Intelligent Systems: Mathematics for AI
Vector calculus and linear algebra - Part II

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Outline

- More linear algebra and vector calculus
  - Cross product of vector
  - Linear maps and matrices
  - Scalar and vector fields

Vector cross product (1)

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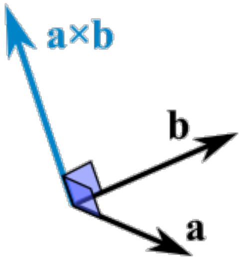
- It is a binary operation on two vectors in $\mathbb{R}^3$ space.
- The vector $\vec{a}x\vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$, thus normal to the plane containing them.



Vector cross product (2)

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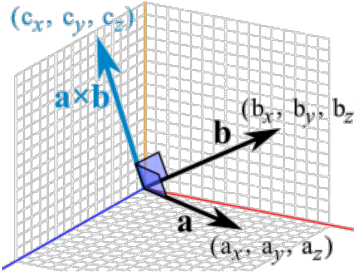
- Can also be calculated from determinant form:

$$c_x = a_y b_z - a_z b_y$$
$$c_y = a_z b_x - a_x b_z$$
$$c_z = a_x b_y - a_y b_x$$



Vector cross product - example

- Give two vectors $\vec{a} = (2, 3, 4)$ and $\vec{b} = (5, 6, 7)$, find the $\vec{c}$ normal to these two vectors?

Linear maps (1)

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  - they are easy to deal with and understand, and,
  - they can be used to formalise the concept of derivative (via tangents).
- We need something similar to operate in higher dimensions.
- Which brings us to linear maps

Linear maps (2)

- A linear map, or linear transformation, is a way to transform vectors from one space to another.
- A map $T : V \to W$ is linear, iff $\forall c \in \mathbb{R}$ and $\forall \vec{x}, \vec{y} \in \mathbb{V}$,

$$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}) \text{ and}$$

$$T(c\vec{x}) = cT(\vec{x})$$

Matrix representation of linear maps

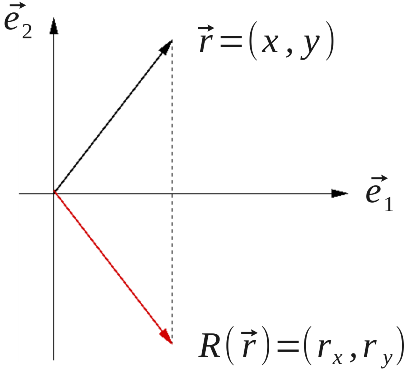
- Vector $\vec{y} = T(\vec{x})$ can be written as,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

- Exercise: Try and derive the above. Will look at it in interactive session.

Linear maps - example

- Let's consider the coordinate space $\mathbb{R}^2$ and define a map R as reflection on the x-axis. What is the matrix representation of this map?

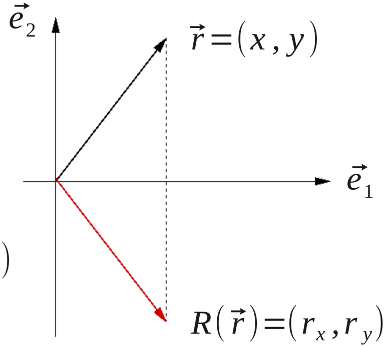


Linear maps - example solution

- Let's consider the coordinate space $\mathbb{R}^2$ and define a map R as reflection on the x-axis. Is this map linear? If so, what is its matrix representation?

$$R: r_x = x$$
$$\quad r_y = -y$$

$$(r_x, r_y) = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} (x, y)$$
$$= (t_{11} x + t_{12} y, t_{21} x + t_{22} y)$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} (x, y)$$



Scalar fields

- Scalar fields and vector fields are useful ways to represent data.
- In scalar fields, we assign a scalar to each point in space, $f : \mathbb{R}^n \to \mathbb{R}$.
- The space may possibly be a physical space, i.e. $n = 2$ or $n = 3$.
- The scalar may represent a dimensionless mathematical number, or a physical quantity, e.g., temperature (C).

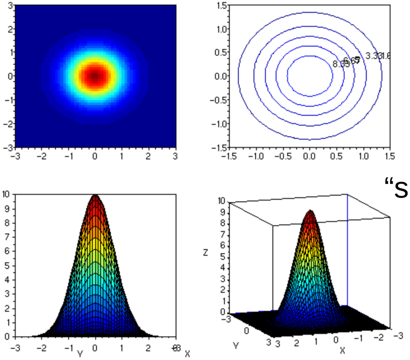
Example scalar fields (1)

- Scalar fields over $\mathbb{R}^2$ space.

$$f(x,y)=e^{-x^2-y^2}$$

“heat map”

“contour lines”



“yz projection”

“surface plot”

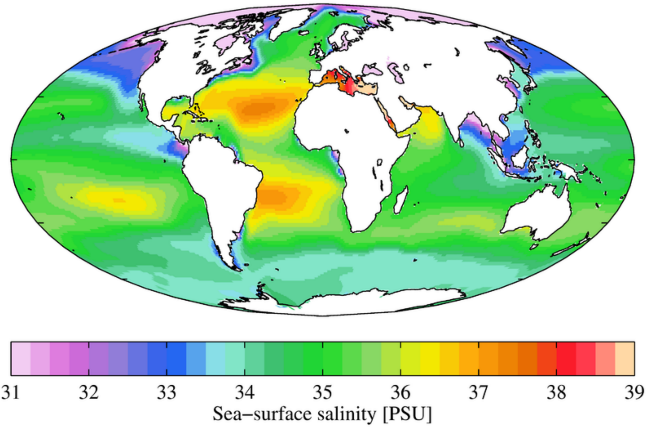
Example scalar fields (2)

- Error function of the output of a neural network; $f(w_1, w_2 \dots w_n)$, where w_i are the weights of the network.

Example scalar fields (3)

- Scalar field of salinity in the ocean.

salinity



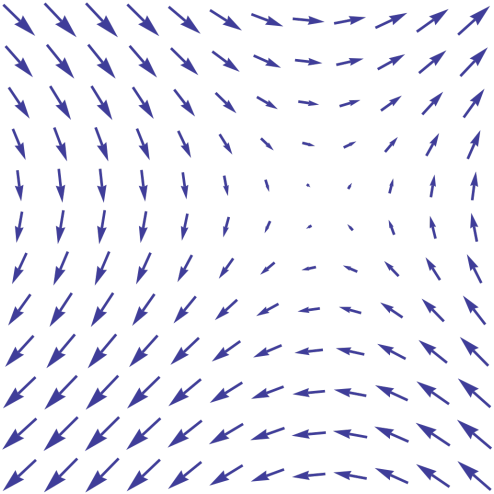
Vector fields

Vector fields

- Assign a vector to each point in space.
- "A vector field in a plane (for instance), can be visualised as a collection of arrows with a given magnitude and direction, each attached to a point in the plane."
- Very useful to understand dynamic systems (e.g., to visualise stationary points in high order differential equations, attractors etc.).

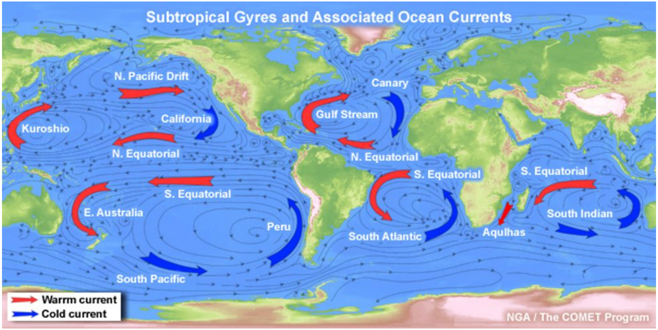
Vector fields - example (1)

- Vector field of $(\sin y, \sin x)$



Vector fields - example (2)

- Ocean currents for small-scale AUV robots:



Summary

- What you should know:
    - Cross product of vectors, and its use
    - Linear maps and matrix representation
    - Scalar and vector fields

Next session

- Next math sessions we take a look at partial differential equations, total differentiation and gradient descent.