Lecture 5

Measurements and instruments

How should we model measurements of an open quantum system? To answer this question, we will have to distinguish two problems:

1. What is the most general way to obtain a classical measurement statistics from a quantum system?

2. How does a general measurement process change the state of a quantum system?

While there is a straightforward way to answer the first question in the open system formalism, the answer to the second question is sometimes less clear. In general, we have to distinguish between *destructive measurements*, where we only obtain classical data, and *non-destructive measurements*, where we also determine how the measurement process changes the quantum state of the system. For destructive measurement we can assume that the quantum system itself got destroyed in the measurement process, or at least that we do not care about it any longer. This will usually lead to an easier mathematical description and there are many cases of physical measurements of quantum systems, e.g., using photon detectors, where we the quantum system is indeed destroyed during the measurement process.

5.1 Destructive measurements

Let us first introduce a specific kind of measure, which is used to model measurements in quantum theory.

Definition 5.1 (POVM). *Let Σ denote a finite alphabet. A function*

$$\mu : \Sigma \to B\left(\mathcal{H}\right)^+$$

is called a positive operator-valued measure *(POVM) if*

$$\sum_{x\in\Sigma} \mu(x) = \mathbb{1}_{\mathcal{H}}.$$

With the notion of a POVM, we can state a postulate of quantum theory that specifies destructive measurements.

Postulate 4 (Destructive measurements). *Consider a quantum system with state space $\mathcal{H}$ in a quantum state $\rho \in D(\mathcal{H})$. For any measurement with outcomes in a finite alphabet Σ, there exists a POVM $\mu : \Sigma \to B\left(\mathcal{H}\right)^+$ such that the probability of measuring outcome $x \in \Sigma$ is*

$$p(x) = \mathrm{Tr}\left[\mu(x)\rho\right].$$

It should be noted that the properties of POVMs guarantee that the numbers $p(x)$ form a probability distribution. Indeed, since $\mu(x)$ is positive semidefinite, we find that $p(x) \geq 0$ for any $x \in \Sigma$, and we also have

$$\sum_{x\in\Sigma} p(x) = \sum_{x\in\Sigma} \mathrm{Tr}\left[\mu(x)\rho\right] = \mathrm{Tr}\left[\rho\right] = 1.$$

It is straightforward to generalize the definition of POVMs to infinite sets of outcomes by using elementary measure theory. However, since we will not need such complications in this course, we will restrict to measurements with a finite number of outcomes.

We should mention a few special cases of POVMs that you might encounter in basic courses on quantum mechanics.

Definition 5.2. *A POVM $\mu : \Sigma \to B(\mathcal{H})^+$ is called a*

- projection-valued measure *(PVM) if the operators $\mu(x)$ are projections.*
- von-Neumann measurement *if we have*

$$\mu(x) = |v_x\rangle\langle v_x|$$

for some orthonormal basis $\{|v_x\rangle\}_{x\in\Sigma}$ of $\mathcal{H}$.

Projection-valued measures are sometimes called *projective measurements* in the literature. If $\{|v_x\rangle\}_{x\in\Sigma}$ is an orthonormal basis of $\mathcal{H}$, then we sometimes say that we *measure a system with respect to the basis* $\{|v_x\rangle\}_{x\in\Sigma}$. By this we mean that we employ the von-Neumann measurement with respect to the basis $\{|v_x\rangle\}_{x\in\Sigma}$. It is common in courses on quantum mechanics that von-Neumann measurements are defined with respect to the eigenbasis of some selfadjoint operator. Such an operator is called an *observable*, and *measuring an observable* again means to employ the corresponding von-Neumann measurement.

It is clear from the definition, that a von-Neumann measurement $\mu : \Sigma \to B(\mathcal{H})^+$ has as many measurement outcomes as the dimension of the complex Euclidean space $\mathcal{H}$. It turns out that the dimension of $\mathcal{H}$ is an upper bound on the number of measurement outcomes of any projective measurement:

Theorem 5.3. *Any projection-valued measure $\mu : \Sigma \to B(\mathcal{H})^+$ satisfies*

$$\mu(x) \perp \mu(y), \ \textit{whenever } x \neq y.$$

In particular, we have $|\Sigma| \leq \dim(\mathcal{H})$.

Proof. Exercises. □

In contrast to projection-valued measures, there is no upper bound on the number of measurement outcomes of a general POVM $\mu : \Sigma \to B(\mathcal{H})^+$. However, we will later see that there is an upper bound for extremal POVMs, which will have important consequences for the amount of classical information that can be stored in a quantum system.

Projection-valued measures and von-Neumann measurements are used to model measurements in the closed system formalism, i.e., the formalism that you might have learned in your first quantum mechanics class. Again, we can implement general POVMs using projection-valued measures on some suitable larger system. The following theorem can be seen as an analogue of the Stinespring dilation we have seen for quantum channels:

Theorem 5.4 (Naimark dilation)**.** *For any POVM $\mu : \Sigma \to B(\mathcal{H})^+$ and any orthonormal basis $\{|v_x\rangle\}_{x\in\Sigma}$ of $\mathbb{C}^\Sigma$, there exists an isometry $V : \mathcal{H} \to \mathcal{H} \otimes \mathbb{C}^\Sigma$ such that*

$$\mu(x) = V^\dagger \left(\mathbb{1}_{\mathcal{H}} \otimes |v_x\rangle\langle v_x|\right) V,$$

for any $x \in \Sigma$.

Proof. We define $V : \mathcal{H} \to \mathcal{H} \otimes \mathbb{C}^\Sigma$ as

$$V = \sum_{x\in\Sigma} \sqrt{\mu(x)} \otimes |v_x\rangle,$$

and note that

$$V^\dagger V = \sum_{x\in\Sigma} \mu(x) = \mathbb{1}_{\mathcal{H}}.$$

Hence, V is an isometry. Finally, we note that

$$V^\dagger \left(\mathbb{1}_{\mathcal{H}} \otimes |v_x\rangle\langle v_x|\right) V = \mu(x),$$

for any $x \in \Sigma$. □

Recall that the Stinespring dilation of quantum channels implied that they can be implemented by unitary time-evolutions. The analogous result for POVMs is the following theorem, which implements a given POVM by a PVM.

Theorem 5.5 (Environment induced measurements)**.** *Consider a POVM $\mu : \Sigma \to B(\mathcal{H})^+$ and some pure state $|\psi_E\rangle\langle\psi_E| \in \mathrm{Proj}\left(\mathbb{C}^\Sigma\right)$. Then, there exists a PVM $\nu : \Sigma \to B\left(\mathcal{H} \otimes \mathbb{C}^\Sigma\right)^+$ such that*

$$\mathrm{Tr}\left[\mu(x)\rho\right] = \mathrm{Tr}\left[\nu(x)\left(\rho \otimes |\psi_E\rangle\langle\psi_E|\right)\right],$$

for any $x \in \Sigma$ and any $\rho \in D(\mathcal{H})$.

Proof. By Theorem 5.4, we can consider the Naimark dilation $V : \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_E$ such that

$$\mu(x) = V^\dagger \left(\mathbb{1}_{\mathcal{H}} \otimes |v_x\rangle\langle v_x|\right) V,$$

for any $x \in \Sigma$. Let $U \in \mathcal{U}\left(\mathcal{H}_A \otimes \mathbb{C}^\Sigma\right)$ denote some unitary operator such that

$$U(\mathbb{1}_{\mathcal{H}} \otimes |\psi_E\rangle) = V,$$

and define $\nu : \Sigma \to B(\mathcal{H})^+$ by

$$\nu(x) = U^\dagger \left(\mathbb{1}_{\mathcal{H}} \otimes |v_x\rangle\langle v_x|\right) U,$$

for every $x \in \Sigma$. This defines a PVM, and we note that

$$\begin{aligned}
\mathrm{Tr}\left[\nu(x)\left(\rho \otimes |\psi_E\rangle\langle\psi_E|\right)\right] &= \mathrm{Tr}\left[\left(\mathbb{1}_{\mathcal{H}} \otimes |v_x\rangle\langle v_x|\right) U \left(\rho \otimes |\psi_E\rangle\langle\psi_E|\right) U^\dagger\right] \\
&= \mathrm{Tr}\left[\left(\mathbb{1}_{\mathcal{H}} \otimes |v_x\rangle\langle v_x|\right) V\rho V^\dagger\right] \\
&= \mathrm{Tr}\left[\mu(x)\rho\right].
\end{aligned}$$

□

5.2 Non-destructive measurements

In the following, let us denote by $\mathrm{CP}\left(\mathcal{H}_A, \mathcal{H}_B\right)$ the set of all completely positive maps from $B(\mathcal{H}_A)$ to $B(\mathcal{H}_B)$. Then, we make the following definition:

Definition 5.6 (Instrument)**.** *Let Σ denote a finite alphabet. A function*

$$\Sigma \ni x \mapsto T_x \in CP\left(\mathcal{H}_A, \mathcal{H}_B\right)$$

is called an instrument *if the linear map $\sum_{x\in\Sigma} T_x$ is a quantum channel. We will usually specify an instrument by the set $\{T_x\}_{x\in\Sigma}$ of linear maps associated to the different symbols in $x \in \Sigma$.*

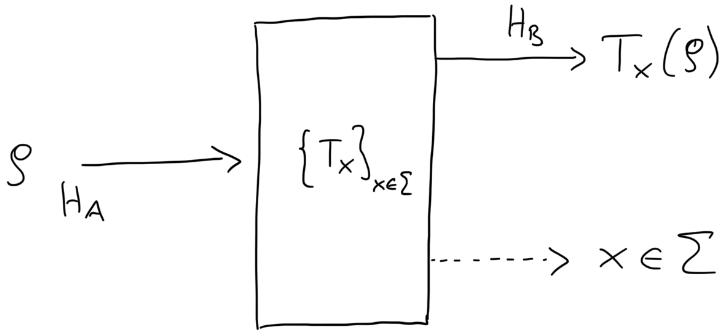

Figure 5.1: *An instrument.*

Instruments can be used to model any type of physical process that has a classical and a quantum output, e.g., a non-destructive measurement where we receive a classical measurement result and the quantum system is not destroyed. See Figure 5.1 for a schematic. We have the following postulate:

Postulate 5 (Non-destructive measurements)**.** *Consider a quantum system with state space $\mathcal{H}_A$ in a quantum state $\rho \in D(\mathcal{H}_A)$. For any physical process that performs a measurement with outcomes in a finite alphabet Σ on the system and transforms it into some quantum system with state space $\mathcal{H}_B$, there exists an instrument $\{T_x\}_{x\in\Sigma} \subset CP(\mathcal{H}_A, \mathcal{H}_B)$ such that the probability of measuring outcome $x \in \Sigma$ is*

$$p(x) = \operatorname{Tr}\left[T_x(\rho)\right],$$

and after $x \in \Sigma$ has been measured the final quantum state is given by

$$\sigma_{post}(x) = \frac{T_x(\rho)}{\operatorname{Tr}\left[T_x(\rho)\right]}.$$

The following example shows that any POVM can describe the classical outcome of an instrument:

Example 5.7 (Instrument for a given POVM)**.** *Consider a POVM $\mu : \Sigma \to B(\mathcal{H})^+$ and decompose each operator as*

$$\mu(x) = A_x^\dagger A_x,$$

for some $A_x \in B(\mathcal{H})$. Then, we can define an instrument $\{T_x\}_{x\in\Sigma}$ by setting

$$T_x(X) = A_x X A_x^\dagger,$$

for any $X \in B(\mathcal{H})$. Clearly, each linear map T_x is completely positive, and we can verify that

$$\sum_{x\in\Sigma} T_x^*(\mathbb{1}_\mathcal{H}) = \sum_{x\in\Sigma} A_x^\dagger A_x = \sum_{x\in\Sigma} \mu(x) = \mathbb{1}_\mathcal{H}.$$

We conclude that $\{T_x\}_{x\in\Sigma}$ defines an instrument, and for every $\rho \in D(\mathcal{H})$ we observe classical outcome $x \in \Sigma$ with probability

$$p(x) = \operatorname{Tr}\left[T_x(\rho)\right] = \operatorname{Tr}\left[A_x \rho A_x^\dagger\right] = \operatorname{Tr}\left[A_x^\dagger A_x \rho\right] = \operatorname{Tr}\left[\mu(x)\rho\right].$$

We obtain the same probability distribution as for the destructive measurement using the POVM μ. ∘

The instrument from the example is often used as the *standard instrument* describing the post-measurement state of a von-Neumann measurement $\mu : \Sigma \to B(\mathcal{H})^+$, i.e., where $\mu(x) = |v_x\rangle\langle v_x|$ is a rank-1 projection. Here, we can define

$$T_x(X) = |v_x\rangle\langle v_x|X|v_x\rangle\langle v_x| = \langle v_x|X|v_x\rangle|v_x\rangle\langle v_x|,$$

for any $x \in \Sigma$. The post-measurement state after measuring outcome $x \in \Sigma$ is

$$\sigma_{post}(x) = |v_x\rangle\langle v_x|,$$

which is the pure state corresponding to the measurement outcome. You might have seen this definition in a class on basic quantum mechanics.

Note that instruments are not uniquely determined by the POVM μ describing their classical output. In fact, for any instrument $\{T_x\}_{x\in\Sigma}$ we could define another instrument which also reproduces the same POVM by composing the completely positive maps T_x with a quantum channel S (which may even depend on x) as depicted in Figure 5.2.

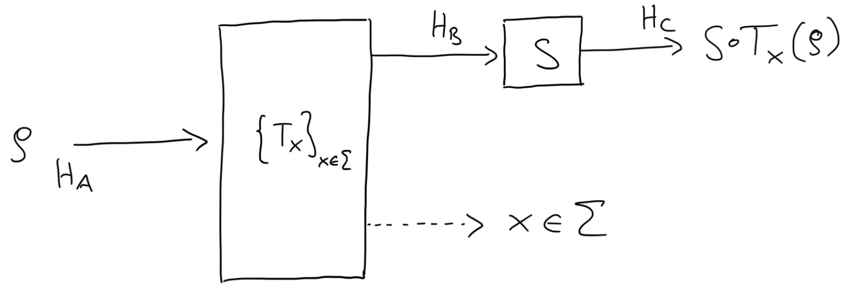

Figure 5.2: *Postprocessing of an instrument.*

Physically, this just means that we can transform the quantum outcome of the instrument using any quantum process without changing the classical outcome. This is not surprising! For von-Neumann measurements this is the only freedom as shown by the next theorem:

Theorem 5.8. *Let $\mu : \Sigma \to B(\mathcal{H})^+$ denote a von-Neumann measurement with $\mu(x) = |v_x\rangle\langle v_x|$. If an instrument $\{T_x\}_{x\in\Sigma} \subset CP(\mathcal{H}_A, \mathcal{H}_B)$ satisfies*

$$\text{Tr}\left[T_x(\rho)\right] = \langle v_x|\rho|v_x\rangle,$$

for every $x \in \Sigma$ and every $\rho \in D(\mathcal{H}_A)$, then there exist quantum states $\{\sigma_x\}_{x\in\Sigma}$ such that

$$T_x(X) = \langle v_x|X|v_x\rangle\sigma_x,$$

for every $X \in B(\mathcal{H}_A)$ and every $x \in \Sigma$.

Proof. Consider an instrument $\{T_x\}_{x\in\Sigma} \subset \text{CP}\left(\mathcal{H}_A, \mathcal{H}_B\right)$ satisfying the assumption from the theorem. Note that any operator $X \in B(\mathcal{H})$ can be written as a linear combination of quantum states. Therefore, we have that

$$\langle T_x^*(\mathbb{1}_{\mathcal{H}_B}), X\rangle_{HS} = \text{Tr}\left[T_x^*(\mathbb{1}_{\mathcal{H}_B})^\dagger X\right] = \text{Tr}\left[T_x(X)\right] = \langle v_x|X|v_x\rangle = \langle |v_x\rangle\langle v_x|, X\rangle_{HS},$$

for every $X \in B(\mathcal{H})$. We conclude that

$$T_x^*(\mathbb{1}_{\mathcal{H}_B}) = |v_x\rangle\langle v_x|.$$

Let us fix $x \in \Sigma$ and let $K_i \in B(\mathcal{H}_A, \mathcal{H}_B)$ for $\{1, \ldots, N\}$ denote the non-zero Kraus operators of the completely positive map T_x. Then, we have

$$\sum_{i=1}^{N} K_i^\dagger K_i = |v_x\rangle\langle v_x|,$$

and hence that $K_i = |w_i^x\rangle\langle v_x|$ for some vectors $|w_i^x\rangle \in \mathcal{H}_A$ satisfying

$$\sum_{i=1}^{N} \langle w_i^x | w_i^x \rangle = 1.$$

We conclude that

$$T_x(X) = \sum_{i=1}^{N} \langle v_x | X | v_x \rangle |w_i^x\rangle\langle w_i^x| = \langle v_x | X | v_x \rangle \sigma_x,$$

for some quantum state $\sigma_x \in D(\mathcal{H}_B)$. $\square$