- $\mathbf{x} \in \mathbb{R}^+ \to \mathbb{R}^2$: Position at time t.
- $\mathbf{x_0} \in \mathbb{R}^2$: Initial position $\mathbf{x_0} = \mathbf{x}(0)$.
- $\mathbf{v_0} \in \mathbb{R}^2$: Initial velocity.
- $\mathbf{a} \in \mathbb{R}^2$: Constant acceleration.
- $\mathbf{d} \in \mathbb{R}^2$: Target.
- $\mathbf{M} \in \mathbb{R}^{2\times 2}$: Transformation matrix

$$\mathbf{M} = \begin{bmatrix}\mathbf{d} & \mathbf{v}_0\end{bmatrix}$$

- $\mathbf{e_k} \in \mathbb{R}^2$: Unit vector for dimension k

$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- $\alpha \in \mathbb{R}^2$: Acceleraction vector $\mathbf{a}$ transformed by $\mathbf{M}^{-1}$.

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$

We assume w.l.o.g that $\mathbf{x}_0 = \mathbf{0}$ and so

$$\mathbf{x}(t) = \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$

To find the $\mathbf{a}$ necessary to get to the target at time t we have

$$\mathbf{d} = \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$
$$\mathbf{a} = \frac{2}{t^2}(\mathbf{d} - \mathbf{v}_0 t)$$

We transform everything by $\mathbf{M}^{-1}$ to get

$$\mathbf{M}^{-1}\mathbf{d} = \mathbf{M}^{-1}\mathbf{v}_0 t + \frac{1}{2}\mathbf{M}^{-1}\mathbf{a}t^2$$
$$\mathbf{e_1} = \mathbf{e_2}t + \frac{1}{2}\alpha t^2$$

Solving this for α gives us

$$\alpha = 2\frac{\mathbf{e_1} - \mathbf{e_2}t}{t^2}$$
$$\alpha = \frac{2}{t^2}\begin{bmatrix} 1 \\ -t \end{bmatrix}$$

Attempt 2

The acceleration in a given dimension is described by the following differential equation

$$x''(t) = -\mu x'(t) + a$$

where μ is the friction coefficient and a is the constant acceleration applied by the creature. This differential equation has the solution

$$x'(t) = C\mathrm{e}^{-\mu t} + \frac{a}{\mu}$$

Using $x'(0) = v_0$ we get

$$C = v_0 - \frac{a}{\mu}$$

and so

$$\begin{aligned}
x'(t) &= (v_0 - \frac{a}{\mu})\mathrm{e}^{-\mu t} + \frac{a}{\mu} \\
x(t) &= x_0 + \int_0^t x'(s)\mathrm{d}s \\
&= x_0 + \int_0^t (v_0 - \frac{a}{\mu})\mathrm{e}^{-\mu t} + \frac{a}{\mu}\mathrm{d}s \\
&= x_0 + (v_0 - \frac{a}{\mu}) \int_0^t \mathrm{e}^{-\mu t}\mathrm{d}s + \int_0^t \frac{a}{\mu}\mathrm{d}s \\
&= x_0 + (v_0 - \frac{a}{\mu}) \left[-\frac{\mathrm{e}^{-\mu t}}{\mu}\right]_0^t + \frac{a}{\mu}t \\
&= x_0 + \frac{1}{\mu}(v_0 - \frac{a}{\mu}) \left[\mathrm{e}^{-\mu s}\right]_t^0 + \frac{a}{\mu}t \\
&= x_0 + \frac{1}{\mu}(v_0 - \frac{a}{\mu}) \left(1 - \mathrm{e}^{-\mu t}\right) + \frac{a}{\mu}t \\
&= x_0 + \frac{1}{\mu}(v_0 - \frac{a}{\mu}) \left(1 - \mathrm{e}^{-\mu t}\right) + \frac{a}{\mu}t \\
&= x_0 + \frac{v_0}{\mu} \left(1 - \mathrm{e}^{-\mu t}\right) + a\left(\frac{t}{\mu} - \frac{1}{\mu^2}\left(1 - \mathrm{e}^{-\mu t}\right)\right) \\
&= x_0 + v_0\alpha(t) + a\left(\frac{t}{\mu} + \frac{1}{\mu}\alpha(t)\right)
\end{aligned}$$

where $\alpha(t) = -\frac{1}{\mu}\left(1 - \mathrm{e}^{-\mu t}\right)$. Collecting both dimensions into vectors we get

$$\mathbf{x}(t) = \mathbf{x}_0 - \alpha(t)\mathbf{v}_0 + \frac{1}{\mu}\left(t + \alpha(t)\right)\mathbf{a}$$

Let $\mathbf{x}_1$ be the difference between the initial position $\mathbf{x}_0$ and the destination. Then we can solve for $\mathbf{a}$.

$$\begin{aligned}
\mathbf{x}_1 &= -\alpha(t)\mathbf{v}_0 + \frac{1}{\mu}\left(t + \alpha(t)\right)\mathbf{a} \\
\mathbf{a} &= \gamma(t)\left(\mathbf{x}_1 + \alpha(t)\mathbf{v}_0\right)
\end{aligned}$$

where $\gamma(t) = \frac{\mu}{t+\alpha(t)}$. The norm of $\mathbf{a}$ is then

$$||\mathbf{a}||^2 = \gamma(t)^2\left(||\mathbf{x}_1||^2 + \alpha(t)^2\,||\mathbf{v}_0||^2 + 2\alpha(t)\mathbf{x}_1\mathbf{v}_0\right)$$