Dipartimento di Matematica Federigo Enriques

Tesi di Laurea Magistrale in Matematica

Topos semantics for three-valued Gödel-Dummett Logic

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Abstract

This work builds upon Professor S.Aguzzoli & P.Codara's paper [1].
Its primary aim is to explore the topos semantics of a particular fragment of super-intuitionistic and fuzzy logic namely *three-valued Gödel-Dummett Logic* $\mathcal{G}_3$. The dual-algebraic semantics of $\mathcal{G}_3$ is given by the sub-category of *finite forests* of height at most two $\mathbb{FF}_2$, a.k.a. *bushes*. Building upon the topos-semantics of *bushes* at the propositional layer, we arrive through a new path at first-order topos semantics for $\mathcal{G}_3$.
In chapter 1 we present this logic after a more general introduction of intuitionistic logic and its principal semantics given by Heyting algebras and Kripke frames, which will come up later on.
In chapter 2 we give a detailed account of *finite forests* and in particular the sub-category of *bushes*, which forms a topos and is dually equivalent to finite three-valued Gödel algebras. Continuing where [1] left off, we provide a novel counterexample to show why $\mathbb{FF}_k$, i.e., *finite forests* of height at most k with k greater than two, fails to be a topos. An original Python code is also used to aid in these constructions.
In chapter 3, following the development outlined in [9] and giving whenever necessary some generalities of topos semantics, we examine the propositional layer of the topos semantics given by *bushes* and propose some results regarding its internal and external logics.
In chapter 4 we follow up from chapter 3 and move on to first order logic. We start by giving an account by [9] of first order topos models and make some considerations on predicates and modeling three-valued Gödel-Dummett logic before taking an alternative type-theoretic approach to quantifiers due to [12] and applying it to our topos of bushes. Conclusive remarks are made on this application and the recovery of first order Gödel-Dummett logic.
In chapter 5, we compare our findings with alternative topos semantics for $\mathcal{G}_3$ given by *variable sets* and *sheaves* and make some new remarks on the

matter.
The conclusion is a short summary of what has been achieved throughout this work, its justification and a few comments on further areas of research.

(Highlighted sections or sub-sections contain the original results made by the author.)

Dedication

To Mum and Dad.
Without their love and support none of this would be possible.

Declaration

I hereby declare that this thesis represents my own work which has been done after registration for the degree of *Laurea Magistrale in Matematica* at *Università degli Studi di Milano*, and has not been previously included in a thesis or dissertation submitted to this or any other institution for a degree, diploma or other qualifications.

Acknowledgements

I want to thank again my family and anyone who helped me in my mathematical journey.
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Contents

Chapter 1

Introduction

1.1 Foreword

This thesis touches on the subjects of category theory and *non-classical* mathematical logic. It aims to study the *Topoi* semantics of a particular *non classical* logic named Gödel-Dummett Logic.
To introduce what Gödel-Dummett Logic is we first give an account (guided by [9] & [10]) of what *Intuitionistic Propositional Logic* and more broadly *Intuitionism* is.

(An introduction to the basic notions of category theory and mathematical logic is not given in this work. For this we refer to [10],[17],[9] among others)

1.2 A Non-Classical World

Intuitionistic or *constructive* logic is commonly described as *classical* logic without Aristotle's *law of excluded middle*, a.k.a. *tertium non datur*:

Def. 1.2.1. (LEM) : $A \vee \neg A$.

Or equivalently without the law of double negation elimination [1]:

Def. 1.2.2. (DNE) : $\neg\neg A \Rightarrow A$.

L.E.J. Brouwer in the first half of the twentieth century observed that (LEM) was abstracted from finite to infinite situations without much justification.
Take the following statement about the existence of infinitely many twin primes a.k.a. the *twin prime conjecture* $\mathcal{P}$ in number theory:

$$\forall x \in \mathbb{N}\ (A(x) \vee \neg A(x))$$
$$A(x) := \exists y \in \mathbb{N}((y > x) \wedge (prime(y) \wedge prime(y+2)))$$

The problem with this assertion is that the *twin prime conjecture* $\mathcal{P}$ like many other unsolved problems in Mathematics has not yet been proven or dis-proven so there is no proof either of $\mathcal{P}$ or $\neg\mathcal{P}$ and hence at this present *state of knowledge* there can be no *constructive* or *effective* proof of $\mathcal{P} \vee \neg\mathcal{P}$. For Brouwer and constructivists alike the claim of $\mathcal{P} \vee \neg\mathcal{P}$ is simply not acceptable. The crucial difference with the classical case is that formulae are only considered *true* when we have proof or direct *evidence of truth*.
This *reductive* notion of truth has not been without controversy. Doing away with (LEM) or (DNE) is problematic as they are so commonly used in mathematical practice.
For example consider a *non-constructive* proof of the statement: "there exist irrational numbers a, b such that a^b is rational".

Proposition 1. $\exists\ a, b \in \mathbb{R} \setminus \mathbb{Q} : a^b \in \mathbb{Q}$.

Proof. $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational.
In the first case we take $a = b = \sqrt{2}$.
In the second case we take $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$ and we are done. $\square$

[1] used for *reductio ad absurdum* proofs.

What's wrong?

Note that in the above proof the status of $\sqrt{2}^{\sqrt{2}}$ is *undefined* and hence no *constructive* proof is given for its rationality or lack thereof.

Definitive evidence of the irrationality of $\sqrt{2}^{\sqrt{2}}$ has been given by the Gelfond-Schneider Theorem which establishes *constructively* that for any complex algebraic numbers $a, b \neq 1$ with b irrational, a^b is in fact a transcendental and hence irrational number. So the original statement has a *constructive* proof.

The *BHK-interpretation* named after Brouwer,Heyting and Kolmogorov provides a framework for *intuitionistic logic*.

In classical logic the meaning of statements is given by stating for example the following *truth conditions*:

- $\phi \wedge \chi$ is true iff ϕ is true and χ is true.
- $\phi \vee \chi$ is true iff ϕ is true or χ is true.
- $\neg\phi$ is true iff ϕ is not true.

In the BHK-interpretation the notion of *proof*[2] replaces that of classical *truth*. The usual logical formulae are now interpreted as:

- No proof of *false* $\perp$ exists.
- a proof of $\phi \ \wedge \ \chi$ consists of a pair of proofs, i.e., "$<$ proof of ϕ ; proof of $\chi >$".
- a proof of $\phi \vee \chi$ is either a proof of one or the other, i.e., "proof of ϕ | a proof of χ".
- a proof of $\phi \Rightarrow \chi$ consists of a *method of converting* a proof of ϕ into a proof of χ
- a proof of $\neg\phi$ consists of a *method of converting* a proof of ϕ into a proof of *false* (in other words "ϕ has no proof").

Moving on to first order logic and quantifiers if we assume the variables to range over a domain D:

[2]not to be construed as syntactic or formal proof but as intuitive/informal proof or convincing mathematical argument.

- a proof of $\forall x A(x)$ is a *construction* that transforms any $d \in D$ into a proof of $A(d)$.
- a proof of $\exists x A(x)$ is a pair $< d$; proof of $A(d) >$.

How do we now formalize Intuitionistic Logic starting from the propositional layer?

1.3 From IPL to CPL

We begin with a presentation of the basic *syntax* of *Intuitionistic Propositional Logic* or **IPL**.
We do this by specifying a *Formal Language* presented as an *alphabet* composed of a denumerable set of propositional variables **Prop** $:= \{.., p, q, r, ..\}$ and the connectives symbols $\neg, \wedge, \vee, \Rightarrow$. [3]
We define *inductively* Formulae **Form** $= \{\alpha, \beta, ..\phi, \chi, \psi, ..\}$ in the following way:

Def. 1.3.1 (propositional formula)**.** **Form** $:= p \mid \bot \mid F \wedge F \mid F \vee F \mid F \Rightarrow F$.

i $\bot \in$ **Form**.

ii If $p \in$ **Prop** then $p \in$ **Form**.

iii If $\alpha, \beta \in$ **Form** then $(\alpha \wedge \beta) \in$ **Form**.

iv If $\alpha, \beta \in$ **Form** then $(\alpha \vee \beta) \in$ **Form**.

v If $\alpha, \beta \in$ **Form** then $(\alpha \Rightarrow \beta) \in$ **Form**.

The constant $\bot$ represents falsity, $\wedge$ conjunction, $\vee$ disjunction and $\Rightarrow$ implication. We also define the following operators for truth, negation and double implication respectively:

Def. 1.3.2. $\top := \bot \Rightarrow \bot$, $\neg\alpha := \alpha \Rightarrow \bot$, $\alpha \Leftrightarrow \beta := (\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.

We characterize ***IPL*** by its axioms.
The axioms for **IPL** are instances of the following schemata:
(Here { A,B,C,..} are meta-variables/placeholders for any propositional formula).

1. $A \Rightarrow (A \wedge A)$.
2. $(A \wedge B) \Rightarrow (B \wedge A)$.
3. $(A \Rightarrow B) \Rightarrow ((A \wedge C) \Rightarrow (B \wedge C))$.
4. $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$.

[3] we could also add symbols for brackets (,)

5. $B \Rightarrow (A \Rightarrow B)$.
6. $(A \wedge (A \Rightarrow B)) \Rightarrow B$.
7. $A \Rightarrow (A \vee B)$.
8. $(A \vee B) \Rightarrow (B \vee A)$.
9. $((A \Rightarrow C) \wedge (B \Rightarrow C)) \Rightarrow ((A \vee B) \Rightarrow C)$.
10. $\neg A \Rightarrow (A \Rightarrow B)$.
11. $((A \Rightarrow B) \wedge (A \Rightarrow \neg B)) \Rightarrow \neg A$.

Having specified the axioms, now we introduce a *Propositional Calculus*[4] for **IPL**. We do this by defining a relation of *entailment* or *(syntactic) derivation* between formulae $\vdash$.
All we need to specify this *derivation* is a single rule of inference that is *the rule of detachment* a.k.a. *Modus Ponens* [MP] .

Def. 1.3.3 (MP)**.** For every $\alpha, \beta \in$ **Form**:
If $\vdash \alpha$ and $\vdash (\alpha \Rightarrow \beta)$, then $\vdash \beta$.
which means: *"From α and $\alpha \Rightarrow \beta$ one can derive β".*

$\mathbf{TH_{IPL}}$, i.e., the *Theorems* for **IPL** or *provable* formulae in **IPL** denoted simply by $\vdash \phi$ [5] are defined as all the formulae that one can derive or *deduce* from $\top$ "truth" or from any instance of the axiom schemata of **IPL** in a *finite* number of steps using [MP].
The relation $\alpha \vdash \beta$ in this context is characterized as $\vdash (\alpha \Rightarrow \beta)$.

The *algebraic Semantics* for **IPL** can be obtained through the construction of a *Lindenbaum-Tarski algebra* $\mathcal{A}$.
$\mathcal{A}$ is formed by taking the equivalence classes of formulae ϕ by the *double-entailment* relation ($\phi \dashv\vdash \chi$ means "$\phi \vdash \chi$ and $\chi \vdash \phi$"):

$$\phi \equiv \chi \text{ iff } \phi \dashv\vdash \chi$$

Furthermore $\mathcal{A}$ admits a well defined ordering given by entailment:

[4] due to *A. Heyting.*
[5] one could also use the notation $\vdash_{\mathbf{IPL}}$. However we will refrain from doing so if the context is unambiguous.

Def. 1.3.4. $[\phi] \leq [\chi]$ iff $\phi \vdash \chi$.

We then define the following operations in $\mathcal{A}$ induced by the logical connectives:

- $1 := [\top]$.
- $0 := [\bot]$.
- $[\phi] \wedge [\chi] := [\phi \wedge \chi]$.
- $[\phi] \vee [\chi] := [\phi \vee \chi]$.
- $[\phi] \Rightarrow [\chi] := [\phi \Rightarrow \chi]$.

Def. 1.3.5. $\mathcal{A} := (\mathbf{Form}/\equiv, \leq)$.

Since for every $\phi \in \mathbf{Form}$ $(\bot \vdash \phi)$ and $(\phi \vdash \top)$ it follows that $\mathcal{A}$ is a bounded *lattice* with minimum $0 = [\bot]$ and maximum $1 = [\top]$.

Proposition 2. *A formula ϕ is* provable $\top \vdash \phi$ *iff* $[\phi] = 1$.

$\mathcal{A}$ with these operations is a *Heyting algebra*.
The class of Heyting algebras will be denoted by **HA**.
What exactly is a *Heyting algebra*?
A *Heyting algebra* $\mathcal{H}$ is first of all a *partially ordered set*, a.k.a. *poset* [6] $(\mathcal{H}, \leq)$ endowed with constants 0,1, binary operations *join* $\vee$ and *meet* $\wedge$ that form a *bounded lattice*. For this the following must hold for all $a, b, c \in \mathcal{H}$:

1. $0 \leq a$.
2. $a \leq 1$.
3. $a \vee b \leq c$ iff $a \leq c$ and $b \leq c$.
4. $c \leq a \wedge b$ iff $c \leq a$ and $c \leq b$.

[6] recall that this means that the ordering is *reflexive,antisymmetric and transitive..* not necessarily *total..*

Def. 1.3.6 (Heyting algebra)**.** The bounded lattice $(\mathcal{H}, \vee, \wedge, 0, 1)$ is said to be a *Heyting algebra* if it admits a binary operator $\Rightarrow$ for which $(a \Rightarrow b) \in \mathcal{H}$, a.k.a. an *exponential element*[7] such that:
For every $a, b, c \in \mathcal{H}$
there exists $(a \Rightarrow b) \in \mathcal{H}$ such that:

$$c \leq (a \Rightarrow b) \text{ iff } (c \wedge a) \leq b.$$

Remark 1. The previous condition on exponential elements corresponds to the categorical notion of an *adjoint pair* between the functors $(- \wedge a)$ and $(a \Rightarrow -)$ which gives rise to *cartesian closed categories*.[8]

This exponential element can also be taken as the *greatest* element with this property, i.e.:

Proposition 3. $a \Rightarrow b \equiv sup\{c \in \mathcal{H} : c \wedge a \leq b\}$.

For an analogue of classical negation, we can define for every $b \in \mathcal{H}$ a *pseudo-complement* of b:

Def. 1.3.7 (pseudo-complement)**.** $\neg b := (b \Rightarrow 0)$.

Remark 2. An equivalent categorical definition for a Heyting algebra, given in [10], is that of a *cartesian closed poset.*

We now specify the environment for the algebraic semantics of **IPL**:

An $\mathcal{H}$-*Interpretation* $\mathcal{I}$: **Prop** $\rightarrow \mathcal{H}$ of **IPL** in a Heyting algebra $\mathcal{H}$, a.k.a. $\mathcal{H}$-*valuation* is an *assignment* of the propositional variables $p, q, r..$ of **Prop** to elements of $\mathcal{H}$ denoted by $[\![p]\!]_{\mathcal{H}}, [\![q]\!]_{\mathcal{H}}, [\![r]\!]_{\mathcal{H}}...$
From now on, if the context is clear $[\![-]\!]_{\mathcal{H}}$ will simply be denoted by $[\![-]\!]$.
The interpretation $\mathcal{I}$ is extended recursively to **Form**:

Def. 1.3.8 (H.A. interpretation)**.** Given an assignment $\mathcal{I}$: **Prop** $\rightarrow \mathcal{H}$ with $p \mapsto [\![p]\!]$,

[7] or *pseudo-complement of a with respect to b.*
[8] see [10] or [17] for details.

$\mathcal{I} : \textbf{Form} \to \mathcal{H}$ is defined as follows:

$$\llbracket \phi \wedge \chi \rrbracket := \llbracket \phi \rrbracket \wedge \llbracket \chi \rrbracket.$$
$$\llbracket \phi \vee \chi \rrbracket := \llbracket \phi \rrbracket \vee \llbracket \chi \rrbracket.$$
$$\llbracket \phi \Rightarrow \chi \rrbracket := \llbracket \phi \rrbracket \Rightarrow \llbracket \chi \rrbracket.$$
$$\llbracket \neg \phi \rrbracket := \neg \llbracket \phi \rrbracket.$$
$$\llbracket \bot \rrbracket := 0.$$
$$\llbracket \top \rrbracket := 1.$$

We can talk about what it means for a formula ϕ to be *valid* in $\mathcal{H}$:

Def. 1.3.9 (validity in HA)**.** ϕ is valid in $\mathcal{H}$ denoted by $\mathcal{H} \models \phi$ when $\llbracket \phi \rrbracket = 1$ for any chosen assignment for the $\mathcal{H}$-interpretation $\mathcal{I}$.

There is a *canonical* interpretation of **IPL** in $\mathcal{A}$ given simply by $\llbracket p \rrbracket := [p]$ and inductively by $\llbracket \phi \rrbracket := [\phi]$ that *validates* only the provable formulae.
If a formula ϕ is *always true* it is called a *tautology*:

Def. 1.3.10 (tautology in **IPL**)**.** A formula ϕ is a *tautology*[9] or *Intuitionistically valid* denoted by $\vDash \phi$ if it is valid in every Heyting algebra $\mathcal{H}$.

Remark 3. If a formula ϕ is a *tautology*, one has in particular in $\mathcal{A}$ that $1 = \llbracket \phi \rrbracket = [\phi]$ and so $\vdash \phi$.

We can show that all axioms are tautologies and that the inference rule (MP) *preserves validity*, i.e., "if $\vDash \phi$ and $\vDash (\phi \Rightarrow \psi)$ then $\vDash \psi$".
This gives us the following result:

Proposition 4. *$\vdash \phi$ iff $\vDash \phi$, i.e., "ϕ is provable*[10] *iff ϕ is* intuitionistically *valid".*

which means that:

Proposition 5.

$$\vdash_{IPL} \alpha \textit{ iff } H \vDash \alpha \textit{ for all } H \in \textbf{HA}.$$

,i.e. , Heyting algebras *provide* sound *and* complete *semantics for* ***IPL***

[9] tautology in **IPL**.
[10] in **IPL**.

What about *Classical Propositional Logic* or **CPL**?
We keep the same syntax and calculus used in the intuitionistic case but add an axiom "12. $(A \vee \neg A)$" which corresponds to the Law of Excluded Middle (LEM).
We use the following notation to show this fact:

$$\mathbf{CPL} = \mathbf{IPL} + (A \vee \neg A)$$

This time a provable formulae in **CPL**, i.e., $\vdash_{CPL} \phi$ [11] can also be derived from any instance of (LEM) and thus:

Proposition 6. $\boldsymbol{TH_{IPL}} \subset \boldsymbol{TH_{CPL}}$.

We will see that this inclusion is strict.

Now, we can construct the Lindenbaum-Tarski algebra $\mathcal{A}'$ as before by taking the equivalence classes of formulae by double entailment and repeating the same definitions for its ordering and operations. The following still holds:

Proposition 7. *A formula* ϕ *is* provable $\vdash \phi$ *iff* $[\phi] = 1$.

$\mathcal{A}'$ is a special case of a *Heyting algebra* called a *Boolean algebra*.
What is a Boolean algebra?

Def. 1.3.11. A Boolean algebra $(\mathcal{B}, \vee, \wedge, \Rightarrow, 0, 1)$ is a Heyting algebra where for every $a \in \mathcal{B}$ the pseudo-complement of a, i.e., $\neg a$ is a *complement* of a which means:

1. $a \wedge (\neg a) = 0$.
2. $a \vee (\neg a) = 1$.

We denote the class of Boolean algebras by **BA**.

Note that with these definitions every Boolean algebra is a Heyting algebra though in general a Heyting algebra need not be Boolean. In fact we will see examples in which this is the case.

The algebraic semantics for **CPL** are given in a very similar way to the classical case. An *Interpretation* or $\mathcal{B}$-valuation $\mathcal{V}$ of **CPL** in a Boolean algebra $\mathcal{B}$ is again an assignment of propositional variables to elements of $\mathcal{B}$

[11] again, from now on we omit to specify $\vdash_{\mathbf{CPL}}$.

denoted by $[\![p]\!], [\![q]\!], [\![r]\!]$.. The interpretation extends recursively to all formulae as before with the addendum $[\![\neg\phi]\!] = \neg \circ [\![\phi]\!]$.

More commonly, we deal with a *valuation* $\mathcal{V}$ in the two-element Boolean algebra of *truth values* $\mathbb{2} = \{0 < 1\}$ which specifies a *value assignment*, i.e., any function $V : \mathbf{Prop} \to \mathbb{2}$. Also by our recursive definition of $\mathcal{V}$:

Remark 4. $V : \mathbf{Prop} \to \mathbb{2}$ is *lifted* in a unique way to $V : \mathbf{Form} \to \mathbb{2}$ a so-called *truth*-function on any formula.

We can talk about what it means for a formula ϕ to be *valid* in $\mathcal{B}$:

Def. 1.3.12 (validity in BA)**.** ϕ is valid in $\mathcal{B}$ denoted by $\mathcal{B} \models \phi$ when $[\![\phi]\!] = 1$ for any chosen assignment for the $\mathcal{B}$-valuation $\mathcal{V}$.

There is a *canonical* interpretation of **CPL** in $\mathcal{A}$ given simply by $[\![p]\!] := [p]$ and inductively by $[\![\phi]\!] := [\phi]$ that *validates* only the provable formulae.

If a formula ϕ is *always true* it is called a *tautology*:

Def. 1.3.13 (tautology in **CPL**)**.** A formula ϕ is a *tautology*[12] or *classically valid* denoted by $\vDash \phi$ if it is valid in every Boolean algebra $\mathcal{B}$.

In a similar fashion as before, we conclude that:

Proposition 8.

$$\vdash_{CPL} \alpha \textit{ iff } B \vDash \alpha \textit{ for all } B \in \mathbf{BA}.$$

,i.e. , Boolean algebras *provide* sound *and* complete *semantics for* Classical Propositional Logic

To sum up:

- Heyting algebras provide a (sound and complete) algebraic semantics for Intuitionistic Propositional Logic (**IPL**).
- $\mathbf{CPL} = \mathbf{IPL} + (A \vee \neg A)$ and $\mathbf{TH_{IPL}} \subset \mathbf{TH_{CPL}}$.
- $\mathbf{BA} \subset \mathbf{HA}$ Boolean algebras are a particular class of Heyting algebras which provide a (sound and complete) algebraic semantics for Classical Propositional Logic (**CPL**).

The canonical examples of Boolean algebras are:

[12] tautology in **CPL**.

Example 9 (Power-Set)**.** $(\mathcal{P}(X), \subseteq)$, i.e., the set of sub-sets, a.k.a. *Power-Set* of a set X with the set-inclusion partial ordering is a *Boolean algebra* with $0 := \emptyset$, $1 := X$, $\vee := \cup$, $\wedge := \cap$ and $\neg := (\ )^{\complement}$.

Example 10 ($\mathbb{2}$)**.** We encountered a special case of the *power-set* Boolean algebra, which corresponds to the power-set of the singleton set $\mathcal{P}(\{*\})$, i.e., the *two-element* Boolean algebra of *truth-values* $\mathbb{2} := \{0 < 1\}$, where 0 and 1 represent $false$ and $true$ respectively.
Note that here the Law of Excluded Middle (LEM) clearly holds:
$\mathbb{2} \models (\alpha \vee \neg\alpha)$.

Example 11 ($free_1(B)$)**.** The Boolean algebra *freely generated*[13] by the element p.

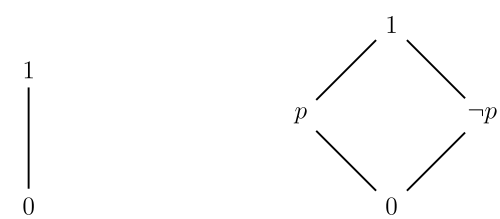


Figure 1.1: $\mathbb{2}$ (left) and $free_1(B)$ (right).

For Heyting algebras the canonical example is:

Example 12 (Open Sets)**.** The lattice of *Open Sets* $(\mathcal{O}, \subseteq)$ (ordered by inclusion) of a topological space (X, τ) is a Heyting algebra with $0 := \emptyset$, $1 := X$, $\vee := \cup$, $\wedge := \cap$ and:
$U \Rightarrow V := (U^{\complement} \cup V)^{\circ}$, i.e., the *largest open sub-set* of $U^{\complement} \cup V$.
This means that whenever W is open, $W \subseteq (U^{\complement} \cup V)^{\circ}$ iff $(U \cap W) \subseteq V$.

The following example shows that not every Heyting algebra need be Boolean $\mathbf{BA} \subsetneq \mathbf{HA}$ and that $\mathbf{TH_{IPL}} \subsetneq \mathbf{TH_{CPL}}$:

Example 13 (The case of $\neg\neg a \neq a$)**.** Note that in the Heyting algebra of Open Sets the *pseudo-complement* of any open set corresponds to the interior of its complement, i.e., $\neg U := (U \Rightarrow \emptyset) = (U^{\complement})^{\circ}$.
Note that in a Boolean algebra "$\neg\neg a = a$" must hold for every element.

[13]this can be seen as the Lindenbaum-Tarski algebra with just one propositional variable.

$a \le \neg\neg a$ holds in $(\mathcal{O}, \subseteq)$ since $\neg\neg a = ((a \Rightarrow 0) \Rightarrow 0)$ and $a \wedge (a \Rightarrow 0) \le 0$ by *evaluation*.

The converse $\neg\neg a \le a$ however in general does not hold.
Take for example the open sets in $[0,1]$ with the induced standard topology.
By unfolding the definition of $\neg$ we obtain : $\neg\neg(0,1) = [0,1] \nsubseteq (0,1)$.

Finally, if we consider (in the induced topology of $[0,1]$) the open set $[0,1/2)$ and $[0,1/2)^{\complement} = (1/2,1]$ we observe that $[0,1/2) \cup [0,1/2)^{\complement} \subsetneq [0,1]$, i.e., LEM $A \vee \neg A$ is not *intuitionistically* valid. In other words:

$$\nvdash_{IPL} A \vee \neg A.$$

Example 14 ($free_1(H)$)**.** The Heyting algebra *freely generated*[14] by the element p.
This forms an infinite lattice called the *Rieger-Nishimura ladder*.

[14]this again can be seen as the Lindenbaum-Tarski algebra with just one propositional variable.

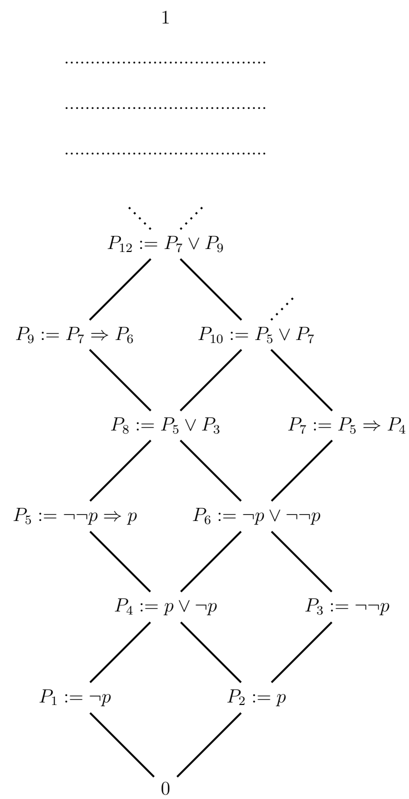


Figure 1.2: $free_1(H)$ displayed as the *Rieger-Nishimura ladder* in which we used auxiliary meta-variables P_n to label the nodes.

1.4 States of Knowledge

We introduce, following [9], a new semantics for IPL given by *S.Kripke* which will give us a better insight into our area of study.
The structure we shall be concerned with is a poset **P** called a *Kripke Frame* which represents a finite set of *possible worlds*, a.k.a. *states of knowledge* with a so-called *temporal ordering.*
(Propositional) Formulae are now interpreted as sub-sets of this poset which represents the states at which the sentence is *true* .
A formula, we shall see, is not true or false per se but rather *true at a certain state of knowledge* and once true remains true in all *future states*, i.e., we have *the persistence of truth in time.*
This accords well with the intuitionistic point of view in which a formula is

true when its truth has been *constructively determined* at some state or *stage* and *constructive knowledge* once established lasts forever.
Note that this temporal ordering of states of knowledge need not be *linear*. This embodies the idea that from the *present* state, there might be more than one possible *future* state. Consider for example a world in which *Fermat's Last Theorem* is determined to be true and one in which it is shown to be false.

Def. 1.4.1 (Kripke Frame)**.** A Kripke Frame $\mathbf{P} := (P, \sqsubseteq)$ is a finite set of *possible worlds* ordered by a partial (so-called *temporal*) ordering $\sqsubseteq$.

Having fixed a Kripke Frame, we introduce the notion of an *hereditary* sub-set of $\mathbf{P}$:

Def. 1.4.2 (hereditary sub-set)**.** $A \subseteq P$ is *hereditary* if it is *upwards-closed* under $\sqsubseteq$, i.e., if $p \in A$ and $p \sqsubseteq q$ we have that $q \in A$.
The collection of all these hereditary up-sets will be denoted by $\mathbf{P}^+$.

Semantics is now given by assigning to each propositional variable a hereditary sub-set by a $\mathbf{P}$-valuation $\mathcal{V}$, i.e., a function $\mathcal{V} : \mathbf{Prop} \to \mathbf{P}^+$.

Remark 5. $\mathcal{V}(\mathbf{p})$ formalizes the idea of *"the set of states at which $\boldsymbol{p}$ is determined to be true"* and being an hereditary sub-set this means that the *knowledge of this truth is persistent in time*.

We define a *Kripke Model* $\mathcal{M}$ and give a formal definition of what it means for a formula to be *true* at a particular state:

Def. 1.4.3 (Kripke Model)**.** A *Kripke Model* based on $\mathbf{P}$ is $\mathcal{M} := (\mathbf{P}, \mathcal{V})$ where $\mathcal{V}$ is a $\mathbf{P}$-valuation.
We define inductively a *forcing relation* $\mathcal{M} \Vdash_w \alpha$
, i.e., *"in $\mathcal{M}$ world w forces formula α"* or also *"the formula α is true in $\mathcal{M}$ at $w \in P$"*:

1. $\mathcal{M} \Vdash_w \mathbf{p}$ iff $w \in \mathcal{V}(\mathbf{p})$.

2. $\mathcal{M} \Vdash_w (\alpha \wedge \beta)$ iff $\mathcal{M} \Vdash_w \alpha$ and $\mathcal{M} \Vdash_w \beta$.

3. $\mathcal{M} \Vdash_w (\alpha \vee \beta)$ iff either $\mathcal{M} \Vdash_w \alpha$ or $\mathcal{M} \Vdash_w \beta$.

4. $\mathcal{M} \Vdash_w (\neg\alpha)$ iff for all s such that $w \sqsubseteq s$ it is not the case that $\mathcal{M} \vDash_s \alpha$.

5. $\mathcal{M} \Vdash_w (\alpha \Rightarrow \beta)$ iff for all s such that $w \sqsubseteq s$, if $\mathcal{M} \vDash_s \alpha$ then $\mathcal{M} \vDash_s \beta$.

With the expression $\mathcal{M} \vDash \alpha$, i.e., α *is true in* $\mathcal{M}$, we indicate that $\mathcal{M} \Vdash_w \alpha$ holds for all $w \in \mathbf{P}$.

Remark 6. The truth of $\neg\alpha$ at state w means that α is never verified at any later stage.

The notion of *validity* is readily given:

Def. 1.4.4 (Kripke Validity)**.** A formula α is valid on the frame $\mathbf{P}$, denoted by $\mathbf{P} \Vdash \alpha$, if for every model $\mathcal{M}$ based on $\mathbf{P}$ we have $\mathcal{M} \Vdash \alpha$.

Let us now consider the hereditary set of states at which α is *true* in $\mathcal{M}$:

Def. 1.4.5. $\mathcal{M}(\alpha) := \{w : \mathcal{M} \Vdash_w \alpha\}$.

Also we introduce the following operations:

Def. 1.4.6. For any hereditary sets S, T:

$$\neg S := \{w : \text{ for all } z,\ w \sqsubseteq z,\ z \notin S\}.$$
$$S \Rightarrow T := \{w : \text{ for all } z,\ w \sqsubseteq z,\ \text{ if } z \in S \text{ then } z \in T\}.$$

With these definitions:

Proposition 15. *For any hereditary* S, T, U*:*
$U \subseteq (S \Rightarrow T)$ *iff* $(S \cap U) \subseteq T$ *and* $\neg S = S \Rightarrow \emptyset$.

The definition of a Kripke Model we gave earlier can be given in an equivalent fashion by requiring that:

(i) $\mathcal{M}(\mathbf{p}) = \mathcal{V}(\mathbf{p})$.

(ii) $\mathcal{M}(\alpha \wedge \beta) = \mathcal{M}(\alpha) \cap \mathcal{M}(\beta)$.

(iii) $\mathcal{M}(\alpha \vee \beta) = \mathcal{M}(\alpha) \cup \mathcal{M}(\beta)$.

(iv) $\mathcal{M}(\neg\alpha) = \neg\mathcal{M}(\alpha)$.

(v) $\mathcal{M}(\alpha \Rightarrow \beta) = \mathcal{M}(\alpha) \Rightarrow \mathcal{M}(\beta)$.

Note that since the intersection $\cap$ and union $\cup$ of hereditary sets is hereditary:

Theorem 16. $(\boldsymbol{P}^{+}, \subseteq)$ *with the operations* $\cap$,$\cup$ *and* $\Rightarrow$ *is a Heyting algebra.*

Any $\mathbf{P}$-valuation $\mathcal{V} : \mathbf{Prop} \to \mathbf{P}^+$ can now be seen as a $\mathbf{P}^+$-valuation for the Heyting algebra $\mathbf{P}^+$.
We have $\mathcal{V}(\mathbf{p}) = \mathcal{M}(\mathbf{p})$ and by extending $\mathcal{V}$ inductively to arbitrary formulae[15]:

Lemma 17. *$\mathcal{M}(\alpha) = \mathcal{V}(\alpha)$ for any formula α.*

Putting all this together:

Proposition 18. *$\mathcal{M} \Vdash \alpha$ iff $\mathcal{M}(\alpha) = P$ iff $\mathcal{V}(\alpha) = P$.*

Also, since P is the top element of the lattice $\mathbf{P}^+$, we obtain a link between *Kripke validity* on the frame $\mathbf{P}$ and *Heyting algebra validity* on $\mathbf{P}^+$:

Theorem 19. *$\boldsymbol{P} \Vdash \alpha$ iff $\boldsymbol{P}^+ \models \alpha$.*

Returning to the Semantics of IPL, we have the following result:

Theorem 20.

$$\vdash_{IPL} \alpha \text{ iff } \boldsymbol{P} \Vdash \alpha \text{ for any frame } \boldsymbol{P}.$$

i.e., Kripke Semantics is sound and complete for IPL.

Soundness comes from the fact that if $\vdash_{IPL} \alpha$ then for any Heyting algebra H we have $H \models \alpha$. For any frame $\mathbf{P}$ we thus have $\mathbf{P}^+ \models \alpha$ and so $\mathbf{P} \Vdash \alpha$.
For Completeness, for $p \in \mathbf{P}$ we introduce the set of sentences *known to be true at* p, a.k.a. the *extension* of p :

Def. 1.4.7 (extension of p)**.** $\Gamma_p := \{\alpha : \mathcal{M} \Vdash_p \alpha\}$ denotes the *extension* of p. This set of sentences satisfies the following properties:

1. (soundness) If $\vdash_{IPL} \alpha$ then $\alpha \in \Gamma_p$.
2. (closure under MP) If $\vdash_{IPL} \alpha \Rightarrow \beta$ and $\alpha \in \Gamma_p$ then $\beta \in \Gamma_p$.
3. (consistency) There exists an α such that $\alpha \notin \Gamma_p$.
4. (primality) If $\alpha \vee \beta \in \Gamma_p$ then $\alpha \in \Gamma_p$ or $\beta \in \Gamma_p$.

Γ_p is also called a *state-description* of the state $p \in \mathbf{P}$ by discerning the sentences known to be true at p.

[15] $\mathcal{V}(\alpha \wedge \beta) := \mathcal{V}(\alpha) \cap \mathcal{V}(\beta)$, $\mathcal{V}(\alpha \vee \beta) := \mathcal{V}(\alpha) \cup \mathcal{V}(\beta)$, $\mathcal{V}(\alpha \Rightarrow \beta) := \mathcal{V}(\alpha) \Rightarrow \mathcal{V}(\beta)$ etc..

Def. 1.4.8 (full set)**.** A set $\Gamma \subseteq \textbf{Form}$ that satisfies the previous conditions 1.-4. is called *full*.

We can now introduce the so-called *canonical model* for IPL:

Def. 1.4.9 (canonical frame)**.** The canonical frame for IPL is the collection of *all* full sets ordered by inclusion $\mathbf{P}_{IPL} := (P_{IPL}, \subseteq)$.

Def. 1.4.10 (canonical model)**.** The canonical model for IPL is given by $\mathcal{M}_{IPL} := (\mathbf{P}_{IPL}, V_{IPL})$ where:

$$V_{IPL} : \mathbf{p}_i \mapsto \{\Gamma : \mathbf{p}_i \in \Gamma\}$$

i.e., the canonical valuation assigns to every proposition the set of full sets that contain it.

If we add the following results:

Lemma 21. *$\mathcal{M} \Vdash_\Gamma \alpha$ iff $\alpha \in \Gamma$.*

Lemma 22 (Lindenbaum)**.** *$\vdash_{IPL} \alpha$ iff α is a member of every full set.*

We obtain the desired completeness result:

Theorem 23. *$\vdash_{IPL} \alpha$ iff $\mathcal{M}_{IPL} \Vdash \alpha$ iff $\boldsymbol{P}_{IPL} \Vdash \alpha$.*

Kripke Semantics can also provide a *topological* interpretation of *intuitionism* as for any frame $\mathbf{P}$ the hereditary sets $\mathbf{P}^+$ form a *topology*:

Proposition 24. *$\boldsymbol{P}^+$ is the Heyting algebra of Open Sets for the topology just described.*
In particular one has $\neg S = (S^{\complement})^\circ$ the largest hereditary subset of $S^{\complement}$ and

Another nice feature of Kripke semantics is that one can determine the validity (or lack thereof) of formulae by looking at the *structure* of the frames.

To illustrate this point, a few simple examples of Kripke Models are illustrated:

Example 25. $\mathcal{T} := \{\mathbf{2}, \mathcal{V}\}$.
Where $\mathbf{2} := \{0 < 1\}$ and $\mathcal{V} : \mathbf{p} \mapsto 1$.
We can ask ourselves if the instance of LEM "$\mathbf{p} \vee \neg\mathbf{p}$" is valid on this frame.
The answer is *no*. To see this:
Note that $\mathcal{T} \nVdash_0 \mathbf{p}$ but $\mathcal{T} \Vdash_1 \mathbf{p}$ with $0 \leq 1$.
So, by definition we also have $\mathcal{T} \nVdash_0 \neg\mathbf{p}$ and so $\mathcal{T} \nVdash_0 (\mathbf{p} \vee \neg\mathbf{p})$ i.e., LEM is not valid on this frame.

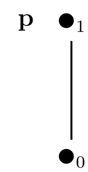


Figure 1.3: Kripke Model $\mathcal{T}$. The propositions *true* at $0, 1$ are listed beside the node labeled by the state $0, 1$.

Example 26. $\mathcal{K} := \{\mathbf{W}, \mathcal{V}\}$.
Where $\mathbf{W} := \{r < w_1 < w_3, r < w_2 < w_3\}$ and $\mathcal{V} : \mathbf{p} \mapsto w_1, w_3, \mathbf{q} \mapsto w_2, w_3$. Similarly as before the LEM is not valid on this frame though its weaker version[16] $\neg\mathbf{p} \vee \neg\neg\mathbf{p}$ holds.

What can we say of the classically valid *pre-linearity axiom*, i.e., "$(\alpha \Rightarrow \beta) \vee (\beta \Rightarrow \alpha)$"?
Since $\mathcal{T} \Vdash_{w_1} \mathbf{p}$, $\mathcal{T} \nVdash_{w_1} \mathbf{q}$ and $\mathcal{T} \Vdash_{w_1} \mathbf{q}$, $\mathcal{T} \nVdash_{w_1} \mathbf{p}$ we have that:
$\nvDash_r (\mathbf{p} \Rightarrow \mathbf{q}) \vee (\mathbf{q} \Rightarrow \mathbf{p})$.
This tells us that the pre-linearity axiom is not valid on this frame and thus is not intuitionistically valid, i.e.,

$$\nvdash_{IPL} (\alpha \Rightarrow \beta) \vee (\beta \Rightarrow \alpha).$$

However this formula is valid for example on the previous frame $\mathcal{T}$.

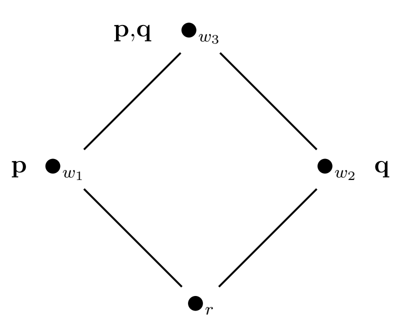


Figure 1.4: Kripke Model $\mathcal{K}$. The propositions *true* at w_i are listed beside the node labeled by the state w_i.

We can also recover *classical* validity:

[16]wLEM := $(\neg\alpha \vee \neg\neg\alpha)$.

Example 27. Recall the classical validity of the form $\mathbb{2} \models \alpha$, i.e., validity on the two-element Boolean algebra $\mathbb{2} = \{0 < 1\}$, means that $[\![\alpha]\!] = 1$ for any truth assignment of the propositional variables $V : \mathbf{Prop} \to \mathbb{2}$. If we now specify:

$\mathcal{C} := \{\{w\}, \tilde{V}\}$.
Where $\mathcal{V} : \mathbf{p} \mapsto w$ iff $V : \mathbf{p} \mapsto 1$.
So classical validity is the same as Kripke-validity on this discrete frame, i.e., $\{w\} \Vdash \alpha$ iff $\mathbb{2} \models \alpha$.

$$\mathbf{p}_i \quad \bullet_w$$

Figure 1.5: Kripke Model $\mathcal{C}$ in which $\mathbf{p}_i$ stands for all the propositions valued at 1.

These examples suggest the following definitions:

Def. 1.4.11 (discrete)**.** The frame $\mathbf{P}$ is *discrete*, i.e., has $z \sqsubseteq w$ iff $z = w$, iff $\mathbf{P} \Vdash (\alpha \vee \neg\alpha)$.

Def. 1.4.12 (directed)**.** The frame $\mathbf{P}$ is *directed*, i.e., if $z \sqsubseteq w$ and $z \sqsubseteq v$ then there exists an s with $w \sqsubseteq s$ and $v \sqsubseteq s$, iff
$\mathbf{P} \Vdash (\neg\alpha \vee \neg\neg\alpha)$.

Def. 1.4.13 (weakly linear)**.** The frame $\mathbf{P}$ is *weakly linear*, i.e., if $z \sqsubseteq w$ and $z \sqsubseteq v$ then either $w \sqsubseteq v$ or $v \sqsubseteq w$, iff
$\mathbf{P} \Vdash (\alpha \Rightarrow \beta) \vee (\beta \Rightarrow \alpha)$.

1.5 An Intermediate and Fuzzy Logic

The following introduction, which draws heavily from [2] & [4] among others, is a recollection of known results. As the name suggests:

Def. 1.5.1 (intermediate propositional logic)**.** An *intermediate*, a.k.a. *super-intuitionistic* propositional logic $\mathcal{L}$ is such that: $\mathbf{Th_{IPL}} \subset \mathbf{Th}_{\mathcal{L}} \subset \mathbf{Th_{CPL}}$, i.e., its theorems include all the **IPL**-theorems and are included in the **CPL**-theorems.

An Intermediate Propositional Logic is Gödel-Dummett Propositional Logic $\mathcal{G}$, obtained by extending the standard **IPL** with the *pre-linearity* axiom we encountered earlier:

Def. 1.5.2. $\mathcal{G} := \mathbf{IPL}\ +\ ((A \Rightarrow B) \vee (B \Rightarrow A))$.

We will also use the following notation to express this fact:

$$\mathbf{IPL} \subset \mathcal{G} \subset \mathbf{CPL}$$

The algebraic semantics of $\mathcal{G}$ is given by the variety, i.e., an equationally definable class, $\mathbb{G}$ of *Gödel algebras*:

Remark 7. For every formula α one has : $\vdash_{\mathcal{G}} \alpha$ iff $G \vDash \alpha$ for all $G \in \mathbb{G}$.

A Gödel algebra G is a Heyting algebra that satisfies *pre-linearity*:

Def. 1.5.3. $G := (G, \vee, \wedge, \Rightarrow, 0, 1)$ such that $\forall x, y, z \in G$:

1. $0 \leq x$.
2. $x \leq 1$.
3. $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$.
4. $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$.
5. $z \leq (x \Rightarrow y)$ iff $(z \wedge x) \leq y$.
6. $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$.

Note that the pre-linearity axiom restricts the space of possible algebraic semantics to Heyting algebras built on top of a total order.

In fact, one can show:

Proposition 28. *Any infinite* chain C *with minimum and maximum elements where:* [17]

$$x \Rightarrow y := \begin{cases} max(C) & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$$

$$x \vee y := max(x, y)$$

$$x \wedge y := min(x, y)$$

$$\neg x := \begin{cases} max(C) & \text{if } x = min(C) \\ min(C) & \text{if } x \neq min(C) \end{cases}$$

provides sound & complete semantics *for all formulae* α*.*

$$\mathcal{G} \vdash \alpha \ \textit{ iff } \ C \models \alpha$$

A special case of the above is:

Def. 1.5.4 (standard model)**.** The standard *model* is the real unit interval $([0, 1], max, min, \Rightarrow, 0, 1)$:

$$\mathcal{G} \vdash \alpha \ \text{ iff } \ [0, 1] \models \alpha.$$

Furthermore, if we take a look at its Kripke semantics we have another characterization:

Proposition 29.

$$\mathcal{G} \vdash \alpha \textit{ iff } \boldsymbol{W} \Vdash \alpha \textit{ for any } \text{weakly linear } \textit{frame } \boldsymbol{W}.$$

[17] memento : $\neg x := x \Rightarrow 0$.

With regards to *first-order logic*:

We fix a first-order language $\mathcal{L}$ with the usual symbols for connectives and quantifiers $\forall, \exists$ and countable sets of predicate symbols $\mathfrak{P}$, function symbols $\mathfrak{F}$ for every arity $k > 0$ and variables $\mathfrak{V}$.

As we just saw, the standard model takes the set $[0, 1] \subset \mathbb{R}$. We define the following:

Def. 1.5.5 (Gödel set)**.** A *Gödel set* is a closed sub-set $\mathcal{V} \subseteq [0, 1]$ containing 0 and 1.

If $\mathfrak{U}$ is the *universe* or *domain* of the *interpretation* $\mathcal{I}$ we extend the language $\mathcal{L}$ to $\mathcal{L}^{\mathfrak{U}}$ with constant symbols $\bar{u}$ [18] for each element $u \in \mathfrak{U}$.

Def. 1.5.6 (first order interpretation)**.** Having fixed a *Gödel set* $\mathcal{V}$.
An *interpretation* $\mathcal{I}$ into $\mathcal{V}$ is given by:

1. a non-empty set $\mathfrak{U} = \mathfrak{U}^{\mathcal{I}}$, i.e., the *universe* of $\mathcal{I}$.
2. a function $\mathfrak{p}^{\mathcal{I}} : \mathfrak{U}^k \to \mathcal{V}$ for each k-ary predicate symbol $\mathfrak{p} \in \mathfrak{P}$.
3. a function $\mathfrak{f}^{\mathcal{I}} : \mathfrak{U}^k \to \mathfrak{U}$ for each k-ary function symbol $\mathfrak{f} \in \mathfrak{F}$. Each new constant symbol $\bar{u}$ is interpreted as the element u.

The first-order semantics are thus:

[18] a.k.a. *the name of u*.

Def. 1.5.7 (first order semantics)**.** Having fixed an interpretation $\mathcal{I}$:
Any term $t = f(\bar{u}_1, .., \bar{u}_k)$ [19] is realized inductively as $\mathcal{I}(t) = f^{\mathcal{I}}(u_1, .., u_k)$.
As for atomic formulae $A \equiv P(t_1, .., t_n)$ we define $\mathcal{I}(A) \equiv P^{\mathcal{I}}(t_1^{\mathcal{I}}, .., t_n^{\mathcal{I}})$ and as for composite formulae:

- $\mathcal{I}(\perp) := 0$.
- $\mathcal{I}(A \wedge B) := min(\mathcal{I}(A), \mathcal{I}(B))$.
- $\mathcal{I}(A \vee B) := max(\mathcal{I}(A), \mathcal{I}(B))$.
- $\mathcal{I}(A \Rightarrow B) := \begin{cases} 1 & \text{if } \mathcal{I}(A) \leq \mathcal{I}(B) \\ \mathcal{I}(B) & else \end{cases}$.
- $\mathcal{I}(\forall x.A(x)) := inf\{\mathcal{I}(A(\bar{u})) : u \in \mathfrak{U}\}$. [20]
- $\mathcal{I}(\exists x.A(x)) := sup\{\mathcal{I}(A(\bar{u})) : u \in \mathfrak{U}\}$.

If $\mathcal{I}(A) = 1$, we say that $\mathcal{I}$ *satisfies* A and denote this by $\models_{\mathcal{I}} A$.
A formula A is said to be *valid in the first order Gödel logic* $\mathcal{G}_{\mathcal{V}}$ whenever $\models_{\mathcal{I}} A$ for all possible interpretations $\mathcal{I}$.

In Classical Logic the truth values are binary $\{0, 1\}$ or $\{true, false\}$.
On the other hand, Intuitionistic and Intermediate Logics provide a framework for arbitrarily many $n > 2$ truth values.

Remark 8. Consider the proposition *"Antarctica is large"*. Binary truth values are clearly unsatisfactory to express its validity. We would be better served with *degrees of truth* from 0 (*false*) to 1 (*true*).

In fact the standard model for $\mathcal{G}$ in 1.5.4 can provide a continuum of truth values between 0 false and 1 true.
Gödel-Dummett Logic is also a so-called *fuzzy logic*. We briefly digress and give a better understanding of this notion following [3]:
"Antarctica is large" or *"The patient is young"* are examples of *fuzzy* propositions which are *true to some degree*.
The standard set used to *encode* the truth degrees is the real unit interval

[19] thanks to the extended language we can restrict ourselves to *ground*-terms, i.e., terms without variables.
[20] $A(u)$ is the formula obtained from $A(x)$ by substituting each *free* occurrence of the variable x in A with u.

$[0, 1]$ with its standard order.
Similarly to the classical case, most fuzzy logics are *truth functional*, i.e., the truth degrees of compound formulae is a function of the truth degrees of the compounds like we saw in (1.5.4).
This is expressed as: *Each connective* c *of arity* n *has a truth function* $f_c : [0, 1]^n \to [0, 1]$ *determining for any formulae* $\phi_1, \phi_2, ..\phi_n$ *the* truth degree *of* $c(\phi_1, \phi_2, .., \phi_n)$ *from the* truth degrees *of* $\phi_1, \phi_2, .., \phi_n$
Moreover, this over-arching principle must be observed: *Each many valued logic must be a generalization of classical two-value logic.*
This means for example that if we introduce a new connective symbol & for *strong* conjunction and denote by $*$ its binary truth function, the following must hold:

$$1 * 1 = 1, \;\; 1 * 0 = 0 = 0 * 1, \;\; 0 * 0 = 0$$

In analogy to the classical case, we would also like for $*$ to be non-decreasing in both arguments, 1 to be its unit element and 0 its zero element. We can thus define a more general $*$ as:

Def. 1.5.8 (t-norm)**.** A *t-norm* is a binary operation $*$ on $[0, 1]$ satisfying:

(i) $*$ is commutative and associative, i.e., for all $x, y, z \in [0, 1]$,

$$x * y = y * x$$
$$(x * y) * z = x * (y * z)$$

(ii) $*$ is non-decreasing in both arguments, i.e.,
for all $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$,

$$x_1 \leq x_2 \text{ implies } x_1 * y \leq x_2 * y$$
$$y_1 \leq y_2 \text{ implies } x * y_1 \leq x * y_2$$

(iii) for all $x \in [0, 1]$ $1 * x = x$ and $0 * x = 0$.

The t-norm $*$ is *continuous* if it is so as a continuous mapping $[0, 1]^2 \to [0, 1]$.

Recall that for the standard model semantics (1.5.4) of Gödel-Dummett Logic the conjunction $p \wedge q$ was realized as $min([\![p]\!], [\![q]\!])$, indeed distinguished examples of continuous t-norms are:

Example 30.

(i) *Łukasiewicz* t-norm: $x * y := max(0, x + y - 1)$

(ii) *Gödel* t-norm: $x * y := min(x, y)$

(iii) *Goguen/Product* t-norm: $x * y := x \cdot y$

With regards to implication, In classical logic $\phi \Rightarrow \psi$ is true iff the truth value of ϕ is less than or equal to the truth value of ψ. This leads us to desire that a truth function $x \Rightarrow y$ should be *non-increasing in x and non-decreasing in y.*
Also we would like a so-called *fuzzy modus ponens* whereby from *lower bounds* of truth degree x of ϕ and $x \Rightarrow y$ truth degree of $\phi \Rightarrow \psi$ one should be able to deduce a truth degree y of ψ. We take the t-norm $*$ to be the operation which *computes the lower bound for y*, i.e.,

$$(\textit{Fuzzy MP}) \text{ If } a \leq x \text{ and } b \leq x \Rightarrow y, \text{ then } a * b \leq y$$

In particular, if $a = x$ and $b = z$ we obtain:

$$\text{If } z \leq x \Rightarrow y, \text{ then } x * z \leq y$$

Also, if we want to define $x \Rightarrow y$ to be *as large as possible* we may require the converse, i.e.,

$$\text{If } x * z \leq y, \text{ then } z \leq x \Rightarrow y$$

which gives the condition that $*$ and $\Rightarrow$ must form an *adjoint pair*:[21]

$$x * z \leq y \text{ iff } z \leq x \Rightarrow y$$

making $x \Rightarrow y$ the *maximal* z satisfying $x * z \leq y$, i.e.,

$$x \Rightarrow y = sup\{z \mid x * z \leq y\}$$

In fact, each continuous t-norm $*$ uniquely determines a so-called *residuum* $x \Rightarrow y$:

Lemma 31. *Let $*$ be a continuous t-norm. There is a unique binary operation $\Rightarrow$ satisfying for all $x, y, z \in [0, 1]$ the condition $x * z \leq y$, iff $z \leq x \Rightarrow y$ determined by $x \Rightarrow y := max\{z \mid x * z \leq y\}$.*

[21] it is no accident that this condition is very similar to the one for pseudo-complements in (1.3.6).

We now have:

Lemma 32. *For each continuous t-norm $*$ and its residuum $\Rightarrow$, the following hold for all $x, y \in [0,1]$:*

$$x \leq y \text{ iff } (x \Rightarrow y) = 1$$
$$(1 \Rightarrow x) = x$$
$$\text{If } x \leq y \text{ then } x = y * (y \Rightarrow x)$$

Returning to the t-norms in (30):

Theorem 33. *The residua of our t-norms are:*

(i) Łukasiewicz *implication:* $x \Rightarrow y = \begin{cases} 1 - x + y & \textit{if } x > y \\ 1 & \textit{if } x \leq y \end{cases}$

(ii) Gödel *implication :* $x \Rightarrow y = \begin{cases} y & \textit{if } x > y \\ 1 & \textit{if } x \leq y \end{cases}$

(iii) Goguen/Product *implication:* $x \Rightarrow y = \begin{cases} y/x & \textit{if } x > y \\ 1 & \textit{if } x \leq y \end{cases}$

The residuum $\Rightarrow$ defines a so-called *pre-complement* $(-)x := x \Rightarrow 0$ which generalizes classical negation:

Lemma 34. *The pre-complements of our t-norms are:*

(i) Łukasiewicz *negation:* $(-)x = 1 - x$

(ii) Gödel *negation:* $(-)x = \begin{cases} 0 & \textit{if } x > 0 \\ 1 & \textit{if } x = 0; \end{cases}$

(iii) Goguen/Product *negation:* $(-)x = \begin{cases} 0 & \textit{if } x > 0 \\ 1 & \textit{if } x = 0; \end{cases}$

We now define the so-called *Basic Many-Valued Logic* $\mathcal{BL}$:
Having fixed a continuous t-norm $*$ we introduce a propositional calculus $PC(*)$ with variables $p_1, p_2, ..$, connectives &,$\rightarrow$ and a constant $\bar{0}$.
Formulae are defined in the usual inductive way: *each propositional variable is a formula,* $\bar{0}$ *is a formula, if* ϕ *and* ψ *are formulae so are* $\phi \,\&\, \psi$ *and* $\phi \rightarrow \psi$.
Further operations are defined as:

$$\phi \wedge \psi := \phi \,\&\, (\phi \rightarrow \psi)$$
$$\phi \vee \psi := ((\phi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \phi) \rightarrow \phi)$$
$$\neg\phi := \phi \rightarrow \bar{0}$$
$$\phi \equiv \psi := (\phi \rightarrow \psi) \,\&\, (\psi \rightarrow \phi)$$
$$\bar{1} := \bar{0} \rightarrow \bar{0}$$

Def. 1.5.9 (axioms of $\mathcal{BL}$)**.** The axioms of our *Basic Logic* are:

(A1) $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$

(A2) $(\phi \,\&\, \psi) \to \phi$

(A3) $(\phi \,\&\, \psi) \to (\psi \,\&\, \phi)$

(A4) $(\phi \,\&\, (\phi \to \psi)) \to (\psi \,\&\, (\psi \to \phi))$

(A5) $(\phi \to (\psi \to \chi)) \to ((\phi \,\&\, \psi) \to \chi)$

(A6) $((\phi \,\&\, \psi) \to \chi) \to (\phi \to (\phi \to \chi))$

(A7) $((\phi \to \psi) \to \chi) \to (((\psi \to \psi)\chi) \to \chi)$

(A8) $\bar{0} \to \phi$

The only deduction rule is *Modus Ponens*.

An *evaluation of propositional variables* is an assignment e of each propositional variable p to a truth value $e(p) \in [0, 1]$.
$*$ and its residuum $\Rightarrow$ become the truth functions of the *strong* conjunction & and implication:

$$\begin{aligned} e(\bar{0}) &:= 0 \\ e(\phi \to \psi) &:= e(\phi) \Rightarrow e(\psi) \\ e(\phi \,\&\, \psi) &:= e(\phi) * e(\psi) \end{aligned}$$

Note that, from the previous definitions, one can show that:

Lemma 35. *For any formulae ϕ, ψ:*

$$\begin{aligned} e(\phi \wedge \psi) &= min(e(\phi), e(\psi)) \\ e(\phi \vee \psi) &= max(e(\phi), e(\psi)) \end{aligned}$$

A formula ϕ is called a *1-tautology* if $e(\phi) = 1$ for each evaluation e.
Note that all axioms of $\mathcal{BL}$ can be shown to be *1-tautologies* in each $PC(*)$ and since *Modus Ponens* preserves 1-tautologies [22], the following result holds:

[22] i.e., If ϕ and $\phi \to \psi$ are 1-tautologies then so is ψ.

Lemma 36. *All formulae provable in $\mathcal{BL}$ are* 1-tautologies *in each $PC(*)$.*

The *algebraization* of $\mathcal{BL}$ can now be achieved:
We introduce the following structure:

Def. 1.5.10 (BL-algebra)**.** A *BL-algebra* is a *residuated prelinear* lattice:
A *residuated lattice* is given by an algebra $(L, \cap, \cup, *, \Rightarrow, 0, 1)$ with binary operations $\cap, \cup, *, \Rightarrow$ and constants $0, 1$ such that:

(i) $(L, \cap, \cup, *, \Rightarrow, 0, 1)$ is a lattice endowed with a partial ordering $\leq$ with largest element 1 and least element 1.

(ii) $(L, *, 1)$ is a commutative *semigroup*[23] with unit element 1.

(iii) $*$ and $\Rightarrow$ form an *adjoint pair*[24].

A residuated lattice becomes a *BL-algebra* iff the following identies hold:

(iv) $x \cap y = x * (x \Rightarrow y)$

(v) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$

Def. 1.5.11. Let $\mathbf{L}$ be a BL-algebra. We can define an $\mathbf{L}$-evaluation by taking an evaluation e of propositional variables and extending it to all formulae in the usual way:

$$
\begin{gathered}
e(\bar{0}) := 0 \\
e(\phi \to \psi) := e(\phi) \Rightarrow e(\psi) \\
e(\phi \,\&\, \psi) := e(\phi) * e(\psi)
\end{gathered}
$$

hence:

$$
\begin{gathered}
e(\phi \wedge \psi) = e(\phi) \cap e(\psi) \\
e(\phi \vee \psi) = e(\phi) \cup e(\psi) \\
e(\neg \phi) = e(\phi) \Rightarrow 0
\end{gathered}
$$

ϕ is called an $\mathbf{L}$-*tautology* if $e(\phi) = 1$ for each $\mathbf{L}$-evaluation e.

For the variety of *BL-algebras* it can be shown that the following hold:

[23]i.e., * is commutative,associative and $1 * x = x$ for any element x.
[24]i.e., $z \leq (x \Rightarrow y)$ iff $x * z \leq y$ for all x, y, z.

Proposition 37. *1. For each t-norm $*$, the real unit interval endowed with the truth functions for the connectives $([0,1], min, max, *, \Rightarrow, 0, 1)$ is a linearly-ordered BL-algebra.*

2. The Lindenbaum-Tarski algebra for $\mathcal{BL}$ is a (not linearly-ordered) BL-algebra.

Theorem 38. *The algebraic semantics of $\mathcal{BL}$ is sound and complete for* BL-algebras *and in particular for* linearly ordered BL-algebras*:*

$$\mathcal{BL} \vdash \phi \text{ iff } \phi \text{ is an } \boldsymbol{L}\text{-tautology for each BL-algebra } \boldsymbol{L}$$
$$\text{iff } \phi \text{ is an } \boldsymbol{L}\text{'-tautology for each linearly ordered BL-algebra } \boldsymbol{L}\text{'}$$

If we extend $\mathcal{BL}$ with the double-negation axiom $\neg\neg\phi \to \phi$ we obtain *Lukasiewicz propositional logic.*
If we add a new binary operation symbol $\odot$ and a couple of new axioms: $\neg\neg\chi \to ((\phi \odot \chi \to \psi \odot \chi) \to (\phi \to \psi))$, $\phi \wedge \neg\phi \to \bar{0}$ we obtain *Product/Goguen logic.*

If we extend $\mathcal{BL}$ with the *idempotency* axiom:

$$\phi \to (\phi \,\&\, \phi)$$

we obtain our Gödel-Dummett logic $\mathcal{G}$, i.e.,

$$\mathcal{G} = \mathcal{BL} + (\phi \to (\phi \,\&\, \phi))$$

The first consequence of this is:

Lemma 39. $\mathcal{G} \vdash (\phi \,\&\, \psi) \equiv (\phi \wedge \psi)$

In other words, strong conjunction & is equivalent to conjunction $\wedge$. Thus we choose to get rid of & one of the two symbols and keep the other.
In fact, recalling that the Gödel t-norm is defined as $x * y := min(x, y)$:

Remark 9. we can define a Gödel algebra $\mathbf{G}$ as a BL-algebra $(G, \cup, \cap, *, \Rightarrow, 0, 1)$ with idempotent multiplication, i.e., satisfying the identity $x * x = x$. Since now $* = \wedge$, we display $\mathbf{G}$ simply as $(G, \cup, \cap, \Rightarrow, 0, 1)$.

A similar approach starting this time from $\mathcal{MTL}$ *monoidal t-norm based logic*, i.e., the logic of *left-continuous t-norms and their residua*, introduced in [18], and adding the *idempotency* axiom produces the same result:

$$\mathcal{G} = \mathcal{MTL} + (\phi \to (\phi \,\&\, \phi))$$

We now return to the main part of our introduction:
In this work we are primarily interested in *finite-valued* Gödel-Dummett Logic $\mathcal{G}_n$ for $n \in \mathbb{N}$.
In 1932's *Zum intuitionistischen aussagenkalkül* [11] *K.Gödel* showed that:

Theorem 40. *IPL cannot be viewed as a system of* many-valued *logic.*

i.e., we cannot find a *finite* set M of *truth values*, a subset $D \subset M$ of designated values [25] and an interpretation of the connectives $\wedge, \vee, \Rightarrow, \neg$ such that: $\vdash_{IPL} \alpha$ iff, for all valuations $\mathcal{V}$ into M, $\mathcal{V}(\alpha) \in D$.

We begin by defining $\mathcal{G}_n$ by adding to the axioms of $\mathcal{G}$ a statement to "limit" the number of distinct elements or *truth values* in a model to at most $n > 0$:

Def. 1.5.12.

$$\mathcal{G}_n := \mathcal{G} \ + \ F_{n+1}$$

Where[26]

$$F_n := \bigvee_{0 \leq i < j < n} (\mathbf{p}_i \Leftrightarrow \mathbf{p}_j)$$

Consider a possible assignment of the propositional variables which maps $\mathbf{p}_i$ and $\mathbf{p}_j$ to the same element e (for some $\mathbf{p}_i \Leftrightarrow \mathbf{p}_j$ in F_n).
Observe that the formula $(a \Leftrightarrow a) \vee b$ is provable in IPL.[27]
It follows that:

Lemma 41. *F_n is satisfied in any* realization *with fewer than n elements which is a model for IPL, i.e., in which every theorem of IPL is satisfied.*

We thus construct a *n-chain* Heyting algebra called C_n.
Its $n > 1$ elements are equidistant points on $[0, 1]$ from 0 to 1, i.e.:

$$\{1 - \frac{1}{k}\}_{k=1}^{n-1} \cup \{1\} = \{0, \frac{1}{n-1}, \frac{2}{n-1}.., 1 - \frac{1}{n-1}, 1\}.$$

The *designated* element is 1:

[25] in the classical case we are used to $M = \{0, 1\}$, $D = \{1\}$ and the usual interpretation of the connectives in the Boolean algebra $\mathbb{2}$.
[26] $\bigvee$ indicates the iterated $\vee$ connective.
[27] recall also that $\vdash_{IPL} (A \vee B)$ iff either $\vdash_{IPL} A$ or $\vdash_{IPL} B$.

Def. 1.5.13 (C_n).

$$C_n := \{0 < \frac{1}{n-1} < \frac{2}{n-1} < .. < 1 - \frac{1}{n-1} < 1\}.$$

$$a \Rightarrow b := \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$$

$$a \vee b := max(a, b)$$

$$a \wedge b := min(a, b)$$

$$\neg a := \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{if } a \neq 0 \end{cases}$$

Recalling that Heyting algebras provide a sound and complete semantics for IPL:
For C_n all theorems of IPL and all formulae F_k with $k > n$ are satisfied since there is no way to choose $n + 1$ values without repeating some of them, i.e.,

Lemma 42. $\mathrm{C}_n \models \mathcal{G}_h$ *for any* $h \geq n$.

On the other hand, F_r with $r \leq n$ are *not* satisfied because it is always possible to choose distinct values for the r variables, i.e.,

Lemma 43. $\mathrm{C}_n \not\models \mathcal{G}_s$ *with* $s < n$.

It follows that no F_n with $n \geq 1$ is provable in IPL and also:
Since $F_n \vdash F_{n+1}$ (F_{n+1} is a "weaker" condition than F_n) and $C_{n+1} \not\models \mathcal{G}_n$ whereas $C_{n+1} \models \mathcal{G}_{n+1}$, we have:[28]

Lemma 44. *For every* $n \in \mathbb{N} : \mathcal{G}_{n+1} \subset \mathcal{G}_n$ *and* $\mathcal{G}_n \nsubseteq \mathcal{G}_{n+1}$.

Note also that we return to classical logic if $n = 2$:

Remark 10. Since any Heyting algebra H with $H \models \mathcal{G}_2$ is such that $H \models F_3$, i.e., H must have only two[29] elements $\bot, \top$, i.e., $H \cong 2$ and $\mathcal{G}_2 = \mathbf{CPL}$.

In fact, the collection $\{\mathcal{G}_n\}_{n \in \mathbb{N}}$ forms a *descending proper chain* of *intermediate logics*:

[28] in general it is the case that if $\mathcal{L} \subseteq \mathcal{L}'$ then the Models of $\mathcal{L}'$ are Models of $\mathcal{L}$, i.e., $Mod(\mathcal{L}') \subseteq Mod(\mathcal{L})$.

[29] we exclude the case of the *degenerate* Heyting algebra in which $\bot \equiv \top$.

Proposition 45. ***IPL*** $\subsetneq \mathcal{G} \subsetneq ... \subsetneq \mathcal{G}_{n+1} \subsetneq \mathcal{G}_n \subsetneq .... \subsetneq \mathcal{G}_3 \subsetneq \mathcal{G}_2 =$ ***CPL***.
From this we obtain:

$$\mathcal{G} = \bigcap_{k \geq 2} \mathcal{G}_k.$$

We can also show that $\mathcal{G}_n$ are *semantically characterized* by these n-chains, i.e.:

Proposition 46. *Let* H *be a Heyting algebra that models* $\mathcal{G}_n$*:*

$$H \models \mathcal{G}_n \ \text{ iff } \ H \cong C_k \text{ for some } k \leq n.$$
$$\mathcal{G}_n \vdash \phi \ \text{ iff } \ C_n \models \phi.$$

Now let us take a closer look at $\mathcal{G}_3$ or *Gödel-Dummett (Propositional) three-valued logic* also known as the logic of *Here and There* which will come up later in this work.
The three *truth values* $\mathfrak{T}$ are the elements of C_3: The designated value 1 which we rename t, a.k.a. true, 0 we rename f, a.k.a. false and the non-classical value $\frac{1}{2}$ we rename $*$, a.k.a. not-false.

The following remark is made by *A.Heyting* in [16]:

Remark 11. The truth values of $\mathcal{G}_3$ can be *thought of* as follows:

- t denotes a *correct* assertion.
- $*$ denotes an assertion that *cannot be false but whose correctness has yet to be proven.* [30]
- f denotes a *false* assertion.

Def. 1.5.14 (truth values for $\mathcal{G}_3$)**.** $\mathfrak{T} := \{0, \frac{1}{2}, 1\} \cong \{f, *, t\}$.

$$\begin{tikzcd}
\mathrm{t} \arrow[d, no head] \\
* \arrow[d, no head] \\
\mathrm{f}
\end{tikzcd}$$

Figure 1.6: $C_3 := \{0, \frac{1}{2}, 1\} \cong \{\mathrm{f} < * < \mathrm{t}\}$.

Given a valuation of the propositional variables into these truth values $V\colon \mathbf{Prop} \to \mathfrak{T}$ with $V : \top \mapsto \mathrm{t}$ and $\bot \mapsto \mathrm{f}$.
In accord with the semantics of C_3, this valuation is extended to propositional

[30] by proof we mean an intuitionistic one.

formulae denoted by $A, B, ..$ as:

$$V(A \wedge B) := min\{V(A), V(B)\}$$
$$V(A \vee B) := max\{V(A), V(B)\}$$
$$V(A \Rightarrow B) := \begin{cases} t & \text{if } V(A) \leq V(B) \\ V(B) & \text{if } V(A) > V(B) \end{cases}$$
$$V(\neg A) := \begin{cases} \mathrm{t} & \text{if } V(A) = \mathrm{f} \\ \mathrm{f} & \text{if } V(A) \neq \mathrm{f} \end{cases}$$

The *truth tables* for the connectives $\neg, \wedge, \vee, \Rightarrow$ are thus given by:

	$\neg$
t	f
$*$	f
f	t

(a)

$\wedge$	t	$*$	f
t	t	$*$	f
$*$	$*$	$*$	f
f	f	f	f

(b)

$\vee$	t	$*$	f
t	t	t	t
$*$	t	$*$	$*$
f	t	$*$	f

(c)

$\Rightarrow$	t	$*$	f
t	t	$*$	f
$*$	t	t	f
f	t	t	t

(d)

Remark 12. Notice that, since $\mathcal{G}_3 \subset \mathcal{G}_2$, the truth tables for t and f are the same as in classical logic.
Also, the classical $p \vee \neg p$ (*Tertium non datur*) here is no longer a *tautology* given that an assignment of $*$ to p yields a truth value of $*$ for "$* \vee (\neg *)$" .

The extension of $\mathcal{G}_3$ to First Order Logic is obtained by applying 1.5.7:

Proposition 47 (first order $\mathcal{G}_3$)**.** *Taking as Gödel set $\mathfrak{X} = \{0, \frac{1}{2}, 1\}$, the interpretation $\mathcal{I}$ with an associated universe $\mathfrak{U}$ is extended to quantified formulae as:*

- $\mathcal{I}(\forall x.A(x)) := min\{\mathcal{I}(A(u)) : u \in \mathfrak{U}\}$.
- $\mathcal{I}(\exists x.A(x)) := max\{\mathcal{I}(A(u)) : u \in \mathfrak{U}\}$.

Chapter 2

Algebras and Forests

In 1.5 we brought up the variety $\mathbb{G}$ of *Gödel algebras* as the algebraic semantics of Gödel-Dummett (Propositional) Logic $\mathcal{G}$.
We saw that *finite-valued* Gödel-Dummett Logic $\mathcal{G}_n$ has a semantic characterization given by the n-chains C_n.
In fact, if we take the *subdirect* product of k-chains with $k \leq n$, the resulting *sub-variety* is called $\mathbb{G}_n$ or *n-valued Gödel algebras* which model the logic $\mathcal{G}_n$.
Using a Stone-like *Duality* between *finite* Gödel algebras and a soon to be introduced category of *Finite Forests* we aim to explore the Topos semantics of a sub-category of *Bushes* and *re-discover* our target logic $\mathcal{G}_3$ at the propositional and first-order levels.
We start by giving the following introduction to *Finite Forests* and their duality with *Finite Gödel algebras* building on the works of [1], [5], [6] and [7] among others.

2.1 Finite Forests

2.1.1 First Steps

We need the following preliminary definitions:

Def. 2.1.1 (*down-set* of $S \subseteq F$)**.** $\downarrow S := \{x \in F \mid x \leq y \text{ for some } y \in S\}$.

Def. 2.1.2 (*up-set* of $S \subseteq F$)**.** $\uparrow S := \{x \in F \mid x \geq y \text{ for some } y \in S\}$.

Now, what do we mean by *finite forests*?

Def. 2.1.3 (*finite forest* F)**.** F is a finite poset in which the *down-set* of each element $x \in F$, i.e., $\downarrow x := \downarrow \{x\}$ is a *chain*[1] with the order inherited from F.

Def. 2.1.4 (*finite tree* T)**.** T is a finite forest with a minimum element called *root* or *bottom*.

Def. 2.1.5 (*sub-forest* G)**.** A sub-forest G of a finite forest F is a *downward-closed sub-poset G of F*, i.e., $G = \downarrow G$.
F in this case is also called a *super-forest* of the finite forest G.

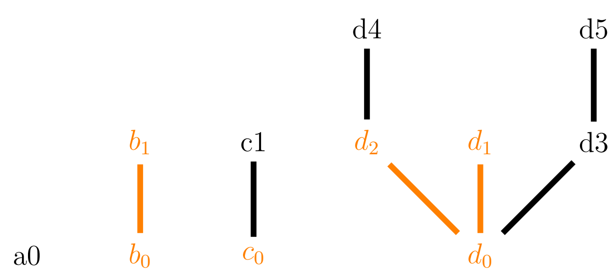


Figure 2.1: A finite forest F and its sub-forest G in which the nodes are ordered bottom to top and displayed accordingly.
The figure is meant to represent the finite poset
$F = \{a_0,\ b_0 < b_1,\ c_0 < c_1,\ d_0 < d_1 < d_4,\ d_0 < d_2,\ d_0 < d_3 < d_5\}$ and its sub-forest
$G = \{b_0 < b_1,\ c_0,\ d_0 < d_1,\ d_0 < d_2\}$.

[1] totally ordered sub-poset.

2.1.2 The Categories $\mathbb{FF}_*$

From now on a slight abuse of notation will be used: "$C \in \mathbb{C}$" instead of "object C in $\mathbb{C}$".
We introduce appropriate *morphisms* or *arrows* between finite forests F and G in the form of *order-preserving open maps*:

Def. 2.1.6 (*open map*)**.** A map $f : F \to G$ is *open* if it carries any down-set of $S \subseteq F$ to a down-set of $T \subseteq G$:$f(\downarrow S) = (\downarrow T)$.

Remark 13. $f(\downarrow S) = (\downarrow T)$ is equivalent to $\forall x \in F : f(\downarrow x) = \downarrow f(x)$.

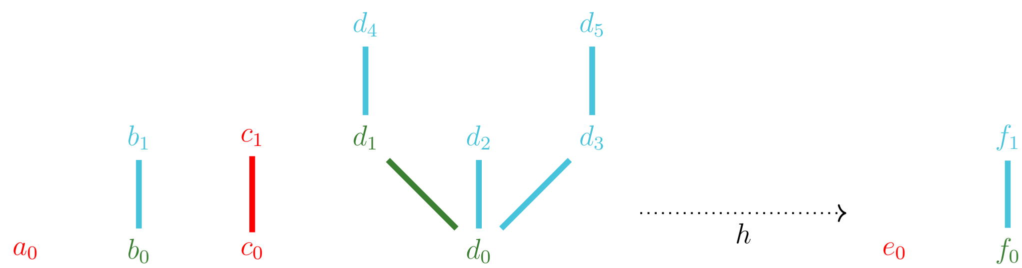


Figure 2.2: An arrow h between forests F (left) and G (right).
The nodes of F are mapped to the nodes of corresponding color in G.
For example $a_0, b_0 \mapsto f_0$ and $b_1 \mapsto f_1$.

We define the category in question:

Def. 2.1.7 (category $\mathbb{FF}$)**.** $\mathbb{FF}$ is the category formed by taking finite forests as objects and order-preserving open maps as arrows.

Of particular interest to us are finite forests of *height* at most $n \geq 0$.

Def. 2.1.8 (*height*)**.** The *height* of a finite forest F is the maximum cardinality of a downset $\downarrow x$ for $x \in F$.

In the example F in figure 2.2 has height 3 whilst G has height 2.

Def. 2.1.9 (category $\mathbb{FF}_n$)**.** The (full) subcategory $\mathbb{FF}_n$ of $\mathbb{FF}$ has finite forests of height at most n as objects and open maps between them as arrows. The objects of $\mathbb{FF}_2$ are called *bushes*.

Also the *finite trees* form the subcategory $\mathbb{T}$.

Let $\mathbb{FF}_*$ stand for either $\mathbb{FF}$ or $\mathbb{FF}_n$ for $n > 0$.
The following properties are valid in $\mathbb{FF}_*$: [2]

Theorem 48 (Terminal)**.** *The* terminal *object is given by the singleton forest denoted as* $\boldsymbol{1} := \{\bullet\}$*.*

Theorem 49 (Initial)**.** *The* initial *object is given by the empty forest denoted as* $\boldsymbol{0} := \{\}$*.*

The *Co-product* or *Sum* between two forests F and G is readily given.

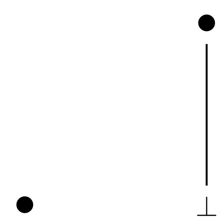


Figure 2.3: Ω.

Theorem 50 (Co-product)**.** *The* Co-product $F + G$ *between* F *and* G *is obtained by taking the* disjoint union *of the two posets and the inclusion maps* $\iota_F : F \hookrightarrow F + G$ *and* $\iota_G : G \hookrightarrow F + G$*.*

For example $\Omega := \mathbf{1} + \mathbf{1}_\perp$ is displayed in figure 2.3.

Corollary 51. *Each forest* F *in* $\mathbb{FF}_*$ *uniquely determines a finite family of trees* $\{F_i\}_{i=1}^N$ *such that* $F = \sum_{i=1}^N F_i$*.* [3]

[2] here we omit to verify that these constructions satisfy the categorical properties of Terminal and Initial objects, Products, Co-products and so on. For reference see [1] & [6].

[3] by $\sum$ we mean the coproduct of the summands.

Now for each finite forest F we write $F_\perp$ as the tree obtained by appending a new bottom/root element as the new minimum.
In fact:

Remark 14. every tree T is of the form $T = T'_\perp$ for some finite forest T'.

(Abuse of notation: oftentimes "=" in these cases is used instead of "≅").
In $\mathbb{FF}$ the *product* $F \times G$ between F and G is defined in a recursive manner
We require the following properties to hold for our product:

Lemma 52.

- *($\times$ by $\boldsymbol{0}$ is $\boldsymbol{0}$)* $\forall F$*:* $F \times \boldsymbol{0} = \boldsymbol{0} = \boldsymbol{0} \times F$
- *($\boldsymbol{1}$ as neutral element of $\times$)* $\forall F$*:* $F \times \boldsymbol{1} = F = \boldsymbol{1} \times F$
- *(distributive law for $\times$)* $\forall F, G, H$*:* $F \times (G + H) = (F \times G) + (F \times H)$

The recursive formula for $F_\perp \times G_\perp$ is given by:

Def. 2.1.10 (Product)**.**

$$F_\perp \times G_\perp := ((F \times G_\perp) + (F \times G) + (F_\perp \times G))_\perp \tag{2.1}$$

The projection maps are also given recursively following the construction of the product object :

Def. 2.1.11 (Projections)**.** The maps $\pi_{F_\perp} : F_\perp \times G_\perp \twoheadrightarrow F_\perp$ and $\pi_{G_\perp} : F_\perp \times G_\perp \twoheadrightarrow G_\perp$ are defined as follows:
Let t_0 be the root of $F_\perp \times G_\perp$ and r_0, s_0 the roots respectively of $F_\perp$ and $G_\perp$.

$$\pi_{F_\perp} : t_0 \mapsto r_0$$
$$\pi_{G_\perp} : t_0 \mapsto s_0$$

Let F_0 and G_0 stand for $F_\perp$ and $G_\perp$.
Recalling the representation of each finite forest as a finite sum of trees, $F_\perp = (\sum_{i=1}^{N} F_i)_\perp$ and $G_\perp = (\sum_{j=1}^{M} G_j)_\perp$.
Each element $x \in (F \times G_0) + (F \times G) + (F_0 \times G)$ belongs to a unique tree $F_i \times G_j$ with $(i + j) > 0$.
Also let $\iota_{F_i} : F_i \hookrightarrow F_\perp$ the set-inclusion of the support of F_i into $F_\perp$ and ι_{G_j} the analogous map for G_j. Then:

$$\pi_{F_\perp} : x \mapsto \iota_{F_i}(\pi_{F_i}(x))$$
$$\pi_{G_\perp} : x \mapsto \iota_{G_j}(\pi_{G_j}(x))$$

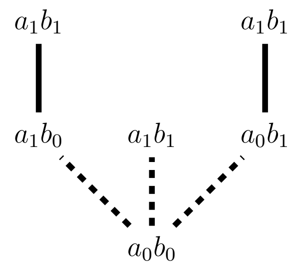


Figure 2.4: The product of $1_\perp = \{a_0 < a_1\} = A$ with $1_\perp = \{b_0 < b_1\} = B$ computed following the recursive formula:
$1_\perp \times 1_\perp = ((1 \times 1_\perp) + (1 \times 1) + (1_\perp \times 1))_\perp = (1_\perp + 1 + 1_\perp)_\perp$.
The labeling of the nodes specifies the projections: a_ib_j indicates that this node is taken to a_i by the 'left' projection π_A and to b_j by the 'right' projection π_B.

Lemma 53. *The object $F_\perp \times G_\perp$ together with $\pi_{F_\perp}$ and $\pi_{G_\perp}$ is proven in [6] to be the desired product object in $\mathbb{FF}$ together with the associated projections.*

Recalling the previous remarks, we have $F = \sum_{i=1}^{N} F_i$ and $G = \sum_{j=1}^{M} G_j$ with each F_i, G_j being a tree equal to $(F_i')_\perp, (G_j')_\perp$ for some finite forests F_i', G_j'.

Theorem 54 (Product in $\mathbb{FF}$)**.** *$\forall F, G \in \mathbb{FF}$ such that $F = \sum_{i=1}^{N} F_i$ and $G = \sum_{j=1}^{M} G_j$ the product is given recursively as:*

$$F \times_{\mathbb{FF}} G = \sum_{i,j=1}^{N,M} (F_i')_\perp \times_{\mathbb{FF}} (G_j')_\perp$$

together with the projection maps π_F and π_G defined as before for each summand.

The product in $\mathbb{FF}_n$ is obtained by *trimming* the product in $\mathbb{FF}$.

Theorem 55 (Product in $\mathbb{FF}_n$)**.** *$\forall F, G \in \mathbb{FF}_n$,*
$F \times_{\mathbb{FF}_n} G$ is the sub-forest of all nodes of height $\leq n$ of $F \times_{\mathbb{FF}} G$ together with the projection maps π_F and π_G restricted *to $F \times_{\mathbb{FF}_n} G$.*

(From now on $\times$ will be used usually instead of $\times_{\mathbb{FF}}$).
For the purpose of this work we only display selected instances of products and projections in the following figures.

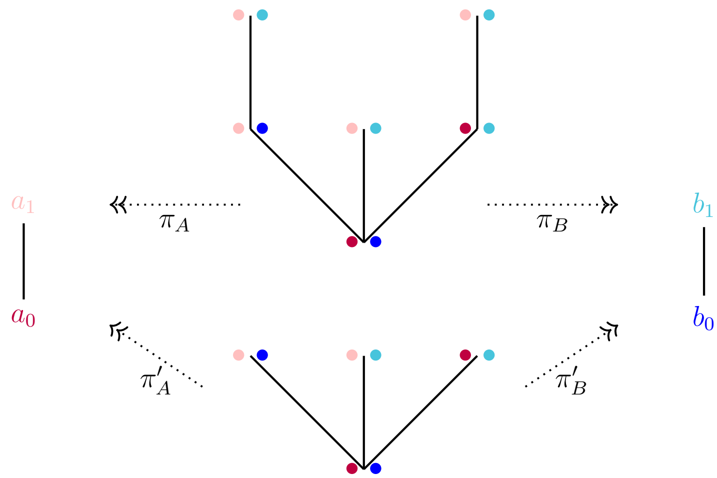


Figure 2.5: At each node of the product forest (top center) $1_\perp \times_{\mathbb{FF}} 1_\perp$ the color of the left and right dot specifies respectively the projection π_A from $1_\perp \times_{\mathbb{FF}} 1_\perp$ to $1_\perp = \{a_0, a_1\}$ (left) and the projection π_B from $1_\perp \times 1_\perp$ to $1_\perp = \{b_0, b_1\}$ (right).
Below we have the product forest for $\mathbb{FF}_2$ (bottom center) $1_\perp \times_{\mathbb{FF}_2} 1_\perp$ and the associated projections π'_A and π'_B.

The product $F \times G$ of two finite forests F and G is *not* the usual cartesian product of the underlying posets.
In fact the cardinality of the underlying set of the product $|F \times G| = 6$ in the case of F and G of the form $1_\perp$ is not the product of the cardinalities of $|F| = 2$ and $|G| = 2$.
(If the context is specified $\times$ will replace $\times_{\mathbb{FF}_n}$).
In the category of *bushes* $\mathbb{FF}_2$ however the underlying set $|A \times B|$ of the product of two bushes A and B *is* the cartesian product $|A| \times |B|$ of the underlying sets of A and B.
In figure figure 2.5 we have $4 = |1_\perp \times 1_\perp| = |1_\perp| \times |1_\perp| = 2 \times 2$.

2.2 The Duality between Algebras and Forests

We now consider the category $\mathbb{G}$ of Gödel algebras and their homomorphisms and the full subcategory $\mathbb{G}_{fin}$ of *finite*[4] Gödel algebras. In turn, for each $n > 0$ the full subcategory $(\mathbb{G}_{\mathbb{n}})_{fin}$ of $\mathbb{G}_{\mathbb{n}}$ contains the *finite n-valued* algebras.
The remarkable result, essentially due to *A.Horn*[5], that we shall use henceforth is the *dual equivalence* between the category $\mathbb{FF}$ of Finite Forests and the category $\mathbb{G}_{fin}$ of finite Gödel algebras realized by the contra-variant functors *Spec* and *Sub*.

Theorem 56 (Finite spectral duality for Gödel algebras)**.**

$$\mathbb{G}_{fin} \simeq \mathbb{FF}^{op}$$

To see how these functors operate we need to introduce the following notions:

Def. 2.2.1 (proper filter)**.** A proper filter $\mathfrak{f}$ for a Gödel algebra $\mathbf{A}$: $\emptyset \neq \mathfrak{f} \subsetneq \mathbf{A}$ is an up-set of $\mathbf{A}$ closed under *meets* i.e., $\forall x, y \in \mathfrak{f} : x \wedge y \in \mathfrak{f}$.

Def. 2.2.2 (prime filter)**.** A prime filter $\mathfrak{p}$ is a proper filter with $0 \notin \mathfrak{p}$ and whenever $(y \vee z) \in \mathfrak{p}$ either $y \in \mathfrak{p}$ or $z \in \mathfrak{p}$.

Def. 2.2.3 (principal filter)**.** $\mathfrak{f}$ is *principal* if $\mathfrak{f} = \uparrow x_{\mathfrak{f}}$ [6] for some $x_{\mathfrak{f}} \in \mathbf{A}$.

Def. 2.2.4 (prime spectrum)**.** The set of prime filters of $\mathbf{A}$ is also called the *prime spectrum* of $\mathbf{A}$, a.k.a. $Spec(A)$ and is partially ordered by reverse-inclusion $\supseteq$.

Def. 2.2.5 (join-irreducible)**.** $x \in \mathbf{A}$ is join-irreducible if $x \neq 0$ and whenever $x = y \vee z$ then either $x = z$ or $x = y$.

The functor *Spec* assigns to an algebra $\mathbf{A}$ the prime spectrum of $\mathbf{A}$ with its reverse ordering.
The functor *Sub* takes the set of sub-forests of $\mathbf{F}$ and forms a finite Gödel algebra with intersection, a new "implication" operator and the empty forest. Both the functors act on morphisms by taking pre-images.

[4] finite cardinality.
[5] see also chapter IX [2].
[6] $\uparrow x := \{y \in \mathbf{A} \mid y \geq x\}$.

Def. 2.2.6 (Spec)**.** $\mathit{Spec} : \mathbb{G}_{fin} \to \mathbb{FF}$

$\mathbf{A} \longmapsto (\mathit{Spec}(\mathbf{A}) = \{\mathfrak{p} \subseteq \mathbf{A} \mid \mathfrak{p} \text{ prime filter}\}, \supseteq)$
$\mathbf{A} \xrightarrow{f} \mathbf{B} \longmapsto \mathit{Spec}(\mathbf{B}) \xrightarrow{f^{-1}\{\}} \mathit{Spec}(\mathbf{A})$

Def. 2.2.7 (Sub)**.** $\mathit{Sub} : \mathbb{FF} \to \mathbb{G}_{fin}$

$\mathbf{F} \longmapsto \mathit{Sub}(\mathbf{F}) = (\{\downarrow G \subseteq F\}, \cup, \cap, \Rightarrow, \emptyset)$ with $H \Rightarrow G := F \setminus \uparrow (H \setminus G)$.
$\mathbf{H} \xrightarrow{h} \mathbf{G} \longmapsto \mathit{Sub}(G) \xrightarrow{h^{-1}\{\}} \mathit{Sub}(H)$

Also, if we restrict the functors to the category of finite n-valued algebras and forests of height n-1:

Theorem 57.

$$(\mathbb{G}_{\mathfrak{n}})_{fin} \simeq \mathbb{FF}^{op}_{n-1}$$

For instance: $(\mathbb{G}_{\mathfrak{3}})_{fin} \simeq \mathbb{FF}^{op}_{2}$*, i.e., the category of bushes.*

We use the following result observed in [6]:

Lemma 58. *In each finite Gödel algebra* $\boldsymbol{A}$*, all filters* $\mathfrak{f}$ *are principal. Every prime filter* $\mathfrak{p}$ *is equal to* $\uparrow x_{\mathfrak{p}}$ *for some join-irreducible element* $x_{\mathfrak{p}}$*.*

To understand how this duality works we start by taking the dual of the *free* Gödel algebra $\mathcal{F}_1$ on one generator x.

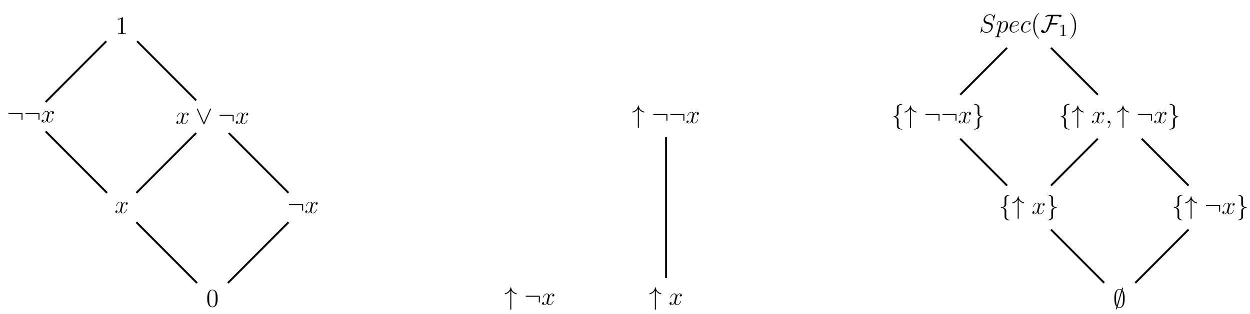


Figure 2.6: $\mathcal{F}_1$ (left), Spec($\mathcal{F}_1$) (center) and Sub(Spec($\mathcal{F}_1$)) (right)

Note that $\Omega := \mathbf{1} + \mathbf{1}_{\perp} = \mathit{Spec}(\mathcal{F}_1)$. This finite forest will be ubiquitous in the upcoming chapters.
What this duality tells us is that:

Remark 15. Each finite Gödel algebra $\mathbf{A}$ is isomorphic to the Gödel algebra of all sub-forests of $(Spec(\mathbf{A}), \supseteq)$ taken as a finite forest.

The knowledge of a dual category to the category of Gödel algebras in which products and co-products are readily computed provides us with a valuable tool for the study of the structure of these algebras. For example:

Remark 16. The free algebra on $k > 0$ generators $\mathcal{F}_k = k \cdot [\mathcal{F}_1]$ is the *k-th co-power* [7] of the free algebra on one generator, as such its dual is $[Spec(\mathcal{F}_1)]^k$ the *k-th power* [8] of $Spec(\mathcal{F}_1)$.

In particular we have:

Remark 17. As a consequence of $(\mathbb{G}_3)_{fin} \simeq \mathbb{FF}_2^{op}$:

$$\mathcal{F}_1 \cong Sub(Spec(\mathcal{F}_1)) = Sub(\mathbf{1} + \mathbf{1}_\perp) = Sub(\Omega).$$

In fact, this duality can be seen as a *generalization* of the finite case of *Stone's Representation Theorem* ([8]) for Boolean algebras whereby *finite Boolean algebras are dually equivalent to finite sets*.
The latter says that classical propositional logic *CPL* can be seen as the logic of *finite sets*:

Remark 18. A formula ϕ of *CPL* seen as a *characteristic function* from a finite set/domain/universe X determines a *subset* $\{x \in X \mid \phi(x) = 1\}$ of elements for which ϕ *is true*.
Similarly, a formula ϕ of $\mathcal{G}$ determines a *sub-forest* of a finite forest for which ϕ *is true*.

To be more precise:

Remark 19. For every formula ϕ which contains propositional variables $p_1, .., p_n$: Finite Forests provide (sound and complete) semantics of Gödel-Dummett Propositional Logic $\mathcal{G}$, i.e., [9]

$$\mathcal{G} \vdash \phi \text{ iff } Sub(Spec(\mathcal{F}_n)) \models_{H.A.} \phi \text{ iff } [\![\phi]\!]_{Sub(Spec(\mathcal{F}_n))} = Spec(\mathcal{F}_n).$$

[7] i.e., the co-product iterated k times over the same object.
[8] i.e., the product iterated k times over the same object.
[9] here $\models_{H.A.}$ refers to Heyting algebra validity.

Using the same *Stone-style* duality as before we obtain:

$$
\begin{tikzcd}
 & 1 \arrow[dl, dash] \arrow[dr, dash] & & & & & & 2 \arrow[dl, dash] \arrow[dr, dash] & \\
x \arrow[dr, dash] & & \neg x \arrow[dl, dash] & & & & \{\uparrow x\} \arrow[dr, dash] & & \{\uparrow \neg x\} \arrow[dl, dash] \\
 & 0 & & & \uparrow \neg x & \uparrow x & & \emptyset &
\end{tikzcd}
$$

Figure 2.7: Free boolean algebra on one generator $\mathcal{B}_1$ (left), the finite forest $\mathbf{2} = \mathbf{1} + \mathbf{1}$ (center) and the Hasse diagram of the sub-forests of $\mathbf{2}$ (right).

In fact:

Remark 20. We re-discover under a different guise the familiar correspondence between the Free Boolean algebra on x $\mathcal{B}_1$ and the two prime filters $\uparrow x$ and $\uparrow \neg x$ of the *Stone Space* $\mathbf{2}$ by observing that:

$$\mathbb{BA}_{fin} \cong (\mathbb{G}_2)_{fin} \simeq \mathbb{FF}_1^{op} = \mathbb{Set}_{fin}^{op}$$

2.3 Forests, Bushes and Topoi

Why *Topoi*?
For classical logic all objects and sub-objects would be represented by sets and sub-sets. This, incidentally, is the case for finite forests of height 1, i.e., $\mathbb{FF}_1$, which is equivalent to the category of finite sets $\mathbb{Set}_{fin}$. However for finite forests of height greater than 1 something different is required.
Topoi as such are a categorical generalization of sets and provide an ideal framework for non-classical logic. In a Topos we wish to abstract notions of sub-sets, elements and set constructions like products and exponentiation.

Recall the definition of an *(elementary) Topos*.

Def. 2.3.1 (Topos)**.** A topos $\mathcal{E}$ is a category $\mathcal{E}$ such that:

1. $\mathcal{E}$ is finitely complete and co-complete.
2. $\mathcal{E}$ has a sub-object classifier.
3. $\mathcal{E}$ has exponential objects.

We have seen that the categories $\mathbb{FF}, \mathbb{FF}_k$ for $k \geq 1$ have initial and terminal objects, finite products and co-products.
Finite completeness follows from the construction of *equalizers* for trees in $\mathbb{T}$ by taking the inclusion-maximal sub-tree contained in the equalizing sub-poset and generalizing to $\mathbb{FF}$.
Finite completeness and co-completeness also follows from the duality with Gödel algebras exploiting the universal algebra fact that locally finite varieties are finitely complete and co-complete.

Lemma 59 (finite forests are finitely complete and co-complete)**.** *$\mathbb{FF}, \mathbb{FF}_k$ for $k \geq 1$ are finitely complete and co-complete.*

What about the *sub-object classifier*?

2.3.1 Sub-object Classifiers

Recall what it means for a category $\mathbb{C}$ to have a *sub-object classifier*:

Def. 2.3.2 (Sub-object classifier)**.** Let $\mathbb{C}$ be a category with a terminal object $\mathbf{1}$. A *sub-object classifier* for $\mathbb{C}$ is an object Ω together with an arrow $true : \mathbf{1} \to \Omega$ that satisfies the following Ω-axiom

Def. 2.3.3 (Ω-axiom)**.** For each sub-object $s : A \rightarrowtail B$ there is a unique *characteristic* arrow $\chi_s : B \to \Omega$ making the following commutative diagram (a.k.a. the characteristic diagram of s) a *pullback* of χ_s and *true* (the unique arrow from A to the terminal object 1 is named $!_A$): In other words, there

$$
\begin{tikzcd}
A \arrow[r, tail, "s"'] \arrow[d, dashed, "!_A"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\chi_s"'] \\
1 \arrow[r, "true"] & \Omega
\end{tikzcd}
$$

Figure 2.8: the characteristic diagram of s.

must be a 1:1 correspondence between sub-objects and characteristic arrows:

$$Sub(B) \cong \mathbb{C}(B, \Omega).$$

For $\mathbb{S}\mathfrak{et}_{fin}$ the sub-object classifier is $\mathbf{2} = \{0, 1\}$ the two element set together with the map $true : \mathbf{1} = \{0\} \to \mathbf{2}\ ,\ 0 \mapsto 1$.
Equivalently in $\mathbb{FF}_1$ the classifier is $\mathbf{2} = \mathbf{1} + \mathbf{1}$ (displayed as the nodes $\mathbf{f}$ and $\mathbf{t}$) and the arrow $true : \mathbf{1} = \{\bullet\} \to \mathbf{2}\ ,\ \bullet \mapsto \mathbf{t}$.
The $\mathbb{FF}_1$-equivalent of each finite set is a *finite anti-chain* represented by a sum of $\mathbf{1}$s.
If we take a sub-set of a finite set say $A = \{a, b, d\} \subset B = \{a, b, c, d, e\}$ where $\subset$ is given by the inclusion arrow ι_A, the *characteristic* arrow χ_A sends the nodes of A to $\mathbf{t}$ and those of $B \setminus A$ to $\mathbf{f}$.
The characteristic diagram in this case is given by: (coloring the domain of *true* and χ_A is meant to show that $\bullet$ and $\bullet$ map to $\mathbf{t}$ and $\mathbf{f}$ respectively in $\mathbf{2}$).

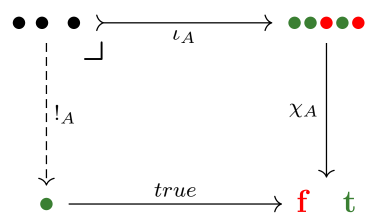


Figure 2.9: Characteristic diagram for $\{a, b, d\} \subset \{a, b, c, d, e\}$.

Generalizing to Finite Forests $\mathbb{FF}$: We know, thanks to [1], that the object $\Omega = \mathbf{1} + \mathbf{1}_\perp = Spec(\mathcal{F}_1)$ *assumes the role* of $\mathbf{2}$ together with an appropriate arrow *true* and is in fact a sub-object classifier for $\mathbb{FF}$.

Theorem 60 (Sub-object classifier for $\mathbb{FF}$)**.** *Ω together with true : $\mathbf{1} \to \Omega$ is the sub-object classifier for $\mathbb{FF}$.*
true is defined as the unique map that carries $\bullet$ to the root of $\mathbf{1}_\perp \subset \Omega$.

Observe that Ω is an object of $\mathbb{FF}_2$ and 1 is the terminal object of every $\mathbb{FF}_k$ $k > 0$ so:

Corollary 61. *Ω and true (as defined for $\mathbb{FF}$) form the sub-object classifier of $\mathbb{FF}_k$ for any $k > 1$.*

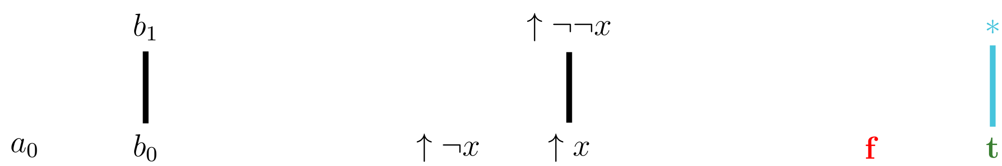


Figure 2.10: We recollect the different representations of Ω as $1 + 1_\perp$ (left), $Spec(\mathcal{F}_1)$ (center) and also give a new one (right) using the labels "**f**","**t**" and "$*$" .

To see why this holds:
Given a sub-object, say $f : F \hookrightarrow G$, which determines a sub-forest $f[F]$ of G, there is a unique $\chi_f : G \to \Omega$ making the characteristic diagram a pullback. An instance of this is given in which $f : \alpha_j \mapsto a_j, \beta_j \mapsto b_j, \delta_0 \mapsto d_0$ for $j = 0, 1$ (the usual coloring notation on the domain applies).

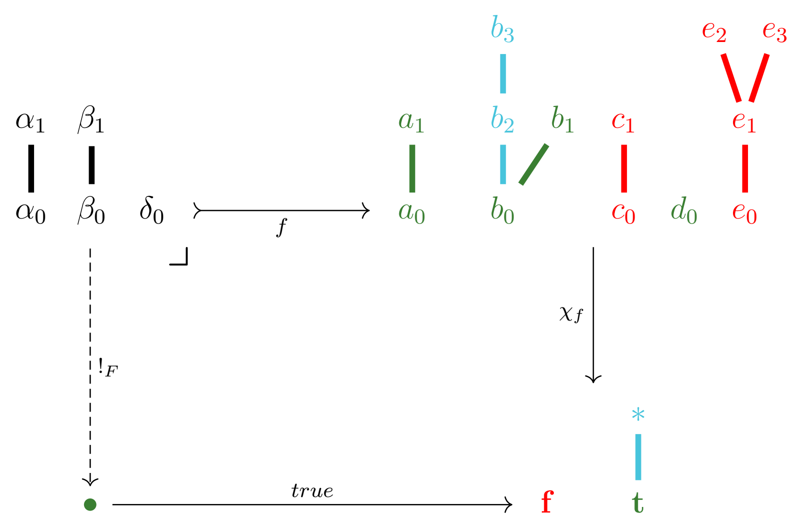


Figure 2.11: Characteristic diagram of $f : F \hookrightarrow G$.

Notice that χ_f assigns the value **t** to the nodes of the sub-forest $f[F]$, the value $*$ to any node that is not in the sub-forest but is in the up-set of $f(F)$ so and the value **f** to any node that is not in the sub-forest nor in its up-set.

2.3.2 Exponentials

We now come to *exponentiation*.

Recall what it means for a category $\mathbb{C}$ to have an *exponential object*:

Def. 2.3.4 (exponential objects)**.** $\forall A, B \in \mathbb{C}$ there exists an exponential object $B^A \in \mathbb{C}$ and a map $eval : B^A \times A \to B$ such that the *universal mapping property* or UMP holds: For any $g : C \times A \to B$ there is a unique $\hat{g} : C \to B^A$ making the following diagram commute:

$$\begin{tikzcd}
B^A \times A \arrow[r, "eval"'] & B \\
C \times A \arrow[u, "\hat{g}\times id_A"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

Figure 2.12: UMP

Remark 21. The UMP can also be read as: $\forall g : C \times A \to B \;\; \exists! \; \hat{g} : C \to B^A$ called an *encoding of* g such that $eval \circ (\hat{g} \times id_A) = g$.

Remark 22. This can also be seen as an *adjunction* $(-) \times A \dashv (-)^A$ between the *product* and *exponent* functors by a fixed object A which gives rise to a 1:1 correspondence for any object B:

$$Hom(C \times A, B) \cong Hom(C, B^A)$$
$$g \overset{1:1}{\longleftrightarrow} \hat{g}$$

Remember also that for finite sets $\mathbb{Set}_{fin}$ the exponential is defined as $B^A := \{f : A \to B\}$ i.e., the set of functions from A to B.
$\hat{g}$ is the result of the operation of *currying* i.e., $\hat{g}(c) := g(c, -) : A \to B$ and *eval* is *evaluation* of a function f in a is $eval(f, a) := f(a)$ so that $eval \circ (\hat{g} \times id_A)(c, a) = eval(\hat{g}(c), a) = (\hat{g}(c))(a) = g(c, a)$.

We now proceed outlining the steps used in [1]:

Do exponential objects exist in $\mathbb{FF}_*$?

If they exist then, since $\mathbb{FF}_*$ is a *distributive* category:

Lemma 62. *For all objects F, G, H in $\mathbb{FF}_*$:*

$$F^{(G+H)} \cong F^G \times F^H$$

Recall that any $F \in \mathbb{FF}_*$ can be written as $F = \sum_{i=1}^N T_i$ for some trees $\{T_i\}_{i=1}^N$, this allows us to reduce the study of the existence of exponentiation to the case F^T for some tree T.

Since $C \cong (C \times 1) \to F$ is adjoint to $C \to F^1$ for any $C \in \mathbb{FF}_*$, we have $F^1 \cong F$ for every $F \in \mathbb{FF}_*$. We now have:

Remark 23. $\forall F, G \in \mathbb{FF}_* : (F+G)^1 \cong F^1 + G^1$.

This can be generalized. It can be shown that $F^T + G^T$ *behaves as* $(F + G)^T$. [10]
What this entails is:

Lemma 63. $\forall F, G \in \mathbb{FF}_*, T \in \mathbb{T}$:

$$(F+G)^T \cong F^T + G^T$$

In fact reducing further the study to the existence of the exponential T^S for both $T, S \in \mathbb{T}$, we now have for $F, G \in \mathbb{FF}_*$:

Theorem 64. *For $F = \sum_{i=1}^n T_i$ and $G = \sum_{j=1}^m S_j$:*

$$F^G \cong \prod_{j=1}^{m} \sum_{i=1}^{n} (T_i^{S_j})$$

[10]i.e., for each $f : H \times T \to F + G$ there is a unique $\hat{f} : H \to F^T + G^T$ such that $(e_F + e_G) \circ (\hat{f} \times id_T) = f$ with e_F, e_G the evaluation maps for F^T and G^T.

Exponentiation for $\mathbb{FF}_1$ has been settled as $\mathbb{FF}_1 \simeq Set_{fin}$.
The next "level up" is $\mathbb{FF}_2$ or the category of *Bushes*.
Each *bush* in $\mathbb{FF}_2 \cap \mathbb{T}$ is of the form $B_\perp$ where B is a finite anti-chain i.e., a finite set.

By the previous considerations, we arrive at the main result of [1]:
(Note that $|A_\perp|$ denotes the cardinality of the underlying poset and $n\mathbf{F}$ is shorthand for the n-th co-power of $\mathbf{F}$).

Theorem 65 (*bushes* have exponential objects)**.** *The following formula holds for all $A_\perp$ and $B_\perp \in \mathbb{FF}_2 \cap \mathbb{T}$:*

$$B_\perp^{A_\perp} \cong |B_\perp|^{|A|}(\ (\ (|B_\perp|^{|A_\perp|} - 1)\mathbf{1})_\perp\ ) \tag{2.2}$$

Using distributivity we can generalize this formula to arbitrary Bushes F and G, written as sums of trees in $\mathbb{FF}_2$, $F = \sum_{i=1}^{m}(F_i)_\perp$ and $G = \sum_{j=1}^{n}(G_j)_\perp$:

$$F^G \cong \prod_{j=1}^{n}\sum_{i=1}^{m}(|(F_i)_\perp|^{|G_j|}(\ (\ (|(F_i)_\perp|^{|(G_j)_\perp|} - 1)\mathbf{1})_\perp\ )) \tag{2.3}$$

The intuition behind this fact is that trees in $\mathbb{FF}_2$ behave with respect to arrows nearly as finite sets. This is because arrows f from $B_\perp$ to a target $C_\perp$ are basically (Set-)functions with the only constraint that the root of $B_\perp$ be sent to the root of $C_\perp$.

Putting together the results in Theorem 65,Corollary 61 & Lemma 59 we obtain:

Corollary 66 (*bushes* form a topos)**.** $\mathbb{FF}_2$/bushes *is an elementary topos.*

We wrap up by giving a few examples of exponentials for Bushes:

Example 67.

$$\Omega^{\mathbf{1}_\perp} \cong \mathbf{1} + 2(3 \cdot \mathbf{1})_\perp.$$
$$(\mathbf{1}_\perp)^\Omega \cong 2(7 \cdot \mathbf{1})_\perp.$$
$$\Omega^\Omega \cong \mathbf{1} + \mathbf{1}_\perp + 2(3 \cdot \mathbf{1})_\perp + 2(7 \cdot \mathbf{1})_\perp.$$

2.3.3 An Exponential Example

Let us examine how these exponential objects work with the example of arrows from $\mathbf{1}_\perp \times \mathbf{1}_\perp$ to $\mathbf{1}_\perp$.
The formula 2.2 says that the exponential object should be:

$$(\mathbf{1}_\perp)^{\mathbf{1}_\perp} \cong 2^1(((2^2 - 1)\mathbf{1})_\perp) = 2(3 \cdot \mathbf{1})_\perp$$

The universal mapping property (UMP) applied to this case states that the following diagram should commute.

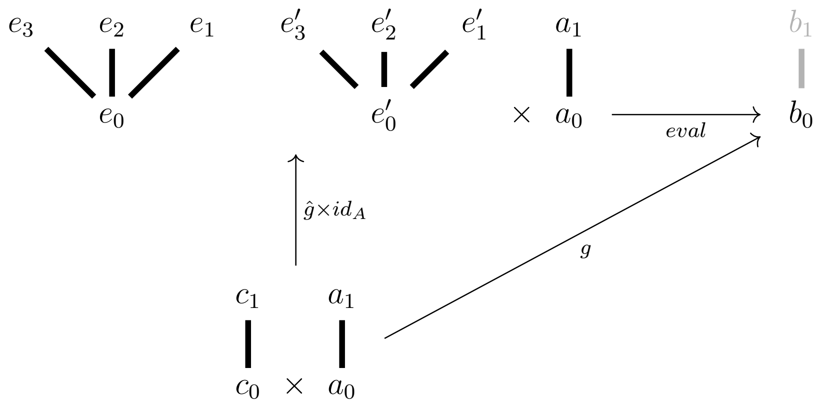


Figure 2.13: UMP diagram for $A = B = C = \mathbf{1}_\perp$. (Bushes)

To understand how to construct the *adjoint* arrow $\hat{g} : \mathbf{1}_\perp \to (\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ of $g : \mathbf{1}_\perp \times \mathbf{1}_\perp \to \mathbf{1}_\perp$ let us take a look at all the possible maps (black dots are mapped to the root b_0 of $B = \mathbf{1}_\perp$ whilst gray dots to b_1):
For example, one of these maps g_1 is:

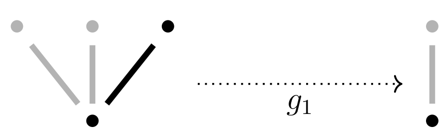


Figure 2.14: $C \times A = \mathbf{1}_\perp \times \mathbf{1}_\perp$ is displayed on the left and $B = \mathbf{1}_\perp$ on the right. The map g_1 sends the nodes on the left object to nodes with matching color on the right object.

Notice that:

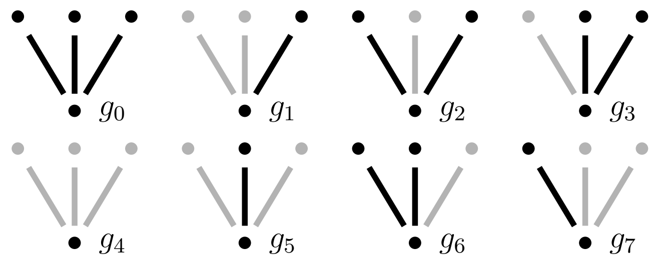


Figure 2.15: All the possible maps $g_j : \mathbf{1}_\perp \times \mathbf{1}_\perp \to \mathbf{1}_\perp$ for j=0,..,7. (Bushes)

Lemma 68. $|Hom(\boldsymbol{1}_\perp \times \boldsymbol{1}_\perp, \boldsymbol{1}_\perp)| = 8$.

This result could be computed by simply counting the number of distinct set-functions from $3 \cdot \mathbf{1}$ to $\mathbf{1}_\perp$ which is 2^3 since as we saw $\mathbf{1}_\perp \times \mathbf{1}_\perp = (3 \cdot \mathbf{1})_\perp$ and there are no constraints on where the non-root elements are assigned.

How to obtain the adjoint maps $\hat{g}_j$?

The *product map* $\hat{g}_j \times id_A$ by definition must satisfy:

1. $\pi_{B^A} \circ (\hat{g}_j \times id_A) = \hat{g}_j$.
2. $\pi_A \circ (\hat{g}_j \times id_A) = id_A$.

From the first condition, the fibers for $i = 0, 1$ $\pi_C^{-1}\{c_i\}$ of $C \times A$ must be mapped to the corresponding fibers of $\pi_{B^A}^{-1}\{\hat{g}_j(c_i)\}$ of $B^A \times A$.
From the second condition, the fibers for $i = 0, 1$ $\pi_A^{-1}\{a_i\}$ of $C \times A$ must be mapped to the corresponding fibers of $\pi_A^{-1}\{a_i\}$ of $B^A \times A$.

The *eval* map can now be constructed on the product $B^A \times A$.

Starting off from the map g_0 which sends every node to the root b_0 we have that $\hat{g}_0 : C = \mathbf{1}_\perp \to \mathbf{1}_\perp^{\mathbf{1}_\perp}$ must send the nodes c_0 and c_1 of C to a root (either e_0 or e_0') of one of the trees $T_1 (\cong (3 \cdot \mathbf{1})_\perp)$ or T_2 $(\cong (3 \cdot \mathbf{1})_\perp)$.

Without loss of generality we can say that $\hat{g}_0$ maps c_0, c_1 to the root e_0 of T_1.
By a similar argument we find that $\hat{g}_1 : c_0 \mapsto e_0$ [11] and $c_1 \mapsto e_1$.
Also $\hat{g}_2 : c_0 \mapsto e_0, c_1 \mapsto e_2$ and $\hat{g}_3 : c_0 \mapsto e_0, c_1 \mapsto e_3$. The rest follow in a

[11] recall that every root must be mapped to a root.

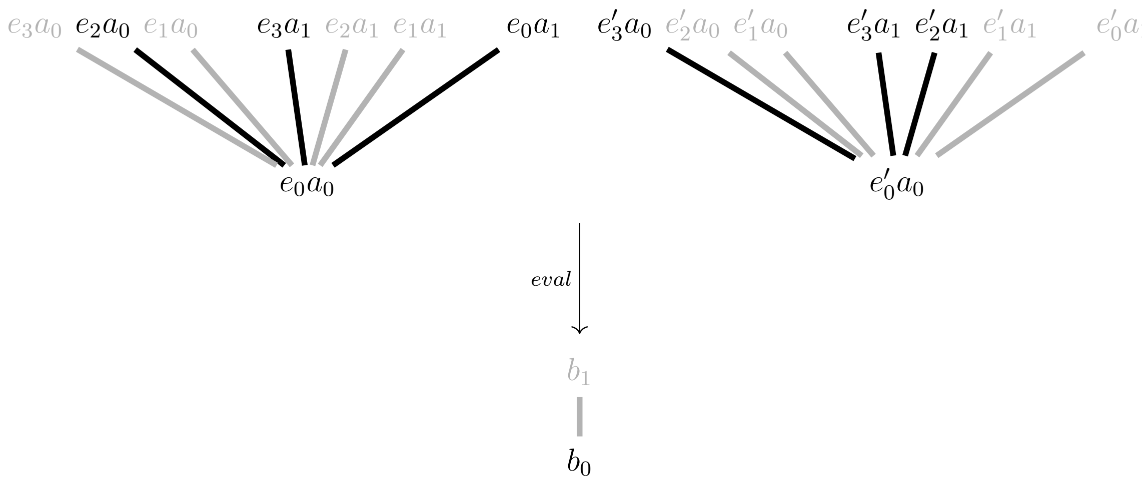


Figure 2.16: The map $eval : (\mathbf{1}_\perp)^{\mathbf{1}_\perp} \times \mathbf{1}_\perp \to \mathbf{1}_\perp$. (Bushes)
(The usual labeling notation on nodes is used for product objects whereby the left and right letters specify the left and right projections respectively).

similar fashion with the maps $\hat{g}_k$ $k = 4, .., 7$ sending c_0 to e'_0 instead.
By requiring that the UMP diagram commute, the fibers for $j = 0, .., 7$ of $g_j^{-1}\{b_i\}$ for $i = 0, 1$ must be preserved by the mapping $\hat{g}_j \times id_A$.

Note that $\hat{g}_j$ is entirely determined by $\hat{g}_j(c_1)$ since we know that c_0 maps to the root of the tree that contains $\hat{g}_j(c_1)$. This suggests the following:

Lemma 69. *The exponential object* $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ encodes *the maps* $\{g_j\}_{j=0}^{7}$.

It is worth pausing here and make a more general categorical consideration which will come in useful later on:

Proposition 70. *In* $\mathbb{FF}_2$/bushes *we have that* $\mathbf{1}_\perp$ *is the representing object i.e.,*

$$\mathbb{FF}_2(\mathbf{1}_\perp, F) \cong |F|$$

whereby $|F|$ *denotes the underlying set of the finite forest* F.

Proof. Let $\mathbf{1}_\perp = \{\perp < \bullet\}$ and $|F| = \{a_j\}_{j=1}^{n}$ where each a_j is either a root r_j or on top of (a unique) one i.e., $r_j < a_j$.
The bijection is realized as we saw earlier by $f_j \leftrightarrow a_j$ where:

$$f_j := \bullet \mapsto a_j \quad \text{and } \perp \mapsto r_j$$

$\square$

We resume our example:
This *encoding* can be visualized by labeling each node that corresponds to $\hat{g}_j(c_1)$ with the map g_j so that the following figure is obtained:

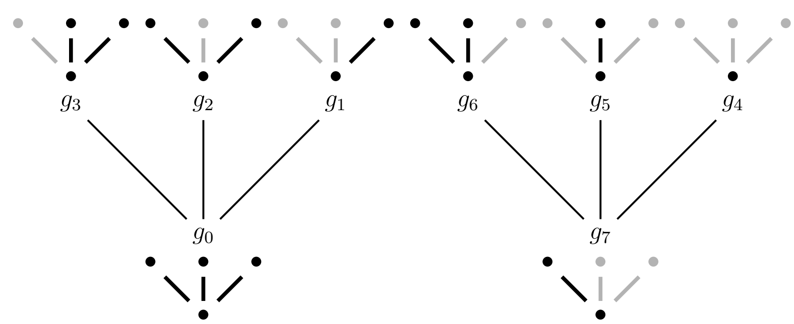


Figure 2.17: The encoded maps in $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$. (Bushes)

The category of *bushes* $\mathbb{FF}_2$ is a *topos.*
What about *higher* finite forests $\mathbb{FF}_k$ with $k \geq 3$?

(Henceforth we denote $\mathbb{FF}_k$ with $k \geq 3$ by $\mathbb{FF}_{k\geq 3}$.)

2.4 When Forests fail to be a Topos

As the title suggests in this section we are going to prove that finite forests $\mathbb{FF}_k$ higher than bushes with $k > 2$ *fail* to be a topos by detailing step by step a novel constructive counter-example.

The failure occurs when we require the existence of exponential objects. Recall that exponentiation for Bushes was obtained thanks to the fact that every tree in $\mathbb{FF}_2$ was of the form $F_\perp$ with F a finite anti-chain or set. Thus every map between trees $B_\perp \to C_\perp$ was essentially a set-function with the only requirement that the root node be mapped to the root of the target. This property is lost when we move to $\mathbb{FF}_k$ with $k > 2$.
In fact, as we shall prove: $\mathbb{FF}_k$ with $k > 2$ generally fails to have exponential objects.

2.4.1 Counter-example for $\mathbb{FF}_3$

Let's start by re-examining the previous example of $g : \mathbf{1}_\perp \times \mathbf{1}_\perp \to \mathbf{1}_\perp$. Though this time $\times$ is taken not to be $\times_{\mathbb{FF}_2}$ but rather $\times_{\mathbb{FF}}$.
Let us take a look at some of the possible maps.

For example, one of these maps g_1 is:

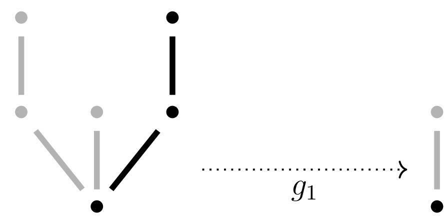


Figure 2.18: As before black dots are mapped to the root b_0 of $B = \mathbf{1}_\perp$ whilst gray dots to b_1.

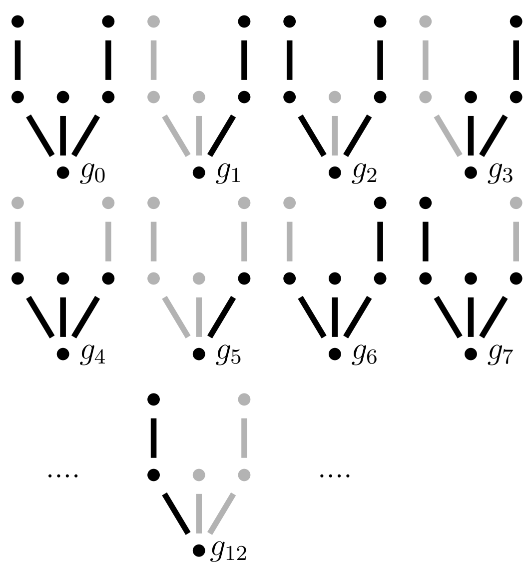


Figure 2.19: The maps $g_j : \mathbf{1}_\perp \times \mathbf{1}_\perp \to \mathbf{1}_\perp$ for $j = 0, .., 17$.

Notice that in this case:

Lemma 71. $|Hom(\boldsymbol{1}_\perp \times \boldsymbol{1}_\perp, \boldsymbol{1}_\perp)| = 18 = |Hom(\boldsymbol{1}_\perp, (\boldsymbol{1}_\perp)^{\boldsymbol{1}_\perp})|$.

The previous result can be obtained in a number of ways. One of them is to count all the possible *extensions* in $\mathbb{FF}$ of the old maps g_j used in $\mathbb{FF}_2$. Another is to count all the possible sub-forests or down-sets of $\mathbf{1}_\perp \times \mathbf{1}_\perp$ since B has only two elements and the pre-image of the root b_0 uniquely determines the map.
The exponential object $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ for $\mathbb{FF}_k$ $k > 2$ is not readily given by some formula as in the case of Bushes. However, we have reason to state that in this case:

Lemma 72. *The* candidate *for* $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ *must be a* super-forest *of the same-name exponential object found for the category of Bushes i.e., of* $2(3 \cdot \mathbf{1})_\perp$.

This is due to the fact that $\mathbf{1}_\perp \times_{\mathbb{FF}_2} \mathbf{1}_\perp$ is by definition a sub-forest (of nodes of height at most 2) of $\mathbf{1}_\perp \times_{\mathbb{FF}} \mathbf{1}_\perp$ and the maps *encoded* by $\mathbf{1}_\perp \times_{\mathbb{FF}} \mathbf{1}_\perp$ are all *extensions* of maps already encoded by $\mathbf{1}_\perp \times_{\mathbb{FF}_2} \mathbf{1}_\perp$.

Remark 24. Another approach to Lemma 72 is obtained by looking at arrows *into* the *candidate* exponential object "$(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$".
We can start by the simple case of an arrow from $\mathbf{1}$ and proceed from there by examining the arrows from $\mathbf{1}_\perp$, $(\mathbf{1}_\perp)_\perp$ and so forth.
Note that effectively this is an application of the *Yoneda Lemma*[12] as we are determining the structure of an unknown object by looking at its *relations*/arrows with other objects C:

$$X \cong (\mathbf{1}_\perp)^{\mathbf{1}_\perp} \text{ iff } \forall C \in \mathbb{FF}_* : Hom(C, X) \cong Hom(C, (\mathbf{1}_\perp)^{\mathbf{1}_\perp}).$$

Now,
by adjunction we have:

$$|Hom(\mathbf{1}, (\mathbf{1}_\perp)^{\mathbf{1}_\perp})| = |Hom(\mathbf{1} \times \mathbf{1}_\perp, \mathbf{1}_\perp)| = |Hom(\mathbf{1}_\perp, \mathbf{1}_\perp)| = 2$$

Since a root can be mapped only to another root this tells us that our candidate for $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ is the sum of two trees $T_0 + T_1$.
Also by the previous remark since there should be precisely 18 maps from $\mathbf{1}_\perp$ to $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ we know that between them these trees have $18 - 2 = 16$ (the 2 constant maps on roots have been subtracted) *first-level branches* i.e., branches from the root node.
The same arguments about *fibers* we used in the case of *bushes* apply here. For example $\hat{g}_0$ must send the nodes of $C = \mathbf{1}_\perp$ to a root say e_0 of the tree T_0 and $\hat{g}_{12}$ to the root e'_0 of T_1.. etc.

[12] see [10] for reference.

The universal mapping property (UMP) in this case states that the following diagram should commute:
(Recall that $A = \{a_0 < a_1\} = \mathbf{1}_\perp, B = \{b_0 < b_1\} = \mathbf{1}_\perp$ and $C = \{c_0 < c_1\} = \mathbf{1}_\perp$,). (Note that our candidate for $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ is purposefully left incomplete as only the first two levels are shown).
The dotted lines in $\mathbf{1}_\perp \times \mathbf{1}_\perp$ show that $\mathbf{1}_\perp \times_{\mathbb{FF}} \mathbf{1}_\perp$ is a super-forest of $\mathbf{1}_\perp \times_{\mathbb{FF}_2} \mathbf{1}_\perp$. The nodes of $\mathbf{1}_\perp \times \mathbf{1}_\perp$ will be labeled with Greek letters and the usual coloring notation will apply to display projections.

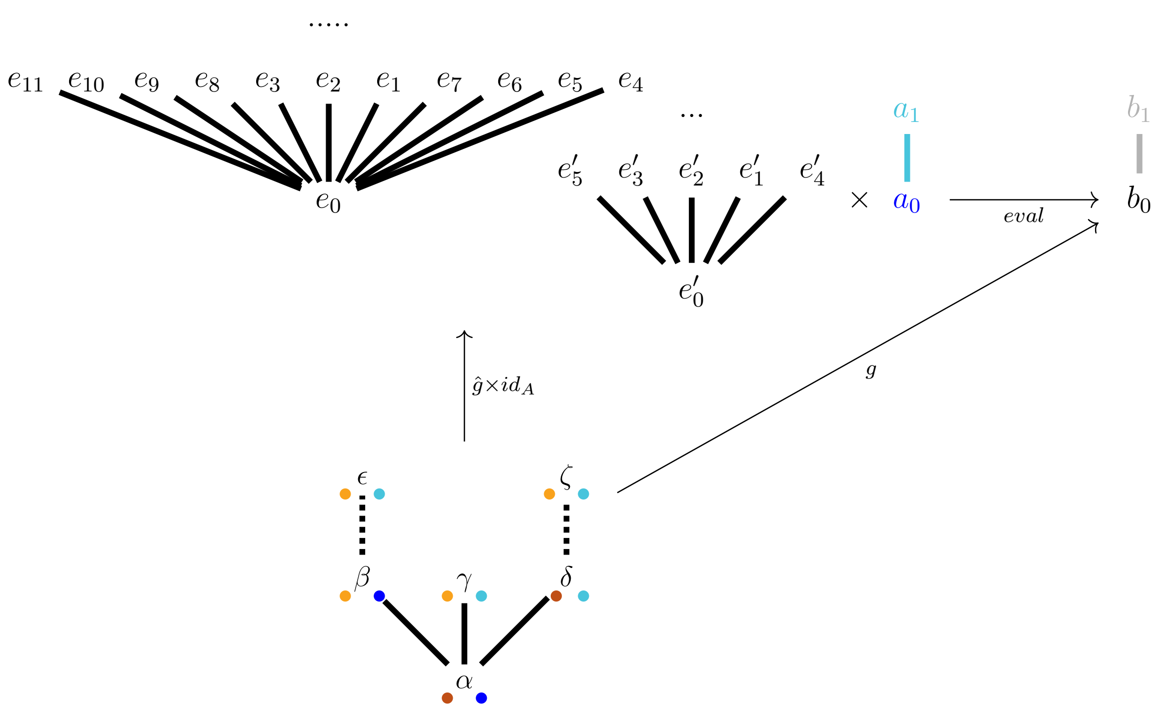


Figure 2.20: UMP diagram for $A = B = C = \mathbf{1}_\perp$.

(From now on the candidate for the exponential object in $\mathbb{FF}_3$ will be denoted by "B^A" or simply by B^A).

The map *eval* can be fully described as we saw in the case of Bushes. However we will concentrate our attention on a specific part of its domain in "$(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$" $\times\, \mathbf{1}_\perp$ namely the first levels of $T_0 \times \mathbf{1}_\perp$ and display only this part in detail leaving the rest purposefully incomplete.

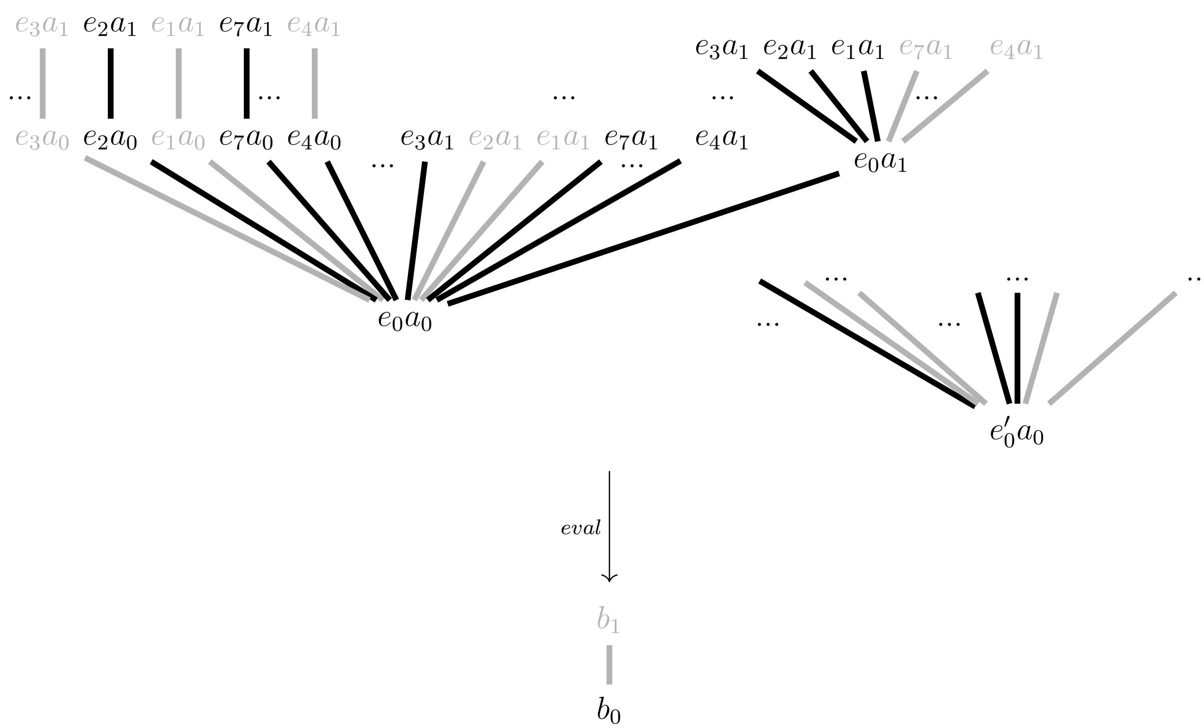


Figure 2.21: The map $eval : B^A \times A \to B : (\mathbf{1}_\perp)^{\mathbf{1}_\perp} \times \mathbf{1}_\perp \to \mathbf{1}_\perp$.
The pre-image of b_1 is colored in gray and that of b_0 in black.
In this case the nodes of the product are labeled with $e_i a_j$ to specify the left and right projections to e_i and a_j respectively.

Remember that, as in the previous case for *bushes*, we would like for the exponential object to encode the maps $\{g_j\}_{j=0}^{17}$. The first two levels of $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ can thus be labeled in a similar fashion as before.

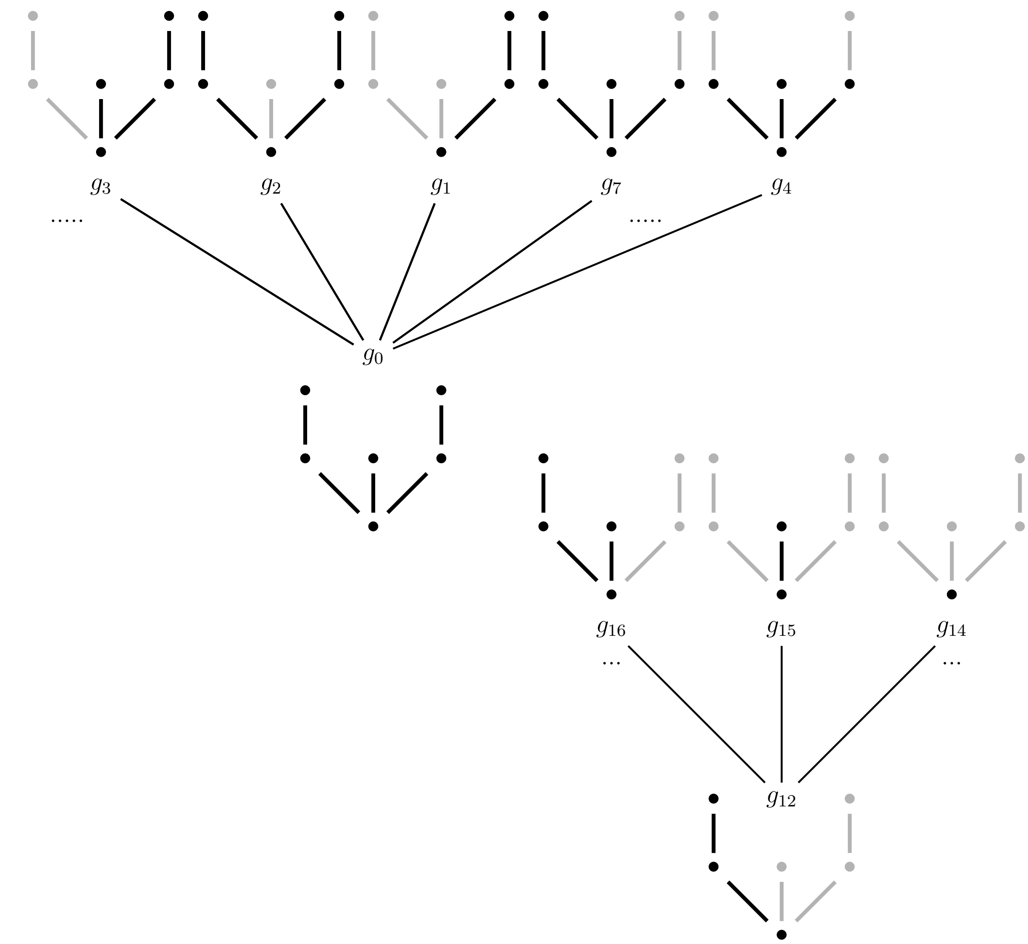


Figure 2.22: The encoded maps $\{g_j\}_{j=0}^{17}$ in $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$.
Not all of them have been displayed.

Now we take a new $C = (\mathbf{1}_\perp)_\perp = \{c_0 < c_1 < c_2\}$ (which is just the old C with a new *child* c_2 of c_1) and consider maps from the new product $C \times A = (\mathbf{1}_\perp)_\perp \times \mathbf{1}_\perp$ to $B = \mathbf{1}_\perp$.
The following holds from Theorem 55:

Lemma 73. *For any $F, G \in \mathbb{FF}$ $F_\perp \times G$ is a super-forest of $F \times G$.*

Remark 25. We have seen that $\mathbf{1}_\perp \times \mathbf{1}_\perp \in \mathbb{FF}_3$ and so $(\mathbf{1}_\perp)_\perp \times_{\mathbb{FF}_3} \mathbf{1}_\perp$ is a super-forest of $\mathbf{1}_\perp \times \mathbf{1}_\perp$.

With this in mind, we examine the new UMP diagram in the $\mathbb{FF}_3$ environment and introduce Greek letters to mark the nodes of $C \times A$ and left and right colored bullets on the product to display the left and right projections. As before we use dotted lines to highlight the new branches of the super-forest.

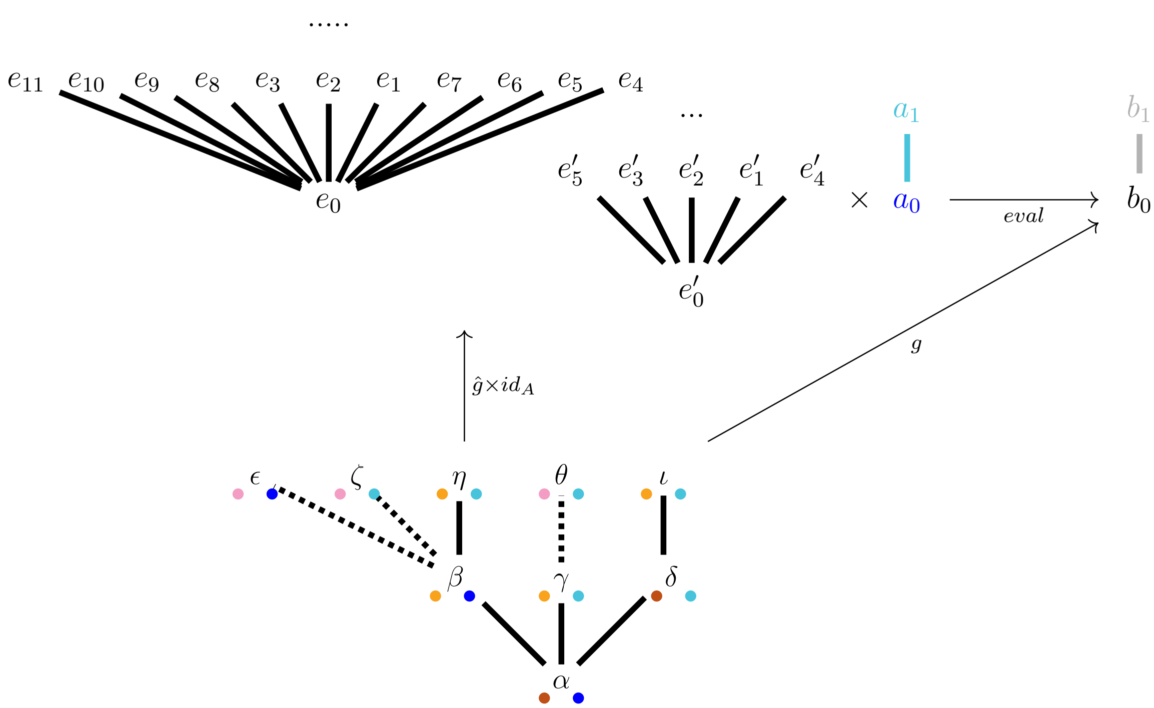


Figure 2.23: UMP diagram for $A = \{a_0 < a_1\}$, $B = \{b_0 < b_1\}$
and $C = \{c_0 < c_1 < c_2\}$.

Remark 26. $(\mathbf{1}_\perp)_\perp \times \mathbf{1}_\perp = ((\mathbf{1}_\perp \times \mathbf{1}_\perp) + (\mathbf{1}_\perp \times \mathbf{1}) + ((\mathbf{1}_\perp)_\perp \times \mathbf{1}))_\perp = ((\mathbf{1}_\perp \times \mathbf{1}_\perp) + \mathbf{1}_\perp + (\mathbf{1}_\perp)_\perp)_\perp$. Recall also that $(\mathbf{1}_\perp)_\perp \times_{\mathbb{FF}_3} \mathbf{1}_\perp$ is the sub-forest nodes of height at most 3.

We focus our attention now on maps from $C = \{c_0 < c_1 < c_2\} \times_{\mathbb{FF}_3} A = \{a_0 < a_1\}$ to $B = \{b_0 < b_1\}$ i.e., $C \times_{\mathbb{FF}_3} A = (\mathbf{1}_\perp)_\perp \times_{\mathbb{FF}_3} \mathbf{1}_\perp$ to $B = \mathbf{1}_\perp$ that send every node of $\{c_0 < c_1\} \times_{\mathbb{FF}_3} \{a_0 < a_1\}$ i.e., $\mathbf{1}_\perp \times \mathbf{1}_\perp$ to the root b_0. We call these *h-maps*.

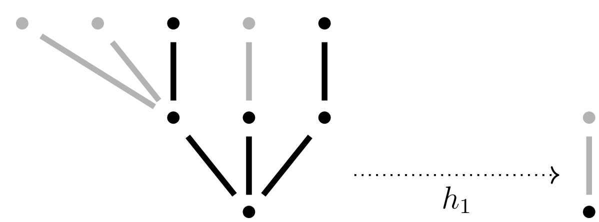
Def. 2.4.1 (*h-map*)**.** An h-map is a map $h : (\mathbf{1}_\perp)_\perp \times \mathbf{1}_\perp \to \mathbf{1}_\perp$ such that the sub-forest $\mathbf{1}_\perp \times \mathbf{1}_\perp$ is mapped to the root of $\mathbf{1}_\perp$.

How many of these maps are there?

Lemma 74. *There are in total 8 h-maps in $\mathbb{FF}_3$.*

This is due to the fact that, once all the nodes of $\mathbf{1}_\perp \times \mathbf{1}_\perp$ are sent to the root, the remaining 3 nodes (labeled from left to right as ϵ, ζ, θ) have no constraints and are incomparable, and are thus free to go to any of the 2 nodes of $\mathbf{1}_\perp$ so $2^3 = 8$.

For example, one of these h-maps h_1 is:



How might these h-maps $\{h_i\}_{i=0}^{7} : C \times A \to B$ be *encoded* in the exponential $(\mathbf{1}_\perp)_\perp^{\mathbf{1}}$?

Lets take a look at $\{\hat{h_i}\}_{i=0}^{7} : C \to B^A$.
Since $(\{c_0 < c_1\} = \mathbf{1}_\perp) \times (\mathbf{1}_\perp = \{a_0 < a_1\})$ is mapped by any h_i to b_0 and this map (recall g_0) was already encoded, it follows from our previous work that any $\hat{h_i}$ must send the nodes c_0, c_1 to the root e_0 of B^A.
What distinguishes these maps is the image of the node c_2 which can be any e_i, $i \in \{1, .., 11\}$. For example we can determine that $\hat{h_0} : c_2 \mapsto e_0$ if we take h_0 as the h-map which sends every node of C to the root b_0.
A simple deduction is made from the fact that there are 8 h-maps that need to be encoded and at the same time there are 11 branches from the root e_0:

Remark 27. There are at least two distinct level-one $i, j \in \{1, .., 11\}$ e_i, e_j nodes $i \neq j$ of B^A that encode the same h-map h_i. We observe that the nodes e_4 and e_7 provide an example of this.

In other words: There are *too many* branches for the h-maps. There are at least two distinct branches that encode the same h-map.
C is again represented as $\{c_0 < c_1 < c_2\}$ and the images of the nodes are displayed as bullets of the same color on B^A.

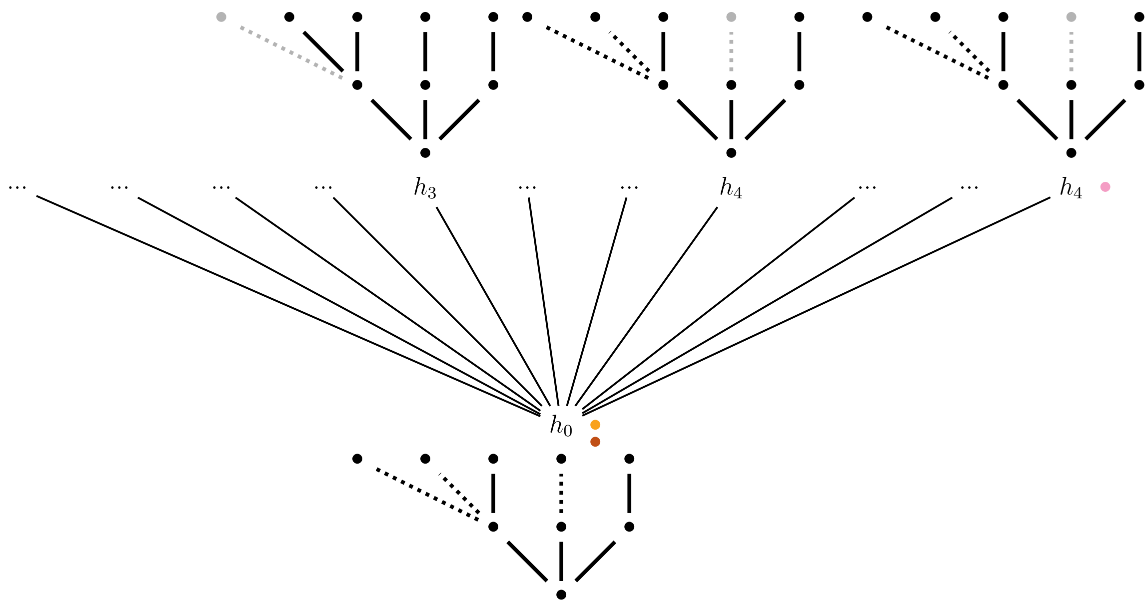


Figure 2.24: Some of the encoded h-maps $\{\hat{h}_i\}_{i=0}^{7}$ in $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$.

What this entails is the following:

Theorem 75. $\mathbb{FF}_3$ *is* not *a topos.*

Proof. Let us assume that $\mathbb{FF}_3$ is in fact a topos and admits exponential objects F^G for all *finite forests* F, G.
The example we just presented is evidence of the lack of uniqueness of adjoint h-maps $\hat{h} : (\mathbf{1}_\perp)_\perp \to (\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ that *encode* $h : (\mathbf{1}_\perp)_\perp \times_{\mathbb{FF}_3} \mathbf{1}_\perp \to \mathbf{1}_\perp$.
Thus $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ does not satisfy the UMP for the h-maps h_i.
Thus $\mathbb{FF}_3$ does not admit the exponential object $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ and we conclude that it cannot be a topos. □

So there is a *uniqueness* problem for exponentiation in $\mathbb{FF}_3$.
However $(\mathbf{1}_\perp)_\perp \times_{\mathbb{FF}_3} \mathbf{1}_\perp$ is a proper sub-forest of $(\mathbf{1}_\perp)_\perp \times_{\mathbb{FF}} \mathbf{1}_\perp$.
How does the situation change if we consider $\mathbb{FF}_k$ with $k > 3$ and $\mathbb{FF}$ in general?

2.4.2 Counter-example for $\mathbb{FF}_{k\geq 3}$

If we move on to $\mathbb{FF}_4$ or any $\mathbb{FF}_{k>3}$ we get the full product, not just a sub-forest of, $C \times A = (\mathbf{1}_\perp)_\perp \times \mathbf{1}_\perp$ given by $((\mathbf{1}_\perp \times \mathbf{1}_\perp) + \mathbf{1}_\perp + (\mathbf{1}_\perp)_\perp)_\perp$.
As before we would like for the following UMP diagram to commute, use Greek letters to mark the nodes of $C \times A$ and colored bullets on the product for the projections.

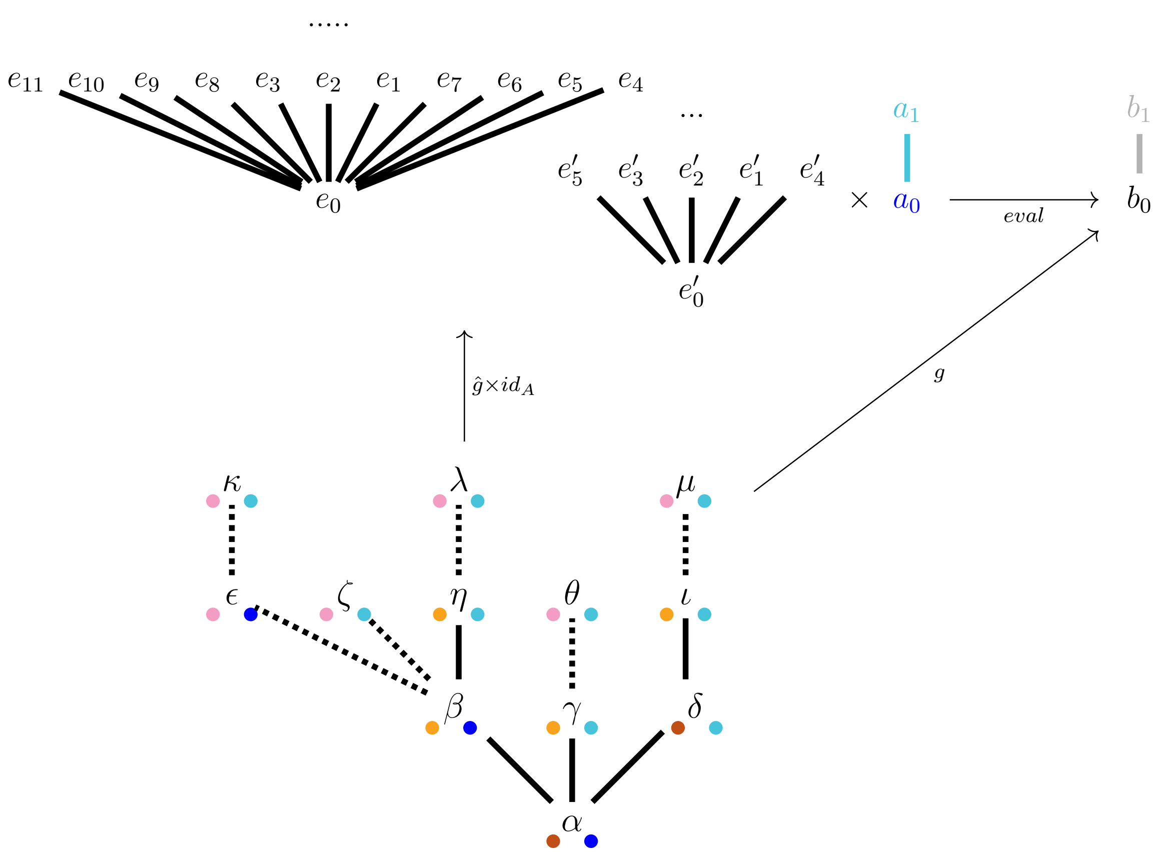
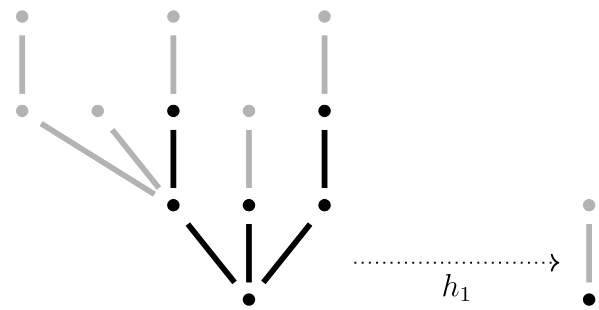


Figure 2.25: UMP diagram for $A = \{a_0 < a_1\}$, $B = \{b_0 < b_1\}$ and $C = \{c_0 < c_1 < c_2\}$.

As before we focus on *h-maps* from $C \times A = (\mathbf{1}_\perp)_\perp \times \mathbf{1}_\perp$ to $B = \mathbf{1}_\perp$. For example, one of these h-maps h_1 is:

We ask again: How many of these maps are there?

Lemma 76. *There are in total 48* h-maps *in* $\mathbb{FF}_{k>3}$.



This can be computed by observing that compared to last time in $\mathbb{FF}_3$ we now have 3 new nodes to add to $C \times A$ so that $48 = 3 \cdot 2^4$.
The first one (reading the forest from left to right) κ is the child of ϵ, the second one λ is the child of η and the third one μ is the child of ι. If we observe the dotted branches we now have (still from left to right) one branch of the form $(\mathbf{1}_\perp)_\perp$ and four of the form $(\mathbf{1}_\perp)$.
Recalling that $|Hom((\mathbf{1}_\perp)_\perp, \mathbf{1}_\perp)| = 3$ and $|Hom(\mathbf{1}_\perp, \mathbf{1}_\perp)| = 2$ a straightforward combinatorial argument tells us that the total number of *h-maps* is $3 \cdot 2^4$.
We ask again: How might these h-maps $\{h_i\}_{i=0}^{47} : C \times A \to B$ be *encoded* in the exponential $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$?

Lets take a look then at $\{\hat{h_i}\}_{i=0}^{47} : C \to B^A$.
The same argument we used last time tells us that any $\hat{h_i}$ must send the nodes c_0, c_1 to the root e_0 of B^A and these maps should be distinguished by the image of the node c_2 which can be any e_i, $i \in \{1, .., 11\}$.
Using the same example, we can determine that $\hat{h_0} : c_2 \mapsto e_0$ if we take h_0 as the h-map which sends every node of C to the root b_0.
In this case however there are 48 maps that need to be encoded with the same 11 branches from the root e_0.
This means that there are too *few* branches to encode all the possible h-maps.

Lemma 77. *There is at least one h-map that cannot be encoded in any level-one node e_i, $i \in \{0, .., 11\}$ of $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$.*

The h-map $?h$ displayed in figure 2.26 is an example of this.
Let $\{\hat{h_i}\}_{i=1}^{11}$ be the maps from C to B^A such that $\hat{h_i} : \{c_0, c_1\} \mapsto e_0$ and $c_2 \mapsto e_i$.
Take for example the h-map h_7 in display.

It should be in $1:1$ correspondence with $\hat{h_7}$.
Note that if we now focus our attention on $\{\beta < \eta < \lambda, \beta < \zeta, \beta < \epsilon < \kappa\}$ i.e., the up-set $\uparrow \beta$ corresponding to $\{c_1 < c_2\} \times \{a_0 < a_1\}$, the restriction of $\hat{h_7}$ already encodes a map which we called $g_7 : \{c_1 < c_2\} \times \{a_0 < a_1\} \to \{b_0 < b_1\}$. The yellow boxes in figure 2.26 are meant to highlight this map.
If we construct a new h-map $?h$ which behaves exactly as g_7 on $\{c_1 < c_2\} \times \{a_0 < a_1\}$ but behaves differently on the two *leaf* nodes θ, μ, then $?h \neq h_7$ and thus cannot be encoded by $\hat{h_7}$.
Furthermore, $?h$ cannot be encoded by $\hat{h_i}$ with $i \neq 7$ since any of these h_i when restricted to $\{c_1 < c_2\} \times \{a_0 < a_1\}$ behaves as g_i with $g_i \neq g_7$.

We are thus in a bind and cannot find a corresponding $\hat{?h}$ for $?h$...

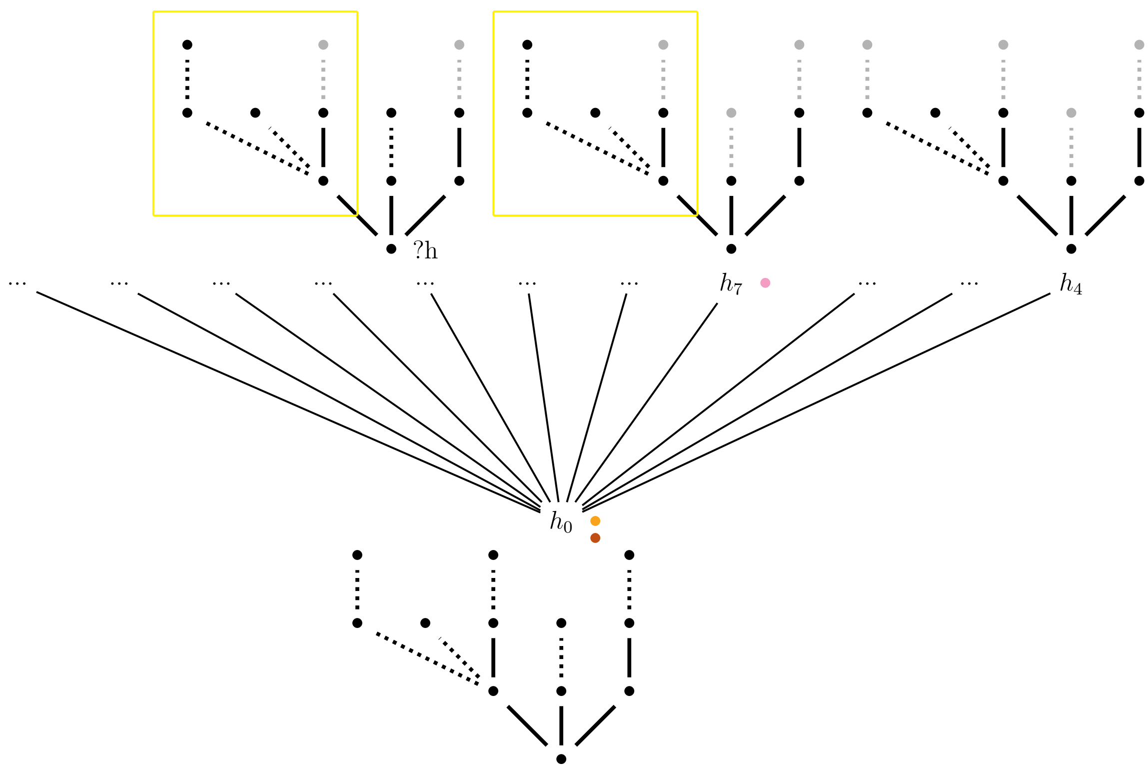


Figure 2.26: Some of the encoded h-maps $\{\hat{h_i}\}_{i=0}^{47}$ in $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$.

What this entails is the following:

Theorem 78. $\mathbb{FF}_{k>3}$ *for* $k > 3$ *is* not *a topos.*

Proof. Let us assume that $\mathbb{FF}_k$ is a topos for some $k > 3$ and admits exponential objects F^G for all *finite forests* F, G.
The example we presented is evidence of the lack of existence of an adjoint h-map $\hat{h} : (\mathbf{1}_\perp)_\perp \to (\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ that *encodes* $h : (\mathbf{1}_\perp)_\perp \times_{\mathbb{FF}_3} \mathbf{1}_\perp \to \mathbf{1}_\perp$.
Thus $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ does not satisfy the UMP for the h-maps h_i.
Thus $\mathbb{FF}_k$ does not admit the exponential object $(\mathbf{1}_\perp)^{\mathbf{1}_\perp}$ and we conclude that it cannot be a topos. $\square$

Summing up we have *uniqueness* (Theorem 75) and *existence* (Theorem 78) problems for exponentiation in $\mathbb{FF}_3$ and $\mathbb{FF}_{k>3}$ respectively. The following results are now immediate:

Corollary 79. $\mathbb{FF}_{k\geq 3}$ *is not a topos.*

Corollary 80. *The only topoi in* $\mathbb{FF}_*$ *are* $\mathbb{FF}_1$ *and* $\mathbb{FF}_2$*.*

Chapter 3

Topos Semantics I

In this section we will explore the topoi-semantics for $\mathbb{FF}_2/bushes$ by following the general approach outlined in [9].
We give, whenever necessary, a general outline from [9] for topoi-semantics and make a few considerations with regards to $\mathbb{FF}_2/bushes$.

Recall the sub-object classifier for $\mathbb{FF}_2/bushes$.
This is given by $\Omega = \{\mathbf{f}, \mathbf{t} < *\}$ and $\mathbf{1} \xrightarrow{true} \Omega$.
For each *sub-object* $f : F \rightarrowtail G$ (which determines a sub-forest $f[F] \subseteq G$) there is a unique *characteristic* arrow $\chi_f : F \to \Omega$ making the following commutative diagram a *pullback* of χ_f and *true*.

$$\begin{tikzcd} F \arrow[r, tail, "s"'] \arrow[d, dashed, "!_F"] \arrow[dr, phantom, very near start, "\lrcorner"] & G \arrow[d, "\chi_f"'] \\ 1 \arrow[r, "true"] & \Omega \end{tikzcd}$$

Figure 3.1: the characteristic diagram of f.

Recall also that $\mathbb{Set}_{fin} \simeq \mathbb{FF}_1$.
In $\mathbb{Set}_{fin}$ and more generally in $\mathbb{Set}$ the sub-object classifier is $\mathbf{2} = \{0, 1\}$ together with $\mathbf{1} = \{0\} \xrightarrow{true} \mathbf{2},\ 0 \mapsto 1$.
This can be translated in $\mathbb{FF}_1$ as $\mathbf{2} = \{\mathbf{f},\ \mathbf{t}\}$ with $\mathbf{1} = \{\bullet\} \xrightarrow{true} \mathbf{2},\ \bullet \mapsto \mathbf{t}$.

3.1 The Propositional Layer

Moving away from the usual environment of sets we wish to recover propositional Logic in the context of topoi.

3.1.1 The Language of Topoi

(Unless otherwise specified, the usual coloring notation to display maps is used. In the domain the fiber/pre-image of a node will have the same color of the node in the co-domain).

We begin the translation from the language of $\mathbb{Set}$ to that of a topos $\mathcal{E}$ from the truth constants $\top$,$\bot$ and the logical connectives $\neg, \wedge, \vee, \Rightarrow$.
In all these cases they will be interpreted as arrows into Ω of the form $\Omega^n \to \Omega$, a.k.a. *Truth-arrows* in which n corresponds to the arity of the connective.

Let's start by translating the constants true/top $\top$ and false/bottom $\bot$ in the context of our topos of *bushes*.

Note that $\top$ has already been introduced as an arrow from the zero-th power of Ω to Ω i.e., $\mathbf{1} = \Omega^0 \to \Omega$.
In any topos $\mathcal{E}$, $\top$ is defined as $\mathbf{1} \xrightarrow{true} \Omega$. In our case:

Def. 3.1.1 ($\top$)**.** $\top$ is $\mathbf{1} \xrightarrow{true} \Omega$ the map $\bullet \mapsto \mathbf{t}$.

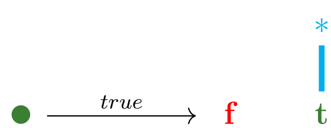


Figure 3.2: $\top : \mathbf{1} \xrightarrow{true} \Omega$. (*bushes*)

This can also be seen as the *characteristic arrow* for $\mathbf{1} \xrightarrow{id} \mathbf{1}$ i.e., for the *maximal sub-object* of $\mathbf{1}$.
This generalizes from $\mathbb{Set}$ in which $\top : \mathbf{1} = \{0\} \xrightarrow{true} \mathbf{2} = \{0, 1\},\ 0 \mapsto 1$ is also the characteristic arrow for $\mathbf{1} \xrightarrow{id} \mathbf{1}$.

In a similar vein, $\perp : \mathbf{1} = \{0\} \xrightarrow{false} \mathbf{2} = \{0,1\},\ 0 \mapsto 0$ in $\mathbb{Set}$ is the characteristic of $\emptyset \xrightarrow{\emptyset} \mathbf{1}$ i.e., of the *minimal sub-object* of $\mathbf{1}$.

It is straightforward to generalize this to an arbitrary topos $\mathcal{E}$ as the characteristic arrow $\perp$ of $0 \xrightarrow{!_0} 1$. In $\mathbb{FF}_2$ this becomes:

Def. 3.1.2 ($\perp$)**.** $\perp$ is $\mathbf{1} \xrightarrow{false} \Omega$ the map $\bullet \mapsto \mathbf{f}$.

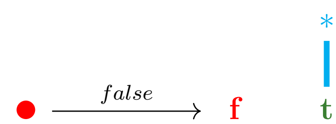


Figure 3.3: $\perp : \mathbf{1} \xrightarrow{false} \Omega$. (*bushes*)

This is the characteristic arrow of the empty sub-forest $\mathbf{0} \xrightarrow{0_1} \mathbf{1}$. Moving on to logical connectives..

Starting with *negation* $\neg$: This is a *unary* operator and we expect a truth-arrow from $\Omega = \Omega^1 \to \Omega$.
In $\mathbb{Set}$ $\mathbf{2} \xrightarrow{\neg} \mathbf{2}$ is the *switch* function $\neg : 0 \mapsto 1, 1 \mapsto 0$. This coincides with the characteristic function of $\perp : \mathbf{1} = \{0\} \xrightarrow{false} \mathbf{2} = \{0, 1\}$ which determines the sub-set $\{\perp(0)\} \subseteq \mathbf{2}$.
In the language of topoi: $\neg$ is the characteristic arrow of $\perp$ as defined previously. So:

Def. 3.1.3 ($\neg$)**.** $\neg : \Omega \to \Omega$ is the characteristic arrow of $\perp : \mathbf{1} \xrightarrow{false} \Omega$.

This means that the following commutative diagram is a pullback in $\mathbb{FF}_2$.

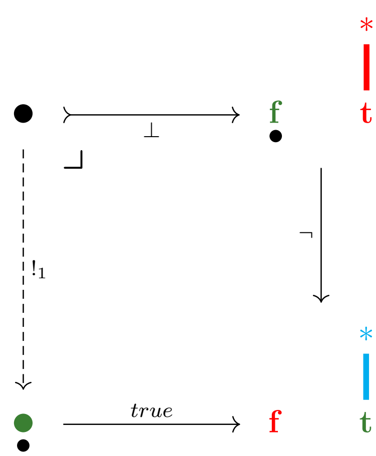


Figure 3.4: the characteristic diagram of $\perp$. The images of $\mathbf{1}$ by $\perp$ and $!_1$ are shown with a black bullet underneath.

In the case of *conjunction* $\wedge$, as a binary operator we expect a truth arrow $\Omega \times \Omega = \Omega^2 \to \Omega$.
For $\mathbb{Set}$ $\wedge$ is the characteristic function of $\{(1,1)\} \subset \mathbf{2} \times \mathbf{2}$ since by the correspondence $0 \leftrightarrow$ **f** and $1 \leftrightarrow$ **t**, $\wedge$ classically outputs **t** only on the pair (**t**,**t**). As an arrow in $\mathbb{Set}$ this sub-set is determined by the *product-function* $\top \times \top : \mathbf{1} \times \mathbf{1} \to \mathbf{2} \times \mathbf{2}$.
We can define $\wedge$ in any topos as the character of the product arrow $\top \times \top$.
In *bushes*:

Def. 3.1.4 ($\wedge$)**.** $\wedge$ is the characteristic arrow of $\top \times \top : \mathbf{1} \times \mathbf{1} \to \Omega \times \Omega$.

Recalling that $\Omega \times \Omega = (\mathbf{1} + \mathbf{1}_\perp) \times (\mathbf{1} + \mathbf{1}_\perp) = \mathbf{1} + \mathbf{1}_\perp + \mathbf{1}_\perp + (\mathbf{1}_\perp \times \mathbf{1}_\perp)$ and $\mathbf{1} \times \mathbf{1} = \mathbf{1}$ the characteristic diagram is the following:

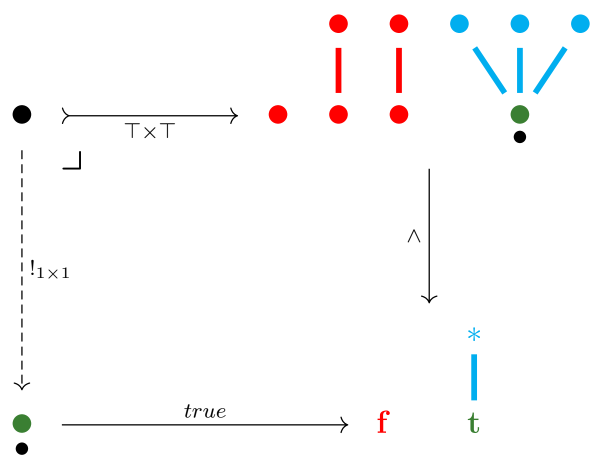


Figure 3.5: the characteristic diagram of $\top \times \top$.

The case of $\vee$ is analogous to $\wedge$:
Starting as usual from $\mathbb{Set}$, $\vee$ is the characteristic function of the sub-set $D = \{(1,1),(0,1)\} \cup \{(1,1),(1,0)\} = \{(0,1),(1,0),(1,1)\} \subset \mathbf{2} \times \mathbf{2}$. The two sub-sets $\{(1,1),(0,1)\}$ and $\{(1,1),(1,0)\}$ are determined by the arrows $id_2 \times \top, \top \times id_2 : \mathbf{2} \to \mathbf{2} \times \mathbf{2}$ so their union D is the image of the *sum-function* $(id_2 \times \top) + (\top \times id_2) : \mathbf{2} + \mathbf{2} \to \mathbf{2} \times \mathbf{2}$.
As such we can define $\vee$ in any topos as the character of the image of the *coproduct-arrow* $f = (id_\Omega \times \top) + (\top \times id_\Omega)$ i.e., the characteristic arrow of $Im(f) \xrightarrow{m} \Omega \times \Omega$ [1]. In *bushes*:

Def. 3.1.5 ($\vee$)**.** $\vee$ is the character of the image of $(id_\Omega \times \top) + (\top \times id_\Omega) : \Omega + \Omega \to \Omega \times \Omega$.

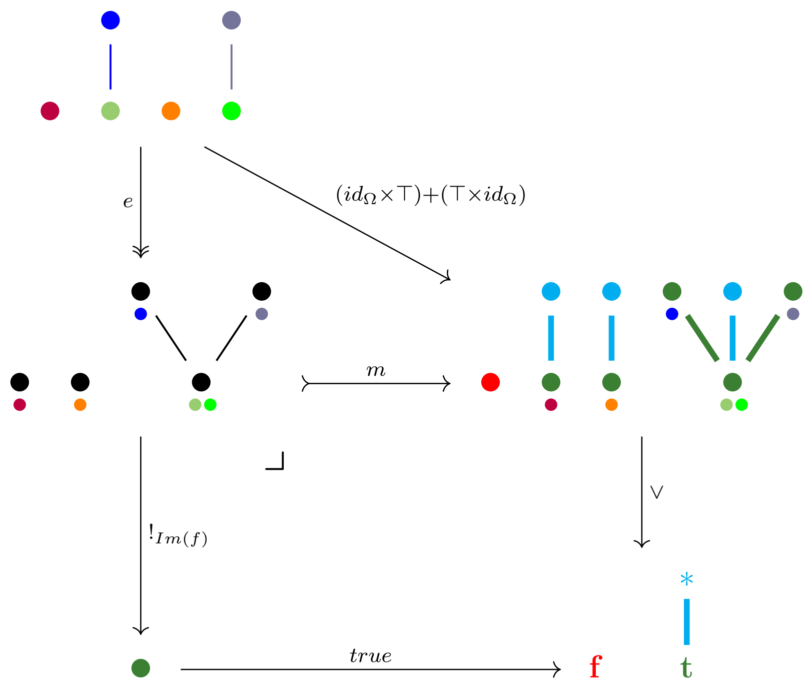


Figure 3.6: the characteristic diagram of $Im((id_\Omega \times \top) + (\top \times id_\Omega))$ with the epi-mono factorization on display.
The images of the nodes $\Omega + \Omega$ are shown with bullets of matching color.

[1] obtained through epi-mono factorization $f = m \circ e$.

Finally we arrive at implication $\Rightarrow$.
Starting from $\mathbb{Set}$ we observe that the sub-set that we want to *characterize* for implication is precisely $\leq = \{(0,0),(0,1),(1,1)\} \subset \mathbf{2} \times \mathbf{2}$ i.e., the partial order relation on the *lattice* $\mathbf{2}$ given by $\leq = \{(x,y) \in \mathbf{2} \times \mathbf{2} : x \leq y\}$. $\leq$ can also be described as $\leq = \{(x,y) \in \mathbf{2} \times \mathbf{2} : x \wedge y = x\}$ i.e., the *equalizer* of $\wedge$ and π_1 the first projection of $\mathbf{2} \times \mathbf{2}$ denoted by $Eq(\wedge, \pi_1) = \leq \overset{e}{\rightarrowtail} \mathbf{2} \times \mathbf{2} \underset{\pi_1}{\overset{\wedge}{\rightrightarrows}} \mathbf{2}$.
This allows us to state that in any topos:
$\Rightarrow$ is the character of the equalizer e of $\wedge$ as previously defined and π_1 the first projection of $\Omega \times \Omega$. As such in *bushes*:

Def. 3.1.6 ($\Rightarrow$)**.** $\Rightarrow$ is the character of e with $Eq(\wedge, \pi_1) = E \overset{e}{\rightarrowtail} \Omega \times \Omega \underset{\pi_1}{\overset{\wedge}{\rightrightarrows}} \Omega$.

The characteristic diagram is thus given by:

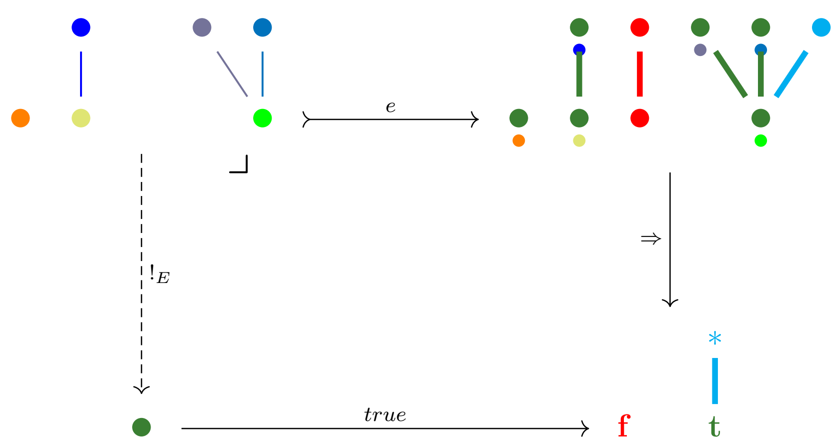


Figure 3.7: the characteristic diagram of e.
The images of the nodes of E are shown with bullets of matching color.

We give some general notions from [9]:

Let's consider a generic topos $\mathcal{E}$.
We defined the *truth-arrows* we need for the semantics of propositional formulae.
The *truth-values* are given by the *hom-set* $\mathcal{E}(\mathbf{1}, \Omega)$.
Note that these truth-values are in fact *generalized elements* of Ω which generalize the set-elements of **2** in $\mathbb{Set}$.
An $\mathcal{E} - valuation$ of a propositional variable is just an assignment of a truth-value. This can be extended inductively to formulae using the connectives we just defined:

Def. 3.1.7 ($\mathcal{E}$ -valuation)**.** An *$\mathcal{E}$-valuation* is a function $V : \mathbf{Prop} \to \mathcal{E}(\mathbf{1}, \Omega)$ which is extended to **Form**:
$\forall \phi, \psi \in \mathbf{Form}$:

- $V(\neg\phi) = \neg \circ V(\phi)$.
- $V(\phi \wedge \psi) = \wedge \circ [V(\phi) \times V(\chi)]$.
- $V(\phi \vee \psi) = \vee \circ [V(\phi) \times V(\chi)]$.
- $V(\phi \Rightarrow \psi) = \Rightarrow \circ [V(\phi) \times V(\chi)]$.

We can talk about *topos-validity* of a formula ϕ if every valuation gives the truth-arrow $\top$:

Def. 3.1.8 ($\mathcal{E}$-validity (propositional))**.** A formula ϕ is $\mathcal{E}$-valid, denoted by $\mathcal{E} \models \phi$ or $\models_{\mathcal{E}} \phi$, when for every $\mathcal{E}$-valuation V, $V(\phi) = \top$.

3.1.2 The Algebra of Sub-Objects

Let $\mathcal{E}$ be a topos and $\mathbf{d} \in \mathcal{E}$ one of its objects.
The truth arrows defined previously can be used to give an algebraic structure to $Sub(\mathbf{d})$ the collection of sub-objects of $\mathbf{d}$.
In the case of $\mathbb{Set}$, consider sub-objects or simply sub-sets A and B of a set D with their characteristic functions $\chi_A, chi_B : D \to \mathbf{2}$ then we have the following:

- $\chi_{A^c} = \neg \circ \chi_A$.
- $\chi_{A \cap B} = \wedge \circ (\chi_A \times \chi_B)$.
- $\chi_{A \cup B} = \vee \circ (\chi_A \times \chi_B)$.

Generalizing to $\mathcal{E}$:

We define the operations in $Sub(\mathbf{d})$ by specifying the *characteristic arrow* $\chi : \mathbf{d} \to \Omega$ of the new sub-object which will then be obtained via the pullback of $\top$ and χ.
Let $f, g \in Sub(d)$ we define the complement of f as:

Def. 3.1.9 $(-f)$**.** The complement of f is the sub-object $-f$ whose characteristic arrow is $\neg \circ \chi_f$.

The intersection of f and g as:

Def. 3.1.10 $(f \cap g)$**.** The intersection of f and g is the sub-object $f \cap g$ whose characteristic arrow is $\wedge \circ (\chi_f \times \chi_g)$.

The union of f and g as:

Def. 3.1.11 $(f \cup g)$**.** The union of f and g is the sub-object $f \cup g$ whose characteristic arrow is $\vee \circ (\chi_f \times \chi_g)$.

The implication of g by f as:

Def. 3.1.12 $(f \Rightarrow g)$**.** The implication of g by f is the sub-object $f \Rightarrow g$ whose characteristic arrow is $\Rightarrow \circ (\chi_f \times \chi_g)$.

Recall that just like $(\mathcal{P}(D), \leq)$ is a partial ordering with set-inclusion, $(Sub(d), \sqsubseteq)$ becomes a partial ordering by *sub-object inclusion* i.e., $f \sqsubseteq g$ if there is an arrow $h : Dom(f) \to Dom(g)$ such that $f = g \circ h$.
We now have the following result:

Proposition 81. $(Sub(d), \sqsubseteq)$ *is a* bounded lattice *in which:*

- $f \cap g$ *is the* greatest lower bound *of* f *and* g *i.e., the* meet *of* f *and* g.
- $f \cup g$ *is the* least upper bound *of* f *and* g *i.e., the* join *of* f *and* g.
- *the arrow from the initial object* $0_d : \boldsymbol{0} \rightarrow \boldsymbol{d}$ *is the* bottom *element.*
- *the identity arrow* $1_d : \boldsymbol{d} \rightarrow \boldsymbol{d}$ *is the* top *element.*

Also the following holds for all $f, g, h \in Sub(\mathbf{d})$ making $\Rightarrow$ a *pseudo-complement*:

Lemma 82. $h \sqsubseteq (f \Rightarrow g)$ *iff* $(f \cap h) \sqsubseteq g$.

We are thus in a position to state that:

Theorem 83. $(Sub(d), \sqsubseteq)$ *is a* Heyting Algebra *with top element* 1_d*, bottom element* 0_d *and* $f \cap g$*,*$f \cup g$ *and* $f \Rightarrow g$ *are respectively the* meet*,* join *and* pseudo-complement *operations.*

We also have:

Theorem 84. $(\mathcal{E}(\boldsymbol{d}, \Omega), \sqsubseteq)$ *is a* Heyting Algebra *with top element* $true_d$ [2]*, bottom element* $false_d$ [3] *and the truth-arrows as operations. This can be seen by the following definitions:*

$$\begin{aligned} \chi_f \sqsubseteq \chi_g & \textit{ iff } (\chi_f \times \chi_g) \textit{ factors through } e : \leq \hookrightarrow \Omega \times \Omega. \\ \chi_f \cap \chi_g :=& \ \ \wedge \circ (\chi_f \times \chi_g). \\ \chi_f \cup \chi_g :=& \ \ \vee \circ (\chi_f \times \chi_g). \\ \neg \chi_f :=& \ \ \neg \circ \chi_f. \\ \chi_f \Rightarrow \chi_g :=& \ \Rightarrow \circ (\chi_f \times \chi_g). \end{aligned}$$

[2]the character of 1_d i.e., the identity arrow on d.
[3]the character of $0_d : 0 \rightarrow d$ i.e., the initial arrow on d.

Recall that the Ω-Axiom from 2.1 gave us a bijection $Sub(d) \cong \mathcal{E}(\mathbf{d}, \Omega)$ in which $f \overset{1:1}{\longleftrightarrow} \chi_f$.
We can show that this bijection is in fact a *Heyting Algebra isomorphism* $\Delta : Sub(d) \cong \mathcal{E}(\mathbf{d}, \Omega)$ where:

$$\begin{aligned}
\Delta(-f) :=& \ \neg \circ \Delta(f). \\
\Delta(f \cap g) :=& \ \wedge \circ (\Delta(f) \times \Delta(g)). \\
\Delta(f \cup g) :=& \ \vee \circ (\Delta(f) \times \Delta(g)). \\
\Delta(f \Rightarrow g) =& \ \Rightarrow \circ(\Delta(f) \times \Delta(g)).
\end{aligned}$$

3.1.3 Soundness and Completeness for CPL and IPL

We proceed by giving a few more general results from [9]:

We have seen that in any topos $\mathcal{E}$ the truth arrow $\top = \chi_{id_{\mathbf{1}}}$ and $\bot = \chi_{0_{\mathbf{1}}}$. If we now focus on the sub-objects of the terminal $\mathbf{1}$ i.e., $Sub(\mathbf{1})$ we can derive the following:

- $\top \wedge \top = \chi_{id_{\mathbf{1}}} \wedge \chi_{id_{\mathbf{1}}} = \chi_{id_{\mathbf{1}} \cap id_{\mathbf{1}}} = \chi_{id_{\mathbf{1}}} = \top$.
- $\top \wedge \bot = \chi_{id_{\mathbf{1}}} \wedge \chi_{0_{\mathbf{1}}} = \chi_{id_{\mathbf{1}} \cap 0_{\mathbf{1}}} = \chi_{0_{\mathbf{1}}} = \bot$.
- $\bot \wedge \top = \chi_{0_{\mathbf{1}}} \wedge \chi_{id_{\mathbf{1}}} = \chi_{0_{\mathbf{1}} \cap id_{\mathbf{1}}} = \chi_{0_{\mathbf{1}}} = \bot$.
- $\bot \wedge \bot = \chi_{0_{\mathbf{1}}} \wedge \chi_{0_{\mathbf{1}}} = \chi_{0_{\mathbf{1}} \cap 0_{\mathbf{1}}} = \chi_{0_{\mathbf{1}}} = \bot$.
- ...

and so forth for $\vee$ and $\Rightarrow$. What we found is that the truth arrows $\top$ and $\bot$ behave *classically* with respect to the $\mathcal{E}$-connectives we defined. [4]

With this in mind and remembering how we defined $\mathcal{E}$-validity, we can give some first results about Soundness and Completeness for *Classical* and *Intuitionistic* Propositional Logic **CPL** and **IPL**.

Namely, *Completeness* for **CPL**:

Theorem 85. *For any topos $\mathcal{E}$, $\alpha \in$ **Form**:*

$$\textit{If } \mathcal{E} \models \alpha \textit{ then } \vdash_{CPL} \alpha.$$

CPL is not always *sound* with respect $\mathcal{E}$-validity..
However, if we restrict ourselves to *bivalent* topoi i.e., topoi with just two truth values i.e., $|\mathcal{E}(\mathbf{1}, \Omega)| = 2$ we have soundness and completeness for **CPL**:

Proposition 86. *If $\mathcal{E}$ is bivalent, then:*

$$\forall \alpha \in \textbf{\textit{Form}} :$$
$$\mathcal{E} \models \alpha \textit{ iff } \vdash_{CPL} \alpha.$$

For example:
$\mathbb{Set}$ *is bivalent as* $\Omega = 2 = \{0, 1\}$ *and thus:*

$$\vdash_{CPL} \alpha \ \textit{ iff } \ \mathbb{Set} \models \alpha.$$

[4] the same result could have been obtained without reference to sub-objects and simply unfolding the definitions, putting one pullback square atop another.

What about *intuitionistic* Logic?

Recall that Heyting Algebras provide a sound and complete semantics for **IPL** i.e.,:

Remark 28. For any Heyting Algebra HA, $\alpha \in \mathbf{Form}$:

$$HA \models \alpha \text{ iff } \vdash_{IPL} \alpha.$$

As we have seen, for any $\mathcal{E}$-object $\mathbf{d}$, there is an isomorphism $Sub(d) \cong \mathcal{E}(\mathbf{d}, \Omega)$ which transfers the H.A. [5] structure of the sub-objects of $Sub(d)$ to the truth arrows of $\mathcal{E}(\mathbf{d}, \Omega)$.
This gives us the following equivalence which links the semantics of topoi to that of Heyting Algebras:
($\models_{\mathcal{E}}$ denotes topos validity whilst $\models_{H.A.}$ Heyting algebra validity).

Proposition 87. *For any topos $\mathcal{E}$, $\alpha \in \mathbf{Form}$:*

$$\models_{\mathcal{E}} \alpha \textit{ iff } \ \mathcal{E}(\mathbf{1}, \Omega) \models_{H.A.} \alpha \textit{ iff } \ Sub(\mathbf{1}) \models_{H.A.} \alpha.$$

To see why this is the case, notice that an $\mathcal{E}$-valuation is an H.A.-valuation for $\mathcal{E}(\mathbf{1}, \Omega)$ which is isomorphic to $Sub(\mathbf{1})$ and that the *unit* $\top$ of the H.A. $\mathcal{E}(\mathbf{1}, \Omega)$ is precisely the truth arrow $\top : \mathbf{1} \to \Omega$ so that $\mathcal{E}$-validity and $\mathcal{E}(\mathbf{1}, \Omega)$-validity amount to the same thing.

This allows us to say that if $\vdash_{IPL} \alpha$, then by soundness for Heyting Algebras $Sub(\mathbf{1}) \models \alpha$ and $\mathcal{E}(\mathbf{1}, \Omega) \models \alpha$ which means $\models_{\mathcal{E}} \alpha$.
What we have shown is, contrary to the case of **CPL**, that topoi provide a sound semantics for **IPL**.

Theorem 88 (Soundness for topos-validity)**.** *For any topos $\mathcal{E}$, $\alpha \in \mathbf{Form}$:*

$$\textit{If } \vdash_{IPL} \alpha \textit{ then } \models_{\mathcal{E}} \alpha.$$

[5] from now on H.A. will be more commonly used instead of *Heyting Algebra*.

3.1.4 External and Internal Logics

Let's take a step back and revisit our definitions for the logical connectives in the topos of *bushes*/$\mathbb{FF}_2$.
Negation $\neg$ for example yielded:

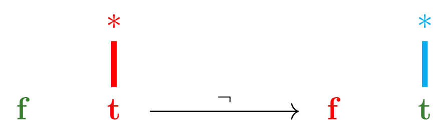


Figure 3.8: $\neg : \Omega \to \Omega$ in $\mathbb{FF}_2$.

Recalling the truth table for $\neg$ in $\mathcal{G}_3$:

	$\neg$
t	f
$*$	f
f	t

Figure 3.9: $\neg$ in $\mathcal{G}_3$.

Looking *from outside* at the $\neg$ arrow for *bushes*, one finds that the nodes t and $*$ are mapped to f and that f is mapped to t.
This is precisely the truth function for $\neg$ in $\mathcal{G}_3$.
Note that $\neg$ is a unary operator and all one had to do was to look at the nodes in order to *visualize* externally the corresponding truth-table. For binary connectives one needs to look at the product object $\Omega \times \Omega$ and its projections π_1, π_2 and label the nodes appropriately:

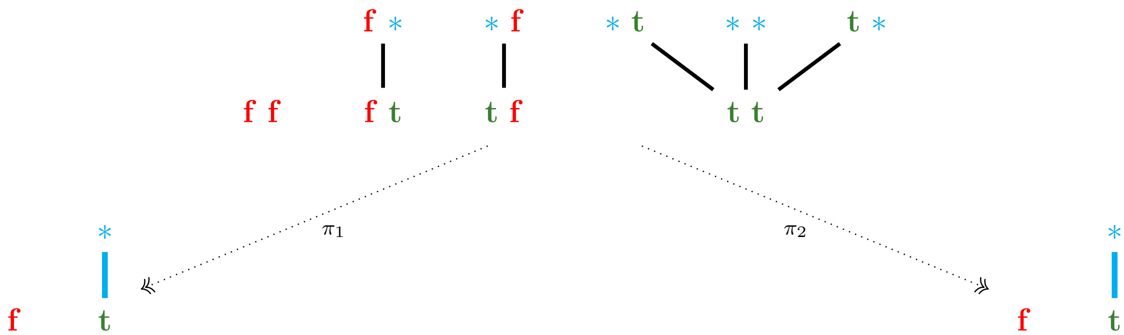
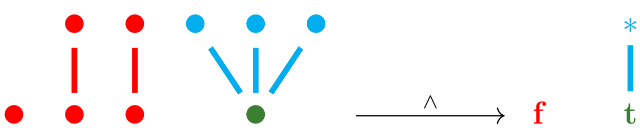


Figure 3.10: $\Omega \times \Omega$ and the projections π_1, π_2 to Ω.
The nodes of the product are labeled in the format lr in which l and r specify the first and second projections respectively:

Keeping this representation in mind, the same phenomenon we observed for the truth-arrow $\neg$ occurs now for the other connectives.

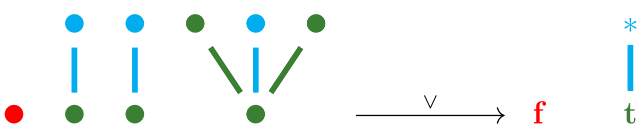
In other words, we re-discover the truth functions for $\mathcal{G}_3$:



(a) $\wedge : \Omega \times \Omega \to \Omega$ in $\mathbb{FF}_2$.

$\wedge$	t	∗	f
t	t	∗	f
∗	∗	∗	f
f	f	f	f

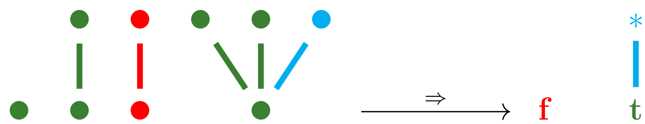
(b) $\wedge$ in $\mathcal{G}_3$.



(a) $\vee : \Omega \times \Omega \to \Omega$ in $\mathbb{FF}_2$.

$\vee$	t	∗	f
t	t	t	t
∗	t	∗	∗
f	t	∗	f

(b) $\vee$ in $\mathcal{G}_3$.



(a) $\Rightarrow : \Omega \times \Omega \to \Omega$ in $\mathbb{FF}_2$.

$\Rightarrow$	**t**	$*$	**f**
t	**t**	$*$	**f**
$*$	**t**	**t**	**f**
f	**t**	**t**	**t**

(b) $\Rightarrow$ in $\mathcal{G}_3$.

From the inside however we just defined truth values for $\mathbb{FF}_2$ as elements of the hom-set $\mathbb{FF}_2(\mathbf{1}, \Omega) = \{\top, \bot\}$ i.e., the set containing the arrow $\top$ for which $\bullet \mapsto$ **t** and $\bot$ for which $\bullet \mapsto$ **f**.
Having just two truth-values, the topos $\mathbb{FF}_2$ of *bushes* is called *bivalent*.
The third node which we labeled $*$ and which externally seems to correspond to the same-name truth-value $*$ *not-false* in $\mathcal{G}_3$ cannot be a truth-value of the form $\mathbf{1} \to \Omega$ since $\mathbf{1}$ as an open map can only be mapped to a root so either to **f** or **t**.
Informally we could say:

Remark 29. $\mathbb{FF}_2/$*bushes* is *internally* bivalent while *from the outside* Ω has three elements.

In other words:

Remark 30. On the propositional level, the *internal* logic of $\mathbb{FF}_2/$*bushes* is classical whilst the *external* logic is that of $\mathcal{G}_3$.

What does it mean if the topos behaves *classically* on the *inside* and *non-classically* on the *outside*?

To clarify this situation, let's take a look at what happens with *double negation.* Recall that in classical logic α *is equivalent to* $\neg\neg\alpha$ so we would expect in the language of topoi for the following to hold: $\neg \circ \neg = id_\Omega$.

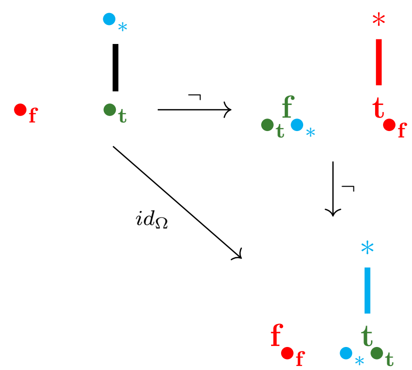


Figure 3.14: $\neg \circ \neg \neq id_\Omega$ in $\mathbb{FF}_2$.
Here we displayed the images of the nodes of Ω in the top left corner as labeled bullets.

This is clearly not the case for our topos.
Note that while the *classical* nodes $\bullet_{\mathbf{t}} \mapsto \mathbf{t}$, $\bullet_{\mathbf{f}} \mapsto \mathbf{f}$ are fixed by $\neg \circ \neg$, the *non-classical* node $\bullet_* \mapsto \mathbf{t}$ is not.[6]

Another approach is to consider the *Algebra of sub-objects.*
When in a topos the algebra of sub-objects for an arbitrary object, which we have seen is a Heyting Algebra, is also a Boolean Algebra:

Def. 3.1.13 (Boolean topos)**.** A topos $\mathcal{E}$ is *Boolean* if for every object $\mathbf{d}$, $(Sub(\mathbf{d}), \sqsubseteq)$ is a Boolean Algebra.

Example 89 (Sets)**.** The prototypical Boolean topos is unsurprisingly $\mathbb{Set}$ where for every set D it is the case that $(Sub(\mathbf{D}), \sqsubseteq) \cong (\mathcal{P}(D), \subseteq)$ which is the *power-set* Boolean Algebra.

We can use the following results from [9] and [12]:

Lemma 90. *For any topos $\mathcal{E}$, $\mathcal{E}$ is Boolean iff $(Sub(\Omega), \sqsubseteq)$ is a Boolean Algebra.*

[6]this corresponds to the fact in $\mathcal{G}_3$ that the negation of $*$ i.e., *not-false* is *false.*

Proposition 91. *A topos $\mathcal{E}$ is Boolean iff $\boldsymbol{1} \xrightarrow{\top} \Omega \xleftarrow{\perp} \boldsymbol{1}$ is a Co-product diagram.*

This would entail $\Omega \cong \mathbf{1} + \mathbf{1}$ and $\neg \circ \neg = id_\Omega$ which are both false for *bushes.*

We also know that $\Omega = Spec(\mathcal{F}_1)$ i.e., the prime spectrum of the free Gödel Algebra on one generator $\mathcal{F}_1$. $Sub(Spec(\mathcal{F}_1))$ are in this case the sub-forests of the prime spectrum of $\mathcal{F}_1$ and by duality we have shown that $Sub(Spec(\mathcal{F}_1)) \cong \mathcal{F}_1$.
$\mathcal{F}_1$ is not a Boolean Algebra ($(x \vee \neg x) \neq 1$) and so, summing up:

Theorem 92 (propositional logic of $\mathbb{FF}_2$)**.**
The topos of bushes $\mathbb{FF}_2$ *is bivalent and non-Boolean.*

By a direct application of (86) we obtain:

Corollary 93.
$$\forall \alpha \in \boldsymbol{Form} : \vdash_{CPL} \alpha \quad \textit{iff} \quad \mathbb{FF}_2 \models \alpha.$$

3.1.5 A few Topoi Examples

To better understand why for topoi *bivalent* and *Boolean* are, in a sense, independent attributes, consider the following examples:
(We omit in both cases the construction of exponential objects and focus on their sub-object classifier and truth-arrows).

Example 94 (Pair of Sets)**.** The topos $\mathbb{Set}^2$ of *Pairs of Sets* has as objects all set pairs $\langle A, B\rangle$ and arrows pairs of set functions $\langle f, g\rangle : \langle A, B\rangle \to \langle C, D\rangle$ with $f : A \to C$ and $g : B \to D$.

One can verify that the sub-object classifier Ω for $\mathbb{Set}^2$ is none-other than $\langle \mathbf{2}, \mathbf{2}\rangle$ with the *true* arrow given by $\langle \top, \top\rangle : \langle\{0\}, \{0\}\rangle \to \langle \mathbf{2}, \mathbf{2}\rangle$.
As such, the *truth values* are precisely the four elements $\{\langle\perp, \perp\rangle, \langle\perp, \top\rangle, \langle\top, \perp\rangle, \langle\top, \top\rangle\}$.

So:

Remark 31. $\mathbb{Set}^2$ fails to be bivalent. It is however a Boolean topos as the sub-objects of Ω form a power-set Boolean Algebra.[7]

Example 95 (Functions between sets)**.** The *functor category* $\mathbb{Set}^{0\to 1}$ of *functions between sets*, with objects the functors $F : \mathbb{2} \to \mathbb{Set}$ from the poset category $\mathbf{2} := \{0 \xrightarrow{\leq_0} 1\}$[8] to $\mathbb{Set}$ and arrows the natural transformations $\tau : F \Rightarrow G$ between these functors, is a topos.
We use the notation F_i, G_j for $F(i), G(j)$. Also f replaces $F(\leq_0)$ and g replaces $G(\leq_0)$.

An arrow τ from the objects F and G is realized in $\mathbb{Set}$ as a commutative diagram:

$$
\begin{tikzcd}
0 \arrow[d] \\
1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F_0 \arrow[r, "\tau_0"'] \arrow[d, "f"] & G_0 \arrow[d, "g"] \\
F_1 \arrow[r, "\tau_1"'] & G_1
\end{tikzcd}
$$

Figure 3.15: The poset category $\mathbb{2}$ (left) and the commutative diagram in $\mathbb{Set}$ (right).

[7]see [9] for details.
[8]$\mathbf{2}$ has only objects 0 and 1 and the only non-identity arrow 01 from 0 to 1.

The terminal object $\mathbf{1}$, one can verify, is the identity function $\{0\} \xrightarrow{id} \{0\}$ on the singleton set $\{0\}$.
The sub-object $\mu : F \Rightarrow G$ is realized again as a commutative diagram and we will assume without loss of generality that the components μ_0, μ_1 (which are injective functions in $\mathbb{Set}$) be set-inclusions $F_0 \subseteq G_0, F_1 \subseteq G_1$ so that f will in fact be the restriction of g to F_0.

$$\begin{tikzcd} F_0 \arrow[r, tail, "\mu_0"] \arrow[d, "f=g\restriction F_0"'] & G_0 \arrow[d, "g"] \\ F_1 \arrow[r, tail, "\mu_1"'] & G_1 \end{tikzcd}$$

Figure 3.16: The sub-object $\mu : F \Rightarrow G$.

Note that an element $x \in G_0$ can be *classified* in three ways:

(i) $x \in F_0$.

(ii) $x \notin F_0$ but $g(x) \in F_1$.

(iii) $x \notin F_0$ and $g(x) \notin F_1$.

For this purpose, we introduce the set $\{0, \frac{1}{2}, 1\}$ and define $\psi : F_0 \to \{0, \frac{1}{2}, 1\}$ by:

$$\psi(x) = \begin{cases} 1 & \text{if (i) holds} \\ \frac{1}{2} & \text{if (ii) holds} \\ 0 & \text{if (iii) holds} \end{cases}$$

This suggests that:
$\Omega(0) := \{0, \frac{1}{2}, 1\}$ and $\Omega(1) := \{0, 1\}$ with $\Omega(\leq_0) := t : 0 \mapsto 0, \frac{1}{2} \mapsto 1, 1 \mapsto 1$.

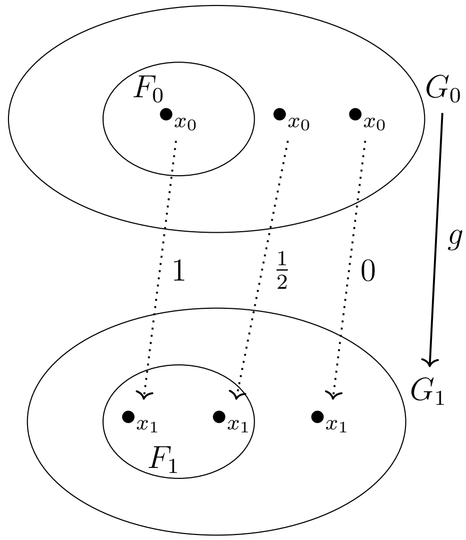


Figure 3.17: The sub-object $F \overset{\mu}{\Rightarrow} G$ and the function ψ.

The sub-object classifier is thus given by $\top : \mathbf{1} \Rightarrow \Omega$:

$$\begin{tikzcd}
\{0\} \arrow[r, "t'"] \arrow[d, "id"'] & \{0, \frac{1}{2}, 1\} \arrow[d, "t"] \\
\{0\} \arrow[r, "true"'] & \{0, 1\}
\end{tikzcd}$$

Figure 3.18: $true : 0 \mapsto 1$, $t' : 0 \mapsto 1$, $t : 0 \mapsto 0, \frac{1}{2} \mapsto 1, 1 \mapsto 1$.

We have in addition to $\top$ other two truth-arrows $*, \bot : \mathbf{1} \Rightarrow \Omega$:

$$\begin{tikzcd}
\{0\} \arrow[r, "*'"] \arrow[d, "id"'] & \{0, \frac{1}{2}, 1\} \arrow[d, "t"] \\
\{0\} \arrow[r, "true"'] & \{0, 1\}
\end{tikzcd}$$

Figure 3.19: $* : \mathbf{1} \Rightarrow \Omega$ with $true : 0 \mapsto 1$, $*' : 0 \mapsto \frac{1}{2}$.

$$\begin{tikzcd}
\{0\} \arrow[r, "f'"] \arrow[d, "id"'] & \{0, \frac{1}{2}, 1\} \arrow[d, "t"] \\
\{0\} \arrow[r, "false"'] & \{0, 1\}
\end{tikzcd}$$

Figure 3.20: $\perp : \mathbf{1} \Rightarrow \Omega$ with $false : 0 \mapsto 0$, $f' : 0 \mapsto 0$.

The characteristic diagram is now a cube instead of the usual square:

The following holds: [9]

Remark 32. $\mathbb{Set}^{0 \to 1}$ is neither bivalent (it has three truth-arrows) nor Boolean.

[9]to prove that $\mathbb{Set}^{0 \to 1}$ is not a Boolean topos requires some further considerations that will be made later on.

$$\begin{tikzcd}
F_0 \arrow[rr, tail, "\tau_0"] \arrow[dr, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dd, dashed] & & G_0 \arrow[dr, "g"] \arrow[dd, dotted, "\psi"] & \\
& F_1 \arrow[rr, tail, "\tau_1"] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dd, dashed, "t'"] & & G_1 \arrow[dd, "\chi_{F_1}"] \\
\{0\} \arrow[rr] \arrow[dr, "id"'] & & \{0, \frac{1}{2}, 1\} \arrow[dr, "t"] & \\
& \{0\} \arrow[rr, "true"'] & & \{0, 1\}
\end{tikzcd}$$

Figure 3.21: The front and back faces of the cube are each pull-backs in $\mathbb{Set}$.

A generalized version of $\mathbb{Set}^{0 \to 1}$, which will prove rather important later on, is that of the functor category of *sets through time*:

Example 96 (Sets through time)**.** Let $\omega := (\omega, \leq)$ be the poset category of natural numbers with their standard ordering $0 \xrightarrow{\leq_0} 1 \xrightarrow{\leq_1} 2 \xrightarrow{\leq_2} ...$ which will be our so-called *moments in time*.
$\mathbb{Set}^{\omega}$, a.k.a. the category of *sets through time* has objects *sequences* of sets and arrows *commutative diagrams* between them:

$$\begin{tikzcd}
F : \arrow[d, squiggly, "\tau"] & F_0 \arrow[r, "F_{01}"'] \arrow[d, dashed, "\tau_0"] & F_1 \arrow[r, "F_{12}"'] \arrow[d, dashed, "\tau_1"] & F_2 \arrow[r] \arrow[d, dashed, "\tau_2"] & ... \arrow[r] & F_m \arrow[r, "F_{m\ m+1}"] \arrow[d, dashed, "\tau_m"] & F_{m+1} \arrow[r] \arrow[d, dashed, "\tau_{m+1}"] & ... \\
G : & G_0 \arrow[r] & G_1 \arrow[r, "G_{12}"'] & G_2 \arrow[r] & ... \arrow[r] & G_m \arrow[r, "G_{m\ m+1}"'] & G_{m+1} \arrow[r] & ...
\end{tikzcd}$$

Figure 3.22: $\tau : F \Rightarrow G$.

It is worth pausing to give an intuition due to *John C. Baez* for the notion of *sets through time*:

Remark 33. Imagine a set of theorems proven by an infallible mathematician at various times $0, 1, 2.. \in \omega$.
As time passes, in steps from 0 to 1 and from 1 to 2 and so on, This set can get new elements as the mathematician produces new theorems.
Also, two distinct theorems can *merge* into one if an equivalence is found between them.

However, we can never remove an element from the set as once a theorem has been proven it remains so forever and cannot be dis-proven.

Notice that: Any non-empty up-set $S \subseteq \omega$ has a minimum m_S and so we have:

$$S = [m_S) = \{m_S, m_S + 1, m_S + 2...\}.$$

i.e., S coincides with the principal up-set generated by its minimum.

Now let us add a symbol for *infinity* to replace the empty-set so that $S = \emptyset$ becomes $S = \{\infty\}$.
So we work with $\omega^+ := \omega \cup \{\infty\}$.

The terminal object, a.k.a. *the infinite telephone pole* is the constant functor $\mathbf{1}$:

$$\forall m \in \omega : \mathbf{1}(m) := \{0\}.$$
$$\mathbf{1}(\leq_m) := id_{\{0\}}.$$

For the sub-object classifier, Ω and $\top : \mathbf{1} \Rightarrow \Omega$ are defined in the following way for each $m \in \omega$: [10]

$$m \;\mapsto\; \Omega_m := [m) = \{m, m+1, .., \infty\}.$$
$$m \xrightarrow{\leq} n \;\mapsto\; [m) \xrightarrow{\Omega_{m\,n}} [n) \text{ where } \Omega_{m\,n}(p) := \begin{cases} n & \text{if } m \leq p \leq n \\ p & \text{if } n \leq p \\ \infty & \text{if } p = \infty \end{cases}.$$
$$\top_m : \{0\} \to [m) \text{ where } \top_m(0) := m.$$

[10]we make use of the same notation we introduced for the previous example.

We can display Ω as:

$$
\begin{tikzcd}
\Omega_m = \arrow[d, "\Omega_{m\ m+1}"'] & \{ & m, \arrow[d, maps to] & m+1, \arrow[dl, maps to] & m+2, \arrow[dl, maps to] & \dots \arrow[dl, maps to, dashed] & \infty \arrow[d, maps to] & \} \\
\Omega_{m+1} = \arrow[d, "\Omega_{m+1\ m+2}"'] & \{ & m+1, \arrow[d, maps to] & m+2, \arrow[dl, maps to] & \dots \arrow[dl, maps to, dashed] & \dots & \infty \arrow[d, maps to] & \} \\
\Omega_{m+2} = \arrow[dd, dotted, "\Omega_{m+2\ n}"'] & \{ & m+2, \arrow[d, maps to, dashed] & \dots \arrow[dl, maps to, dashed] & \dots & \dots & \infty \arrow[dd, maps to, dashed] & \} \\
 & & {} \arrow[d, maps to, dashed] & {} \arrow[dl, maps to, dashed] & \dots & \dots & & \\
\Omega_n = & \{ & n, & n+1, & \dots & \dots & \infty & \}
\end{tikzcd}
$$

Figure 3.23: Ω displayed in a few of its components.

The characteristic arrow χ_τ of a sub-object $\tau : F \Rightarrow G$ (as before w.l.o.g. we assume $F_m \subseteq G_m$) is given by the so-called *time till truth*:

$$(\chi_\tau)_m(x) := \begin{cases} min\{n : n \geq m |\ G_{m\,n}(x) \in F_n\} & \text{if such } n \text{ exists.} \\ \infty & \text{if } G_{m\,n}(x) \notin F_n \text{ for every } n \geq m. \end{cases}$$

This is rendered visually as:

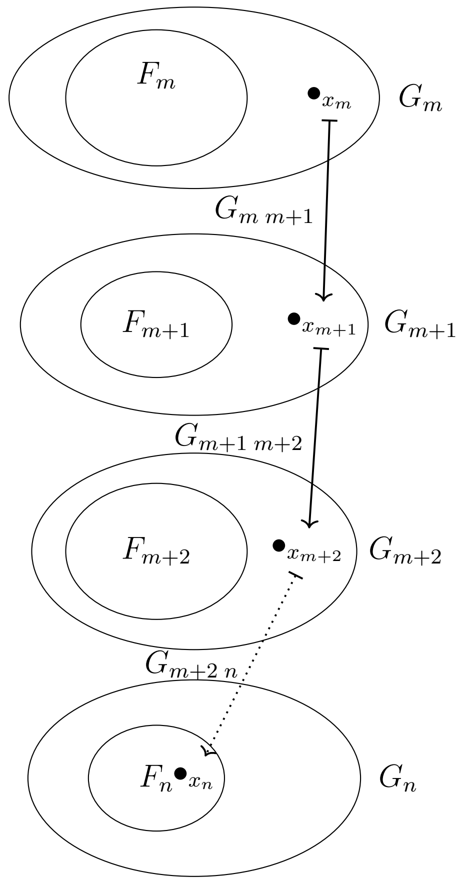


Figure 3.24: Here $(\chi_\tau)_m(x) = n$.
The sub-object $F \overset{\tau}{\Rightarrow} G$ is displayed in some of its components $F_m \subseteq G_m$, $F_{m+1} \subseteq G_{m+1}$, and $F_n \subseteq G_n$ and the transition function $G_{m\,n}$ is shown in its constituents $G_{m\,n} = G_{m\,m+1} G_{m+1\,m+2} G_{m+2\,n}$.

Notice that we now have a countable infinity of *truth-values* between $\perp : \mathbf{1} \Rightarrow \Omega$:

$$\forall m \in \omega : (\perp)_m(0) := \{\infty\}.$$

and $\top : \mathbf{1} \Rightarrow \Omega$ given by $\{*_q\}_{q \in \omega}$ where each $*_q : \mathbf{1} \Rightarrow \Omega$ is given by:

$$\forall m \in \omega : (*_q)_m(0) := \begin{cases} q & \text{if } m \leq q. \\ m & \text{if } m > q. \end{cases}.$$

Chapter 4

Topos Semantics II

4.1 First Order

We move on from the *Propositional* level to the *Predicate* or *First Order* level of logic following [9]'s *first-principles* approach before changing course to [12]'s *BKJ Semantics* in order to better realize quantifiers in our topos of study.
For our purposes we have chosen at this stage not to add multiple constants or predicates and to omit function symbols[1]. Also we start off with a single *sort* or *type* of variables.

We have also decided, for the purpose of this work, not to mention Soundness & Completeness Theorems which generalize to first order level soundness & completeness of topoi (122) with respect to intuitionistic logic.
We leave as references chapter XI of [9] and part II of [12].

[1]any function symbol could be substituted with a relation or predicate symbol which specifies the function's *graph*.

4.1.1 First Order Logic and Topoi

In order to interpret an elementary language $\mathcal{L}$ in a topos $\mathcal{E}$ it is necessary first to reformulate Tarski Semantics for $\mathcal{L}$-terms and $\mathcal{L}$-formulae (which from now on will be called just 'terms' and 'formulae') in a given *context*.

Def. 4.1.1 (*context*)**.** We specify a *context* for a formula ϕ by fixing an integer $m \geq 1$ which will be called *appropriate* to ϕ if all the variables that occur in ϕ free or bound are all elements of the list $\{x_1, .., x_m\}$. We will refer to this ϕ as a *formula-in-context*.
Similarly a *term-in-context* is a term $\mathbf{t}$ in which all occurrences of variables in $\mathbf{t}$ belong to $\{x_1, .., x_m\}$.

We re-define satisfaction for ϕ by m-length sequences $\{\mathbf{a}_1, .., \mathbf{a}_m\}$ by requiring that $\mathcal{M} \vDash \phi[\mathbf{a}_1, .., \mathbf{a}_n]$ iff $\mathcal{M} \vDash \phi[\mathbf{y}]$ for some assignment $\mathbf{y}$ for which $\mathbf{y}_i = \mathbf{a}_i$ whenever x_i is free in ϕ.
Note that given an $\mathcal{L}$-model $\mathcal{M} = \langle \mathbf{A}, \mathfrak{P}, \mathfrak{c} \rangle$ and a *context* $m \geq 1$, each formula-in-context ϕ determines a sub-set $\phi^m \subseteq \mathbf{A}^m$ namely the set of all m-tuples satisfying ϕ:

$$\phi^m = \{(a_1, .., a_m) : \mathcal{M} \vDash \phi[a_1, .., a_m]\}.$$

Note also that by this definition:

$$\begin{aligned}
(\neg\phi)^m &= \mathbf{A} \setminus \phi^m \\
(\phi \wedge \psi)^m &= \phi^m \cap \psi^m. \\
(\phi \vee \psi)^m &= \phi^m \cup \psi^m. \\
(\phi \Rightarrow \psi)^m &= (\mathbf{A} \setminus \phi^m) \cup \psi^m. \\
& etc..
\end{aligned}$$

This is a translation of $\mathcal{L}$-formulae into sub-sets or sub-objects of the product domain $\mathbf{A}^m$ which in turn can be replaced by their characteristic functions $[\![\phi^m]\!] : \mathbf{A}^m \to \mathbf{2}$.
With this in mind, we can finally generalize from $\mathbb{Set}$ to a generic topos.

Let $\mathcal{E}$ be a topos and a fixed $\mathcal{E}$-object $\mathfrak{a}$ [2].

We first give some preliminary definitions:

[2] the fixed object corresponds to a fixed type or sort.

Def. 4.1.2 ($\Delta_{\mathfrak{a}}$ and $\delta_{\mathfrak{a}}$)**.** $\Delta_{\mathfrak{a}} : \mathfrak{a} \rightarrowtail \mathfrak{a} \times \mathfrak{a}$ is the product arrow $id_{\mathfrak{a}} \times id_{\mathfrak{a}}$.
$\delta_{\mathfrak{a}} : \mathfrak{a} \times \mathfrak{a} \to \Omega$ is the characteristic arrow of $\Delta_{\mathfrak{a}}$.

Def. 4.1.3 ($true_{\mathfrak{o}}$)**.** For any $\mathcal{E}$-object $\mathfrak{o}$,
$true_{\mathfrak{o}}$ is the composite arrow $true \circ !_{\mathfrak{o}}$
(where $!_{\mathfrak{o}}$ as usual is the unique arrow from $\mathfrak{o}$ to the terminal $\mathbf{1}$).

We are finally ready to define a topos-model or $\mathcal{E}$-model for First Order Logic: (From now on we fix an appropriate $m \geq 1$ context).

Def. 4.1.4 ($\mathcal{E}$-model)**.** [3] An $\mathcal{E}$-model for $\mathcal{L}$ is a structure $\mathfrak{M}$=$\langle \mathfrak{a}, \mathfrak{p}, \mathfrak{f}_c \rangle$ where

I $\mathfrak{a}$ is an $\mathcal{E}$-object for the *domain* that is *not empty* i.e., $\mathcal{E}(\mathbf{1}, \mathfrak{a}) \neq \emptyset$.

II $\mathfrak{p} : \mathfrak{a}^n \to \Omega$ is an $\mathcal{E}$-arrow for the *predicate/relation.* [4]

III $\mathfrak{f}_c : \mathbf{1} \to \mathfrak{a}$ is an $\mathcal{E}$-*element* of $\mathfrak{a}$ for the *particular individual.*

Remark 34. Notice that if the arity of the predicate-arrow is $n = 0$ we recover *propositions*, if the arity is $n = 1$ we obtain unary *predicates* and if $n > 1$ n-ary *relations.*

We *realize* i.e., interpret the terms $\mathbf{t}$ as arrows $\mathfrak{a}^m \to \mathfrak{a}$:

Def. 4.1.5 ($[\![\mathbf{t}]\!]^m$)**.**

$$[\![\mathbf{t}]\!]^m = \begin{cases} pr_i^m : \mathfrak{a}^m \to \mathfrak{a} & \text{if } \mathbf{t} \text{ is the variable } x_i \\ \mathfrak{f}_c \circ !_{\mathfrak{a}} : \mathfrak{a}^m \to \mathfrak{a} & \text{if } \mathbf{t} \text{ is the constant } \mathbf{c} \end{cases}$$

For $m > 1$ the m variables in context $\{x_i\}_{i=1}^m$ are realized by the m projections pr_i^m from $\mathfrak{a}^m$ to $\mathfrak{a}$.
If $m = 1$ the only variable in context $x = x_1$ is realized by the identity arrow $id_{\mathfrak{a}}$.

We now define for each $\mathcal{L}$-formula ϕ a realization/interpretation $[\![\phi]\!]^m$ as an arrow $\mathfrak{a}^m \to \Omega$:
(From now on $[\![\phi]\!], [\![\mathbf{t}]\!]$ will be used instead of $[\![\phi]\!]^m, [\![\mathbf{t}]\!]^m$ if the context is already specified and there is no ambiguity).

Def. 4.1.6 ($[\![\phi]\!]$)**.** The atomic formulae admit the following *realizations*:

1. $[\![\mathbf{t} \approx \mathbf{u}]\!] = \delta_{\mathfrak{a}} \circ ([\![\mathbf{t}]\!] \times [\![\mathbf{u}]\!])$.

$$\begin{tikzcd} \mathfrak{a}^m \arrow[r, "[\![\mathbf{t}]\!]\times[\![\mathbf{u}]\!]"'] \arrow[dr, "[\![\mathbf{t}\approx\mathbf{u}]\!]"'] & \mathfrak{a}^2 \arrow[d, "\delta_{\mathfrak{a}}"'] \\ & \Omega \end{tikzcd}$$

$$\begin{tikzcd} a^m \arrow[r, "[\![\mathbf{t}_1]\!]\times\ldots\times[\![\mathbf{t}_n]\!]"] \arrow[dr, "[\![\mathbf{P}(\mathbf{t}_1,..,\mathbf{t}_n)]\!]"'] & a^n \arrow[d, "p"'] \\ & \Omega \end{tikzcd}$$

2. $[\![\mathbf{P}(\mathbf{t}_1, .., \mathbf{t}_n)]\!] = \mathfrak{p} \circ ([\![\mathbf{t}_1]\!] \times ... \times [\![\mathbf{t}_n]\!])$.

The rest follow by induction:

3. $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \wedge [\![\psi]\!] = \wedge \circ ([\![\phi]\!] \times [\![\psi]\!])$.

$$\begin{tikzcd} a^m \arrow[r, "[\![\phi]\!]\times[\![\psi]\!]"] \arrow[dr, "[\![\phi\wedge\psi]\!]"'] & \Omega \times \Omega \arrow[d, "\wedge"'] \\ & \Omega \end{tikzcd}$$

4. $[\![\phi \vee \psi]\!] = [\![\phi]\!] \vee [\![\psi]\!] = \vee \circ ([\![\phi]\!] \times [\![\psi]\!])$.
5. $[\![\neg\phi]\!] = \neg \circ [\![\phi]\!]$.
6. $[\![\phi \Rightarrow \psi]\!] = [\![\phi]\!] \Rightarrow [\![\psi]\!] = \Rightarrow \circ([\![\phi]\!] \times [\![\psi]\!])$.

We can now define $\mathcal{E}$-validity for a formula ϕ starting by what it means for $\mathfrak{M}$ to *model* ϕ denoted by $\mathfrak{M} \vDash_{\mathcal{E}} \phi$.
Let $\phi = \phi(x_{i_1}, ..x_{i_n})$ be a formula-in-context and take any arrow $g : \mathfrak{a}^n \to \mathfrak{a}$

[3] note that we can generalize this definition by taking multiple objects in I (multiple sorts), multiple predicates in II and generalized elements in III.

[4] we will assume $0 \leq n \leq m$.

,we construct a product arrow $f : p_1 \times .. \times p_m$ where: (pr^n_k as usual denotes the k-th projection from $\mathfrak{a}^n$)

$$p_i = \begin{cases} pr^n_k : \mathfrak{a}^n \to \mathfrak{a} & \text{if } j = i_k \text{ for some } \ 1 \leq k \leq n. \\ g & \text{otherwise.} \end{cases}$$

$\mathfrak{M}$ is thus an "$\mathcal{E}$-model of $\phi = \phi(x_{i_1}, ..x_{i_n})$" i.e., $\mathfrak{M} \vDash_{\mathcal{E}} \phi$ if:

Def. 4.1.7 ($\mathfrak{M}$ models $\phi(x_{i_1}, ..x_{i_n})$)**.**

$$\mathfrak{M} \vDash_{\mathcal{E}} \phi \text{ iff } [\![\phi]\!]_{\mathfrak{M}} = true_{\mathfrak{a}^n}.$$

Where the arrow $[\![\phi]\!]_{\mathfrak{M}} = true_{\mathfrak{a}^n} : \mathfrak{a}^n \to \Omega$ is defined as $true_{\mathfrak{a}^m} \circ f$:

$$\begin{tikzcd} \mathfrak{a}^n \arrow[r, "f"] \arrow[dr, "{[\![\phi]\!]_{\mathfrak{M}}}"'] & \mathfrak{a}^m \arrow[d, "{[\![\phi]\!]}"] \\ & \Omega \end{tikzcd}$$

By the categorical properties of the arrows $true_{\mathfrak{d}}$ [5] we find that:

Remark 35. $[\![\phi]\!]_{\mathfrak{M}} = true_{\mathfrak{a}^n}$ iff $[\![\phi]\!] = true_{\mathfrak{a}^m}$.

Thus:

Def. 4.1.8 ($\mathcal{E}$-validity)**.** Let $\phi = \phi(x_{i_1}, ..x_{i_n})$ be a formula-in-context, then:

$$\mathfrak{M} \vDash_{\mathcal{E}} \phi \ \text{ iff } \ [\![\phi]\!] = true_{\mathfrak{a}^m}.$$

The formula ϕ is $\mathcal{E}$-valid i.e., $\vDash_{\mathcal{E}} \phi$ if for every $\mathcal{E}$-model $\mathfrak{M}$ one has $\mathfrak{M} \vDash_{\mathcal{E}} \phi$.

4.1.2 Another look at $\mathcal{G}_3$ through Predicates

We continue making some considerations on our topos of *bushes*/$\mathbb{F}\mathbb{F}_2$ like the following:
Recall our discourse about external and internal topos-logic in the propositional case, we left off with the result about *bushes*/$\mathbb{F}\mathbb{F}_2$ being bivalent and non-Boolean.
The culprit was the definition of propositional semantics for topoi.

(a) $\top : \mathbf{1} \to \Omega$. (b) $\bot : \mathbf{1} \to \Omega$.

The propositional truth-values are given by the two arrows $\top$ and $\bot$ of $\mathbb{FF}_2(\mathbf{1}, \Omega)$ that pick out the two roots 't' and 'f' of $\Omega = \mathbf{1} + \mathbf{1}_\bot$:

However, as in the case of $\mathbb{Set}$, truth values are meant to be *generalized elements* of the sub-object classifier Ω. The arrows from $\mathbf{1}$ to Ω are clearly insufficient to obtain the top element $*$ of $\mathbf{1}_\bot \subset \Omega$. This is remedied if we take arrows from $\mathbf{1}_\bot$. The object $\mathbf{1}_\bot$ in fact is shown to be a *representing object* for the category of *bushes.*

The hom-set $\mathbb{FF}_2(\mathbf{1}_\bot, \Omega)$ has exactly three arrows which we denote by $\mathfrak{p}_t, \mathfrak{p}_f, \mathfrak{p}_*$ each determined by the image of the top element of $\mathbf{1}_\bot$.

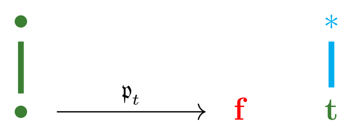


Figure 4.2: $\mathfrak{p}_t : \mathbf{1}_\bot \to \Omega$.

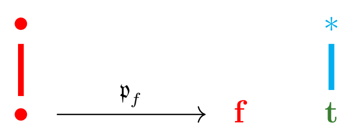


Figure 4.3: $\mathfrak{p}_f : \mathbf{1}_\bot \to \Omega$.

Note that these arrows can be viewed as realizations of unary *predicates* which we name: $\mathbf{P}_t, \mathbf{P}_f, \mathbf{P}_*$.
Consider the following $\mathcal{E}$-model $\mathfrak{X}$ with specified context $m = 1$:

[5] by using the fact that any arrow "that factors through *true* is *true*".

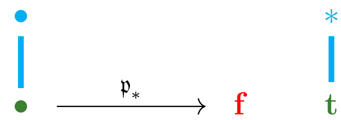


Figure 4.4: $\mathfrak{p}_* : \mathbf{1}_\perp \to \Omega$.

Example 97. (m=1)

$$\mathfrak{X} = \langle \mathbf{1}_\perp, \{\mathfrak{p}_t, \mathfrak{p}_f, \mathfrak{p}_*\}, \mathfrak{f}_c \rangle.$$

$$[\![x]\!] = id_{\mathbf{1}_\perp} : \mathbf{1}_\perp \to \mathbf{1}_\perp.$$
$$[\![\mathbf{c}]\!] = \mathfrak{f}_c \circ !_{\mathbf{1}_\perp} : \mathbf{1}_\perp \to \mathbf{1}_\perp \text{ i.e., the constant map on the root of } \mathbf{1}_\perp.$$
$$[\![\mathbf{P}_t(x)]\!] = \mathfrak{P}_t \circ [\![x]\!] = \mathfrak{P}_t : \mathbf{1}_\perp \to \Omega.$$
$$[\![\mathbf{P}_f(x)]\!] = \mathfrak{P}_f \circ [\![x]\!] = \mathfrak{P}_f : \mathbf{1}_\perp \to \Omega.$$
$$[\![\mathbf{P}_*(x)]\!] = \mathfrak{P}_* \circ [\![x]\!] = \mathfrak{P}_* : \mathbf{1}_\perp \to \Omega.$$

Note that by construction $\mathfrak{P}_t = true_{\mathbf{1}_\perp}$ and so we have $\mathfrak{X} \vDash_{\mathcal{E}} \mathbf{P}_t(x)$. If we now compose these predicates with truth-arrows, we obtain:

$$[\![\neg\mathbf{P}_t(x)]\!] = \neg \circ [\![\mathbf{P}_t(x)]\!] = \neg \circ \mathfrak{P}_t = \mathfrak{P}_f = [\![\mathbf{P}_f(x)]\!].$$
$$[\![\neg\mathbf{P}_*(x)]\!] = \neg \circ [\![\mathbf{P}_*(x)]\!] = \neg \circ \mathfrak{P}_* = \mathfrak{P}_f = [\![\mathbf{P}_f(x)]\!].$$
$$[\![\neg\mathbf{P}_f(x)]\!] = \neg \circ [\![\mathbf{P}_f(x)]\!] = \neg \circ \mathfrak{P}_f = \mathfrak{P}_t = [\![\mathbf{P}_t(x)]\!].$$

$$[\![\mathbf{P}_*(x) \wedge \mathbf{P}_t(x)]\!] = \wedge \circ ([\![\mathbf{P}_*(x)]\!] \times [\![\mathbf{P}_t(x)]\!]) = \mathfrak{P}_*.$$
$$etc..$$

$$[\![\mathbf{P}_*(x) \vee \mathbf{P}_t(x)]\!] = \vee \circ ([\![\mathbf{P}_t(x)]\!] \times [\![\mathbf{P}_*(x)]\!]) = \mathfrak{P}_t.$$
$$etc..$$

$$[\![\mathbf{P}_t(x) \Rightarrow \mathbf{P}_*(x)]\!] = \Rightarrow \circ([\![\mathbf{P}_t(x)]\!] \times [\![\mathbf{P}_*(x)]\!]) = \mathfrak{P}_*.$$
$$etc..$$

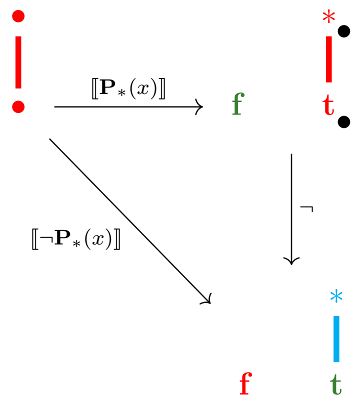


Figure 4.5: $[\![\neg\mathbf{P}_*(x)]\!] : \mathbf{1}_\perp \to \Omega$. The usual coloring notation is applied for this arrow. The images of the nodes of $\mathbf{1}_\perp$ by $[\![\mathbf{P}_*(x)]\!]$ are also shown as black bullets.

Remark 36. Again we find the (propositional) truth functions of $\mathcal{G}_3$. Though this time *internally* by composition of the *new* truth arrows for First Order Logic that we defined. In this case, these arrows are of the form $\mathbf{1}_\perp \to \Omega$ which is precisely what we need for generalized elements of $\mathbb{FF}_2$.

Notice that $\mathbf{P}_t$,$\mathbf{P}_*$ and $\mathbf{P}_f$ can also be viewed as characteristic arrows for the sub-forests $\{\mathbf{1}_\perp\}$, $\{\perp\}$ and $\emptyset$ respectively.
This leads to the following consideration:

Remark 37. What we found with these predicates is an application of the duality between *bushes* $\mathbb{FF}_2$ and $(\mathbb{G}_3)_{fin}$.
$Sub(\mathbf{1}_\perp)$ is isomorphic (as Gödel Algebras) to C_3 the 3-element chain which semantically characterizes our logic $\mathcal{G}_3$ i.e., $C_3 \models \mathcal{G}_3$.

4.2 What about Quantifiers?

To treat First Order Logic and *quantifiers* we leave aside the method in [9] we introduced earlier and instead apply a new approach outlined in [12], inspired by *type theory* and the *Curry-Howard* correspondence between objects and types, and arrive at a few conclusions.

4.2.1 A Type-Theoretic Approach

We introduce some new jargon:
(The notation $\langle a, b\rangle$ is equivalent to $a \times b$.)

- The *special* type Ω is the object Ω.
- Variables of type A_i i.e., $x_i : A_i$ are realized as *indeterminate* arrows $1 \xrightarrow{x_i} A_i$.
- From $(a : A)$ $1 \xrightarrow{a} A$ and $(b : B)$ $1 \xrightarrow{b} B$ one obtains $(\langle a, b\rangle : A \times B)$ $1 \xrightarrow{\langle a,b\rangle} A \times B$.
- $a = a'$ denotes *internal* equality[6] and is realized as[7] $1 \xrightarrow{\langle a,a'\rangle} A \times A \xrightarrow{\delta_A} \Omega$.
- $\cdot = \cdot$ instead denotes *external* equality[8] between arrows in the topos.
- $\mathfrak{T} \models p$ means that the topos $\mathfrak{T}$ satisfies the proposition p: In which case $p \cdot = \cdot \top$ as arrows in $\mathfrak{T}$.

Def. 4.2.1 (realization of $\phi(x)$)**.** A predicate formula $\phi(x)$ which takes as argument an $x : A$ is realized as $1 \xrightarrow{x} A \xrightarrow{f} \Omega$ i.e., $\ulcorner\phi(x)\urcorner \equiv fx$ where $f = \ulcorner\phi\urcorner$ is the realization of the predicate ϕ.

Generalizing to an arbitrary context:

Def. 4.2.2 (realization of $\psi(x_1, .., x_n)$)**.** $\psi(x_1, x_2, ..., x_n)$ with $h = \ulcorner\psi\urcorner$ the realization of the n-ary predicate ψ and $x_i : A_i$ is realized as
$1 \xrightarrow{\langle x_1,..,x_n\rangle} A_1 \times ... \times A_n \xrightarrow{h} \Omega$.

[6] as a first order relation.
[7] recall that δ_A characteristic arrow of $\langle id_A, id_A\rangle$.
[8] as equality of arrows.

Def. 4.2.3 (realization of $\phi(a)$)**.** We extend this notion to $\ulcorner \phi(a) \urcorner \equiv fa$ realized as $C \xrightarrow{a} A \xrightarrow{f} \Omega$ (by a slight abuse of notation) where a is one of the *generalized elements* of A at stage C.

Generalizing to an arbitrary context:

Def. 4.2.4 (realization of $\psi(a_1, .., a_n)$)**.** $\psi(a_1, a_2, ..., a_n)$ with generalized elements $C \xrightarrow{a_i} A_i$ at stage C is realized as $C \xrightarrow{\langle a_1, a_2 .., a_n \rangle} A_1 \times ... \times A_n \xrightarrow{h} \Omega$.

Def. 4.2.5 (truth at a stage)**.** $\phi(a)$ *holds at stage* C or C *forces* $\phi(a)$ denoted by $C \Vdash \phi(a)$ if the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[dr, dotted, "!_C"'] & A \arrow[r, "f"] & \Omega \\ & 1 \arrow[ur, "\top"'] & \end{tikzcd}$$

Figure 4.6: $f\ a\cdot = \cdot \top\ !_C$

In general: $\psi(a_1, .., a_n)$ *holds at stage* C or C *forces* $\psi(a_1, .., a_n)$ denoted by $C \Vdash \psi(a_1, .., a_n)$ if the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, dashed, "\langle a_1, a_2, .., a_n \rangle"] \arrow[dr, dotted, "!_C"'] & A_1 \times .. \times A_n \arrow[r, "h"] & \Omega \\ & 1 \arrow[ur, "\top"'] & \end{tikzcd}$$

Figure 4.7: $h\ \langle a_1, a_2, .., a_n \rangle \cdot = \cdot \top\ !_C$

As a consequence of these definitions the following hold:

Proposition 98. *1. If $C \Vdash \phi(a)$ and $D \xrightarrow{h} C$ then $D \Vdash \phi(ah)$.* [9]

2. If $h : D \twoheadrightarrow C$ is an epi and $D \Vdash \phi(ah)$, then $C \Vdash \phi(a)$.

We wish to characterize *truth* in a topos as *truth* at all stages and for all generalized elements.

We can do better by restricting the stages:

[9]by ah we mean $a \circ h$ generalized element of A at stage D.

Def. 4.2.6 (generating set)**.** A set $\mathcal{C}$ of objects of $\mathfrak{T}$ is a *generating set* if for any two arrows $f, g : A \to B$ we have $f\cdot = \cdot g$ iff for all $C \in \mathcal{C}$ and all $C \xrightarrow{h} A$ $fh\cdot = \cdot gh$.

Def. 4.2.7 (truth in $\mathfrak{T}$)**.** A formula $\phi(x)$ is *true* in a topos $\mathfrak{T}$ denoted by $\models_{\mathfrak{T}} \phi(x)$ iff for all objects $C \in \mathcal{C}$ and all generalized elements $C \xrightarrow{a} A$ of A at stage C, $C \Vdash \phi(a)$.

We also give a preliminary definition that will come in useful:

Def. 4.2.8 (indecomposable)**.** The object C is *indecomposable* if for all arrows $D \xrightarrow{k} C$ and $E \xrightarrow{l} C$ such that $[k, l] : D + E \twoheadrightarrow C$ [10] is an epi, either k or l is an epi.

The so-called *Beth-Kripke-Joyal* Semantics, which we will use, are given by:

Def. 4.2.9 (BKJ Semantics)**.** Given $C \xrightarrow{a} A$ generalized element of the topos $\mathfrak{T}$:

(i) $C \Vdash a$ (in case $A = \Omega$) iff $a\cdot = \cdot\top!_C$.

(ii) $C \Vdash \top$ always holds. [11]

(iii) $C \Vdash \bot$ iff $C \cong 0$ i.e., is an initial object in $\mathfrak{T}$. [12]

(iv) $C \Vdash \phi(a) \wedge \psi(a)$ iff $C \Vdash \phi(a)$ and $C \Vdash \psi(a)$.

(v) $C \Vdash \phi(a) \vee \psi(a)$ iff there is an epi $[k, l] : D + E \twoheadrightarrow C$ such that $D \Vdash \phi(ak)$ and $E \Vdash \psi(al)$.

(vi) $C \Vdash \phi(a) \Rightarrow \psi(a)$ iff for all $D \xrightarrow{h} C$ if $D \Vdash \phi(ah)$ then $D \Vdash \psi(ah)$.

(vii) $C \Vdash \neg\phi(a)$ iff for all $D \xrightarrow{h} C$ if $D \Vdash \phi(ah)$ then $D \cong 0$.

However, if C is indecomposable, then we can replace (v) with the much simpler:

(v)' $C \Vdash \phi(a) \vee \psi(a)$ iff either $C \Vdash \phi(a)$ or $C \Vdash \psi(a)$.

---

[10] the notation $[k, l]$ is equivalent to $k + l$ the unique arrow from the co-product.

[11] this can be also thought as *there is always an arrow-witness from C.*

[12] in our case of *bushes* this would be the empty forest which can only give the *trivial* generalized element.

What interests us above all are the semantics for quantifiers where variables range over some sort. We give the definitions for the cases of unary predicates and binary relations and leave implicit the successive generalizations for arbitrary arities.
(In this case we quantify over $x : A$) :

(vii) $C \Vdash \forall_{x:A}\phi(x)$ iff, for all generalized elements $C \xrightarrow{a} A$, then $C \Vdash \phi(a)$.

(viii) $C \Vdash \exists_{x:A}\phi(x)$ iff there is a generalized element $C \xrightarrow{a} A$ such that $C \Vdash \phi(a)$.

(If we introduce an additional variable or *parameter* $y : B$ one has:)

$$\ulcorner\phi(y,x)\urcorner \equiv g\langle y,x\rangle$$
$$\ulcorner\phi(y,a)\urcorner \equiv g\ \langle y!_C, a\rangle \text{ with } \ulcorner\phi\urcorner \equiv g : B \times A \to \Omega$$

(vii)' $C \Vdash \forall_{y:B}\psi(y,a)$ iff, for all $D \xrightarrow{h} C$ and $D \xrightarrow{b} B$, then $D \Vdash \psi(b, ah)$.

(viii)' $C \Vdash \exists_{y:B}\psi(y,a)$ iff there is an epi $h : D \twoheadrightarrow C$ and a $D \xrightarrow{b} B$ such that $D \Vdash \psi(b, ah)$.

4.2.2 Quantifying Predicates

Let's re-interpret the predicate symbols p_t, p_*, p_f in this environment. These are still arrows in $\mathbb{FF}_2$ of the form $\mathfrak{p}_t, \mathfrak{p}_*, \mathfrak{p}_f : \mathbf{1}_\perp \to \Omega$.

Remark 38. The predicates in question are $p_t(x), p_*(x), p_f(x)$ with $x : \mathbf{1}_\perp$ variable of type $\mathbf{1}_\perp$.

Remembering from (70) that in our topos $\mathbb{FF}_2$ we have that $\mathbf{1}_\perp$ is also the representing object i.e., $\mathbb{FF}_2(\mathbf{1}_\perp, F) \cong |F|$, we can restrict ourselves to the only stage given by $\mathbf{1}_\perp$ and let the *generating set* be $\mathcal{C} := \{\mathbf{1}_\perp\}$.

Example 99. What does it mean for say $p_*(x)$ to be *true* at stage $\mathbf{1}_\perp$ for some generalized element $\mathbf{1}_\perp \xrightarrow{a_0} \mathbf{1}_\perp$ i.e., $\mathbf{1}_\perp \Vdash p_*(a_0)$? We require the following diagram to commute.

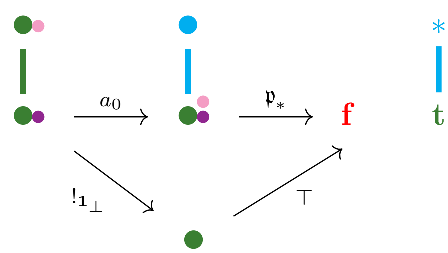


Figure 4.8: $\mathfrak{p}_* a_0 \cdot = \cdot \top !_{\mathbf{1}_\perp}$

Of course, taking the other generalized element $\mathbf{1}_\perp \xrightarrow{a_1} \mathbf{1}_\perp$ the diagram fails to commute so $\mathbf{1}_\perp \nVdash p_*(a_1)$.

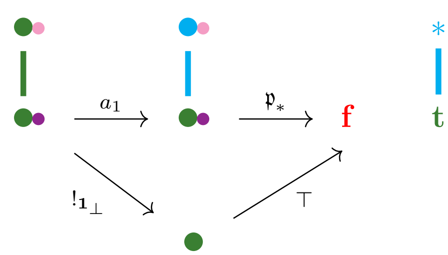


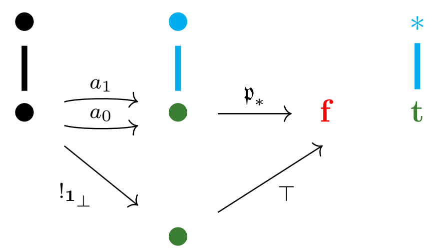
Figure 4.9: $\mathfrak{p}_* a_1 \cdot \neq \cdot \top !_{\mathbf{1}_\perp}$

By the previous considerations we conclude that:

$$\not\models_{\mathbb{FF}_2} p_*(x). \quad \not\models_{\mathbb{FF}_2} p_f(x). \quad \models_{\mathbb{FF}_2} p_t(x).$$

What happens now if we quantify over $(x : \mathbf{1}_\bot)$?

Example 100. For $\mathbf{1}_\bot \Vdash \forall_{x:1_\bot} p_*(x)$ we need to check whether $\mathbf{1}_\bot \Vdash p_*(a)$ for all $\mathbf{1}_\bot \xrightarrow{a} \mathbf{1}_\bot$.



We find, without surprises, that:

$$\not\models_{\mathbb{FF}_2} \forall_{x:1_\bot} p_f(x). \quad \not\models_{\mathbb{FF}_2} \forall_{x:1_\bot} p_*(x). \quad \models_{\mathbb{FF}_2} \forall_{x:1_\bot} p_t(x).$$

Moving on to existentials:
For $\mathbf{1}_\bot \Vdash \exists_{x:1_\bot} \mathfrak{p}_*(x)$ we need to check whether $\mathbf{1}_\bot \Vdash p_*(a)$ for some $\mathbf{1}_\bot \xrightarrow{a} \mathbf{1}_\bot$. We find:

$$\not\models_{\mathbb{FF}_2} \exists_{x:1_\bot} p_f(x). \quad \models_{\mathbb{FF}_2} \exists_{x:1_\bot} p_*(x). \quad \models_{\mathbb{FF}_2} \exists_{x:1_\bot} p_t(x).$$

4.2.3 Quantifying Relations

In a new example, we introduce a *relation* symbol r.
This is realized as an arrow $\mathfrak{r}$ from $\mathbf{1}_\perp \times \mathbf{1}_\perp$ to Ω (x and y have both the same type $1_\perp$) of the following form:

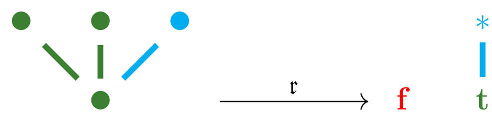


Figure 4.10: $\mathbf{1}_\perp \times \mathbf{1}_\perp \xrightarrow{\mathfrak{r}} \Omega$:

Example 101. Fixing a generalized element $\mathbf{1}_\perp \xrightarrow{a'} \mathbf{1}_\perp$ we aim to establish if $\mathbf{1}_\perp \Vdash \forall_{x:1_\perp} r(x, a')$.
Unfolding the previous definitions we require that the following commutative diagram for all arrows $D \xrightarrow{h} \mathbf{1}_\perp$ and $D \xrightarrow{a} \mathbf{1}_\perp$ results in $D \Vdash r(a,\ a'h)$ i.e., $\mathfrak{r}\langle a,\ a'h\rangle \cdot = \cdot \top !_D$.
Notice however that since $\mathbf{1}_\perp$ is a representing object for *bushes* (70) any arrows $D \xrightarrow{h} \mathbf{1}_\perp$ and $D \xrightarrow{a} \mathbf{1}_\perp$ admit a family of *liftings* $\{\mathbf{1}_\perp \xrightarrow{e_j} D\}_{e_j \in D}$ for every node of D [13] and corresponding families of arrows $\{\mathbf{1}_\perp \xrightarrow{h_j} \mathbf{1}_\perp\}_{e_j \in D}$ and $\{\mathbf{1}_\perp \xrightarrow{a_j} \mathbf{1}_\perp\}_{e_j \in D}$ such that $\forall e_j \in D : he_j = h_j$ and $ae_j = a_j$.
All this to say that, without loss of generality, we may assume $D = \mathbf{1}_\perp$. This is motivated by the fact that *truth* of the formula is obtained when every node of D is sent to the node t of Ω by $\mathfrak{r}\langle a, a'h\rangle$ and this is the same as requiring that every image of $\mathbf{1}_\perp$ into D is sent to t.

$$
\begin{tikzcd}
 & \mathbf{1}_\perp \arrow[r, "a'"] & \mathbf{1}_\perp & \\
D = \mathbf{1}_\perp \arrow[ur, shift left=2] \arrow[ur, "\forall h"] \arrow[ur, shift right=2] \arrow[rr, dashed, "\langle a, a'h\rangle"'] \arrow[dr, shift left=2] \arrow[dr, "\forall a"'] \arrow[dr, shift right=2] & & \mathbf{1}_\perp \times \mathbf{1}_\perp \arrow[u, squiggly, ->>] \arrow[dl, squiggly, ->>] \arrow[r, "\mathfrak{r}"] & \Omega \\
 & \mathbf{1}_\perp & &
\end{tikzcd}
$$

---

[13] e_j picks out the node of D by the image of the top node of $\mathbf{1}_\perp$.

Let's start by examining $\forall_{x:1_\perp} r(x, a_0)$: We discover that:

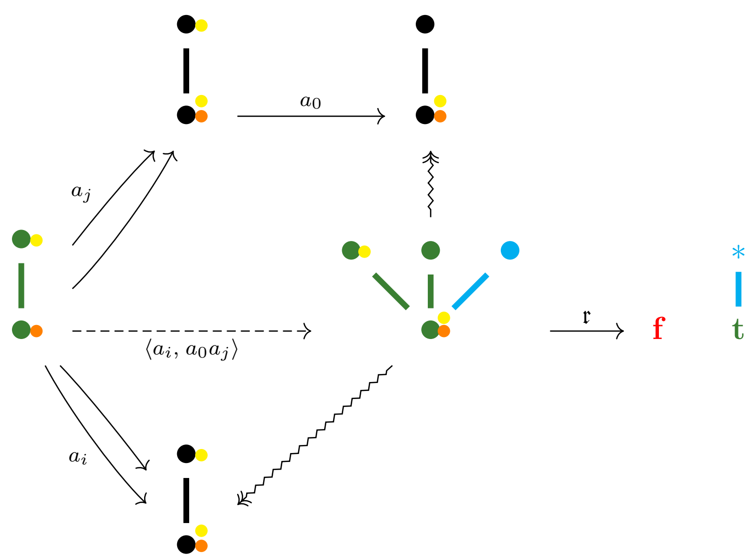


Figure 4.11: The images of a_i, a_j with $i, j \in \{0, 1\}$ are displayed with bullets of matching color.

$$\models_{\mathbb{FF}_2} \forall_{x:1_\perp} r(x, a_0).$$
$$\not\models_{\mathbb{FF}_2} \forall_{x:1_\perp} r(x, a_1).$$

Moving on to existential quantification, we aim to realize in analogy to $\forall$, the formula $\exists_{x:\mathbf{1}_\perp} r(x, a_1)$:

Example 102. If we unfold the definitions, in order for $\exists_{x:\mathbf{1}_\perp} r(x, a_1)$ to be true at stage $\mathbf{1}_\perp$ there must exist an epi $e : D \twoheadrightarrow \mathbf{1}_\perp$ and an arrow $D \xrightarrow{a} \mathbf{1}_\perp$ such that $D \Vdash r(a,\ a_1 e)$.
Let $D = \mathbf{1}_\perp$,$e = a_1$ and $a = a_1$.

$$\begin{tikzcd}
 & \mathbf{1}_\perp \arrow[r, "a_1"] & \mathbf{1}_\perp & \\
D = \mathbf{1}_\perp \arrow[ur, two heads, "e=a_1"] \arrow[rr, dashed, "\langle a_1, a_1 a_1 \rangle"'] \arrow[dr, "a=a_1"'] & & \mathbf{1}_\perp \times \mathbf{1}_\perp \arrow[u, squiggly, two heads] \arrow[dl, squiggly, two heads] \arrow[r, "\mathfrak{r}"] & \Omega \\
 & \mathbf{1}_\perp & &
\end{tikzcd}$$

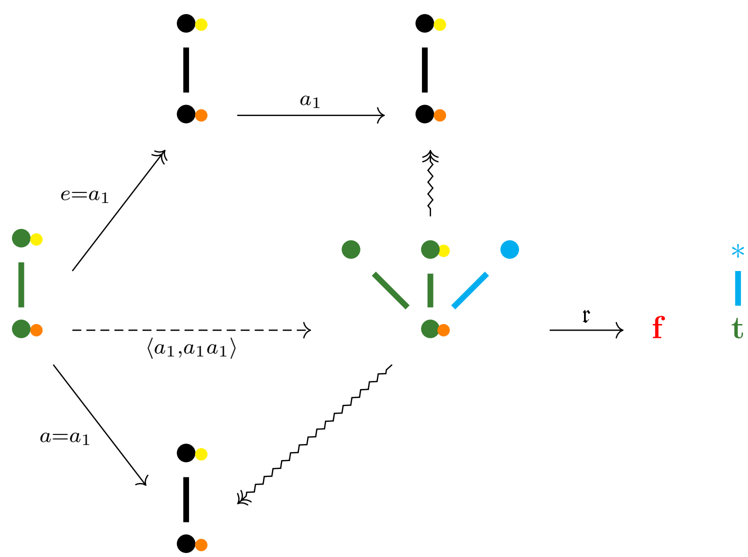


Figure 4.12: The usual coloring notation is applied.

This results in:

$$\models_{\mathbb{FF}_2} \exists_{x:\mathbf{1}_\perp} r(x, a_1).$$
$$\models_{\mathbb{FF}_2} \exists_{x:\mathbf{1}_\perp} r(x, a_0).$$

We present a final example in which we introduce a new type $A' := \mathbf{1}_\perp + \mathbf{1}$ and a new relation r' which *extends* r and is realized by $\mathbf{1}_\perp \times \mathbf{A}' \xrightarrow{\mathfrak{r}'} \Omega$.

Example 103. We want to check the validity of $\exists_{x:1_\perp} r(x, a_2')$:

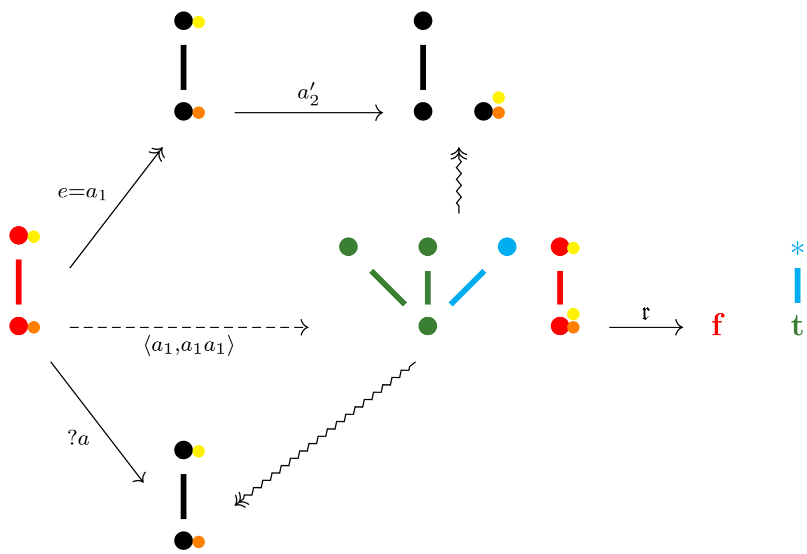


Figure 4.13: In this case $?a$ can be either a_0 or a_1.

This results in:

$$\not\models_{\mathbb{FF}_2} \exists_{x:1_\perp} r'(x, a_2').$$

4.2.4 Recovering First-Order $\mathcal{G}_3$

Let's take a look back at what we found using BKJ Semantics. Notice that:

Remark 39. In order to establish the validity of the universal quantified predicate $\forall_{x:1_\perp} p(x)$ we have to *check* a *finite* number of generalized elements a_i. In fact, this was equivalent to checking if $\mathbf{1}_\perp \Vdash p(a_0)$ and $\mathbf{1}_\perp \Vdash p(a_1)$ i.e., if $\mathbf{1}_\perp \Vdash p(a_0) \wedge p(a_1)$ which is equivalent to establishing if $\models_{\mathbb{FF}_2} p(a_0) \wedge p(a_1)$.

Also:

Remark 40. The same phenomenon occurs in order to establish the validity of a universal quantified relation $\forall_{x:1_\perp} r(x, a_0)$. We also have to *check* a *finite* number of generalized elements a_i, a_j. In fact, this was equivalent to checking if $\mathbf{1}_\perp \Vdash r(a_1, a_0)$ and $\mathbf{1}_\perp \Vdash r(a_0, a_0)$ i.e., if $\mathbf{1}_\perp \Vdash r(a_0, a_0) \wedge r(a_1, a_0)$ which in turn is equivalent to establishing if $\models_{\mathbb{FF}_2} r(a_0, a_0) \wedge r(a_1, a_0)$.

This suggests that:

Lemma 104. *The semantics of a universally quantified formula in $\mathbb{FF}_2$ correspond to finitary*[14] *conjunction of instanced formulae.* [15]

In a similar manner:

Remark 41. In order to establish the validity of the existential quantified predicate $\exists_{x:1_\perp} p(x)$ we have to *check* a *finite* number of generalized elements a_i. In fact, this was equivalent to checking if either $\mathbf{1}_\perp \Vdash p(a_0)$ or $\mathbf{1}_\perp \Vdash p(a_1)$ i.e., if $\mathbf{1}_\perp \Vdash p(a_0) \vee p(a_1)$ which is equivalent to establishing if $\models_{\mathbb{FF}_2} p(a_0) \vee p(a_1)$.

Now, a categorical consideration:

Lemma 105. *$\mathbf{1}_\perp$ is* indecomposable*.*

This becomes quite apparent as the epis in $\mathbb{FF}_*$ are the surjective arrows. Assuming $[k, l] : D + E \twoheadrightarrow \mathbf{1}_\perp$ is an epi, if neither $D \xrightarrow{k} \mathbf{1}_\perp$ nor $E \xrightarrow{l} \mathbf{1}_\perp$ are epis then they must both be constant maps into the root of $\mathbf{1}_\perp$ and so $[k, l]$ must be a constant map into the root of $\mathbf{1}_\perp$ which brings us to a contradiction.

[14] *bushes*, recall, are *finite* forests.

[15] the instances to consider are those of $\phi(a_i, ..)$ where $x_i : A_i$ is the variable being quantified over and a_i is a generalized element of A_i.

Remark 42. In order to establish the validity of the existential quantified formula $\exists_{x:1_\perp} r(x, a_0)$ we have to *check* a *finite* number of generalized elements a_i, a_j.
In fact, this was equivalent to checking if either $\mathbf{1}_\perp \Vdash r(a_1, a_0)$ or $\mathbf{1}_\perp \Vdash r(a_0, a_0)$ i.e., if $\mathbf{1}_\perp \Vdash r(a_0, a_0) \vee r(a_1, a_0)$ which in turn is equivalent to establishing if $\models_{\mathbb{FF}_2} r(a_0, a_0) \vee r(a_1, a_0)$.

This suggests, analogously to the previous case:

Lemma 106. *The semantics of an existentially quantified formula in $\mathbb{FF}_2$ corresponds to a generalized finitary disjunction of instanced formulae.*

Finally, let us observe that:

Remark 43. $\mathbb{FF}_2$-validity of instanced atomic formulae like $\phi(a_1, .., a_n)$ is always reduced to $\mathbb{FF}_2$-validity at the stage $\mathbf{1}_\perp$ which in turn corresponds to checking if the fiber of t by $\top!_{1_\perp}$ coincides with the maximal sub-forest of $\mathbf{1}_\perp$ i.e., $(\top!_{1_\perp})^{-1}[t] = \{\mathbf{1}_\perp\}$.

Recall now that $Sub(\mathbf{1}_\perp) \cong C_3$ and semantically characterizes $\mathcal{G}_3$.

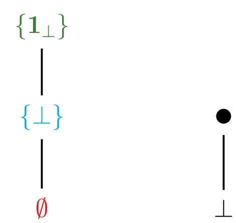


Figure 4.14: $Sub(\mathbf{1}_\perp)$ (left) and $\mathbf{1}_\perp$ (right).

Remark 44. We can define an assignment $\mathcal{A}$ from instanced atomic formulae ϕ to the Gödel set $\mathfrak{X} = \{0, \frac{1}{2}, 1\}$ whereby if we consider the sub-forest of $\mathbf{1}_\perp$ determined by $(\top !_{1_\perp})^{-1}[t]$:

$$\mathcal{A}(\phi) := \begin{cases} 1 & \text{if } (\top !_{1_\perp})^{-1}[t] = \{\mathbf{1}_\perp\} \\ \frac{1}{2} & \text{if } (\top !_{1_\perp})^{-1}[t] = \{\perp\} \\ 0 & \text{if } (\top !_{1_\perp})^{-1}[t] = \emptyset \end{cases} .$$

Recall also that the semantics of first order $\mathcal{G}_3$ in 1.5.7 whereby $\forall$ and $\exists$ where essentially interpreted as generalized $\wedge$ and $\vee$, i.e., the interpretation of $\forall x.A(x)$ and $\exists x.A(x)$ was respectively the *min* and *max* of the interpretations of the instances $A(u)$ where u ranged over some domain or universe $\mathfrak{U}$.

This suggests, summing up all these results, the following:

Theorem 107 (first-order logic of $\mathbb{FF}_2$)**.**
The first-order logic of the topos of $\mathbb{FF}_2$/bushes *corresponds to first-order three-valued Gödel-Dummett Logic on* finite *domains.*

In other words:

Remark 45. The topos of *bushes* provides first order finite models for three-valued Gödel-Dummett Logic.

Chapter 5

Topos Semantics III

5.1 The Logic of Variable Sets

We now give an account of the logic of variable sets as outlined in [9] and by so doing present another approach to the topos semantics of $\mathcal{G}_3$.

Informally speaking, in the *classical* world the truth of a statement $\phi(x)$ regarding some *thing* x determines once and for all a set $\{x : \phi(x)\}$ of all things of which the statement is true.
However, in the *non-classical* world the truth-value of a statement is not *absolute* but rather, *context-dependent*.
As we saw in the Introduction varies according to the states of knowledge at some particular time. Similarly as we did before, we say that ϕ determines for each state p a set ϕ_p of all the things of which the statement is known to be true at p called the *extension* of ϕ at p.

Def. 5.1.1 (ϕ_p)**.** The extension of ϕ at p is denoted by:

$$\phi_p := \{x : \phi(x) \textit{ is known at } p \textit{ to be true}\}.$$

We also require, as we saw for Kripke Semantics, that what is known now to be true remain true in the future, i.e., that truth *persist* in time.

Formalizing this construction:

Def. 5.1.2 ($F : \mathbf{P} \to \mathbb{Set}$)**.** Given a frame $\mathbf{P}$ seen as a pre-order category, the assignments $p \mapsto \phi_p$ and $p \to q \mapsto \phi_p \subseteq \phi_q$:

$$\phi_p := \{x : \phi(x) \textit{ is known at } p \textit{ to be true}\}.$$
$$\text{If } p \sqsubseteq q \text{ then } \phi_p \subseteq \phi_q$$

yield a functor $F : \mathbf{P} \to \mathbb{Set}$.

The category $\mathbb{Set}^{\mathbf{P}}$ has as objects these functors which can be seen as the *variable sets* $\{\phi_p\}_{p\in\mathbf{P}}$, i.e., the *extensions* of ϕ at each stage.
The remarkable result about this category is:

Proposition 108. $\mathbb{Set}^{\boldsymbol{P}}$ *is a topos.*

This comes from a more general fact:

Theorem 109. *For any* small *category* $\mathcal{C}$ *the (functor)* category of diagrams $\mathbb{Set}^{\mathcal{C}}$ *is a topos.*

In chapter 3 (3.1.5) we saw a few instances of this category first with $\mathbf{P} = \mathbf{2} = 0 \xrightarrow{\leq_0} 1$, i.e., *functions between sets* and secondly with $\mathbf{P} = \omega = 0 \xrightarrow{\leq_0} 1 \xrightarrow{\leq_1} 2 \xrightarrow{\leq_2} ...$, i.e., *sets through time.*

5.1.1 Back to Kripke Frames

When we introduced a Kripke Frame $\mathbf{P}$ as a finite poset $(P, \sqsubseteq)$ of *possible worlds*, we constructed $\mathbf{P}^+$ the collection of *up-sets*, a.k.a. *hereditary sub-sets* of the Frame and found out it could be made a Heyting algebra.
Having fixed a Frame, we focus on *principal* up-sets :

Def. 5.1.3 (principal up-set)**.** The principal up-set generated by an element $p \in \mathbf{P}$ is:

$$[p) := \{q : p \sqsubseteq q\}.$$

, i.e., the elements of the Frame *above* p in the ordering $\sqsubseteq$.

The operations introduced to make $\mathbf{P}^+$ a Heyting algebra can now be characterized as:

Lemma 110. *For any $S, T \in \boldsymbol{P}^+$:*

$$S \Rightarrow T = \{p : S \cap [p) \subseteq T\}.$$
$$\neg S = \{p : [p) \cap S = \emptyset\}.$$

If we restrict $\sqsubseteq$ to $[p)$, we can talk about the *principal set generated by an element* $q \in [p)$ denoted by $[q)_p$:

Def. 5.1.4. $[q)_p := [p) \cap [q)$.

More generally if $S \subseteq P$ we can *relativize* S to $[p)$:

Def. 5.1.5. $S_p := S \cap [p)$.

What we can obtain of particular interest is:

Proposition 111. *The poset $([p)^+, \subseteq)$ of up-sets of $[p)$ ordered by inclusion forms a* sub-directly irreducible *Heyting algebra with the operations defined for any up-sets $S, T \subseteq [p)$:*

$$S \cap_p T := S \cap T.$$
$$S \cup_p T := S \cup T.$$
$$S \Rightarrow_p T := \{q : q \in [p) \text{ and } S \cap [q)_p \subseteq T\}.$$
$$\neg_p S := \{q : q \in [p) \text{ and } [q)_p \cap S = \emptyset\}.$$

The notable result is that:

If we start from an up-set $S \subseteq P$ we may choose to first relativize S to $[p)$ and then apply the operations of the H.A.[1] $([p)^+, \subseteq)$ or first apply the corresponding operations of the H.A. $\mathbf{P}^+$ and then relativize to obtain the same result.

Formally:

Lemma 112. *for any $S, T \in \boldsymbol{P}^+$,*

$$(S_p) \cap_p (T_p) = (S \cap T)_p.$$
$$(S_p) \cup_p (T_p) = (S \cup T)_p.$$
$$\neg_p(S_p) = (\neg S)_p.$$
$$(S_p) \Rightarrow_p (T_p) = (S \Rightarrow T)_p.$$

In fact one can prove that:

Proposition 113. *The assignment $S \mapsto S_p$ is a surjective H.A.* homomorphism *from $\boldsymbol{P}^+$ to $[p)^+$.*

[1]H.A. stands for Heyting algebra.

5.1.2 Topos Structure

We fix some notation:

Remark 46. For a functor $F : \mathbf{P} \to \mathbb{Set}$ we denote F_p for $F(p)$ and F_{pq} for the transition map between F_p and F_q when $p \sqsubseteq q$.

Let's use what we just proved to provide a sub-object classifier for $\mathbb{Set}^{\mathbf{P}}$.

The terminal object is given by (a generalization of what we saw in the chapter 3):

Lemma 114 (terminal object for $\mathbb{Set}^{\mathbf{P}}$)**.** *The terminal object is given by the* constant *functor* $1 : \boldsymbol{P} \to \mathbb{Set}$ *where every component* $1_p := \{0\}$ *and the transition maps are the identity* $1_p q = id_0$*.*

The functor we have in mind $\Omega : \mathbf{P} \to \mathbb{Set}$ is the following: [2]

Lemma 115 (sub-object classifier for $\mathbb{Set}^{\mathbf{P}}$)**.** *The sub-object classifier* Ω *is defined by the following assignments:*

$$p \mapsto [p)^+.$$

$$p \sqsubseteq q \mapsto [p)^+ \xrightarrow{\Omega_{pq}} [q)^+ \text{ where } \Omega_{pq} : S \mapsto S_q = S \cap [q)^+.$$

The truth arrow true, *i.e.,* $\top : 1 \Rightarrow \Omega$ *in its components* $\{\top_p\}_{p \in \boldsymbol{P}}$ *is given by the assignment of the maximal or* unit *element of each* $[p)^+$*, i.e.,:*

$$\top_p(0) := [p).$$

A sub-object $\tau : F \Rightarrow G$ in its components, again w.l.o.g., can be assumed to be a set of inclusions $\{\tau_p : F_p \hookrightarrow G_p\}_{p \in P}$.
The characteristic arrow of τ is given by:

Lemma 116 (characteristic arrow in $\mathbb{Set}^{\mathbf{P}}$)**.** $\chi_\tau : G \Rightarrow \Omega$ *for each* $x \in G_p$*:*

$$(\chi_\tau)_p(x) := \{q : p \sqsubseteq q \text{ and } G_{p\,q}(x) \in F_q\}.^3$$

By this definition we have:

$$F_p = \{x : (\chi_\tau)_p(x) = [p)\}.$$

[2]This is a particular case of a more general construction for $\mathbb{Set}^{\mathcal{C}}$ for $\mathcal{C}$ small.

[3]One can check first that χ_τ is a natural transformation and secondly that $(\chi_\tau)_p(x)$ is an up-set in $\mathbf{P}$.

We already defined *true*, i.e., $\top : 1 \Rightarrow \Omega$ which picks out the *unit* element $[p)$ from each H.A. $[p)^+$ and can now define the rest of the truth arrows.
Note that the initial object is given by:

Lemma 117 (initial in $\mathbb{Set}^{\mathbf{P}}$)**.** *The initial object* $0 : \boldsymbol{P} \to \mathbb{Set}$ *is the constant functor:*

$$\forall p \in P : 0_p := \emptyset.$$
$$\forall p \sqsubseteq q \in P : 0_{p\,q} := id_{\emptyset}.$$

The unique arrow into the terminal, i.e., $!_0 : 0 \Rightarrow 1$ is made up of inclusions $\emptyset \hookrightarrow \{0\}$ for each $p \in P$.
The character of this arrow is defined as the truth arrow *false*:

Def. 5.1.6 (*false* in $\mathbb{Set}^{\mathbf{P}}$)**.** *false*, i.e., $\bot : 1 \Rightarrow \Omega$ for each $p \in P$ is:

$$\begin{aligned} \bot_p(0) = \{q : p \sqsubseteq q \text{ and } 1_{p\,q}(0) \in 0_q\} = \\ = \{q : p \sqsubseteq q \text{ and } 1_{p\,q}(0) \in \emptyset\} = \emptyset. \end{aligned}$$

, i.e., $\bot$ picks out the *zero* element from each H.A. $[p)^+$.

The negation arrow $\neg : \Omega \Rightarrow \Omega$ is the character of $\bot$ where $\bot_p : \{\emptyset\} \subseteq \Omega_p$ [4]:

Def. 5.1.7 (negation in $\mathbb{Set}^{\mathbf{P}}$)**.** $\neg : \Omega \Rightarrow \Omega$ in its components $p \in P$ is defined as $\neg_p : \Omega_p \to \Omega_p$ on $S \subseteq \Omega_p$ as:

$$\begin{aligned} \neg_p(S) = \{q : p \sqsubseteq q \text{ and } \Omega_{p\,q} \in \{\emptyset\}\} = \\ = \{q : p \sqsubseteq q \text{ and } S \cap [q) = \emptyset\} = \\ = [p) \cap \neg S = \\ = (\neg S)_p. \end{aligned}$$

Remark 47. There appears to be conflicting notation in the form of $\neg_p$ for the component in p of the natural transformation $\neg$ and the pseudo-complement in the H.A. $[p)^+$.
Notice however from the result $\neg_p(S) = (\neg S)_p$ that these operations are *compatible* and so the notation is actually consistent.
This phenomenon appears for all the other truth arrows.

[4] We identify $\bot_p$ with $\{\emptyset\}$.

Notice for instance that in $\mathbb{Set}^{\mathbf{P}}$ products are defined *component-wise*, i.e., $(F \times G)_p := F_p \times G_p$ and $(F \times G) : p \sqsubseteq q \mapsto F_{pq} \times G_{pq}$.

The conjunction arrow $\wedge : \Omega \times \Omega \Rightarrow \Omega$ is thus defined as the character of $\top \times \top : 1 \Rightarrow \Omega \times \Omega$ where $(\top \times \top)_p(0) = ([p), [p))$.

We make similar considerations for implication and disjunction obtaining:

$$\wedge_p(S, T) = (S \wedge T)_p.$$
$$\Rightarrow_p (S, T) = (S \Rightarrow T)_p.$$
$$\vee_p(S, T) = (S \vee T)_p.$$

Let's take stock of what we just learned:

Remark 48. The components of the *truth-arrows* in $\mathbb{Set}^{\mathbf{P}}$ are essentially the same as the corresponding connectives on the Heyting algebras we defined from $\mathbf{P}$.
This suggests, as we foretold, that the logic of Variable Sets is *intuitionistic*.

5.1.3 Validity and Applications

We want now to clarify the link we envisioned between topos validity in $\mathbb{Set}^{\mathbf{P}}$ and H.A. validity on $[p)^+$.

The main result about this is the following, which links topos, Kripke and H.A. validity:

Theorem 118 (Validity Theorem)**.** *The notation $\models_{\mathcal{E}}, \models_{K.}, \models_{H.A.}$ stands for respectively topoi, Kripke and Heyting algebra validity.*
For any (propositional) formula ϕ:

$$\mathbb{Set}^{\boldsymbol{P}} \models_{\mathcal{E}} \phi \;\; \textit{iff} \;\; \boldsymbol{P} \models_{K.} \phi \;\; \textit{iff} \;\; P^+ \models_{H.A.} \phi.$$

Furthermore,
from what we know about topos validity:

Proposition 119.

$$\mathbb{Set}^{\boldsymbol{P}} \models_{\mathcal{E}} \phi \;\; \textit{iff} \;\; \mathbb{Set}^{\boldsymbol{P}}(1, \Omega) \models_{H.A.} \phi; \;\; \textit{iff} \;\; Sub(1) \models_{H.A.} \phi.$$

A sketch of the proof of the *Validity Theorem* is given:

Remark 49. Let $\mathcal{M} = (\mathbf{P}, V)$ be a Kripke Model based on $\mathbf{P}$ with a valuation $V : \mathbf{Prop} \to P^+$.
A $\mathbb{Set}^{\mathbf{P}}$-valuation $V' : \mathbf{Prop} \to \mathbb{Set}^{\mathbf{P}}(1, \Omega)$ can be constructed by defining each component of $V'(\mathbf{r}) : 1 \Rightarrow \Omega$:

$$V'(\mathbf{r})_p(0) := V(\mathbf{r}) \cap [p) = V(\mathbf{r})_p.$$

, i.e., $V'(\mathbf{r})_p$ picks out the states in $[p)$ at which $\mathbf{r}$ is true in $\mathcal{M}$.
In fact, from this one can show something stronger:

$$V'(\phi)_p(0) = \mathcal{M}(\phi)_p.$$

, i.e., $V'(\phi)_p$ picks out the states in $[p)$ at which ϕ is true in $\mathcal{M}$.
If $\mathbb{Set}^{\mathbf{P}} \models \phi$ then $V'(\phi) = \top$ and so for each state p: $V'(\phi)_p(0) = \mathcal{M}(\phi)_p = [p)$ meaning $\phi = P$. In other words, $\mathbf{P} \models \phi$.

Remark 50. For the converse, let $V' : \mathbf{P} \to \mathbb{Set}^{\mathbf{P}}$ be a $\mathbb{Set}^{\mathbf{P}}$-valuation.
Each arrow $V'(\mathbf{r}) : 1 \Rightarrow \Omega$ determines a collection of up-sets of $[q)$ $V'(\mathbf{r})_q(0)$

for each stage $q \in P$.
We define a $\mathbf{P}$-valuation $V : \mathbf{Prop} \to \mathbf{P}^+$ by taking their union at all stages:

$$V(\mathbf{r}) := \bigcup_{q \in P} V'(\mathbf{r})_q(0).$$

, i.e., $p \in V(\mathbf{r})$ iff for some q we have $p \in V'(\mathbf{r})_q(0)$.
If we now define a $\mathbb{Set}^{\mathbf{P}}$-valuation V'' from the $\mathbf{P}$-valuation V in the same manner as before:

$$V''(\mathbf{r})_p(0) = V(\mathbf{r}) \cap [p).$$

This just gives us back the original V', i.e.,

$$V(\mathbf{r}) \cap [p) = V'(\mathbf{r})_p(0)$$

In a similar manner if we start from a $\mathbf{P}$-valuation V, define as before V' a $\mathbb{Set}^{\mathbf{P}}$-valuation with $V'(\mathbf{r})_p(0) = V(\mathbf{r})_p$ and try to construct a $\mathbf{P}$-valuation:

$$\bigcup_{p \in P} V'(\mathbf{r})_p(0) = \bigcup_{p \in P} V(\mathbf{r})_p = V(\mathbf{r}).$$

We return back to the original V.

We may conclude:

Remark 51. There exists a bijection between $\mathbb{Set}^{\mathbf{P}}$-valuations and $\mathbf{P}$-valuations.

One of the first consequences of the Validity Theorem is the *characterisation* of topos-valid sentences:

Proposition 120. *Take the canonical Kripke Frame $\boldsymbol{P}_{IPL}$. We now know that:*

$$\vdash_{IPL} \phi \;\; \textit{iff} \;\; \boldsymbol{P}_{IPL} \models_{K.} \phi \;\; \textit{iff} \;\; \mathbb{Set}^{\boldsymbol{P}_{IPL}} \models_{\mathcal{E}} \phi.$$

From this we get *Completeness* for topos-validity:

Theorem 121 (Completeness Theorem for topos-Validity)**.** *If ϕ is valid on every topos $\mathcal{E}$, i.e., $\mathcal{E} \models \phi$, then ϕ is an intuitionistic tautology, i.e., $\vdash_{IPL} \phi$.*

Together with the result about *Soundness* for topos-validity, we conclude:

Theorem 122 (Soundness and Completeness Theorem for topos-Validity)**.** *For all formulae ϕ and topoi $\mathcal{E}$:*

$$\vdash_{IPL} \phi \;\; \textit{iff} \;\; \models_{\mathcal{E}} \phi.$$

In other words: *sentences valid on all topoi are precisely the IPL theorems* or *topoi provide a sound and complete semantics for IPL.*

Furthermore, the Validity Theorem turns gives us a very interesting application for Gödel-Dummett Logic.
Recall from the introduction that $\mathcal{G} = \bigcap_{k\geq 2} \mathcal{G}_k$ and that $\mathcal{G}_n \vdash \phi$ iff $C_n \models \phi$.

We can give a topos-semantics characterization for the family $\{\mathcal{G}_n\}_{n\geq 2}$:

Proposition 123. *Let* $\boldsymbol{N}$ *be the N-chain Kripke frame,* $\forall N \geq 1$*:*

$$C_{N+1} \models_{H.A.} \mathcal{G}_{N+1} \;\; \textit{iff} \;\; \boldsymbol{N} \models_{K.} \mathcal{G}_{N+1} \;\; \textit{iff} \;\; \mathbb{Set}^{\boldsymbol{N}} \models_{\mathcal{E}} \mathcal{G}_{N+1}.$$

This is because $N^+ \cong C_{N+1}$.
For $\mathcal{G}_3$ this translates into:

Corollary 124.

$$\mathcal{G}_3 \vdash \phi \;\; \textit{iff} \;\; C_3 \models_{H.A.} \phi \;\; \textit{iff} \;\; \boldsymbol{2} \models_{K.} \phi \;\; \textit{iff} \;\; \mathbb{Set}^{\boldsymbol{2}} \models_{\mathcal{E}} \phi.$$

Recalling now the examples made in 3.1.5:

This also allows us to motivate the assertion we made about the category of *functions between sets* $\mathbb{Set}^{\mathbf{2}}$ not being Boolean.
Note that if we identify the poset category $\mathbf{2} = 0 \xrightarrow{\leq_0} 1$ with the 2-chain Kripke frame $\mathbf{2} = 0 \leq 1$ we have the following evidence for being non-Boolean:

Proposition 125. $\boldsymbol{2} \not\models_{K.} \alpha \vee \neg\alpha$ *holds for the two-element Kripke frame and thus:*

$$\mathbb{Set}^{\boldsymbol{2}} \not\models_{\mathcal{E}} \alpha \vee \neg\alpha.$$

, i.e., the law of excluded middle is not valid in the topos $\mathbb{Set}^{\boldsymbol{2}}$*.*

We conclude with a noteworthy result from *Dummett & Segerberg* reported in [9]:
Applying what we learned for variable sets we discover:

Proposition 126. *Let* ω *be the linear Kripke frame on natural numbers, i.e.,* $\omega := \{0 \leq 1 \leq 2..\}$*:*

$$\omega \models_{K.} \alpha \;\; \textit{iff} \;\; \mathcal{G} \vdash \alpha$$
$$\mathcal{G} \vdash \alpha \;\; \textit{iff} \;\; \mathbb{Set}^{\omega} \models_{\mathcal{E}} \alpha$$

What this tells us is:

Proposition 127. $\mathcal{G}$ *is the logic of* sets through time*.*

Furthermore, the structure of ω which corresponds to *discrete time* can be altered to correspond to *continuous time*:

Proposition 128.

$$\omega \models \alpha \;\; \textit{iff} \;\; \mathbb{Q} \models \alpha \;\; \textit{iff} \;\; \mathbb{R} \models \alpha$$

In fact, if **C** is *any* infinite chain:

Proposition 129.

$$\mathbb{Set}^{\boldsymbol{C}} \models_{\mathcal{E}} \alpha \;\; \textit{iff} \;\; \mathcal{G} \vdash \alpha.$$

5.2 Sheaf Semantics

To conclude, we give yet another approach to topoi-semantics of $\mathcal{G}_n$ using *sheaves on locales* as seen in [13] and building on the work of [14].

We have been working with *covariant* functor categories of the form $\mathbb{Set}^{\mathbb{C}}$ with $\mathbb{C}$ a small category like the poset category $\mathbf{P}$.
We could have considered *contra-variant* functor categories like *presheafs*:

Def. 5.2.1 (presheaf)**.** A presheaf $F : \mathbb{C}^{op} \to \mathbb{Set}$ is a contra-variant set-valued functor.

Note that since $(-)^{op}$ is an involution [5]:

Lemma 130. *Any functor category* $\mathbb{Set}^{\mathbb{C}}$ *is a presheaf* $\mathbb{Set}^{(\mathbb{C}^{op})^{op}}$*.*

Traditionally *pre-sheaves* were used in Topology as functors from $\mathcal{O}(X)$ the lattice of open subsets of a space X to $\mathbb{Set}$.

One can generalize from $\mathcal{O}(X)$ to more general lattices called *locales*:

Def. 5.2.2 (Locale)**.** We say a lattice is *complete* if every $S \subseteq \mathcal{L}$ has a join $(\bigvee_{a\in S} a) \in \mathcal{L}$ [6] and meet $(\bigwedge_{a\in S} a) \in \mathcal{L}$ [7].
A *locale* $\mathcal{L}$ is a *complete* lattice in which arbitrary joins distribute over finite meets, i.e., for an arbitrary indexing set I and elements $a_i, b \in \mathcal{L}$:

$$a \wedge (\bigvee_{i\in I} b_i) = \bigvee_{i\in I} (a \wedge b_i)$$

Example 131 (the locale $O(X)$)**.** Given a topological space $(X, \mathcal{O}(X))$, the locale of *open subsets* $(\mathcal{O}(X), \subseteq)$ is a *sub-lattice* of the locale of subsets $(\mathcal{P}(X), \subseteq)$ that is closed under arbitrary joins [8] and finite meets [9].

Note that by this definition a locale can be seen as a *co-complete* and small category in which for each $b \in \mathcal{L}$ every functor $(- \wedge b)$ preserves co-limits, i.e., the distributive condition for arbitrary joins. This is equivalent [10] to each functor $(- \wedge b)$ having a right adjoint $(b \Rightarrow -)$, so that:

[5] formally this is an endo-functor in $\mathbb{Cat}$ that is an involution, i.e., $((-)^{op})^{op}$ is the identity functor *id*.

[6] i.e., a *least upper bound*.

[7] i.e., a *greatest lower bound*.

[8] an arbitrary union of opens is open.

[9] any finite intersection of opens is open.

[10] by the Adjoint Functor Theorem, see [10].

Proposition 132.

$$b \Rightarrow c = \bigvee \{a \in \mathcal{L} \mid a \wedge b \leq c\}$$

$\mathcal{L}$ is a locale iff $\mathcal{L}$ is a complete Heyting algebra.

A H.A. is complete *if it is so as a lattice.*

Example 133 (the locale C_n)**.** Every n-chain C_n, as a complete Heyting algebra, is a locale.

If we consider the corresponding poset category on a locale, the presheaves on a locale $\mathcal{L}$ are thus:

Def. 5.2.3 (presheaf)**.** A presheaf on a locale $\mathcal{L}$ is a contra-variant functor $F : \mathcal{L} \to \mathbb{Set}$.
If $v \xrightarrow{\leq_{uv}} u$ in $\mathcal{L}$, the action of the transition map $F(u) \xrightarrow{F(\leq_{uv})} F(v)$ on elements is denoted by $x \mapsto x \restriction_v$. [11]
Notice that the *functoriality* of F is given by:

1. $\forall u \in \mathcal{L}, \forall x \in F(u)\ x \restriction_u = x$.

2. $\forall w \leq v \leq u \in \mathcal{L}\ \forall x \in F(u)\ x \restriction_w = (x \restriction_v) \restriction_w$.

Letting F be a presheaf on a locale $\mathcal{L}$, we introduce the following notion:

Def. 5.2.4 (compatible family)**.** Taking an arbitrary family $(u_i)_{i \in I}$ in $\mathcal{L}$, a family of elements $\{x_i \in F(u_i)\}_{i \in I}$ is *compatible* when:

$$\forall i, j \in I : x_i \restriction_{u_i \wedge u_j} = x_j \restriction_{u_i \wedge u_j} .$$

On a locale $\mathcal{L}$, when does a presheaf become a *sheaf* ?

Def. 5.2.5 (sheaf)**.** A presheaf is a *sheaf* when, given a so-called *covering* of $u \in \mathcal{L}$, i.e., $u = (\bigvee_{i \in I} u_i)$ and a compatible family $(x_i \in F(u_i))_{i \in I}$, there exists a unique element, a.k.a. *gluing* $x \in F(u)$ such that $x \restriction_{u_i} = x_i$ for each $i \in I$.
In short, given some *covering* there exists a unique *gluing* for every *compatible family.*

[11] the reason for this notation will become soon apparent.

Example 134 (continuous functions)**.** Let (X, τ) and (Y, σ) be topological spaces and fix $U \in \tau$.
If we consider the set $\mathcal{C}(U, Y)$ of *continuous* functions $f : U \to Y$ and choose the usual restriction mappings $\restriction_V$ of a function to a subset $V \subseteq U$, this yields a presheaf $\mathcal{C}(-, Y)$ on the locale $(\mathcal{O}(X), \subseteq)$.
Furthermore, this is a *sheaf* since a compatible family $\{f_i : U_i \to Y\}$ on an open covering $U = \bigcup_{i \in I} U_i$ entails that any f_i and f_j with $i, j \in I$ coincide on $U_i \cap U_j$, i.e., $f_i \restriction_{U_i \wedge U_j} \equiv f_j \restriction_{U_i \wedge U_j}$.
The unique *gluing* is given as the *collation* of all the functions in the family:

$$f : \bigcup_{i \in I} U_i \to Y, \quad x \mapsto f_i(x) \quad \text{if } x \in U_i.$$

Remark 52. An obvious counter-example to show that *not every presheaf is a sheaf* is given by taking $Y = \mathbb{R}$ with the standard topology.
The presheaf $\mathcal{B}(-, \mathbb{R})$ of *bounded* functions fails to be a sheaf since the collation of functions that are bounded may yield an unbounded function.

Let $\mathcal{H}$ be a complete H.A., or equivalently a locale, and $\mathbb{C}_{\mathcal{H}}$ be the corresponding poset category.
We now take pre-sheaves on $\mathbb{C}_{\mathcal{H}}$ and define the following category:

Remark 53. We use the following notation: $\mathbf{H}$ instead of $\mathbb{C}_{\mathcal{H}}$.

Def. 5.2.6 (presheaf & sheaf categories)**.** The category $\mathbf{PreSh}(\mathbf{H})$ has objects the pre-sheaves on $\mathbf{H}$ and arrows given by natural transformations between them.
The category $\mathbf{Sh}(\mathbf{H})$ is the category with objects sheaves on $\mathbf{H}$ and arrows given by natural transformations between them.

We use the following results from [13]:

Proposition 135. *If H is a complete Heyting algebra, then $\mathbf{Sh}(\mathbf{H})$ is a topos.*

Proposition 136. *If H is a complete H.A., then:*

$$H \cong Sub_{\mathbf{Sh}(\mathbf{H})}(1).$$

We are now in a position to *semantically characterize* using sheaves the family of intermediate logics $\{\mathcal{G}_N\}_{N\geq 2}$:
(The following result is the main conclusion of [14])

Theorem 137 (Sheaf semantics for $\mathcal{G}_N$)**.** *For every $N \geq 2$ and (propositional) formula ϕ:*

$$\mathcal{G}_N \vdash \phi \;\; \textit{iff} \;\; C_N \models_{H.A.} \phi$$
$$\textit{iff} \;\;\; Sub_{\mathbf{Sh(C_N)}}(1) \models_{H.A.} \phi \;\; \textit{iff} \;\;\; \mathbf{Sh}(\mathbf{C_N}) \models_{\mathcal{E}} \phi.$$

$$\mathcal{G}_N \vdash \phi \;\; \textit{iff} \;\; \mathbf{Sh}(\mathbf{C_N}) \models_{\mathcal{E}} \phi.$$

As an immediate application of the above:

Corollary 138.
$$\mathcal{G}_3 \vdash \phi \;\; \textit{iff} \;\; \mathbf{Sh}(\mathbf{C_3}) \models_{\mathcal{E}} \phi.$$

5.2.1 Sheaves and Variable Sets

We make a few concluding remarks about the relationship between topoi-semantics of sheaves and variable sets.

Note that in a sheaf F on a locale $\mathcal{L}$ the bottom element $0 \in \mathcal{L}$ is the *zero*-th join and:

Remark 54.
$$0 = \bigvee_{i \in \emptyset} u_i.$$
, i.e., the *empty covering* of 0.
The empty family $(x_i \in F(u_i))_{i \in \emptyset}$ is trivially compatible and admits a unique gluing $* \in F(0)$.

What this implies is:

Lemma 139. *$F(0)$ is a singleton set $F(0) = \{*\}$.* [12]

This allows us to link the semantic characterizations of $\mathcal{G}_n$ on variable sets $\mathbb{Set}^{\mathbf{N}}$ (or equivalently pre-sheaves on $\mathbf{N}^{op}$) and sheaves on $\mathbf{C_n}$:

Proposition 140. *For every $N \geq 2$:*

$$\mathcal{G}_N \vdash \phi \;\; \textit{iff} \;\; \mathbf{Sh}(\mathbf{N}) \models_{\mathcal{E}} \phi$$
$$\textit{iff} \;\; C_N \models_{H.A.} \phi \;\; \textit{iff} \;\; \mathbf{N-1} \models_{K.} \phi \;\; \textit{iff} \;\; \mathbb{Set}^{\mathbf{N-1}} \models_{\mathcal{E}} \phi;$$

Recall that $\mathcal{G}_2 = CPL$, i.e., classical propositional logic, C_2 is the Boolean algebra of binary truth values, $\mathbf{1}$ is the one-world Kripke frame and that $\mathbb{Set}^{\mathbf{1}} \cong \mathbb{Set}$ is a bivalent and boolean topos.
For $N = 2$ we recover classical logic:

Proposition 141.

$$\mathcal{G}_2 \vdash \phi \;\; \textit{iff} \;\; \mathbf{Sh}(\mathbf{C_2}) \models_{\mathcal{E}} \phi$$
$$\textit{iff} \;\; C_2 \models_{H.A.} \phi \;\; \textit{iff} \;\; \mathbf{1} \models_{K.} \phi \;\; \textit{iff} \;\; \mathbb{Set}^{\mathbf{1}} \models_{\mathcal{E}} \phi.$$

[12] this is a common result found in [13] among others.

Since the poset category $\mathbf{N} \cong \mathbf{C_N}$ for any $N \in \mathbb{N}$:

Lemma 142. *For every* $N \geq 2 : \mathbf{Sh}(\mathbf{N}) \cong \mathbb{Set}^{\mathbf{N-1}}$.

To see why this isomorphism holds, recall that $\mathbb{Set}^{\mathbf{N}}$, i.e., variable sets on $\mathbf{N}$ are the same as pre-sheaves on $\mathbf{N^{op}}$ and consider the case of $N = 3$:
Let's consider the sheaves on the 3-chain $C_3 = \{u_0 \leq u_1 \leq u_2\}$:

Proposition 143.

$$\mathcal{G}_3 \vdash \phi \;\; \textit{iff} \;\; \mathbf{Sh}(\mathbf{C_3}) \models_{\mathcal{E}} \phi$$
$$\textit{iff} \;\; C_3 \models_{H.A.} \phi \;\; \textit{iff} \;\; \mathbf{2} \models_{K.} \phi \;\; \textit{iff} \;\; \mathbb{Set}^{\mathbf{2}} \models_{\mathcal{E}} \phi.$$

Remark 55. The image of a sheaf F of this form is given by sets and restriction maps between them $F_2 \xrightarrow{\restriction_1} F_1 \xrightarrow{\restriction_0} F_0 = \{*\}$ where $\restriction_0 = !_{F_1}$ is the unique map on the singleton set, i.e., the terminal object in $\mathbb{Set}$.
Arrows between these sheaves are, as in the case of variable sets, natural transformations $\tau : F \Rightarrow G$ which form commutative diagrams.
Any covering $u = \bigvee_{i \in I} u_i$ with $I \subseteq \{0, 1, 2\}$ in the chain C_3 must have an element $u_{i'}$ such that $u_{i'} = u$ and of course all elements $u_i \leq u$.
What this means for compatible families $\{x_i \in F_i\}_{i \in I}$ is that given $i, j \in I$ if $i \geq j$ then $\restriction_{u_j} : x_i \mapsto x_j$.
Considering the *forest structure* of the variable sets in question, $\{x_i \in F_i\}_{i \in I}$ is nothing more than a bunch of nodes on a *branch*. The highest node in this case $x_{u_{i'}} = x_u$ corresponds to the gluing.

The following figure gives an example of this situation:

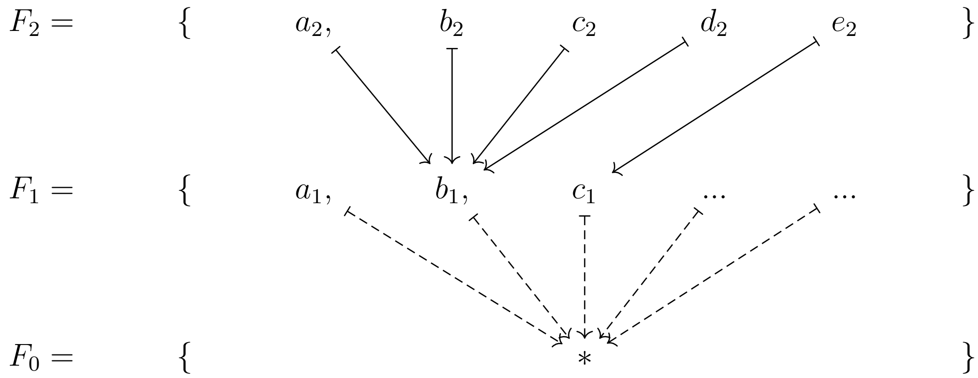


Figure 5.1: The image of a sheaf F on C_3. Branches like $\{b_2, b_1, *\}$ and $\{e_2, c_1\}$ are compatible families with gluings respectively b_2 and e_2.

Remark 56. If we remove the bottom level of F_0, which is present in every sheaf of this form, what we are left with is a presheaf on C_2.
Vice-versa, if we start from a presheaf X on C_2, add a new set $X_0 := \{*\}$ and transition function $X_1 \xrightarrow{c_*} X_0$, i.e., the constant function on the singleton set $x \mapsto *$ we return to sheaves on C_3.
What this implies is that there is a one to one correspondence between variable sets on C_2 and sheaves on C_3 which is functorial and establishes an isomorphism.

Remark 57. The 3-chain C_3 in a topological context can be thought of as the locale of open subsets of the *Sierpinski Space* $\mathcal{S} := \{0, 1\}$, i.e., $\mathcal{O}(\mathcal{S}) := \emptyset \subseteq \{1\} \subseteq \{0, 1\}$.

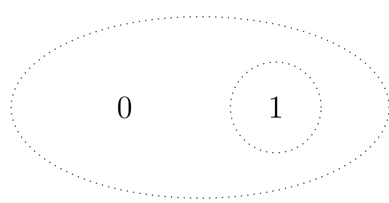


Figure 5.2: The *Sierpinski* topology on the space $\{0, 1\}$ where the only non-trivial open sub-set is $\{1\}$.

This gives us the following characterization inspired by [15]:
Sheaves over the Sierpinski space $\boldsymbol{Sh}(\mathcal{S})$, *a.k.a. the* Sierpinski topos *is equivalent to the category of presheaves over* $\boldsymbol{2}$ *or* $\boldsymbol{PSh}(\boldsymbol{2})$ *which in turn is equivalent to* functions between sets.

Proposition 144.

$$\boldsymbol{Sh}(\mathcal{S}) \simeq \boldsymbol{PSh}(\boldsymbol{2}) \simeq \mathbb{Set}^{\boldsymbol{2}}.$$

$$\mathcal{G}_3 \vdash \phi \;\; \textit{iff} \;\; \mathbf{Sh}(\mathcal{S}) \models_{\mathcal{E}} \phi.$$

In other words,

Proposition 145. $\mathcal{G}_3$ *is the logic of the Sierpinski topos.*

5.2.2 Forests and Variable Sets

Here we give some new insight about the relationship between *forests* and *variable sets*.

Remember from 3.1.5 that $\mathbb{Set}^{0\to 1}$/*functions between sets* is a tri-valent and non-Boolean topos.

Restricting ourselves to the sub-category $\mathbb{Set}^{0\to 1}_{fin}$, i.e., *functions between finite sets*, let's revisit the truth-arrows $\top, *, \bot : \mathbf{1} \Rightarrow \Omega$:

$$
\begin{tikzcd}
\{0\} \arrow[d, "id"] \arrow[rrrr, squiggly, bend left, "t'"] \arrow[drr, phantom, "\overset{\top}{\Longrightarrow}"] & & \{0, \arrow[d, "t"] & \frac{1}{2}, \arrow[d, "t"] & 1\} \arrow[dl, "t"] \\
\{0\} \arrow[rrr, squiggly, bend right, "true"'] & & \{0, & 1\} &
\end{tikzcd}
$$

Figure 5.3: $\top : \mathbf{1} \Rightarrow \Omega$ with $true : 0 \mapsto 1$, $t' : 0 \mapsto 1$.

$$
\begin{tikzcd}
\{0\} \arrow[d, "id"] \arrow[rrr, squiggly, bend left, "*'"] \arrow[drr, phantom, "\overset{*}{\Longrightarrow}"] & & \{0, \arrow[d, "t"] & \frac{1}{2}, \arrow[d, "t"] & 1\} \arrow[dl, "t"] \\
\{0\} \arrow[rrr, squiggly, bend right, "true"'] & & \{0, & 1\} &
\end{tikzcd}
$$

Figure 5.4: $* : \mathbf{1} \Rightarrow \Omega$ with $true : 0 \mapsto 1$, $*' : 0 \mapsto \frac{1}{2}$.

$$
\begin{tikzcd}
\{0\} \arrow[d, "id"] \arrow[rr, squiggly, bend left, "f'"] \arrow[drr, phantom, "\overset{\perp}{\Longrightarrow}"] & & \{0, \arrow[d, "t"] & \frac{1}{2}, \arrow[d, "t"] & 1\} \arrow[dl, "t"] \\
\{0\} \arrow[rr, squiggly, bend right, "false"'] & & \{0, & 1\} &
\end{tikzcd}
$$

Figure 5.5: $\perp : \mathbf{1} \Rightarrow \Omega$ with $false : 0 \mapsto 0$, $f' : 0 \mapsto 0$.

This *structure* is very similar to what we saw in *bushes* or *finite forests*.

In fact, the *translation* from $\mathbb{Set}^{0\to 1}_{fin}$ to $\mathbb{FF}_2$ is readily given in this case: (The usual coloring notation is applied.)

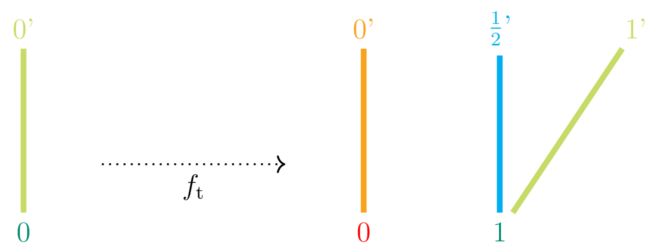


Figure 5.6: $f_t : \mathbf{1}_\perp \to \mathbf{1}_\perp + (2 \cdot \mathbf{1})_\perp$ as $\mathbb{FF}_2$-arrow where $0 \mapsto 1, 0' \mapsto 1'$.

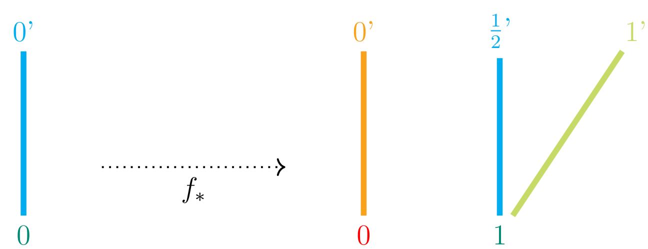


Figure 5.7: $f_* : \mathbf{1}_\perp \to \mathbf{1}_\perp + (2 \cdot \mathbf{1})_\perp$ as $\mathbb{FF}_2$-arrow where $0 \mapsto 1, 0' \mapsto \frac{1}{2}'$.

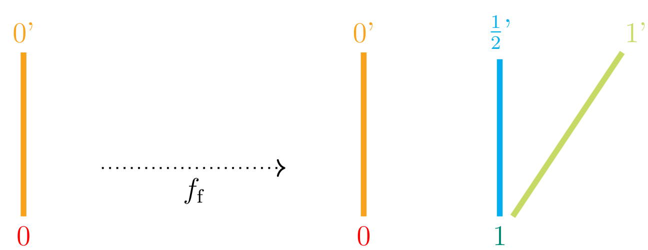


Figure 5.8: $f_{\mathrm{f}} : \mathbf{1}_\perp \to \mathbf{1}_\perp + (2 \cdot \mathbf{1})_\perp$ as $\mathbb{FF}_2$-arrow where $0 \mapsto 0, 0' \mapsto 0$.

The objects are translated as follows:

- At levels 0 and 1 the elements of the sets F_0, F_1 and G_0, G_1 become distinct nodes.
- The transition functions $f : F_0 \to F_1$ and $g : G_0 \to G_1$ specify the partial ordering by requiring $\forall_{a \in F_0}\ f(a) \leq a$ and $\forall_{b \in G_0}\ g(b) \leq b$.
  What we are left with is two finite forests F and G.

As for the arrows:

- The natural transformation $\tau : F \Rightarrow G$ in its components τ_0, τ_1 determines the image of an arrow f_τ for each node.
  Notice that the naturality of τ, i.e., $g\tau_0 = \tau_1 f$ makes f_τ an order-preserving and open map, i.e., an arrow in $\mathbb{FF}$.

This method can be generalized from the poset category **2**, i.e., $0 \xrightarrow{\leq_0} 1$ to a generic functor category of *variable sets*, a.k.a. *finite sets through finite time* $\mathbb{Set}_{fin}^{\mathbf{N}}$ where **N** is the analogous poset category **N**, i.e., $0 \xrightarrow{\leq_0} 1 \xrightarrow{\leq_1} 2.. \xrightarrow{\leq_{N-1}} N$ and provides a translation from $\mathbb{Set}_{fin}^{\mathbf{N}}$ to $\mathbb{FF}_{\mathbb{N}}$.
So:

Remark 58. There is a *translation* available from the categories $\mathbb{Set}_{fin}^{\mathbf{N}}$ to the category of finite forests $\mathbb{FF}$.

At first glance there seems to be a *reverse translation* available from $\mathbb{FF}_{\mathbb{N}}$ to $\mathbb{Set}_{fin}^{\mathbf{N}}$.
Consider the following example:

Example 146. Let $f : \mathbf{1} + \mathbf{1}_\perp \times \mathbf{1}_\perp \to \mathbf{1}_\perp + (\mathbf{1}_\perp)_\perp$ be an arrow in $\mathbb{FF}_{\mathbb{3}}$:

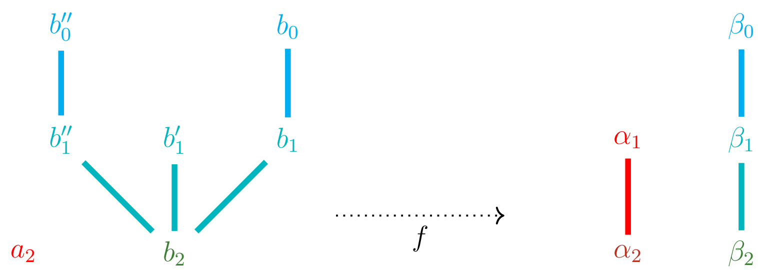


Figure 5.9: The usual coloring notation is used.
So $f : a_2 \mapsto \alpha_2$, $b_1, b'_1, b''_1 \mapsto \beta_1$ and $b_0, b''_0 \mapsto \beta_0$.

If we follow the translation steps in *reverse*, we obtain:

$$\begin{tikzcd}
\{b''_0, \arrow[d, mapsto] & & b_0\} \arrow[d, mapsto, "f_0"] \arrow[r, rightsquigarrow, "\tau_0"'] & \{\beta_0 \arrow[d, mapsto, "g_0"'] & \} \\
\{b''_1, \arrow[dr, mapsto] & b'_1, \arrow[d, mapsto] & b_1\} \arrow[dl, mapsto, "f_1"] \arrow[r, rightsquigarrow, "\tau_1"'] & \{\beta_1, \arrow[d, mapsto, "g_1"'] & \gamma_1\} \arrow[d, mapsto] \\
\{a_2, & b_2 & \} \arrow[r, rightsquigarrow, "\tau_2"'] & \{\beta_2, & \gamma_2\}
\end{tikzcd}$$

Figure 5.10: The transition maps are displayed as $F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} F_2$ and $G_0 \xrightarrow{g_0} G_1 \xrightarrow{g_1} G_2$

The nodes at each level correspond to sets at a particular time [13] and the edges between them indicate the transition maps.
Notice that, differently from 2.1, finite forests seem to *grow* from top to bottom [14] where the transition maps f_0, f_1 need not be injective [15] or surjective [16].

This is reflected in the fact that every transition function from say F_m to F_n with $m \leq n$ is an arbitrary set-function and, as we just said, need not be injective or surjective.
This simple fact gives the *forest structure* we observed for variable sets.

[13]in this case either 0,1 or 2.

[14]the notation reflects this as the top-most nodes have a 0 for subscript

[15]this is seen for example with $b_1, b'_1, b''_1 \mapsto \beta_1$.

[16]this corresponds for example to the emergence of $b'_1 \in F_1$ which is not in the image of f_0.

The reverse translation from $\mathbb{FF}_{\mathbb{N}}$ to $\mathbb{Set}^{\mathbf{N}}$, however, breaks down when we consider arrows from finite forests of different height like in this simple case:

Example 147. Let $!_{\mathbf{1}_\perp}$ be the only arrow from $\mathbf{1}_\perp$ to the terminal $\mathbf{1}$:

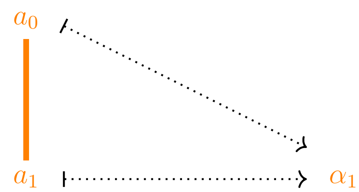


Figure 5.11: $\mathbf{1}_\perp \xrightarrow{!_{\mathbf{1}_\perp}} \mathbf{1}$.

We would need the following to be a commutative diagram:

$$\begin{tikzcd}
\{a_0\} \arrow[r, squiggly, "?\tau_0"'] \arrow[d, mapsto, "f_0"] & \emptyset \arrow[d, dashed, mapsto, "g_0=\emptyset_{G_1}"] \\
\{a_1\} \arrow[r, squiggly, "\tau_1"'] & \{\alpha_1\}
\end{tikzcd}$$

Figure 5.12: $f_0 : a_0 \mapsto a_1$ and $\tau_1 : a_1 \mapsto \alpha_1$ with $g_0 = \emptyset_{G_1}$ the (unique) empty map from $\emptyset$ to $G_1 = \{\beta_1\}$.

But:

Remark 59. There exists no map from a non-empty set like $\{a_0\}$ to the empty-set $\emptyset$ which would be the component τ_0 of $\tau : F \Rightarrow G$.

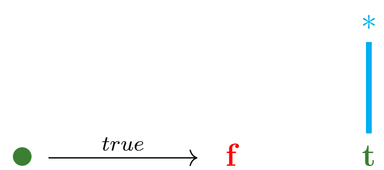
We conclude with the following considerations:

Remark 60. The *translation* $\tilde{F}$ from $\mathbb{Set}^{\mathbf{N}}$ to $\mathbb{FF}_{\mathbb{N}}$ defines an assignment for objects and morphisms and is functorial.[17]
The *reverse translation* from $\mathbb{FF}_{\mathbb{N}}$ to $\mathbb{Set}^{\mathbf{N}}$ fails to be an assignment for all $\mathbb{FF}_{\mathbb{N}}$-arrows.

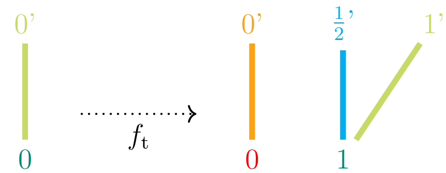
Remark 61. If, using our imperfect translation, we *compare* the arrows in the two categories one notices for instance that the terminal object in $\mathbb{Set}^{\mathbf{N}}$ is $\{0\} \xrightarrow{id} \{0\} \xrightarrow{id} ... \xrightarrow{id} \{0\}$ a.k.a the *finite telephone pole* whilst the terminal

[17] the functor $\tilde{F}$ is such that $\tilde{F}(id_A) = id_{\tilde{F}(A)}$ and $F(fg) = F(f)F(g)$.

object in $\mathbb{FF}_*$ is the singleton forest $\mathbf{1}$.
For $N > 1$ the *telephone pole* in $\mathbb{FF}_N$, i.e., $((\mathbf{1}_{\perp_1})_{\perp_2}..)_{\perp_{N-1}}$ is of course *not* terminal as there can be many distinct arrows into it.
Furthermore if we compare the sub-object classifiers (both seen as forests) between $\mathbb{FF}_2/bushes$ and $\mathbb{S}\mathrm{et}^{\mathbf{2}}$ we find a rather different structure:



(a) $\top : \mathbf{1} \xrightarrow{true} (\mathbf{1} + \mathbf{1}_\perp)$.



(b) $f_t : \mathbf{1}_\perp \to \mathbf{1}_\perp + (2 \cdot \mathbf{1})_\perp$.

We conclude with the following observation:

Remark 62. Having fixed an $N > 0$, the arrows in $\mathbb{FF}_N$ that can be translated into natural transformations in $\mathbb{S}\mathrm{et}^{\mathbf{N}}$ must have the following requirements:

- the finite forests in the domain and co-domain must have the same height equal to N.
- each node must be sent to a node of equal height.

In a nutshell, for $N > 1$:

Remark 63. There are *more* arrows in $\mathbb{FF}_N$ than in $\mathbb{S}\mathrm{et}^{\mathbf{N}}$.

Remark 64. Though remarkably similar, for any $N > 0$ the categories of *Variable Finite Sets* $\mathbb{S}\mathrm{et}^{\mathbf{N}}$ and *finite forests of height at most N* $\mathbb{FF}_N$ in fact have very different structures.
The most relevant distinction for our concerns is that: $\mathbb{S}\mathrm{et}^{\mathbf{N}}$ is a topos for all N, while $\mathbb{FF}_N$, as we have proven through the lengthy counterexample in 2.4, is a topos only for $N \leq 2$.

Chapter 6

Conclusive Remarks

We take stock of what has been achieved in this work:

The primary aim was to explore the topos semantics of $\mathcal{G}_3$. This has been done extensively thanks to the dual-algebraic semantics of $\mathcal{G}_3$ given by the sub-category of *finite forests* known as $\mathbb{FF}_2/bushes$.
In fact, citing [1]:

Remark 65. we have seen that the category of *bushes* represents a sort of *best of both worlds* semantics as it already completely characterizes $\mathcal{G}_3$ at the propositional level through its duality with finite three-valued *Gödel algebras* and at the same time is a *topos* which provides a path to develop first order semantics for $\mathcal{G}_3$ based on *bush*-concepts instead of *sets*.

We used the tools of Categorical Logic available in [9] & [12] to arrive at the following results:
At the propositional layer we found in 3.1.4:

Proposition. *The topos of* bushes/$\mathbb{FF}_2$ *is bivalent and non-Boolean.*

In other words: *internally* the propositional logic of *bushes* is classical whilst *externally* it is not.
However, moving on to the first-order predicate level, we remedied this fact and recovered $\mathcal{G}_3$ *internally* in 4.1.2.

Recall from 1.5 that: First-order semantics for $\mathcal{G}_3$ was defined in the usual way using set-concepts and interpreted the quantifiers $\forall$ and $\exists$ as generalized $\wedge$ and $\vee$, i.e., in this case *min* and *max* of truth-values.

What we found in 4.2, from the perspective of the first-order semantics of *bushes*, is that: *universal and existential quantification are equivalent to finite conjunction and disjunction over generalized elements*, i.e.,:

Proposition. *The first-order logic of the topos of* $\mathbb{FF}_2$/bushes *corresponds to first-order three-valued Gödel-Dummett Logic on* finite *domains.*

Remark 66. This justifies our approach building up from the propositional to the first order layer of topos-semantics of *bushes* as it *recovers* the first-order set-based semantics for $\mathcal{G}_3$ defined in 1.5.

In chapter 5, we compared our findings with alternative (propositional) topos-semantics for $\mathcal{G}_3$ given by *variable sets* and *sheaves on locales* found in [9] and [14].
We reviewed the fact that $\mathcal{G}$ *is the logic of sets through time* $\mathbb{Set}^{\omega}$ and observed the following corollary for *bushes* linking topos $\models_{\mathcal{E}}$, Kripke $\models_{K.}$ and Heyting algebra $\models_{H.A.}$ semantics:

Corollary.

$$\mathcal{G}_3 \vdash \phi \;\; \textit{iff} \;\; C_3 \models_{H.A.} \phi \;\; \textit{iff} \;\; \mathbf{2} \models_{K.} \phi \;\; \textit{iff} \;\; \mathbb{Set}^{\mathbf{2}} \models_{\mathcal{E}} \phi.$$

A topos semantics for $\mathcal{G}_3$ is thus given by $\mathbb{Set}^{\mathbf{2}}$, i.e., the category of *functions between sets*.
With regards to *sheaves on locales*, we proposed the following:
The 3-chain C_3 in a topological context can be thought of as the locale of open subsets of the *Sierpinski Space*.
This gives us the following characterization inspired by [15] which links Sheaves over the Sierpinski space $\mathbf{Sh}(\mathcal{S})$, a.k.a. the *Sierpinski topos*, to the category of presheaves over $\mathbf{2}$ or $\mathbf{PSh}(\mathbf{2})$, which in turns is equivalent to the category of *functions between sets*.

$$\mathbf{Sh}(\mathcal{S}) \simeq \mathbf{PSh}(\mathbf{2}) \simeq \mathbb{Set}^{\mathbf{2}}.$$
$$\mathcal{G}_3 \vdash \phi \;\text{ iff }\; \mathbf{Sh}(\mathcal{S}) \models_{\mathcal{E}} \phi.$$

In other words:

Proposition 148. $\mathcal{G}_3$ *is the logic of the Sierpinski topos.*

What is the link between *variable sets* and *finite forests*?

It is the case, as we saw in 5.2.2, that for any $N > 0$ objects and arrows in $\mathbb{Set}^{\mathbf{N}}_{fin}$ can be easily translated into objects and arrows in $\mathbb{FF}_{\mathbb{N}}$.
However, the converse translation breaks down for arbitrary arrows in $\mathbb{FF}_*$.
In fact back in 2.1 we introduced the categorical structure of *finite forests* and, as it turns out, though strikingly similar, is remarkably different from that of *variable sets*: For $N > 1$:

Remark 67. The most relevant distinction for our concerns is that: $\mathbb{Set}^{\mathbf{N}}$ is a topos for all N, while $\mathbb{FF}_N$, as we have proven through the lengthy counterexample in 2.4, is a topos only for $N \leq 2$.

Recall also the observation made in 5.2.2:

Remark 68. In a nutshell, for $N > 1$, there are *more* arrows in $\mathbb{FF}_N$ than in $\mathbb{Set}^{\mathbf{N}}$.

This gives a new insight for the phrase *"best of both worlds"* used to describe the category of *bushes*:

Remark 69. $\mathbb{FF}_{\mathbb{2}}$/*bushes* has a *richer arrow structure* than its *variable set/presheaf* counterpart $\mathbb{Set}^{\mathbf{2}}$ and, unlike all the other finite forests of greater height, has a topos structure.

We conclude with a short selection of proposals for further areas of inquiry:

- A semantic approach using *Lawvere Theories* for Gödel-Dummett Logic and other fuzzy logics like *Nilpotent-Minimum Logic* $\mathcal{NM}$.
- A deeper categorical understanding of the *non-toposness* of $\mathbb{FF}_{\mathbb{k}\geq\mathbb{3}}$.
- An improvement and expansion of the Python code used.
- Ind/Pro-finite completion of *bushes* and associated topos semantics.

The link to the Python implementation of the tool used for computing operations between finite forests and the number of arrows between them in chapter 2 is given:

https://github.com/albertpaner/finiteforest.git

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