P(Y=1|do(T=1)) \\
&= \sum_{z=0,1} P(Y=1, Z=z|do(T=1)) \\
&= \sum_{z=0,1} P(Y=1|Z=z, do(T=1))\ P(Z=z|do(T=1)) \\
&= \sum_{z=0,1} P(Y=1|Z=z, do(T=1))\ P(Z=z) \quad \text{ as } (Z \perp\!\!\!\perp A)_{G_{DO(T)}} \\
&= \sum_{z=0,1} P(Y=1|Z=z, T=1)\ P(Z=z) \quad \text{ as } (Y \perp\!\!\!\perp T|Z)_{G_{NULL(T)}}
\end{aligned}$$

$\mathbb{E}[Y^0]$ derivation using the interventional graph probabilities:

$$\begin{aligned}
\mathbb{E}[Y^0] &= P_{m0}(Y=1) \\
&= \sum_{z=0,1} P_{do(T=0)}(T=0, Y=1, Z=z) \\
&= \sum_{z=0,1} P_{do(T=0)}(Y=1|T=0, Z=z)\ P_{m0}(T=0, Z=z) \\
&= \sum_{z=0,1} P_{do(T=0)}(Y=1|T=0, Z=z)\ P_{do(T=0)}(Z=z|T=0) P_{do(T=0)}(T=0) \\
&= \sum_{z=0,1} P_{do(T=0)}(Y=1|T=0, Z=z)\ P_{do(T=0)}(Z=z)\ P_{do(T=0)}(T=0) \\
&= \sum_{z=0,1} P_{do(T=0)}(Y=1|T=0, Z=z)\ P_{do(T=0)}(Z=z) \\
&= \sum_{z=0,1} P(Y=1|T=0, Z=z)\ P(Z=z)
\end{aligned}$$

$\mathbb{E}[Y^0]$ derivation using DO-Calculus:

$$\begin{aligned}
\mathbb{E}[Y^0] &= P(Y=1|do(T=0)) \\
&= \sum_{z=0,1} P(Y=1, Z=z|do(T=0)) \\
&= \sum_{z=0,1} P(Y=1|Z=z, do(T=0))\ P(Z=z|do(T=0)) \\
&= \sum_{z=0,1} P(Y=1|Z=z, do(T=0))\ P(Z=z) \quad \text{ as } (Z \perp\!\!\!\perp A)_{G_{DO(T)}} \\
&= \sum_{z=0,1} P(Y=1|Z=z, T=0)\ P(Z=z) \quad \text{ as } (Y \perp\!\!\!\perp T|Z)_{G_{NULL(T)}}
\end{aligned}$$

D Adjustment formula, Backdoor and Towards Necessity Criterion

Consider a simple causal target $P(B|do(A))$ in some graph G in which a set of variables Z is a valid adjustment set, according to the Adjustment formula, the causal effect of A on B is given by

$$P(B=b|do(A=a)) = \sum_{z} P(A=a|B=b, Z=z)\ P(Z=z)$$

where z ranges over all combinations of the values that the set of variables Z can take.

Effectively, the Adjustment Formula allows us to transform an expression involving the do-operator into an expression based on the observed probability distributions.

The Backdoor Criterion allows us to identify a set of variables in a graph G that satisfy the adjustment properties.
According to the Backdoor Criterion, given two variables A and B in a DAG G, a set of variables Z is a valid adjustment set if:

- Z blocks all paths between A and B where the edge connected to A is directed at A.
- No descendants of A are in Z.

While the Backdoor criterion helps us identify valid adjustment sets, it is not complete. Indeed, there are valid adjustment sets that do not meet the backdoor criterion. The "Towards necessity" criterion captures all valid adjustment sets.

In what follows, we will derive the Adjustment formula using Do-calculus, and we will see why the Backdoor criterion identifies valid adjustment sets.

$$\begin{aligned}
P(B|do(A)) &= \sum_z P(B, Z = z|do(A)) \\
&= \sum_z P(B|Z = z, do(A))\ P(Z = z|do(A)) \\
&= \sum_z P(B|Z = z, do(A))\ P(Z = z) \quad \text{ if } (Z \perp\!\!\!\perp A)_{G_{DO(A)}} \\
&= \sum_z P(B|Z = z, A)\ P(Z = z) \quad \text{ if } (B \perp\!\!\!\perp A|Z)_{G_{NULL(A)}}
\end{aligned}$$

The second step is based on the law of total probability. The third step applies Rule 3 of Do-calculus and holds if Z and A are d-separated in a graph without incoming edges to A. The last step applies Rule 2 of Do-calculus. Such a rule can be applied as long as Z d-separates A and B in the graph where all outgoing edges of A are removed. Any such set Z is a valid adjustment set. It is trivial to prove that if a set Z satisfies the Backdoor criterion, it also satisfies the adjustment properties.