Theorem Corollary Example

# Mathematics and Statistics Brush-up: Continuity and Convexity

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# Continuity

## Continuity

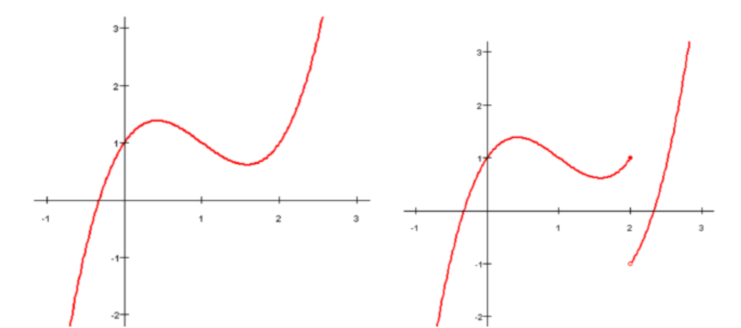
A map $f : X \to \mathbb{R}$ is said to be **continuous at a point** $x$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$\forall y \in X \quad s.t. \quad d(x, y) < \delta \implies d(f(x), f(y)) < \varepsilon$$

If $f$ is not continuous at $x$, we say that $f$ is discontinuous at $x$. If $f$ is continuous at every point in a set $X$, we say that $f$ is continuous on $X$. Notice that the definition given above is equivalent to:

$$f(N_\delta(x)) \subseteq N_\varepsilon(f(x))$$

# Examples



**Exercise**
*let* $f : \mathbb{R} \to \mathbb{R}$ *be* $f(x) = x^2$. *Show that f(x) is continuous everywhere:*

# Uniformly continuous (global continuity)

Notice that continuity is a local property. To achieve global continuity, a *unique* $\delta$ regardless what $x$ is.

A function $f : X \to \mathbb{R}$ is said to be **uniformly continuous** if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$\forall y, x \in X \quad s.t. \quad d(x, y) < \delta \longrightarrow d(f(x), f(y)) < \varepsilon$$

Note that this is equivalent to sat that for uniform continuity we require a $\delta > 0$ such that $\delta$ is not a function of $x$. Obviously, a function is not uniformly continuous if it is not continuous.

# Proposition

**Proposition:** If $f : X \to Y$ is continuous and $X$ is compact, then it is uniformly continuous.

# Other definitions of continuity

A function $f : X \subset \mathbb{R} \to Y \subset \mathbb{R}$ is **continuous (sequences definition)** if and only if for any sequence $(x_n)$ converging to $\bar{x} \in \mathbb{R}$, the sequence $(f(x_n))$ converges to $f(\bar{x}) \in \mathbb{R}$

# Other definitions of continuity

**Proposition:** Let $f : X \subset \mathbb{R} \to Y \subset \mathbb{R}$. The following statements are equivalent:

1. $f$ is continous.
2. for all sets $O \subseteq Y$ open, the set called inverse image, $f^{-1}(O) \subseteq X$, is open.
3. for all sets $S \subseteq Y$ closed, the set $f^{-1}(S) \subseteq X$, is closed.

That is a function $f$ is continuous if and only if the preimages of open (closed) sets are open (closed).

# Intermediate value theorem

**Theorem**
*Intermediate Value Theorem: If* $f : \mathbb{R} \to \mathbb{R}$ *is continuous on the closed interval* $[a, b]$ *and* $k \in \mathbb{R}$ *is a scalar between* $f(a)$ *and* $f(b)$*, then there exists a* $c \in [a, b]$ *such that* *f(c)=k.*

# The Weierstrass's theorem

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*Weierstrass's theorem: If $X$ is a compact metric space and $f : X \to \mathbb{R}$ is a continuous function, then there exists maximum and minimum of $f$ in $X$.*

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*Weierstrass's theorem: If $X$ is a compact metric space and $f : X \to \mathbb{R}$ is a continuous function, then there exists maximum and minimum of $f$ in $X$.*

**Proof.**

Notice that the set spanned by the images of the whole domain $X$ under $f$, is closed and bounded by continuity of $f$. Since $f(X)$ is bounded, both $sup\{f(x) : x \in S\}$ and $inf\{f(x) : x \in S\}$ exist. Since the supremum and the infimum belong to the closure $(cl(f(X))$ and f(X) is closed, then they must belong to f(X) as well. ■

# Convexity

# Convexity

For any two points $x, y \in \mathbb{R}^n$

- the set $\{\lambda x + (1-\lambda)y : \lambda \in \mathbb{R}\}$ is called **line** through $x$ and $y$.
- the set $\{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}$ is called **segment** between $x$ and $y$.

Moreover, for all $m \in \mathbb{N}$ and for all $x^1, \ldots, x^m \in \mathbb{R}^n$, their **convex combination** are points of the following form: $\sum_{i=1}^m \lambda_i x^i$, with $\lambda_i > 0$ for all $i$ and $\sum_{i=1}^m \lambda_i = 1$. thus the segment between $x$ and $y$ is the set of all convex combinations of $x$ and $y$.

A set $S \subset \mathbb{R}^n$ is said to be **convex** if for all $x, y \in S$ and for all $\lambda \in [0,1]$ we have that

$$(\lambda x + (1-\lambda)y) \in S$$

This implies that a set is convex if and only if it contains all the segments between any pair of points the set itself. Equivalently, a set $S$ is convex if and only if every combination of points of $S$ lies in $S$.

# Convex hull

Given a set $S \in \mathbb{R}^n$, the smallest convex set that contains $S$ is called **convex hull** of $S$, denoted by $co(S)$. An analogous definition of convex hull states it is the set of all convex combinations of points in the set $S$:

$$co(S) = \{\lambda_1 x_1 + \cdots + \lambda_k x_k : x_i \in S, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1\}$$

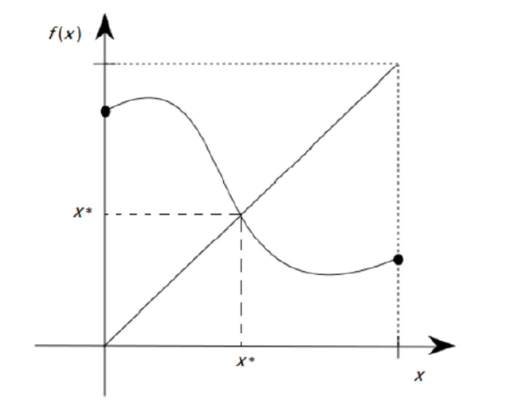
Notice that any closed convex set can be written as the convex hull of itself.

# Properties convex sets

- the intersection of any collection of convex sets is a convex set.
- The linear combination of convex sets is convex: Let $S$ and $T$ be convex subsets in $X \subseteq \mathbb{R}^n$ and $\alpha$ and $\beta$ be real numbers. Then, the set $Z = \alpha S + \beta T = \{z \in Z : \alpha x + \beta y,\ \ x \in S, y \in Y\}$ is convex.

# Brouwer's fixed-point theorem

**Theorem**
**Brouwer's fixed-point theorem** *Every continuous function $f$ from a convex compact set $X \subset \mathbb{R}^n$ to itself ($f : X \to X$) has a* **fixed point** $x^*$ *such that* $f(x^*) = x^*$.



# Convex functions

Let $S$ be a convex subset in a real vector space. A function $f : S \to \mathbb{R}$ is said to be **convex (strictly convex)** if for all $x, y \in S, \lambda \in [0, 1]$

$$f(\lambda x + (1-\lambda)y) \underset{(<)}{\leq} \lambda f(x) + (1-\lambda)f(y)$$

And a function $f : X \to \mathbb{R}$ is said to be **concave (strictly concave)** if for all $x, y \in S, \lambda \in [0, 1]$

$$f(\lambda x + (1-\lambda)y) \underset{(>)}{\geq} \lambda f(x) + (1-\lambda)f(y)$$

That is a function is convex (concave) if $-f$ is concave (convex).

## Epigraph and Hypograph

Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$, then:

- The **epigraph** of $f$ is defined as: $epif := \{(x, y) \in \mathbb{R}^{n+1} : x \in S, y \geq f(x)\}$
- The **hypograph** of $f$ is defined as: $hypf := \{(x, y) \in \mathbb{R}^{n+1} : x \in S, y \leq f(x)\}$

Notice that any function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its epigraph is a convex set. Conversely, $f$ is concave if and only if its hypograph is a convex set. We define:

# Contour sets

- The **upper contour set** of a function $f$ consists of all of the points *in the domain* of $f$ at which the value of the function is at least a certain value (say $\alpha \in \mathbb{R}$).

$$U_f(\alpha) = \{x \in X : f(x) \geq \alpha\}.$$

- The **lower contour set** of a function $f$ consists of all of the points *in the domain* of $f$ at which the value of the function is no more than a certain value ($\alpha \in \mathbb{R}$)

$$L_f(\alpha) = \{x \in X : f(x) \leq \alpha\}$$

# Quasi-convex and quasi-concave functions

Let $S$ be a convex subset of $\mathbb{R}^n$, then a real function $f : S \to \mathbb{R}$ is said to be

- **quasi-convex** if for all $x, y \in S$ and $\lambda \in [0, 1]$ then:

$$f(\lambda x + (1 - \lambda)y) \leq max\{f(x), f(y)\}$$

- **quasi-concave** if for all $x, y \in S$ and $\lambda \in [0, 1]$ then:

$$f(\lambda x + (1 - \lambda)y) \geq min\{f(x), f(y)\}$$

Notice that a function $f$ is quasi-concave if $U_f(\alpha)$ is a convex set $\forall \alpha \in \mathbb{R}$.
Analogously, a function $f$ is said quasi-convex if $L_f(\alpha)$ is a convex set $\forall \alpha \in \mathbb{R}$.

## Prop: Increasing functions are both quasi-ccv and quasi-cvx

- **Proposition:** Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Then, $f$ is both quasi-concave and cuasi-convex.

**Proof.**
Consider $x, y \in \mathbb{R}, \lambda \in (0, 1)$. Assume, without l.o.g, that $x > y$. Then

$$x > \lambda x + (1 - \lambda)y > y$$

Since $f$ is increasing, then

$$f(x) \geq f(\lambda x + (1 - \lambda)y \geq f(y)$$

Since $f(x) = max\{f(x), f(y)\}$ then from the first inequality, $f(y) \leq max\{f(x), f(y)\}$, so $f$ is quasi-convex. Then, since $f(y) = min\{f(x), f(y)\}$, then, from the second inequality, $f(x) \geq min\{f(x), f(y)\}$, so $f$ is quasi-concave. ∎