# Intelligent Systems: Mathematics for AI
# Extending differentiation to multivariate calculus - Part I

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# Announcement

- Fourth COMP2208 math coursework sheet is now available at `https://secure.ecs.soton.ac.uk/notes/comp2208/problems/prob4_2019.pdf`.
- It is due in four weeks on Mon Dec. 16 by 16:00.
- Submission instructions remain the same.

# Outline

- Moving from $y = f(x)$ to $y = f(\vec{x})$ and even $\vec{y} = f(\vec{x})$.
- In Part I of multi-variate calculus:
  - **Partial derivatives**
- Later:
  - Total derivatives
  - Rules for differentiation
  - Directional derivatives and gradients
  - Gradient descent

# Motivation for partial differential equations (1)

- Imagine a surface – such as a metal sheet which is heated at one end.
- How does the heat flow across the surface?
- How can we model this to make sure the temperature at a specific point on the sheet does not cross a safety threshold (e.g., for pressure tests).

# Motivation for partial differential equations (2)

- Equation for heat conduction through a solid:

$$\frac{\partial u}{\partial t} = \alpha(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})$$

- where $\alpha$ is the diffusivity of the medium, and $u$ is the temperature (see heat diffusion animation).

# Partial derivatives (1)

- Suppose $f(x, y)$ is a function of more than one variable.
- Example: $z = f(x, y) = x^2 + xy + y^2$
- We want to measure "how much the function changes when we change variables a bit"

# Partial derivatives (2)

- Informally, "partial derivative with respect to a variable = derivative with regard to this variable while treating all other variables as constants".

# Partial derivatives (3)

- See animation of $\frac{\partial Z}{\partial x}$ and $\frac{\partial Z}{\partial y}$ for 2D function $Z = f(x, y)$.

# Partial derivatives (4)

- This results in a formal definition:

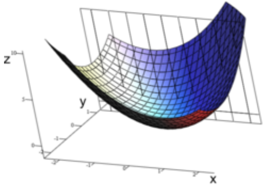
$$\frac{\partial f(x_1, \dots, x_n)}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n)}{h} - \lim_{h \to 0} \frac{f(x_1, \dots, x_i, \dots, x_n)}{h}$$

- What we do is, calculate derivative with respect to $x_i$ while treating the other variables $x_j : j \in \{1 \dots n\} \land j \neq i$ as constants.

# Partial derivatives – Example (1)

- Given $f(x, y) = x^2 + xy + y^2$, find $\frac{\partial f(x,y)}{\partial x}$?

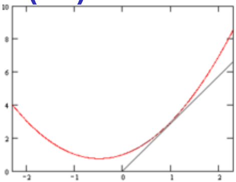
# Partial derivatives – Example (1) solution




- $f(x, y) = x^2 + xy + y^2$
- Formally:

$$\frac{\partial f(x, y)}{\partial x} = 2x + y$$

# Partial derivatives – Example (1) solution



- $f(x, y) = x^2 + xy + y^2$
- Geometrically: Go into the subspace(=plane) in which $y = a = const.$ and calculate normal derivative.
- $f(x) = x^2 + ax + a^2, \frac{df(x)}{dx} = 2x + a$

# Partial derivatives – Example (2)

$$f(x, y, z) = x^2 + xy + y^2 + z\sin(x) + 5$$

- $\frac{\partial f(x,y,z)}{\partial x} = ?$
- $\frac{\partial f(x,y,z)}{\partial y} = ?$
- $\frac{\partial f(x,y,z)}{\partial z} = ?$

# Partial derivatives – Example (2) - solution

$$f(x, y, z) = x^2 + xy + y^2 + z\sin(x) + 5$$

- $\frac{\partial f(x,y,z)}{\partial x} = 2x + y + zcos(x)$
- $\frac{\partial f(x,y,z)}{\partial y} = x + 2y$
- $\frac{\partial f(x,y,z)}{\partial z} = \sin(x)$

# Partial derivatives – Example (3)

$f(r, \phi) = e^{-r^2} \sin \phi$

- $\frac{\partial f(r,\phi)}{\partial r} = ?$
- $\frac{\partial f(r,\phi)}{\partial \phi} = ?$

# Partial derivatives – Example (3) - solution

$f(r, \phi) = e^{-r^2} \sin \phi$

- See scanned notes.

# Partial derivatives – Example (4)

$$f(x, y) = \cos\left(\frac{4}{x}\right) e^{x^2 y - 5y^3}$$

- $\frac{\partial f(x,y)}{\partial x} = ?$
- $\frac{\partial f(x,y)}{\partial y} = ?$

# Partial derivatives – Example (4) - solution

$$f(x, y) = \cos\left(\frac{4}{x}\right) e^{x^2 y - 5y^3}$$

- See scanned notes.

# Partial derivatives – Example (5)

$$f(x,y,z) = x^2y - 10y^2z^3 + 43x - 7\tan(4y)$$

- $\frac{\partial f(x,y,z)}{\partial x} = ?$
- $\frac{\partial f(x,y,z)}{\partial y} = ?$
- $\frac{\partial f(x,y,z)}{\partial z} = ?$

# Partial derivatives – Example (5) - solution

$f(x, y, z) = x^2y - 10y^2z^3 + 43x - 7\tan(4y)$

- See scanned notes.

# Partial derivatives – Example (6)

$$f(s,t) = t^7 \ln\left(s^2\right) + \frac{9}{t^3} - \sqrt[7]{s^4}$$

- $\frac{\partial f(s,t)}{\partial s} = ?$
- $\frac{\partial f(s,t)}{\partial t} = ?$

# Partial derivatives – Example (6) - solution

$f(s,t) = t^7 \ln\left(s^2\right) + \frac{9}{t^3} - \sqrt[7]{s^4}$

- See scanned notes.

# Partial derivatives – Example (7)

Given $x^3z^2 - 5xy^5z = x^2 + y^3$, use implicit differentiation to find,

- $\frac{\partial z}{\partial x} = ?$
- $\frac{\partial z}{\partial y} = ?$

# Partial derivatives – Example (7) - solution

Given $x^3z^2 - 5xy^5z = x^2 + y^3$, use implicit differentiation to find,

- $\frac{\partial z}{\partial x} =?$
- $\frac{\partial z}{\partial y} =?$

- See scanned notes.

# Next session

- Differentiability in multivariate calculus.