Mathematics and Statistics Brush-up: Integration

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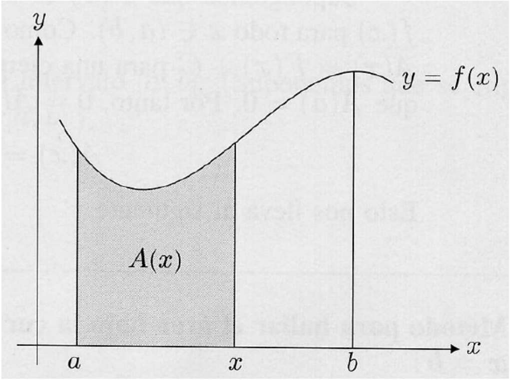
The fundamental theorem of calculus

The fundamental theorem of calculus is a theorem that links the concept of differentiating a function with the concept of integrating a function. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and non-negative ($f \geq 0$). We want to find the area $A(x)$ between the graph of the function and the horizontal axis on $[a, x] \subseteq [a, b]$



The fundamental theorem of calculus

Let's increase x by an infinitesimal amount Δx. The increase in the area is $A(\Delta x) = A(x + \Delta x) - A(x)$. note that:

- if $f(x)$ is increasing:

$$f(x)\Delta x \leq A(x + \Delta x) - A(x) \leq f(x + \Delta x)\Delta x$$

- if $f(x)$ is decreasing:

$$f(x)\Delta x \geq A(x + \Delta x) - A(x) \geq f(x + \Delta x)\Delta x$$

The fundamental theorem of calculus

In an increasing (decreasing) function, divide the 3 terms in the previous expression by Δx

$$f(x) \underset{(\geq)}{\leq} \frac{A(x+\Delta x) - A(x)}{\Delta x} \underset{(\geq)}{\leq} f(x+\Delta x)$$

And take the limit when Δx goes to 0

$$\underset{\Delta x \to 0}{lim} f(x) \underset{(\geq)}{\leq} \underset{\Delta x \to 0}{lim} \frac{A(x+\Delta x) - A(x)}{\Delta x} \underset{(\geq)}{\leq} \underset{\Delta x \to 0}{lim} f(x+\Delta x)$$

The fundamental theorem of calculus

Note that the middle term is the definition of a derivative

$$\lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x} = A'(x)$$

Also note that since f is continuous it holds that $\lim_{\Delta x \to 0} f(x + \Delta x) = f(x)$. Thus, we arrive to the following expression

$$f(x) \underset{(\geq)}{\leq} A'(X) \underset{(\geq)}{\leq} f(x)$$

That is

$$A'(x) = f(x) \qquad \text{(Fundamental Theorem of Calculus I)}$$

$\blacksquare$

The fundamental theorem of calculus

the first part of *the fundamental theorem of calculus* and tells us how we can find the area under a curve using antidifferentiation: Finding the area between the graph of f and the horizontal axis on the interval $[a, b]$ is equivalent to find the F such that $F' = f$.

The indefinite integral, antiderivative or primitive

a function F is a **primitive** or **antiderivative** or **indefinite integral** of a function f if $F' = f$. We write the indefinite integral of f as

$$\int f(x)dx$$

- **Lemma 1:** If F is a primitive of f, then $G = F + C$ where C is a constant, is also a primitive of f (i.e. $G' = f$).
- **Lemma 2:** If F and G are primitives, then $\forall x,\ G(x) - F(x) = C$

Fundamental theorem of calculus II

From lemma 2 note that

$$A(x) = F(x) + C \qquad \text{where } F' = f$$

The area -the integral- of a point is equal to 0; $A(a) = 0$. Substituting point a for x in previous equation

$$0 = A(a) = F(a) + C \longrightarrow C = -F(a)$$

and therefore,

$$A(x) = F(x) - F(a), \qquad \text{where } F' = f$$

So that the area A in interval $[a, b]$ is

$$A = F(b) - F(a) \qquad F' = f \qquad \text{(Fundamental Theorem of Calculus II)}$$

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The definite integral

The **definite integral** of a continuous function f on the interval $[a, b]$ is

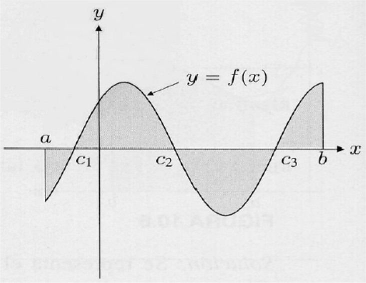
$$\int_a^b f(x)dx = F(b) - F(a) \equiv [F(x)]_a^b \equiv F(x)|_a^b, \qquad \text{where} F' = f$$

Areas in negative intervals

As for now, we work for the case $f(0) \geq 0$. We define the area or integral A of a continous function $f(x)$ on $[a, b]$ where for all points in the interval $f(x)$ is negative as

$$A = \int_a^b [-f(x)]dx = -\int_a^b f(x)dx \geq 0/$$

Note that an area cannot be negative.



Areas in negative intervals

To find the area A between the graph of a continuous function f and the horizontal axis on the interval $[a, b]$ where the function takes both positive and negative values; we will sum the integrals from the positive areas and the "inverted" integrals from the negative ones.

Properties of integrals

Properties of the indefinite integral of a continuous function f:

1. $\int cf(x)dx = c\int f(c)dx$, where c is a constant.
2. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

Note that in a definite integral the variable appearing as the argument of the function f is a *mute* variable, that is:

$$\int_a^b f(x)dx = \int_a^b f(z)dz = F(b) - F(a), \text{ where F'=f}$$

Properties of the definite integral of a continuous function f:

- 1.
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$
- 2.
$$\int_a^a f(x)dx = 0$$
- 3.
$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \quad \text{where } c \text{ is a constant.}$$
- 4.
$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$
- 5.
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad \text{when } a \leq c \leq b$$

- 6.
$$\frac{\partial}{\partial x}\int_a^x f(z)dz = f(x) \quad \text{and} \quad \frac{\partial}{\partial x}\int_x^b f(z)dz = -f(x)$$
- 7. If $f(x) \geq g(x)$ for all $x \in [a, b]$, then:
$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$
- 8.
$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx, \text{for } a > b$$
- 9. Cauchy-Schwartz Inequality:
$$\left[\int_a^b |f(x)g(x)|dx\right]^2 \leq \left[\int_a^b [f(x)]^2 dx\right] \cdot \left[\int_a^b [g(x)]^2 dx\right]$$
$$\left[\int_a^b |f(x)g(x)|dx\right] \leq \left[\int_a^b [f(x)]^2 dx\right]^{1/2} \cdot \left[\int_a^b [g(x)]^2 dx\right]^{1/2}$$

Immediate integrals

- $\int f(x)dx = F(x) + C$
- $\int 0dx = C$
- $\int 1dx = x + C$
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } x \neq -1$
- $\int \frac{1}{x}dx = ln|x| + C \quad \text{for } x \neq 0$
- $\int e^x dx = e^x + C$
- $\int a^x dx = a^x \frac{1}{lna} + C$
- Inverse chain rule
  $\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + C$
- $\int \frac{f'(x)}{f(x)}dx = ln|f(x)| + C$
- $\int f'(x)e^x dx = e^{f(x)} + C$
- $\int sinxdx = -cosx + C$
- $\int cosxdx = sinx + C$
- $\int lnxdx = xlnx - x + C$

Example

$$\int x\underbrace{(x^2+4)}_{f(x)}{}^{1/2}dx = \int \frac{1}{2}\underbrace{2x}_{f'(x)}\underbrace{(x^2+4)^{1/2}}_{[f(x)]^{1/2}}dx = \frac{1}{2}\left[\frac{(x^2+4)^{3/2}}{3/2}\right]+C =$$

$$= \frac{1}{3}(x^2+4)^{3/2}+C$$

Example

$$\int e^{3x+2}dx = \int \frac{1}{3}\cdot 3e^{3x+2}dx = \frac{1}{3}\int 3e^{3x+2}dx = \frac{1}{3}e^{3x+2}+C$$

Integration by parts

The **integration by parts** for definite integrals is defined by the following equation

$$\int_a^b F(x)g(x)dx = F(x)G(x)|_a^b - \int_a^b G(x)f(x)dx$$

Proof.
Let F and G be primitive functions, then

$$\frac{\partial F(x)G(x)}{\partial x} = f(x)G(x) + F(x)g(x)$$

Computing the indefinite integral of both

$$F(x)G(x) + C = \int f(x)G(x)dx + \int F(x)g(x)dx$$

So that

$$\int F(x)|_a^b g(x)dx = F(x)G(x)|_a^b - \int_a^b f(x)G(x)dx$$

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Example: Integration by parts

Example
Let $x > 0$, compute

$$\int (lnx)dx$$

Note that we can define $F(x) = lnx$ and $g(x) = 1$, so that $f(x) = \frac{1}{x}$ and $G(x) = x$. then,

$$\int (lnx)dx = F(x) \cdot G(x) - \int f(x)G(X)dx + C =$$

$$= lnx \cdot x - \int \frac{1}{x} \cdot xdx + C = lnx \cdot x - \int 1dx + C =$$

$$= x(lnx - 1) + C$$

Improper integrals

We call **improper integrals** the integrals of a continuous function f defined on non-closed intervals.

- integral on the interval $[a, \infty)$: $\int_a^\infty f(x)dx = \lim\limits_{b \to \infty} \int_a^b f(x)dx$
- integral on the interval $(\infty, b]$: $\int_\infty^a f(x)dx = \lim\limits_{a \to \infty} \int_a^b f(x)dx$
- integral on the interval (∞, ∞): $\int_\infty^\infty f(x)dx = \lim\limits_{a \to \infty} \int_a^0 f(x)dx + \lim\limits_{b \to \infty} \int_0^b f(x)dx$

Improper integrals

- Integral on the right-semiclosed interval $(a, b]$: $\int_{a^+}^{b} f(x)dx = \lim_{z^+} \int_{z}^{b} f(x)dx$, where $z > a$.
- Integral on the open interval (a, b): $\int_{a^+}^{b^-} f(x)dx \equiv \lim_{z^+} \int_{z}^{c} f(x)dx + \lim_{z \to b^-} \int_{c}^{z} f(x)dx$, with $c \in (a, b)$

All the previous limits might fail to exist: They could be equal to $\infty - \infty$ or be equal to $\pm\infty$. In the latter case, we sat that the improper integral **diverges**.

Examples

Example
$\int_{0+}^{1} \frac{1}{x} dx = \lim_{z \to 0^+} [ln|x|]_z^1 = ln1 - \lim_{z \to 0^+} ln|x| = 0 - (-\infty) = \infty,$ So that the improper integral diverges.

Example
$\int_{1}^{\infty} \frac{1}{x} dx = \lim_{b \to \infty} [ln|x|]_1^b = \lim_{b \to \infty} ln|x| - ln1 = \infty - 0 = \infty,$ So that the improper integral diverges.

Example
$\int_{1}^{\infty} \frac{1}{x^2} dx = \lim_{b \to \infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b \to \infty} \left(-\frac{1}{x}\right) \ \left(-\frac{1}{1}\right) = 0 + 1 = 1$

Integration with respect to several variables

We worked for the case of a function $f : \mathbb{R} \to \mathbb{R}$, where the integration of a function f is the area between the graph of the function and the x-axis on an interval $[a, b]$. Now, consider functions $f : \mathbb{R}^n \to \mathbb{R}$ defined on higher dimensional spaces. First, suppose a function $y = f(x_1, x_2)$ on the three-dimensional Cartesian plane (i.e. $n = 2$). Then, its integral will be the volume of the region between the surface defined by the function and the plane $A = [a_1, b_1] \times [a_2, b_2]$ which contains its domain. If there are more variables, a multiple integral will yield hypervolumes of multidimensional functions.

The multiple integral

The **multiple integral** of a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ defined on a **rectangular** or **hyperrectangular** domain A such that $A = [a_1, b_1] \times [a_2, b_2] \times \ldots [a_n, b_n]$ is defined as:

$$\begin{aligned}\int_A f(x_1, x_2, \ldots, x_n) d(x_1, x_2, \ldots, x_n) &= \int_{a^n}^{b^n} \cdots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n \\ &= \int_{a^n}^{b^n} \ldots \left[\int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} f(x_1, x_2, \ldots, x_n) dx_1 \right] dx_2 \right] \ldots dx_n\end{aligned}$$

Properties of multiple integrals

Multiple integrals over a rectangle have many properties common to those of integrals of functions of one variable (linearity, commutativity, monotonicity, and so on). To minimize notation, Let's state the properties for a function $z = f(x, y)$ on the rectangular $A = [a, b] \times [c, d]$ (yet they generalize for any $n \in 2.3, 4, 5, \dots, N$).

- $$\int_a^b \int_c^d f_1(x) f_2(y) dx dy = \left[\int_a^b f_1(x) dx \right] \cdot \left[\int_c^d f_2(y) dy \right]$$
- $$\int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx \quad \text{(Fubini's theorem)}$$
- $$\frac{\partial^2}{\partial x \partial y} \int_c^y \int_a^x f(t_1, t_2) dt_1 dt_2 = f(x, y)$$

Example: Multiple integrals

Example

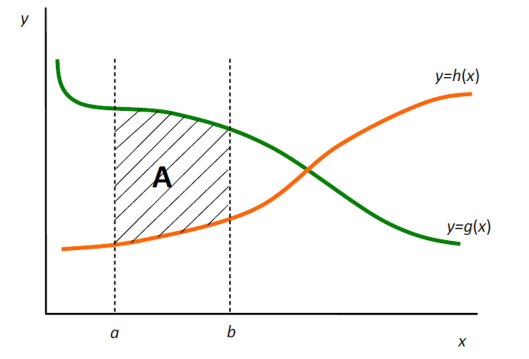
$$\int_D 2x + 3y + 4d(x, y) \quad \text{where } D = [0, 1] \times [0, 2]$$

$$\int_0^2 \left[\int_0^1 2x + 3y + 4dx \right] dy = \int_0^2 \left[x^2 + 3yx + 4x \right]_0^1 dy$$

$$= \int_0^2 5 + 3ydy = \left[\frac{3}{2} y^2 + 5y \right]_0^2 = \frac{3}{2} 4 + 10 = 16$$

Integrals over non-rectangular areas

We have been working for the case A is a rectangular region. To perform an integral over a region that is not rectangular, we have to express each of the bounds of the inner integral as a function of the outer variable. Consider the following non-rectangular region A



Integrals over non-rectangular areas

then, we compute the integral of $f(x, y)$ on A in the following manner:

$$\begin{aligned}\int_A f(x, y) d(x, y) &= \int_a^b \int_{h(x)}^{g(x)} f(x, y) dy dx \\ &= \int_a^b \left[\int_{h(x)}^{g(x)} f(x, y) dy \right] dx\end{aligned}$$

Note that we express the non-linear bounds in the inner integral as a function of the outer variable.

Exercise: integrals over non-rectangular areas

Exercise

$$\int_C \frac{1}{y} d(x,y) \quad where\ C := \{(x,y) : 0 \leq x \leq y, 0 \leq y \leq 1, x + y \geq 1/2\}$$

Differentiation of integrals

On the differentiability and continuity of an integral

$$H(z) := \int_a^z f(t)ft, \quad \text{Where } z \in [a, b]$$

1. If f is continuous at x, then H is differentiable at x and $H'(x) = f(x)$.
2. If f is discontinuous at x, then H is not differentiable at x
3. The function H is continuous on $[a, b]$,

$$\lim_{z \to x} H(z) = \lim_{z \to x} \int_a^z f(t)dt = \int_a^x f(t)dt = H(x) \quad \forall x \in [a, b]$$

Differentiation of integrals

- **Proposition:** Let $f(x, y)$ be a function such that the partial derivative $\frac{\partial f(x,y)}{\partial y}$ exists and is continuous. then

$$\frac{d}{dy}\int_a^b f(x, y)dx = \int_a^b \frac{\partial f(x, y)}{\partial y} dx$$

  thus, one may interchange the integral and partial differential operators.

Proof.

$$\begin{aligned}
\frac{d}{dy}\int_a^b f(x, y)dx &= \lim_{h\to 0} \frac{\int_a^b f(x, y+h)dx \int_a^b f(x, y)dx}{h} = \\
&= \lim_{h\to 0} \frac{\int_a^b [f(x, y+h) - f(x, y)]dx}{h} = \\
&= \int_a^b \lim_{h\to 0} \left[\frac{f(x, y+h) - f(x, y)}{h}\right] dx = \\
&= \int_a^b \frac{\partial f(x, y)}{\partial y} dx
\end{aligned}$$

- **Proposition: Liebniz rule.** Let $f(x, y)$ be a function such that the partial derivative $\frac{\partial f(x,y)}{\partial y}$ exists and is continuous, and $a(y)$ and $b(y)$ be differentiable functions. Then,

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(x, y)dx = \int_{a(y)}^{b(y)} \frac{\partial f(x, y)}{\partial y}dx + f(b(y), y) \cdot b'(y) - f(a(y), y) \cdot a'(y)$$