Mathematics and Statistics Brush-up: Preliminaries, Set theory, and Topology

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Preliminaries

Preliminaries: Definitions and proofs

- A good definition of a property X is a statement that lets one know unambiguously whether any given object y in the universe has property x or not.
- a **proof** is a sequence of statements or propositions, each of which is properly formed and correctly justified by those before it.
- We use **implication** to join the links in the "chains" that consitute our proofs. We say proposition A implies B, denoted as $A \longrightarrow B$, where A is called the hypothesis and B the conclusion. To express implication we also say
    - "if A then B"; "B follows from A";
    - B if A", "A only if B";
    - "A is sufficient for B; B is necessary for A.

Preliminaries: Implication

Implication is defined by the *truth table*

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

- We may think of an **implication as a rule** that is true if it is being obeyed and false if it is being broken.
- The rule is not broken if (F $\longrightarrow$ T) or by (F $\longrightarrow$ F) and therefore the implication holds (**vacuously true**).

Example
If it rains, you must carry an umbrella.

Preliminaries: Logically equivalent

If two statements always have the same truth value, we say that they are **logically equivalent** and we denote them as $A \longleftrightarrow B$. We express logically equivalent conditions with

- A *if and only if* B
- $A(B)$ is a *necessary and sufficient* condition for $B(A)$

Preliminaries: Types of proofs

To prove the statement $A \longrightarrow B$ there are four fundamental approaches:

- **Direct proof (by construction)**: the conclusion is established by logically combining the axioms, definitions, and earlier theorems. Assume that A is true. Use A to show that B must be true.
- **Proof by contradiction**: consists in showing that if some statement is assumed true, a logical contradiction (a statement that is both true and false) occurs, hence the statement must be false.
- **Proof by contraposition**: The conclusion is established by the logically equivalent contrapositive statement: "if not A then not B.
- **Proof by induction**: First show that a single "base case" is proved. Then, show that an "induction rule" is proved that establishes that any arbitrary case implies the next case.

Set Theory

Basic set Operators

Let A, B be sets.

- A is a subset of B, denoted $A \subseteq B$ if $a \in A \longrightarrow a \in B$
- Two sets are **equal**, $A = B$ if $A \subseteq B$ and $B \subseteq A$.
- If $A \subseteq B$ and $A \neq B$, then we write $A \subset B$.

Basic Set Operators

- The **union** of 2 sets A, B, defined as $A \cup B$ is the set formed by all the elements present in *at least one of the sets* (i.e. $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- The **intersection** of two sets, denoted $A \cap B$, is the set formed by all the elements present *in both sets* (i.e. $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Let A , B and X be sets such that $A, B \subseteq X$.

- The **difference** of set A minus B, denoted $A - B$ or $A \setminus B$ is defined as $A - B = \{x : x \in Aandx \notin B\}$.
- **Cartesian Product** is $A \times B = \{(a, b) : a \in A, b \in B\}$.
- The **complement** of a set A, denoted A^c is $A^c = \{x \in X : x \notin A\} = X - A$

De Morgan's Laws and Complement Laws

Let A, B be sets in a universe X. The following identities capture important properties of absolute complements:

- De Morgan's Laws

$$(A \cup B)^c = A^c \cap B^c \qquad (A \cap B)^c = A^c \cup B^c$$

- Complement laws:

$$\begin{aligned} A \cup A^c &= X & A \cap A^c &= \emptyset \\ \emptyset^c &= X & X^c &= \emptyset \\ A \subseteq B \longrightarrow B^c &\subseteq A^c \end{aligned}$$

The power set

Let A be a set. The **power set** of A, denoted as $\mathcal{P}(A)$ or 2^A, is the set of all subsets of A: $\mathcal{P}(A) = \{x : x \subseteq X\}$.

- Example: The power set of $A = \{1, 2, 3\}$ is $\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.
- Note that the power set of A always includes the empty set $\emptyset$ and itself A.

Relations

Let A and B be sets. any **binary relation**, $R \subseteq A \times B$, relates elements of A to elements of B. Some basic relations are:

- The **domain** of R is given by: $dom(R) = \{x \in A : \exists y \in B \text{ and } (x, y) \in R\}$.
- The **range** of R is given by: $range(R) = \{y \in B : \exists x \in A \text{ and } (x, y) \in R\}$
- The **inverse** of R is given by:
  $R^{-1} = \{(x, y) \in B \times A : \exists y \in B$ and $x \in A$, such that $(y, x) \in A \times B$.
- **function** is binary relation $f \subseteq X \times Y$ if and only if for each $x \in A$, there exists a *unique* $y \in B$, such that Xfy. The convention is then to say $f : A \to B$ and $y = f(x)$.

Some properties of relations

Typical (sometimes desirable) properties of a relation R on a set A:

- R is **complete** if for all $a, b \in R$, aRb or bRa.
- R is **reflexive** if for all $a \in A$, aRa.
- R is **transitive** if aRb and $bRc \iff aRc$.
- R is **symmetric** if $aRb \iff bRa$.
- R is **asymmetric** if $aRb \iff \text{not} bRa$.
- R is **antisymmetric** ifaRb & $bRa \iff a = b$

Relations

Let R be a relation on A, then

- R it is called an **equivalence relation** if it is reflexive, symmetric and transitive.
- R is called an **order relation** if it is reflexive, transitive and not symmetric. The pair (A, R) is then said to be an **ordered set**.
- R is a **weak order** if it is transitive, reflexice and complete. In economics we usually think of *preferences* as weak orders.

R

Topology

Metric spaces

Let X be a non-empty space. A function d: $X \times X \to \mathbb{R}_+$ is called a **metric** or distance function defined on X if for all $x, y, z \in X$ satisfies the following properties:

1. Positivity: $d(x, y) \geq 0$, for all $x, y \in X$
2. Non-degenerated: $d(x, y) = 0 \iff x = y$
3. Symmetry: $d(x, y) = d(y, x)$
4. Triangular inequality: $d(x, y) \leq d(x, z) + d(z, y)$

If d is a distance function on X, then the couple (d, X) defines a **metric space**. A well-defined distance between pairs of elements in a given space induces what is called a **topological space**.

Example: the discrete metric

Example

The **discrete metric** on X defines points as either equal or different:

$$\begin{aligned} d(x, y) &= 0 \quad \text{if } x = y \\ d(x, y) &= 1 \quad \text{if } x \neq y \end{aligned}$$

Show that the discrete metric defines a metric space.

Example: the discrete metric

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$$d(x,y) = 0 \quad \text{if } x = y$$
$$d(x,y) = 1 \quad \text{if } x \neq y$$

Show that the discrete metric defines a metric space. **Solution:** Note that properties of (1)-(2)-(3), are satisfied by definition of the metric. To check property (4), select three generic points x, y, z and note that

1. If $x = y$ then $d(x,y) = 0 \leq d(x,z) + d(z,y)$
2. If $x \neq y$ it must be that
   - either $x \neq z \implies d(x,y) = 1 \leq 1 + d(z,y)$
   - or $y \neq z \longrightarrow d(x,y) = 1 \leq d(x,z) + 1$

Metric spaces: Standard metrics

Some standard metrics for a set $X \in \mathbb{R}^n$ are:

- The **taxi or block distance**: $d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|$
- The **Euclidean Distance**: $d_2(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$
- The **maximum distance**: $d_\infty = \underset{i}{max}|x_i - y_i|$
- For any real number $p \geq 1$, the **p-distance** is defined by: $d_p(x, y) = (\sum_{i=1}^{n}|x_i - y_i|^p)^{\frac{1}{p}}$

Example:

Example

Let $n = 2$, $y = (0,0)$ and $x = (x_1, x_2)$ be the points such that $d(x,y) = 1$. Represent them graphically under the following metrics:

1. $d(x,y) = d_1(x,y) = \sum_i |x_i - y_i|$
2. $d(x,y) = d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
3. $d(x,y) = d_\infty(x,y) = max_i |x_i - y_i|$

Note that using different metrics we find different locations of the points.

Notion of proximity: Open balls and neighborhood

Among other properties, metrics endow sets with a notion of proximity. That is, with a precise definition of what it means for points to be close together. Given any metric space (d, X) and any point $x_0 \in X$, the **open ball** of radius ε around x_0 is defined as

$$B_\varepsilon(x_0) = \{x \in X : d(x, x_0) < \varepsilon\}$$

Let (d, X) be a metric space. For all $x \in X$ and $\varepsilon \in \mathbb{R}_{++}$, the **neighborhood** of a point x_0 is

$$N_\varepsilon(x_0) = \{x \in X : d(x, x_0) < \varepsilon\}$$

Equivalently, $N_\varepsilon(x_0)$ is a neigborhood of x_0 if there exists an open ball such that $B_\varepsilon(x_0) \subseteq N_\varepsilon(x_0)$. Notice that the neighborhood on a point x_0 not only depends on x and but also (and especially) on the metric d and the space X.

Notion of proximity: Open sets

- Given any metric space (d, X), a set $S \subseteq X$ is said to be **open** or **open in** X if it is a neighborhood of all its points. That is, for all

$$\forall x \in S \quad \exists \varepsilon \text{ such that} N_\varepsilon(x) \subseteq S.$$

- By negation a set $s \in X$ is **not open** if it exists $x \in S$, for which it does not exist ε such that $N_\varepsilon(X_0) \subseteq S$.
- The **interior** of a set $S \subseteq X$, $int(S)$ is the largest open set contained in S. Equivalently, it can be defined as the union of all the open subsets of S.

Notion of proximity: Propositions for open sets

- A set S in (d, X) is open if and only if $S = int(S)$.
- Real intervals of the form (a, b) are open sets in the metric space $(d_2, \mathbb{R})$.
- The union $\cup_{i \in I} O_i$ of an arbitrary family of open intervals is open.
- the insersection $\cap_{i \in \{1,2,\dots,n\}} O_i$ of a finite family of intervals is open.

Notion of proximity: Closed sets

- Given any metric space (d, X), A set $S \subseteq X$ is **closed** or **closed in** X if its complement S^c is open.
- A point $x \in X$ is a boundary point of $S \subseteq X$ if every neighborhood of X contains points of S and points of S^c. The **boundary** of a set S, denoted $bn(S)$, is the set of all its boundary points, that is is the set of points which can be approached both from S and from the outside of S.
- The **closure** of a set S is the set and its boundary $\overline{S} = S \cup bn(S)$.

Notion of proximity: Propositions for closed sets

- A set S in a topological space (d, X) is closed if and only if the set S contains its boundary $bn(S)$.
- a set S in (d, X) is closed if and only if it coincides with its closure (i.e. $S = \overline{S}$).
- The set $S = [a, b] \subset \mathbb{R}$ is closed in $(d_2, \mathbb{R})$.
- the set $\{x\}$ is closed in $(d_2, \mathbb{R})$.

Open and closed sets

Notice that the notion of openness and closeness are not mutually exclusive: any set might be open, closed, both, or neither. In any topological space (d, X)

- the entire space is simultaneously open and closed (clopen).
- the empty set is simultaneously open and closed (clopen).

A **sequence** (x^n) in a set X is a function $x : \mathbb{N} \to X$ of finite or infinite *ordered* list of elements of X $(x^1, x^2, x^3, \ldots)$.
Given a sequence (x^n) in a metric space (d, X), (z^m) is a **subsequence** of (x^n) if there exists a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that for all m, $z^m = x^{f(m)}$.

Example

- $x^n = (1 + x, 1 + x + x^2, 1 + x + x^2 + x^3, \ldots)$
- $x^n = \frac{n-1}{n+1}$

Bounded sequence and convergence

- A sequence of real numbers (x^n) is said **bounded** if $\exists M \in \mathbb{R}$ for which every term x_n satisfies $|x_n| \leq M$.
- Given the metric space $(d, \mathbb{R})$, a sequence (x^n) is said to **converge** to a **limit point** $\overline{x} \in \mathbb{R}$ under the metric $d(x, y)$ if

$$\forall \varepsilon > 0, \exists \overline{n} \in \mathbb{N} : n > \overline{n} \longrightarrow d(x_n, \overline{x}) < \varepsilon$$

Equivalently, a sequence (x^n) converges of $\overline{x}$ if (x^n) eventually and forever after enters any neigborhood of $\overline{x}$. That is for any neigborhood N_ε of $\overline{x}$ there exists $\overline{n} \in \mathbb{N}$ such that for all $n > \overline{n}, x_n \in N_\varepsilon(\overline{x})$.

Example
$x^n = \frac{n-1}{n+1}$ is a convergent sequence.

Propositions and closed sets

some propositions for sequences:

- A Sequence x^n can converge to at most one limit.
- The sequence $x^n = (x_n^1, x_n^2, \dots x_n^k)$ converges to $x = (x^1, x^2, \dots x^k)$ in $(d_p, \mathbb{R})$ if and only if each series x_n^i converges to x_i in $(d, \mathbb{R})$.

Closed set (sequential definition): A set S is closed if and only if any convergent sequence formed by elements of S has limit in S. Otherwise, a set S is not closed if there exists at least one convergent sequence formed by elements of S that has a limit outside S.

Cauchy Sequence

A sequence (x^n) is called **Cauchy** if for all $\varepsilon > 0$, there exists $\overline{n}$, such that for all $k, l > \overline{n}$ implies $d(x_k, x_l) < \varepsilon$.

- **Proposition:** Every convergent sequence is Cauchy.

Proof.
Let x^n be a convergent sequence with limit point $\overline{x}$. Let $\varepsilon > 0$. Then there exists an $\overline{n} \in \mathbb{N}$ such that $d(x_n, \overline{x}) < \frac{\varepsilon}{2}$ for all $n \geq \overline{n}$. Thus, for all $k, l > \overline{n}$ it must be that:

$$d(x_k, x_l) \leq d(x_k, \overline{x}), + d(x_l, \overline{x}) < \varepsilon$$

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Completeness and Open Covers

- A set S is said to be **complete** if every Cauchy sequence of S converges to a point in S. Intuitively, a space is complete if there are no "points missing" from it (inside or at the boundary).
- Let (d, X) be a metric space and $S \subseteq X$. A class of subsets of X is said to **cover** S if $S \subseteq \cup_{i \in I} O_i$. If the sets O_i are open, then we say that O is an **open cover**.

Compactness

A metric space (d, X) is said to be **compact (total boundedness definition)** if every open cover has a finite subset that also covers X. A subset S of X is said to be compact in X if every open cover of S has a finite subset that also covers S.

A metric space (d, X) is said to be **compact (sequence definition)** if every sequence has a subsequence that converges to some element of X. a set $S \subseteq X$ is said to be compact if every sequence in S has a subsequence that converges to some limit in S.

Notice that if X is finite, then any sequence must have a convergent subsequence.

Propositions:

- the real space, $\mathbb{R}$, endowed with the Euclidean norm, is not a compact metric space.
- the interval $X = (0, 1)$ is not compact.

Notice that every compact metric space is both closed and bounded. However, closed and bounded sets are not necessarily compact for some metric spaces.

Theorem
(Heine-Borel's theorem) *Consider the metric space* $(d, \mathbb{R}^n)$ *where* $n \in \mathbb{N}$ *and* d *is a non-discrete metric function. Then, any subset* $S \subset \mathbb{R}^n$ *is compact if and only if it is closed and bounded.*

Boundedness

- Let (d, X) be a metric space. A subset $S \subseteq X$ is said to be bounded in X if $\exists \varepsilon > 0$ and for all $x \in S$ such that $S \subseteq N_\varepsilon(x)$.
- A set $S \subseteq \mathbb{R}$ is said **bounded from above** if $\exists k \in \mathbb{R}$ such that $k > s$ for all $s \in S$. Analogously, S is said **bounded from below** if $\exists k \in \mathbb{R}$ such that $k > s$ for all $s \in S$. A set S is bounded if it has both upper and lower bounds.
- Finite sets are bounded.

Supremum, Infimum

Suppose that S is a set of real numbers and is bounded above. Then, there is a number $M \in \mathbb{R}$ called **supremum**, **sup** or **least upper bound** such that:

1. M is a lower bound of S.
2. Given any $\varepsilon > 0$, there exists $s \in S$ such that $s < m + \varepsilon$.

Analogously, suppose S is bounded below. Then there is a number $m \in \mathbb{R}$ called **infimum**, **inf**, or **greatest lower bound** such that:

1. M is a lower bound of S.
2. Given any $\varepsilon > 0$, there exists $s \in S$ such that $s < m + \varepsilon$.

Maximum, minimum

The **maximum (minimum)** of a set S is its largest (smallest) element if such an element exists. If a set has a maximum (minimum), that is a supremum (infimum) for that set. The converse is true if and only if the supremum (infimum) belongs to the set. By construction, in any bounded set both the least upper bound and the greatest lower bound are well-defined and finite, though they might not belong to the set itself (for example if the set is open).