Intelligent Systems: Mathematics for AI
Extrema, Interpolation and Roots - Part II

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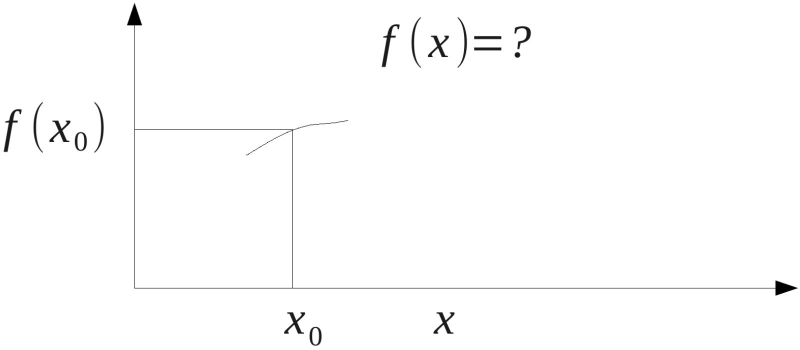
Outline

- Interpolation, Taylor series
- Newton's method

Linear Interpolation (1)

- Sometimes we need the value of a function $f(x)$ at some point x, but only know it exactly at a near by point x_0.
- How can we estimate $f(x)$ from $f(x_0)$.



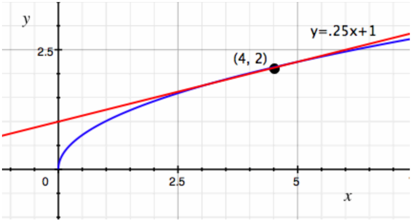
Linear Interpolation (2)

- Sometimes we need the value of a function $f(x)$ at some point x, but only know it exactly at a near by point x_0.
- How can we estimate $f(x)$ from $f(x_0)$.
- Use linear interpolation.
- Example: what we know is $x_0 = 4, f(x_0) = 2$. Can we use it to approximate $f(4.2)$, i.e. $\sqrt{4.2}$?

Linear Interpolation (3)

- Example: what we know is $x_0 = 4, f(x_0) = 2$. Can we use it to approximate $f(x = 4.2)$, i.e. $\sqrt{4.2}$?
- Tangent line at x_0 is $y = f(x_0) + f'(x_0)(x - x_0)$ (red line in plot).

- $f(x) = \sqrt{x}, f'(x) = 1/(2\sqrt{x})$
- $\sqrt{4.2} \approx \sqrt{4} + 1/4(4.2 - 4)$
- $\sqrt{4.2} \approx 2.05$, $(\sqrt{4.2} = 2.0493...)$



More generally: Taylor series

- In general, there are "higher order" corrections.
- If the function f is infinitely differentiable, we can develop it into a Taylor series around a point x_0.

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots$$

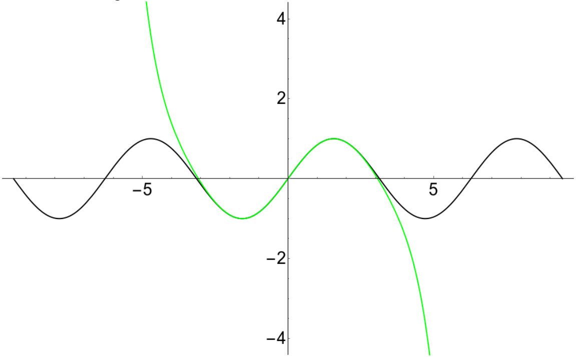
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

- Need x to be sufficiently close to x_0.
- Be wary, series does not always converge!

Example using Taylor series for interpolation

- Approximation (green line) of the sine function (black line) around $x_0 = 0$.

$$f(x) = \sin x$$
$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(n)(x) = -\cos x$$



- Using Taylor series with $x_0 = 0$, we get,
- $f(x) \approx \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

Newton-Raphson method (1)

Our problem:

- We want to find solutions of some (non-)linear equation $g(x^*) = A$.
  - Also for optimisation problems, if we want to find $g'(x*) = 0$, this is the same class of problem.

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- We want to find solutions of some (non-)linear equation $g(x^*) = A$.
  - Also for optimisation problems, if we want to find $g'(x*) = 0$, this is the same class of problem.
- Can easily do it analytically for linear g and quadratic g, but this is not so easy in the general case!

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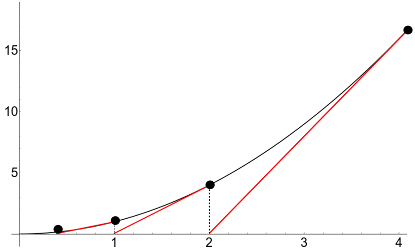
- We want to find solutions of some (non-)linear equation $g(x^*) = A$.
- $1st$ step: If we can solve $f(x*) = 0$ for any f we can also solve $g(x*) = A$.
- $2nd$ step: want to find a numerical approximation of the solution. Let's assume f is differentiable.

Newton-Raphson method (3)

Strategy to find root of f(x), i.e. solve x, for f(x)=0:

1. Start at some x_0.

$$f(x) = x^2, x_0 = 4$$



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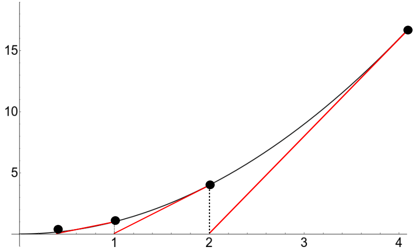

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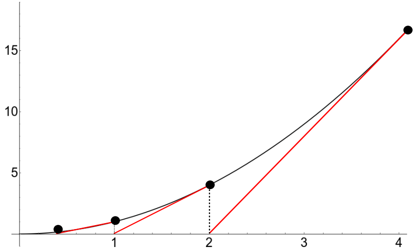

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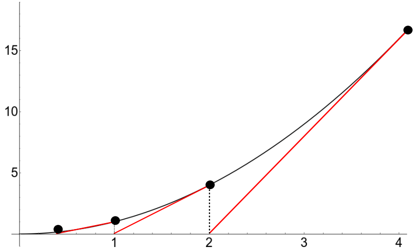

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5. Goto Step-2.

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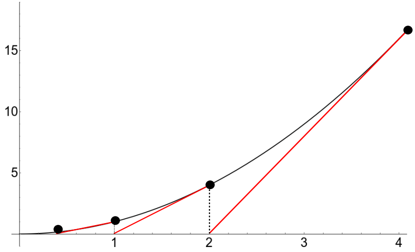

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- Root x^* of tangent (i.e. where tangent intercepts x-axis) is: $0 = f(x_n) + f'(x_n)(x^* - x_n)$.
- So $x^* = x_n - f(x_n)/f('x_n)$.
- Use x^* as next value of x_n, i.e.,
  $\boxed{x_{n+1} = x_n - f(x_n)/f'(x_n)}$

Newton-Raphson method - Example (1)

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- Remember: $x_{n+1} = x_n - f(x_n)/f'(x_n)$
- However: $f'(x) = 2x \rightarrow x_{n+1} = x_n - x_n^2/(2x_n) \rightarrow x_{n+1} = \frac{1}{2x_n}$

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- However: $f'(x) = 2x \rightarrow x_{n+1} = x_n - x_n^2/(2x_n) \rightarrow x_{n+1} = \frac{1}{2x_n}$
- $x_0 = 2 \rightarrow x_1 = 1 \rightarrow x_2 = 1/2 \rightarrow x_3 = 1/4 \rightarrow x_n = 2^{-n+1} \rightarrow 0$ as $n \rightarrow \infty$

Newton-Raphson method - Example (2)

- Apply Newton-Raphson method to find the x-value at the point of intersection of $2\cos x$ and $3x$ starting from $x_0 = \pi/6$.

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- Apply Newton-Raphson method to find the x-value at the point of intersection of $2\cos x$ and $3x$ starting from $x_0 = \pi/6$.
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Newton-Raphson method - Example (3)

- Consider the function $f(x) = x^2 - 2x - 1$.
  Want to find its root starting from $x_0 = 2$.

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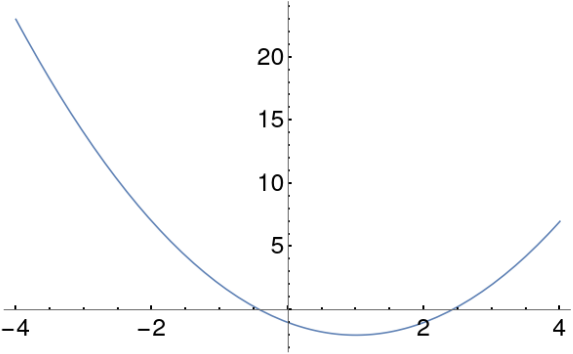
Newton-Raphson method - Example (3)

- Consider the function $f(x) = x^2 - 2x - 1$. Want to find its root starting from $x_0 = 2$.
- Remember: $x_{n+1} = x_n - f(x_n)/f'(x_n)$
- Use another starting point say $x_0 = -2$ to find the second root.

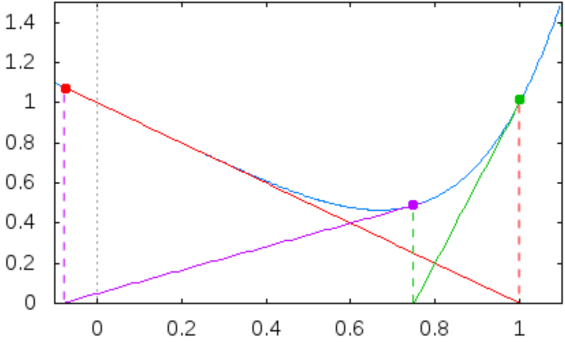


Newton-Raphson method - Example (4)

Please note – method does not always converge!

- Can get stuck around local extrema, the tangent at which does not cross the x-axis.

$f(x) = (x^5 - x + 1)$, $x_0 = 0$



- Problematic points are extrema – $f'(x) = 0$. Then choose another starting point.

Summary

What is essential to remember:

- Interpolation:
  - Linear interpolation
  - Taylor series
  - Newton-Raphson method for solving difficult equations

Next session

- Next math session we take a look at scalars & vectors and linear algebra.