Logical implications between fundamental properties of relativistic quantum theories

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April 5, 2024

Abstract

In order to clarify the issues concerning the compatibility of quantum mechanics with special relativity, a consistency condition constraining any relativistic quantum theory is formulated. It turns out to be equivalent to the locality of physics as well as, in the context of quantum field theory, microcausality, thereby revealing that these are actually two redundant hypotheses. It also promotes an epistemic interpretation of the wavefunction collapse and provides a new proof of the non-measurability of fermionic fields. Retracing in detail the history of this topic reveals the essential contribution of quantum information theory to the foundations of quantum mechanics.

Keywords: relativistic quantum mechanics, quantum information theory, causality, locality, microcausality.

Acknowledgements

I would like to gratefully thank my PhD supervisor Dimitri Petritis for the great freedom he grants me in my research, while being nevertheless always present to guide me. I also thank my friend Dmitry Chernyak for careful rereading and sharp remarks.

Statements and Declarations

Funding: no funding was received for conducting this study.
Competing Interests: the author has no competing interests to declare that are relevant to the content of this article.
Ethics approval, Consent, Data, Materials and code availability, Authors' contribution statements: not applicable.

1 Introduction

The aim of this paper is to clarify and present rigorously a lot of well-known, but sometimes poorly presented, facts about relativistic quantum theories. Most results are not new, but the perspective is. In so doing, unexpected questions and conclusions will emerge, as well as a better understanding of the interrelationships between the fundamental properties of these theories.

Standard quantum field theory (QFT) textbooks usually focus on constructing the quantum fields and the dynamics in the Fock space, but are rarely concerned with issues relative to measurements, as though it were a purely non-relativistic topic. More generally, most of them don't check the compatibility of quantum mechanics (QM) with the disappearance of the notion of instantaneity in special relativity, and there have been surprisingly few works addressing this problem until the 2000s, when quantum information theory entered the game.

Although this paper is not directly concerned with the measurement problem, we will sometimes get close to interpretational issues. This is why the annex §A is dedicated to some considerations on the status of the wavefunction collapse in QM. The interested reader could read it first, though it can also be skipped, since most of the results obtained in this paper are independent of any particular interpretation.

In section §2, two possible sources of instantaneities in QM are exhibited but, in the light of §A, only one of them is *a priori* problematic. To deal with the latter, a consistency condition called (C), constraining the statistics of measurements outcomes in any relativistic quantum theory, is formulated and is shown to be necessary to prevent faster-than-light communication and non-covariance (§2.1). This condition takes a precise mathematical form, known as the no-communication theorem, which can be derived under some (generally accepted) assumption on the unitary evolution operator of two isolated systems. (C) being satisfied allows, and even fosters, an epistemic interpretation of the wavefunction collapse, seen as a non-physical process (§2.2). We extensively retrace the history of this topic and compare our treatment with the existing literature in §2.3.

At this point, we try to go further than the previous works by embracing a more axiomatic point of view. Rather than considering it as a theorem, we argue that (C) is the truly fundamental postulate of relativistic quantum theories because it is required by special relativity. We then look for the complete logical structure between (C) and the previously introduced properties, as well as with the locality of physics on one hand (section §3) and with microcausality and the spin-statistics theorem in the context of QFT on the other hand (section §4). An important result is Theorem 3.1, which extends a previously known result [4, Theorem 7] to the infinite dimensional (separable) case. Our study will also reveal rather unexpectedly that locality and microcausality are actually two redundant hypotheses of QFT; it will also provide us with an original proof that the Dirac field can in no sense be measured.

2 The conditions (C) and (MC)

2.1 Literal and mathematical formulation of the conditions

There are two different sources of instantaneity in QM that could cause troubles when trying to build a relativistic quantum theory. When two subsystems are entangled, they must be considered as a whole[1], therefore:

- any physical evolution (for instance a non-selective measurement (6)) on the first instantaneously affects the whole state no matter how far the other part may be,
- any measurement performed on the first allows the observer to apply a collapse (5) on the whole state, which is generally presented as an instantaneous update.

However, according to the understanding of the collapse defended in §A.2, the second item is not problematic at all, because the epistemic update of one particular observer has no reason to be constrained by the speed of light. As Bell wrote [7]: 'When the Queen dies in London (may it long be delayed) the Prince of Wales, lecturing on modern architecture in Australia, becomes instantaneously King', and he could have added: from the Queen and her entourage's point of view. On the other hand, what has to be

[1]It has been experimentally confirmed that non-local correlations are satisfied (almost) immediately: in [35], the authors present an incredibly precise experimental test which has allowed them to 'set a lower bound on the speed on quantum information to $10^7 c$, *i.e.* seven orders of magnitude larger than the speed of light.'

constrained by special relativity are the physically predictable effects, so that no experiment conducted on the King can determine faster than light whether the Queen is alive. For this to be prevented in QM despite the non-locality of the entanglement phenomenon, the following consistency condition has to hold:

(C) *For all quantum systems composed of two entangled subsystems, any physical evolution of the first must leave invariant* ***the statistical results*** *of any measurement on the second, if the two are spacelike separated.*

In the sequel, we will also consider the following more specific consistency condition (dealing only with measurements), that will subsequently be referred to as the condition (MC), because it is easier to manipulate mathematically and will turn out to be equivalent to (C):

(MC) *For all quantum systems composed of two entangled subsystems, any ideal projective measurement of the first must leave invariant* ***the statistical results*** *of any measurement on the second, if the two measurements are spacelike separated.*

The letter (C) may stand for 'consistency condition', but also 'causality' and 'covariance', which are two different things. Indeed, if (MC) were not satisfied (*a fortiori* (C)), the theory would face two types of inconsistencies:

- *Non-covariance* — Consider two entangled quantum systems that violate the condition (MC). Then there exists an experimental protocol concerning the second system that yields different statistical results depending on whether a certain measurement has been performed on the first system or not, such that the two measurements are spacelike separated. Thus one can find a reference frame in which the measurement on the first system happens *before* the other measurement, and another reference frame in which it happens *after*. Consequently, the statistical predictions of the theory depend on the reference frame.
- *Causal paradoxes* — Consider a reference frame in which Alice and Bob are apart, motionless, and share N entangled pairs that violate the condition (MC) as well as two synchronized clocks. At $t = 0$, Alice performs the suitable measurement on each of the N subsystems if she wants to communicate the bit 0, or do nothing if she wants to communicate the bit 1. At $t = 0^+$, Bob performs the corresponding measurement on each of the subsystems in his possession: the statistical distribution he obtains allows him to distinguish if Alice has sent the bit 0 or 1, with an error margin arbitrarily small when N goes larger. It is well known that such faster-than-light communications can induce causal loops like in the grandfather paradox.

The condition (MC) can be given a precise mathematical formulation. Let $\mathcal{S}_1 + \mathcal{S}_2$ be two entangled systems described by a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, prepared in a state ρ. Let also $\hat{A}$ (resp. $\hat{B}$) be an observable of $\mathcal{H}_1$ (resp. $\mathcal{H}_2$). **In this paper, we will only consider separable Hilbert spaces and observables that are compact hermitian operators** (stronger conditions could be formulated, but at least this weaker condition must hold), so that we can use their spectral decomposition $\hat{A} = \sum_{x\in\mathrm{spec}\hat{A}} x\Pi_x^{(1)}$ (resp. $\hat{B} = \sum_{y\in\mathrm{spec}\hat{B}} y\Pi_y^{(2)}$) where the $\Pi_x^{(1)}$ (resp. $\Pi_y^{(2)}$) are the spectral projectors of the observable. If an ideal projective measurement of A is performed on $\mathcal{S}_1$, the whole state evolves to $\sum_{x\in\mathrm{spec}\hat{A}}(\Pi_x^{(1)} \otimes \mathbb{1}_2)\rho(\Pi_x^{(1)} \otimes \mathbb{1}_2)$ (non-selective operation (6)). After this, in a given reference frame, the system evolves according to a unitary operator U (see Fig. 1). Since the state of $\mathcal{S}_2$, obtained by tracing over $\mathcal{S}_1$, fully characterizes the probabilities of any measurement on $\mathcal{S}_2$, the invariance of the statistics of $\mathcal{S}_2$ under the measurement of $\mathcal{S}_1$ is expressed by:

Mathematical formulation of (MC)

(MC) $$\forall\rho, \forall U, \forall\hat{A}, \quad \mathrm{tr}_1\left(\sum_{x\in\mathrm{spec}\hat{A}} U(\Pi_x^{(1)} \otimes \mathbb{1}_2)\rho(\Pi_x^{(1)} \otimes \mathbb{1}_2)U^\dagger\right) = \mathrm{tr}_1(U\rho U^\dagger).$$

Or, equivalently, due to the universal property of the partial trace:

$$\forall \rho, \forall U, \forall \hat{A}, \forall \hat{B}, \forall y_0 \in \mathrm{spec}\hat{B},$$
$$\sum_{x \in \mathrm{spec}\hat{A}} \mathrm{tr}\left(U(\Pi_x^{(1)} \otimes \mathbb{1}_2)\rho(\Pi_x^{(1)} \otimes \mathbb{1}_2)U^\dagger(\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)})\right) = \mathrm{tr}\left(U\rho U^\dagger(\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)})\right). \tag{1}$$

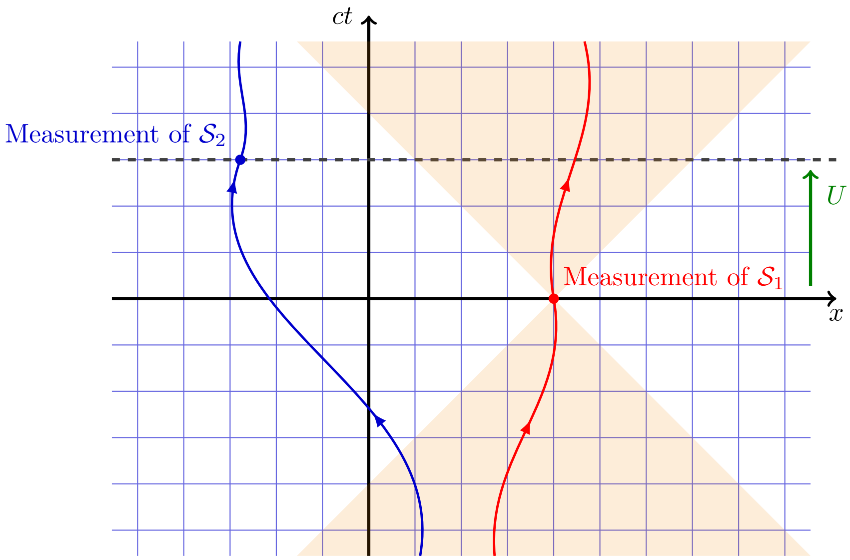

Figure 1: Spacelike measurements

Concerning (C), the most general physical evolution that $\mathcal{S}_1$ may undergo is a unitary V in $\mathcal{H}_1 \otimes \mathcal{H}_\mathcal{E}$ (possibly non-trivial only on $\mathcal{H}_1$), where $\mathcal{E}$ stands for any external third system. It is not restrictive to consider them initially non-entangled, hence in an initial state $\rho \otimes \rho_\mathcal{E} \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{S}(\mathcal{H}_\mathcal{E})$, up to integrating some initial entanglement in the unitary V. Afterwards, $\mathcal{S}_1 + \mathcal{S}_2$ evolves, as previously, according to a unitary U in $\mathcal{H}_1 \otimes \mathcal{H}_2$. For the theory to be consistent, whether V has been applied or not must not modify the statistics of $\mathcal{S}_2$, hence the following condition:

Mathematical formulation of (C)

(C) $$\forall \rho, \forall \rho_\mathcal{E}, \forall U, \forall V, \quad \mathrm{tr}_1\left(U\, \mathrm{tr}_\mathcal{E}\left[(V \otimes \mathbb{1}_2)(\rho_\mathcal{E} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2)\right]U^\dagger\right) = \mathrm{tr}_1\left(U\rho U^\dagger\right).$$

Or, equivalently, due to the universal property of the partial trace:

$$\forall \rho, \forall \rho_\mathcal{E}, \forall U, \forall V, \forall \hat{B}, \forall y_0 \in \mathrm{spec}\hat{B},$$
$$\mathrm{tr}\left(U\, \mathrm{tr}_\mathcal{E}\left[(V \otimes \mathbb{1}_2)(\rho_\mathcal{E} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2)\right]U^\dagger(\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)})\right) = \mathrm{tr}\left(U\rho U^\dagger(\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)})\right).$$

Clearly, when $\mathcal{E}$ is a measurement apparatus that causes perfect and immediate decoherence of $\mathcal{S}_1$ in the eigenbasis of $\hat{A}$, then $\mathrm{tr}_\mathcal{E}\left[(V \otimes \mathbb{1}_2)(\rho_\mathcal{E} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2)\right] = \sum_{x \in \mathrm{spec}\hat{A}}(\Pi_x^{(1)} \otimes \mathbb{1}_2)\rho(\Pi_x^{(1)} \otimes \mathbb{1}_2)$ (recall that perfect decoherence in a given basis amounts, by definition, to extracting the diagonal part of the density matrix in this basis) and we recover the previous condition (MC). Therefore **(C) implies (MC)**. As noticed in [4], it is equivalent to check (C) or (MC) only on initial states of the form $\rho = \rho_1 \otimes \rho_2$ (product states).

Remark 2.1. on the unavoidable probabilistic nature of QM. In the literal formulations of (C) and (MC) given above, we have highlighted the fact that these conditions only constrain the statistical results of measurements. This is because special relativity only imposes the covariance of what is predictable by a given theory; and QM, being a probabilistic theory, only predicts the statistics. Similarly, the ability to transmit information depends on the best theory available to the communicators. If they can't do better than probabilistic predictions, Alice has to be able to modify the statistical results of a repeated experiment on Bob's side in order to send him a bit.

Alternatively, if they had a deterministic (hidden-variable) theory superseding QM, only one run could suffice. Such a theory should then be constrained by a stronger version of (C) where the word 'statistical' is removed. But this is precisely forbidden by Bell's theorem [6] which implies that such a theory would necessarily display non-local features. More precisely, either the outcome or parameter independence assumption would have to be violated [23], allowing in any case for superluminal signalling and covariance issues. See also [16] for a simple analytic example of non-locality in Bohmian mechanics compared to the standard QM treatment, where it is made clear that no probabilistic knowledge other than the one given by the wavefunction is compatible with special relativity. Quantum entanglement seemed spooky to Einstein because of the conviction that nothing in the physical universe could in principle be unpredictable, *i.e.* inaccessible to human physics,[2] but it is actually the contrary: in a world where entanglement exists, QM is the only non-spooky theory! This remark proves that we will never be able to build a deterministic theory supplanting QM (at least not without a radical change in our physicists' paradigms). This is also why the terms added to the Schrödinger equation in collapse models will always be stochastic, 'because otherwise [they] would allow for faster-than-light communication' [3].

2.2 (F) implies (C)

Let's now check that the condition (C) (*a fortiori* (MC)) is indeed satisfied in any relativistic quantum theory provided it satisfies the following factorization property (F), generally accepted in QM:

(F) *For all pairs of isolated systems*[3] $\mathcal{S}_1$ *and* $\mathcal{S}_2$*, the unitary evolution operator of* $\mathcal{S}_1 + \mathcal{S}_2$ *takes the factorized form* $U = U_1 \otimes U_2$*.*

Let's keep the notations introduced in §2.1. It is clear that the systems are isolated between the two measurements, since they are spacelike separated; therefore, assuming (F), we may write $U = U_1 \otimes U_2$ and:

$$\begin{aligned}
\mathbb{P}(B = y_0) &= \operatorname{tr}\Big(U \operatorname{tr}_{\mathcal{E}}\Big[(V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2)\Big] U^\dagger (\mathbb{1}_1 \otimes \Pi^{(2)}_{y_0})\Big) \\
&= \operatorname{tr}\Big(\operatorname{tr}_{\mathcal{E}}\Big[(V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2)\Big](\mathbb{1}_1 \otimes U_2^\dagger \Pi^{(2)}_{y_0} U_2)\Big) \\
&= \operatorname{tr}\Big((V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2)(\mathbb{1}_{\mathcal{E}} \otimes \mathbb{1}_1 \otimes U_2^\dagger \Pi^{(2)}_{y_0} U_2)\Big) \\
&= \operatorname{tr}\Big(\rho_{\mathcal{E}} \otimes (\rho U_2^\dagger \Pi^{(2)}_{y_0} U_2)\Big) \\
&= \operatorname{tr}\Big(U \rho U^\dagger (\mathbb{1}_1 \otimes \Pi^{(2)}_{y_0})\Big)
\end{aligned}$$

where we have used the fact that $\operatorname{tr}(\rho_{\mathcal{E}}) = 1$ in the last step, and the property of the partial trace $\operatorname{tr}(\operatorname{tr}_1(A_{12})B_2) = \operatorname{tr}(A_{12}(\mathbb{1}_1 \otimes B_2))$ in the third step. [To prove this equality, it is enough by linearity to check it for pure tensors $A_{12} = A_1 \otimes A_2$; it then simply reads: $\operatorname{tr}(\operatorname{tr}_1(A_1 \otimes A_2)B_2) = \operatorname{tr}(\operatorname{tr}(A_1)A_2B_2) = \operatorname{tr}(A_1)\operatorname{tr}(A_2B_2) = \operatorname{tr}(A_1 \otimes A_2B_2) = \operatorname{tr}((A_1 \otimes A_2)(\mathbb{1}_1 \otimes B_2))$.] Thus **(F) implies (C)**.

We have therefore established that any relativistic quantum theory which satisfies (F) is consistent with an instantaneous effect of physical interactions on an entangled state, even though the notion of

[2]This metaphysical stance is exemplified in his famous 'God does not play dice'; indeed, note how positivist this remark is: it is not that God necessarily plays dice, but only that the human mind may not be able to access better than statistically the way God plays.

[3]We say that two systems are isolated if no particles of $\mathcal{S}_1$ and $\mathcal{S}_2$ can meet each other. The possibility to write U as $U_1 \otimes U_2$ is sometimes taken as a definition for being isolated, or we could have replaced 'isolated' by 'spacelike separated' in all this section §2, but our choice will become clear when discussing locality in §3. Of course, the main difficulty with our definition is that the concept of particle is ill-defined. In particular, Unruh [33] has showed that the very presence of a particle is frame-dependent. Also, should we consider virtual particles? How to include gravitational interactions in our treatment if the graviton actually doesn't exist?

instantaneity is frame dependent. Different observers may write different states for the entangled pair, but they agree on the statistics. Accepting that there is one wavefunction per observer (for a probability distribution always depends on what is known, recall especially note 10 in §A.2 and Bell's example in §2.1) solves a lot a spooky problems, because then the wavefunction doesn't have to be covariant, only the physics has to. Also, what precedes constitutes the proof that an entangled pair cannot be used to convey information in QM, known as the no-communication theorem.

Remark 2.2. Rigorously speaking, the above computation suffices to exclude the possibility of faster-than-light communication, but lacks an additional argument to ensure covariance. See annex B for the details.

2.3 Previous works

The above verification of (C) is straightforward but essential to establish the compatibility between QM and special relativity. Of course, some previous works have already investigated these topics, although remarkably few. Bloch (1967) [8] pointed out some apparent inconsistencies and new intuitions, notably: 'it appears that either causality or Lorentz covariance of wavefunctions must be sacrificed'. In reply, Hellwig and Kraus (1970) [17] published a paper in which they clarified Bloch's ideas, checked a simple version (MC) (equation (6)) based on an assumption called locality (commutation of projectors associated to spacelike measurements), and then proposed that the effects of measurements (selective and non-selective) should be implemented along the past light cone so that their description becomes covariant. A general proof of (MC), presented as a no-communication theorem, was detailed by Ghirardi, Rimini and Weber (1980) in [15]. The following year, Aharonov and Albert (1981) [1] explained how non-local measurements can actually be implemented based on local interactions only. As a consequence, they claim that 'the proposal that the reduction be taken to occur covariantly along the backward light cone (...) or along *any* hypersurface other than $t = 0$ will fail' (although we must confess that we didn't understand their argument), and that 'the covariance of relativistic quantum theories (...) resides *exclusively* in the experimental probabilities, and not in the underlying quantum states. The states themselves make sense only within a given frame'. In a long footnote, Malament (1996) [21] actually proved the logical equivalence between a simple version of (MC) and the fact that spacelike measurements projectors commute, embracing a more logician perspective than his predecessors. Surprisingly enough, all these works were concerned with (MC), which is after all a very artificial statement compared to (C). The reason may be that these studies were sparkled by foundational considerations on the wavefunction collapse and on the measurement problem; in addition, the theory of decoherence was still in its infancy, so that a measurement was less easily understood as a mere particular case of entangling interaction.

Then came the quantum information era. Beckman, Gottesman, Nielsen and Preskill's article (2001) [4] was a milestone: the two systems were now called Alice and Bob's parts, the evolution operators were renamed quantum operations, and a distinction between operations which are causal (not allowing communication) and localizable (realizable with local unitaries) was introduced. As explained in the introduction, the authors' aim was to contribute to the long-standing problem of characterizing the set of observables in QFT, plagued by several issues such as Sorkin's impossible measurements [31] already pointed out by Dirac [13]. The fact that (F) implies (C) is clearly stated from the beginning and the converse is also proved in finite dimension (Theorem 7), but they go well beyond and their framework is more general than ours since it allows for an ancilla system that can be sent between Alice to Bob. In particular, they solved a problem raised in [1] concerning the realizable non-local measurements. Schumacher and Westmoreland (2005) [29] exhibited three ways of expressing locality (Locality (III) being very similar to (C)), and showed their mathematical equivalence. Many works followed, and the condition (C) is now well-known in the quantum information literature (although the major reference in the domain [24] does not mention it), especially for those working on causal decomposition [20] and quantum causal models [2]. Nonetheless, these works are not necessarily widespread among other research communities.

In the context of QFT, Polo-Gómez, Garay and Martín-Martínez (2022) [27] proved the condition (C) (equation (29)) based on the microcausality hypothesis, used to derive a relation not very different from (F) (equation (28)). They also argued that selective measurements cannot be compatible with special relativity unless one accepts to define as many wavefunctions as there are observers and that the update occurs along the future light cone. Contrary to the latter and in accordance with [1], we consider that, as soon as the wavefunction is understood to be observer dependent (hence, in particular, frame dependent), it is pointless and even awkward to ask for frame independent updates. Why would a given observer in her reference frame wait some time before using all the information she has (in the case of a selective measurement), or before taking into account the evolution she knows has happened on $\mathcal{S}_1$ to write down the whole state of $\mathcal{S}_1 + \mathcal{S}_2$?

As a concluding remark for this section, we note that none of these works have completely pursued the logician point of view that we now propose to explore in the sequel.

3 Locality

3.1 (L') implies (L) and (F)

In §2.2, we have seen that (F) ensures a theory to be consistent with special relativity. The property (F) is directly linked to a fundamental principle of physics, namely the locality hypothesis (L) which states the absence of interactions at a distance:

(L) *Two localized particles can interact*[4] *at a given time only if they are located at the same point in space.*

In QFT, this hypothesis has a more specific formulation:

(L') *The interaction Hamiltonian of any system can be written in the form* $H_{int}(t) = \int_{\mathbb{R}^3} H_{int}(\vec{x}, t)\mathrm{d}^3x$ *where* H_{int} *is a field of operators defined on the whole spacetime.*

Why is (L') called a locality hypothesis? Suppose one wants to compute the coefficient $S_{\alpha\beta}$ of the S-matrix, where $|\alpha\rangle$ is the state composed of two particles localized in $\vec{x}_1$ and $\vec{x}_2$, *i.e.*

$$|\alpha\rangle = \int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^{3/2}} e^{-i\vec{p}_1\cdot\vec{x}_1} a^\dagger(\vec{p}_1) \int \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^{3/2}} e^{-i\vec{p}_2\cdot\vec{x}_2} b^\dagger(\vec{p}_2)\,|0\rangle\,.$$

Under (L'), the Born approximation reads: $S_{\alpha\beta} = -i\int \mathrm{d}^4x\,\langle\alpha|H_{int}(x)|\beta\rangle$. Furthermore, even without specifying $H_{int}(x)$, one knows by covariance that the latter can only be built from the different quantum fields of the theory, and that if an interaction is possible it necessarily contains two quantum fields of the type of the particles of the state $|\alpha\rangle$, and so includes at least an expression of the form $\int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}} e^{-ipx} u(\vec{p}) a^\dagger(\vec{p}) \int \frac{\mathrm{d}^3p'}{(2\pi)^{3/2}} e^{-ip'x} v(\vec{p}') b^\dagger(\vec{p}')$ where u and v are objects that depend on the nature of the quantum fields. Using the (anti-)commutation relation for the operators a and b, one finds:

$$\langle\alpha|H_{int}(x)|\beta\rangle = \iint \mathrm{d}^3\vec{p}_1\mathrm{d}^3\vec{p}_2[\ldots]e^{i\vec{p}_1\cdot\vec{x}_1}e^{i\vec{p}_2\cdot\vec{x}_2}e^{-i(p_1+p_2)x} \propto \delta^{(3)}(\vec{x}-\vec{x}_1)\delta^{(3)}(\vec{x}-\vec{x}_2) \propto \delta^{(3)}(\vec{x}_1-\vec{x}_2).$$

This means that two localized particles can interact at a given time only if they are located at the same point in space, namely (L). The electromagnetic force between two charged particles, for example, is an interaction 'at a distance' only because (virtual) photons are exchanged between them. Furthermore, under (L), two isolated[5] systems $\mathcal{S}_1$ and $\mathcal{S}_2$ can not interact, therefore the total Hamiltonian reads $H = H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2$ with H_1 and H_2 the internal Hamiltonians of $\mathcal{S}_1$ and $\mathcal{S}_2$. Therefore, the unitary evolution operator of $\mathcal{S}_1 + \mathcal{S}_2$ is $U = e^{itH} = e^{itH_1} \otimes e^{itH_2} = U_1 \otimes U_2$. We have just shown that **(L') implies (L) which implies (F)**.

3.2 (F) implies (L) and justifies (L')

Conversely, suppose that (F) is true. Then, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are two isolated systems, the total Hamiltonian of $\mathcal{S}_1 + \mathcal{S}_2$ reads:

[4] Again, as in note 3, this term can be confusing. One can think of possible interactions as allowed vertices in Feynman diagrams, but what about the gravitational interaction? If the latter is to be background independent [22], then there is no pre-existing spacetime in which to define a notion of locality...

[5] Recall that we defined 'isolated' as the fact that no particles of $\mathcal{S}_1$ and $\mathcal{S}_2$ meet each other.

$$H = \frac{1}{i}\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{1}{i}\frac{\mathrm{d}U_1 \otimes U_2}{\mathrm{d}t}\Big|_{t=0} = \frac{1}{i}\frac{\mathrm{d}U_1}{\mathrm{d}t}\Big|_{t=0} \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \frac{1}{i}\frac{\mathrm{d}U_2}{\mathrm{d}t}\Big|_{t=0} = H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2 \Rightarrow H_{int} = 0,$$

hence **(F) implies (L)**. Now, it is natural to imagine (this is not a proof!) that in a theory satisfying (L), the interaction Hamiltonian of an arbitrary system should be composed of a combination of local operators of the form $\int_{\mathcal{D}} H_{int}(\vec{x}, t)$ where $\mathcal{D}$ is an integration domain *a priori* unknown. However, to preserve the translation invariance of the laws of physics, $\mathcal{D}$ must be the whole space $\mathbb{R}^3$. Thus **(L) justifies the locality hypothesis in QFT (L')**.

3.3 (MC) implies (F)

We need our theories to satisfy (C). In §2.2, we have shown that (F) $\Rightarrow$ (C) and remarked that (F) is generally postulated in QM. But, unlike (F), (C) is a consistency condition directly required by physical considerations, so it seems after all more natural to postulate the latter. One can then try to determine the set of unitaries compatible with (C), which would yield a constraint on any unitary evolution operator associated to a bipartite system $\mathcal{S}_1 + \mathcal{S}_2$ with $\mathcal{S}_1$ and $\mathcal{S}_2$ spacelike separated. It turns out that the more restrictive condition (MC) suffices to deduce (F), as shown in the following theorem. The same result was already proved in [4, Theorem 7] using a completely different and less algebraic argument, but our proof extends it to the infinite (separable) dimensional case.

Theorem 3.1 (**(MC)** $\Rightarrow$ **(F)**)**.** *Let $U(t)$ be the unitary evolution operator, expressed in a fixed reference frame $\mathcal{R}$, of a quantum system $\mathcal{S}_1 + \mathcal{S}_2$ composed of two isolated subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$, described in a consistent relativistic quantum theory. Then there exist two unitary operators $U_1(t)$ and $U_2(t)$ such that:*

$$U(t) = U_1(t) \otimes U_2(t)$$

Proof. The two subsystems being isolated, it is possible to divide the temporal axis of $\mathcal{R}$ into small time intervals such that $\mathcal{S}_1$ and $\mathcal{S}_2$ are spacelike separated[6] during any of these time intervals. On each interval, the theory satisfies (in particular) the condition (MC) by consistency, and it suffices to show there the factorization result. From now on, for the sake of clarity, we will not write the parameter t which plays no role in the proof anymore. Let's denote $\mathcal{H}_1$ and $\mathcal{H}_2$ the Hilbert spaces associated with $\mathcal{S}_1$ and $\mathcal{S}_2$, and assume first that they are finite dimensional with $n_1 = \dim(\mathcal{H}_1)$ and $n_2 = \dim(\mathcal{H}_2)$. One can write U in the following generic form:

$$U = \sum_{\substack{i \\ 1 \leqslant k,l \leqslant n_2}} \alpha_{ikl} T_i \otimes |k\rangle \langle l| \tag{2}$$

with $(|k\rangle \langle l|)_{1 \leqslant k,l \leqslant n_2}$ the canonical basis of $\mathcal{L}(\mathcal{H}_2)$ associated with an orthonormal basis $(|k\rangle)_{1 \leqslant k \leqslant n_2}$, and $(T_i)_i$ a basis of $\mathcal{L}(\mathcal{H}_1)$. When replacing U in the expression (1) for the condition (MC), one gets for all hermitian operators $\hat{A}$ and $\hat{B}$ (automatically compact in finite dimension) of $\mathcal{H}_1$ and $\mathcal{H}_2$ and for all $y_0 \in \mathrm{spec}\hat{B}$:

$$\forall \rho, \quad \mathrm{tr}\left(\rho \sum_{\substack{i,k,l \\ j,k',l'}} \Big[\sum_{x \in \mathrm{spec}\hat{A}} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i\Big] \otimes \alpha_{ikl}\overline{\alpha_{jk'l'}} |l'\rangle \langle k'| \Pi_{y_0}^{(2)} |k\rangle \langle l| \right) = 0$$

$$\Rightarrow \sum_{\substack{i,k,l \\ j,k',l'}} \Big[\sum_{x \in \mathrm{spec}\hat{A}} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i\Big] \otimes \alpha_{ikl}\overline{\alpha_{jk'l'}} |l'\rangle \langle k'| \Pi_{y_0}^{(2)} |k\rangle \langle l| = 0.$$

Note that $\hat{B}$ may be chosen arbitrarily, as well as $\Pi_{y_0}^{(2)}$. In particular, for any pair $\{k_1, k_2\} \subset [\![1, n_2]\!]$ and $\mu, \nu \in \mathbb{C}$ such that $|\mu|^2 + |\nu|^2 = 1$, one can define $\Pi_{y_0}^{(2)}$ to be the projector on the vector $\mu |k_1\rangle + \nu |k_2\rangle$, that is $|\mu|^2 |k_1\rangle \langle k_1| + \mu\overline{\nu} |k_1\rangle \langle k_2| + \overline{\mu}\nu |k_2\rangle \langle k_1| + |\nu|^2 |k_2\rangle \langle k_2|$. Inserting into the previous equation divides it in four sums:

[6] Meaning that every point of the spacetime region spanned by $\mathcal{S}_1$ is spacelike separated to every point of the spacetime region spanned by $\mathcal{S}_2$.

$$\forall k_1,k_2,\quad \sum_{\substack{i,l\\ j,l'}}\Big[\sum_{x\in\mathrm{spec}\hat{A}}\Pi_x^{(1)}T_j^\dagger T_i\Pi_x^{(1)}-T_j^\dagger T_i\Big]$$
$$\otimes\left[|\mu|^2\alpha_{ik_1l}\overline{\alpha_{jk_1l'}}+\mu\overline{\nu}\alpha_{ik_2l}\overline{\alpha_{jk_1l'}}+\overline{\mu}\nu\alpha_{ik_1l}\overline{\alpha_{jk_2l'}}+|\nu|^2\alpha_{ik_2l}\overline{\alpha_{jk_2l'}}\right]|l'\rangle\langle l|=0.$$

The particular cases $\mu=1,\nu=0$ or $\mu=0,\nu=1$ imply that the first and fourth terms actually always vanish. Setting $\mu=\nu=\frac{1}{\sqrt{2}}$ or $\mu=\frac{1}{\sqrt{2}},\nu=\frac{i}{\sqrt{2}}$ leads to:

$$\begin{cases}\displaystyle\sum_{\substack{i,l\\ j,l'}}\Big[\sum_{x\in\mathrm{spec}\hat{A}}\Pi_x^{(1)}T_j^\dagger T_i\Pi_x^{(1)}-T_j^\dagger T_i\Big]\otimes\Big[\frac{1}{2}\alpha_{ik_2l}\overline{\alpha_{jk_1l'}}+\frac{1}{2}\alpha_{ik_1l}\overline{\alpha_{jk_2l'}}\Big]|l'\rangle\langle l|=0\\ \displaystyle\sum_{\substack{i,l\\ j,l'}}\Big[\sum_{x\in\mathrm{spec}\hat{A}}\Pi_x^{(1)}T_j^\dagger T_i\Pi_x^{(1)}-T_j^\dagger T_i\Big]\otimes\Big[-\frac{i}{2}\alpha_{ik_2l}\overline{\alpha_{jk_1l'}}+\frac{i}{2}\alpha_{ik_1l}\overline{\alpha_{jk_2l'}}\Big]|l'\rangle\langle l|=0\end{cases}$$

and taking appropriate linear combinations of these shows that the second and third terms vanish as well. Therefore:

$$\forall k,k',\quad\sum_{\substack{i,l\\ j,l'}}\alpha_{ikl}\overline{\alpha_{jk'l'}}\Big[\sum_{x\in\mathrm{spec}\hat{A}}\Pi_x^{(1)}T_j^\dagger T_i\Pi_x^{(1)}-T_j^\dagger T_i\Big]\otimes|l'\rangle\langle l|=0$$
$$\Rightarrow\quad\forall k,k',l,l',\quad\sum_{i,j}\alpha_{ikl}\overline{\alpha_{jk'l'}}\Big[\sum_{x\in\mathrm{spec}\hat{A}}\Pi_x^{(1)}T_j^\dagger T_i\Pi_x^{(1)}-T_j^\dagger T_i\Big]=0$$

because $(|l'\rangle\langle l|)_{1\leqslant l,l'\leqslant n_2}$ is a basis of $\mathcal{L}(\mathcal{H}_2)$. This being true for all hermitian operators $\hat{A}$, the operator $\sum_{i,j}\alpha_{ikl}\overline{\alpha_{jk'l'}}\,T_j^\dagger T_i=(\sum_i\alpha_{ik'l'}T_i)^\dagger(\sum_i\alpha_{ikl}T_i)$ is diagonal in every orthonormal bases of $\mathcal{H}_1$, so it is a dilation:

$$\forall k,k',l,l',\exists\lambda_{kk'll'}\in\mathbb{C}:\quad(\sum_i\alpha_{ik'l'}T_i)^\dagger(\sum_i\alpha_{ikl}T_i)=\lambda_{kk'll'}\mathbb{1}.$$

Obviously, $\lambda_{kk'll'}\neq 0$, otherwise $\sum_i\alpha_{ikl}T_i=0$ (for $\mathcal{L}(\mathcal{H}_2)$ is a C^*-algebra) which would contradict the linear independence of the $(T_i)_i$. Moreover, by unicity of the inverse, we have for all k and l, $\frac{1}{\lambda_{k1l1}}\sum_i\alpha_{ikl}T_i=\frac{1}{\lambda_{1111}}\sum_i\alpha_{i11}T_i$ and since the $(T_i)_i$ are linearly independent:

$$\forall k,l,\exists\beta_{kl}\in\mathbb{C}:\forall i,\quad\alpha_{ikl}=\beta_{kl}\,\alpha_{i11}.$$

It is now possible to factorize:

$$U=\sum_{\substack{i\\ 1\leqslant k,l\leqslant n_2}}\beta_{kl}\alpha_{i11}T_i\otimes|k\rangle\langle l|=\left(\sum_i\alpha_{i11}T_i\right)\otimes\left(\sum_{1\leqslant k,l\leqslant n_2}\beta_{kl}\,|k\rangle\langle l|\right)=U_1\otimes U_2$$

where one can identify U_1 and U_2 with the evolution operators of $\mathcal{S}_1$ and $\mathcal{S}_2$, which are necessarily unitary since $U=U_1\otimes U_2$ is.

Let's now turn to the infinite dimensional (separable) case. The already cited proof given in [4, Theorem 7] works only for $n_1,n_2<+\infty$ because it relies on the Schmidt decomposition of the unitary operator U, which does not necessarily exist in infinite dimension, because U is not a Hilbert-Schmidt operator since $\|U\|_{HS}=\mathrm{tr}(U^\dagger U)=\mathrm{tr}(\mathbb{1})=+\infty$ [28].

Our proof, however, remains valid. Indeed, if now $\mathcal{H}_1$ and $\mathcal{H}_2$ are infinite dimensional and separable, we can still consider $(|k\rangle)_{k\in\mathbb{N}}$ and $(T_i)_i$ to be countable bases of $\mathcal{H}_1$ and $\mathcal{L}(\mathcal{H}_2)$ respectively. Therefore, $(|k\rangle\langle l|\otimes T_i)_{k,l,i\in\mathbb{N}}$ is a countable basis of $\mathcal{L}(\mathcal{H}_1)\otimes\mathcal{L}(\mathcal{H}_2)$ and we we can still decompose U as in (2). (MC) being true for all compact hermitian operators, we can choose $\hat{A}$ to be diagonal in any orthonormal basis, so that we still conclude that $(\sum_i\alpha_{ik'l'}T_i)^\dagger(\sum_i\alpha_{ikl}T_i)$ must be diagonal in every orthonormal bases, hence it is a dilation. The rest of the proof involves no difficulty. □

A consequence of this theorem is that (F), (C) and (MC) imply themselves circularly, therefore **(F), (C) and (MC) are logically equivalent**. Another unexpected corollary is the following: while (C) and (MC) do not seem *a priori* to be symmetrical with respect to $\mathcal{S}_1$ and $\mathcal{S}_2$, their equivalence to (F) implies such a symmetry. Said differently, $\mathcal{S}_1$ does not allow to communicate to $\mathcal{S}_2$ if and only if the converse is true.

4 Microcausality

In addition to the locality hypothesis, another deep postulate of QFT is the microcausality hypothesis (M).

(M) *For all quantum fields Φ and spacelike intervals $x-y$, $[\Phi(x), \Phi^\dagger(y)]_\pm = 0$ where $[\ ,\ ]_\pm$ stands for an anti-commutator or a commutator depending on the fermionic or bosonic nature of Φ.*

Usually, standard QFT textbooks justify this hypothesis by invoking, for once, the concept of measurement [26, p.28] [18, p.106] [36, p.121], but they generally make do with the affirmation that two spacelike measurements must be independent, without more explanations. Not only is the argument too vague, but it is hard to see how the relation could differ according to the fermionic or bosonic nature of the field. Weinberg, on the other hand, makes an interesting remark:

> 'The condition [(M)] is often described as a causality condition, because if $x-y$ is spacelike then no signal can reach y from x, so that a measurement of Φ at point x should not be able to interfere with a measurement of Φ or $\Phi^\dagger$ at point y. Such considerations of causality are plausible for the electromagnetic field, any one of whose components may be measured at a given spacetime point, as shown in a classic paper of Bohr and Rosenfeld [9]. However, we will be dealing here with fields like the Dirac field of the electron that do not seem in any sense measurable. The point of view taken here is that [(M)] is needed for the Lorentz invariance of the S-matrix, without any ancillary assumptions about measurability or causality.' [34, p.198]

As Weinberg notes himself, microcausality is only a sufficient condition for the invariance of the S-matrix:

> 'Theories of this class [satisfying (M)] are not the only ones that are Lorentz invariant, but the most general Lorentz invariant theories are not very different. In particular, there is always a commutation condition something like [(M)] that needs to be satisfied. This condition has no counterpart for non-relativistic systems, for which time-ordering is always Galilean-invariant. *It is this condition that makes the combination of Lorentz invariance and QM so restrictive.*' [34, p.145]

However, Weinberg's argument heavily relies on the use of normal-ordered fields. Arguably, this writing is only a computation convenience without physical meaning. Its main purpose is to get rid of the infinite constants that appear in certain computations, by making finite all the matrix elements of the operators manipulated. This operation is justified when the divergences have no influence in the considered context. Alternatively, the prediction of the Casimir effect or the Lamb shift by the vacuum energy is only possible *without* normal-ordering (one substantially uses the fact that $\langle 0|\Phi^2|0\rangle \neq 0$ for a non-ordered Φ, see [18, p.111]). It seems that a stronger argument is needed to justify the microcausality hypothesis (M). Note that (M) is especially crucial to prove the famous spin-statistics theorem (S):

(S) *Scalar and vector fields correspond to bosons, while Dirac fields correspond to fermions.*

Here is how most QFT textbooks proceed to establish (S). Since we don't know the result yet, we have to compute both commutators and anti-commutators. A short calculation first shows that when Φ is a Dirac field, we have:

$$[\Phi(x), \Phi^\dagger(y)]_\pm = \Delta_+(x-y) \mp \Delta_+(y-x) \tag{3}$$

depending on Φ describing fermions or bosons, *i.e.* $[a(\vec{p}), a^\dagger(\vec{p'})]_\pm = \delta^{(3)}(\vec{p}-\vec{p'})$, and where $\Delta_+(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p_0} e^{ip_\mu x^\mu}$ is shown to be a Lorenz invariant quantity. Likewise, when Φ is a scalar or a vector field:

$$[\Phi(x), \Phi^\dagger(y)]_\pm = \Delta_+(x-y) \pm \Delta_+(y-x) \tag{4}$$

depending on Φ describing fermions or bosons. It is quite easy to see that when $x-y$ is spacelike, $\Delta_+(x-y)-\Delta_+(y-x)=0$, while $\Delta_+(x-y)+\Delta_+(y-x)$ is not identically zero [34, p.202]. All these remarks, in addition to (M), imply (S). Not the reverse: these computations generally don't justify (M) but only the fermionic or bosonic nature of a field, whereas one can intuitively feel that they contain much more information.

We now present a simultaneous proof of the microcausality hypothesis and the spin-statistics theorem that relies only on the consistency of the theory and on the above (anti-)commutators computations.

Theorem 4.1 ((**MC**) $\Rightarrow$ (**M**) **and** (**S**))**.** *Let Φ be a quantum field. Then, for all spacelike intervals $x-y$, $[\Phi(x),\Phi^\dagger(y)]_\pm = 0$ where $[\ ,\ ]_\pm$ stands for an anti-commutator or a commutator depending on Φ being a Dirac field (and in that case it describes a fermion) or a scalar or vector field (and in that case it describes a boson).*

Proof. Let's first suppose that Φ is effectively a measurable field, in other words $\Phi(x)$ is an observable for all x. As stated by Weinberg in the previous quote, it is for example the case for the electromagnetic field that describes the photon. Recalling our choice to restrict the discussion to compact hermitian operators in §2.1, one can write its spectral decomposition $\Phi(x)=\sum_{\lambda\in\text{spec }\Phi(x)}\lambda\Pi_\lambda^{(x)}$ (rigorously speaking, one should work with smeared fields). Contrary to the previous sections, there is now only one Hilbert space, the Fock space $\mathcal{H}_{Fock}$, and the system's state is given by a density matrix $\rho\in\mathcal{S}(\mathcal{H}_{Fock})$. For all spacelike intervals $x-y$, a measurement of $\Phi(x)$ doesn't affect the statistics of a measurement of $\Phi(y)$ if and only if the following condition, variant of (MC), is satisfied:

$$\forall\rho,\forall\mu\in\text{spec }\Phi(y),\quad \mathrm{tr}\left(\Big(\sum_{\lambda\in\text{spec }\Phi(x)}\Pi_\lambda^{(x)}\rho\Pi_\lambda^{(x)}\Big)\Pi_\mu^{(y)}\right)=\mathrm{tr}(\rho\Pi_\mu^{(y)})$$

where we implicitly moved to a reference frame $\mathcal{R}$ in which $x^0=y^0$, so as to avoid to introduce the (non-covariant) unitary evolution operators[7]. It yields:

$$\begin{aligned}
&\forall\rho,\forall\mu\in\text{spec }\Phi(y),\quad \mathrm{tr}\left(\rho\Big(\sum_{\lambda\in\text{spec }\Phi(x)}\Pi_\lambda^{(x)}\Pi_\mu^{(y)}\Pi_\lambda^{(x)}-\Pi_\mu^{(y)}\Big)\right)=0\\
\Rightarrow\quad &\forall\mu\in\text{spec }\Phi(y),\quad \sum_{\lambda\in\text{spec }\Phi(x)}\Pi_\lambda^{(x)}\Pi_\mu^{(y)}\Pi_\lambda^{(x)}=\Pi_\mu^{(y)}\\
\Rightarrow\quad &\sum_{\lambda\in\text{spec }\Phi(x)}\Pi_\lambda^{(x)}\Phi(y)\Pi_\lambda^{(x)}=\Phi(y).
\end{aligned}$$

Thus, $\Phi(y)$ is (block) diagonal in the eigenbasis of $\Phi(x)$, so they are codiagonalizable and $[\Phi(x),\Phi(y)]=0$. This relation *a priori* holds in the frame $\mathcal{R}$, but when applying the appropriate representation of the Lorenz group under which Φ transforms, one sees that $\Phi(x)$ and $\Phi(y)$ commute in all reference frames.

If now Φ is not supposed hermitian anymore, we still know that $\Phi\Phi^\dagger$ is. Applying what precedes to $\Phi\Phi^\dagger$ instead of Φ, we obtain that for all $x-y$ spacelike, $[\Phi(x)\Phi^\dagger(x),\Phi(y)\Phi^\dagger(y)]=0$. Moreover, as Φ is not hermitian, one shows as usual that for all x and y, $[\Phi(x),\Phi(y)]=[\Phi^\dagger(x),\Phi^\dagger(y)]=0$ [8]. But on the other hand, it is also possible to compute the commutator $[\Phi(x)\Phi^\dagger(x),\Phi(y)\Phi^\dagger(y)]$ using the commutation relations (3) and (4). Suppose for instance that Φ is a Dirac field. Since we don't know yet if it is a boson or a fermion, let's distinguish the possible cases:

- if the particle described by Φ is a fermion, we have (3) with the upper signs. Then:

[7] We are aware that there still doesn't exist a proper and consensual mathematical framework for the implementation of measurements in QFT [12]). This is probably why the same computation leads to oddities if one doesn't move to the special reference frame $\mathcal{R}$ beforehand. Still, we reckon that this simple reasoning actually captures something of physical interest.

[8] This is because the only commutators that can appear between a creation and an annihilation operator are $[a(\vec{p}),a^{c\dagger}(\vec{p'})]$ or $[a^c(\vec{p}),a^\dagger(\vec{p'})]$ (with the label c standing for the antiparticle) which are zero if $a(\vec{p})\neq a(\vec{p})^c$.

$$\begin{aligned}\Phi(x)\underbrace{\Phi^\dagger(x)\Phi(y)}_{\substack{=-\Phi(y)\Phi^\dagger(x)\\ -\Delta_+(x-y)\\ +\Delta_+(y-x)}}\Phi^\dagger(y) &= -\Phi(x)\Phi(y)\Phi^\dagger(x)\Phi^\dagger(y)+\Big(\Delta_+(y-x)-\Delta_+(x-y)\Big)\Phi(x)\Phi^\dagger(y)\\ &= -\Phi(y)\underbrace{\Phi(x)\Phi^\dagger(y)}_{\substack{=-\Phi^\dagger(y)\Phi(x)\\ +\Delta_+(x-y)\\ -\Delta_+(y-x)}}\Phi^\dagger(x)+\Big(\Delta_+(y-x)-\Delta_+(x-y)\Big)\Phi(x)\Phi^\dagger(y)\\ &= \Phi(y)\Phi^\dagger(y)\Phi(x)\Phi^\dagger(x)+\Big(\underbrace{\Delta_+(y-x)-\Delta_+(x-y)}_{=0 \text{ if } x-y \text{ spacelike}}\Big)(\Phi(y)\Phi^\dagger(x)+\Phi(x)\Phi^\dagger(y)),\end{aligned}$$

- if the particle described by Φ is a boson, we have (3) with the lower signs. Then

$$\begin{aligned}\Phi(x)\underbrace{\Phi^\dagger(x)\Phi(y)}_{\substack{=\Phi(y)\Phi^\dagger(x)\\ -\Delta_+(x-y)\\ -\Delta_+(y-x)}}\Phi^\dagger(y) &= \Phi(x)\Phi(y)\Phi^\dagger(x)\Phi^\dagger(y)-\Big(\Delta_+(x-y)+\Delta_+(y-x)\Big)\Phi(x)\Phi^\dagger(y)\\ &= \Phi(y)\underbrace{\Phi(x)\Phi^\dagger(y)}_{\substack{=\Phi^\dagger(y)\Phi(x)\\ +\Delta_+(x-y)\\ +\Delta_+(y-x)}}\Phi^\dagger(x)-\Big(\Delta_+(x-y)+\Delta_+(y-x)\Big)\Phi(x)\Phi^\dagger(y)\\ &= \Phi(y)\Phi^\dagger(y)\Phi(x)\Phi^\dagger(x)+\Big(\underbrace{\Delta_+(x-y)+\Delta_+(y-x)}_{\substack{\text{non-identically zero}\\ \text{if } x-y \text{ spacelike}}}\Big)(\Phi(y)\Phi^\dagger(x)-\Phi(x)\Phi^\dagger(y)).\end{aligned}$$

As a consequence, one recovers the commutation relation $[\Phi(x)\Phi^\dagger(x), \Phi(y)\Phi^\dagger(y)] = 0$ imposed by the condition (MC) if, and only if, Φ is a fermionic field and in this case, the relation (3) implies that $\{\Phi(x), \Phi^\dagger(y)\} = 0$ for all spacelike intervals $x - y$. These are indeed the statements (M) and (S). When Φ is a scalar or vector field, the proof is similar. □

Remark 4.2. Note that this proof makes use, as usually in physics, of the identification between the notion of observable in the mathematical sense (hermitian operator) and in the physical sense (quantity measurable by a concrete experimental protocol). However, the second is far more restrictive, since in practice we can only measure a few very specific observables. Rigorously speaking, only the physical notion of observable is constrained by (MC), since the latter must ensure the absence of inconsistencies between *actual* measurements. Therefore, in the above proof, although $\Phi\Phi^\dagger$ is hermitian, one could question the legitimacy of imposing it (MC). Of course, any mathematical observable could be in principle measured by applying a suitable unitary evolution to the system that would map its eigenbasis to the eigenbasis of a physically measurable observable. But is it satisfactory to rely on the idea that all unitary evolutions are *a priori* feasible, even though we will never be able to implement them? Nonetheless, it is still possible to adapt the proof by replacing $\Phi\Phi^\dagger$ by an undoubtedly physical observable, such that a function of the components of $\hat{T}_{\mu\nu}$ or even the charge $\hat{Q}$, that one can express in terms of Φ. For example, for a Dirac field, $\hat{T}^\mu_\mu = m\Phi^\dagger\gamma^0\Phi$, would allow a quite similar proof.

What precedes has an unexpected consequence, expressed in the corollary below.

Corollary 4.3. *A fermionic field is not measurable.*

Proof. In the previous proof, we have seen that if a field is an observable (*i.e.* $\Phi = \Phi^\dagger$), then the condition (MC) implies that for all $x - y$ spacelike, $[\Phi(x), \Phi(y)] = 0$. But if Φ is a fermionic field, it also satisfies $\{\Phi(x), \Phi(y)\} = 0$. Adding these two relations yields for all $x - y$ spacelike, $\Phi(x)\Phi(y) = 0$, which is too strong a constraint for a quantum field. In particular, $\Phi(x)\Phi(y)\,|0\rangle$ is the state containing two localized particles at x and y; in any case it is a non-zero vector of the Fock space[9]. □

[9]This result is not new, and can also be derived, for instance, by the fact that the equation $\Phi = \Phi^\dagger$ is neither covariant nor

5 Conclusion

In this paper, we have investigated the logical interrelationships between fundamental properties in relativistic QM. Such a work is not only useful to found quantum theory on clear bases, but it also delivered unexpected results, lead to various philosophical remarks and revealed how the recent development of quantum information theory entailed significant progress in this topic.

Our starting point was the apparent incompatibility between special relativity and two kinds of instantaneities that seem to appear in QM. By the time it was historically developed, quantum theory was not built to integrate special relativity, so that it is always surprising to contemplate their 'peaceful coexistence' (expression coined by Shimony in [30]). We have formulated a mathematical consistency condition that proved to share deep logical links with other fundamental postulates in physics; it also promoted an epistemic interpretation of the wavefunction collapse, clarified the status of microcausality and the derivation of the spin-spatistics theorem in QFT, and implied the non-measurability of the Dirac field. The following diagram summarizes all the logical relationships established in this paper:

$$\begin{tikzcd}
 & & & \boxed{\text{Special relativity}} \arrow[dd, squiggly] & & \\
 & & & & & \text{(M)} \arrow[dl, Rightarrow] \arrow[dd, Rightarrow] \\
\text{(L')} \arrow[r, Rightarrow, shift right] & \text{(L)} \arrow[l, dotted, Rightarrow, shift right] \arrow[r, Leftrightarrow] & \text{(F)} \arrow[r, Leftrightarrow] & \text{(C)} \arrow[r, Leftrightarrow] & \text{(MC)} \arrow[dr, Rightarrow] & \\
 & & & & & \text{(S)}
\end{tikzcd}$$

where the dotted arrow stands for 'justifies' rather than 'implies'. The implication (M) $\Rightarrow$ (MC) is not proved in this paper but in the aforementioned [27]. Let's emphasize again that (C) (*a fortiori* (MC)) is not an arbitrary postulate but a mere consistency criterion that must be valid for any relativistic quantum theory: it 'costs nothing' to be assumed because it stems from the constraints of special relativity. Surprisingly, it becomes obvious on this diagram that the two fundamental postulates (L') *and* (M) of QFT are actually redundant, since *locality implies microcausality.*

A The measurement problem

It is not the aim of this paper to address thoroughly the interpretational issues of QM, but some basic considerations will be needed at some point. The material for this section is based on an ongoing article where we propose a detailed investigation of the measurement problem.

A.1 Selective measurement, non-selective measurement

When discussing the implementation of measurements in QM, it is crucial to distinguish clearly between selective and non-selective measurement. The former is what is usually referred to as the 'wavefunction collapse'. When measuring an observable $\hat{A}$, of spectral decomposition $\hat{A} = \sum_{x \in \mathrm{spec}\hat{A}} x\Pi_x$, relative to a system in a pure state (resp. mixed state) $|\Psi\rangle$ (resp. ρ), if the outcome is the eigenvalue x_0, then the system's state is standardly postulated to evolve as:

$$|\Psi\rangle \longrightarrow \frac{\Pi_{x_0}|\Psi\rangle}{\|\Pi_{x_0}|\Psi\rangle\|} \qquad \text{resp.} \quad \rho \longrightarrow \frac{\Pi_{x_0}\rho\Pi_{x_0}}{\mathrm{tr}(\rho\Pi_{x_0})}. \tag{5}$$

Note that the implementation of such a selective measurement requires to know, or to assume, the outcome. Alternatively, a non-selective measurement describes the update of the state without distinguishing the actual outcome: it is only concerned about the statistics that would be obtained if the experiment were repeated. Except in specific cases, the result is always a mixed state:

$$\rho \longrightarrow \sum_{x \in \mathrm{spec}\hat{A}} \Pi_x \rho \Pi_x. \tag{6}$$

independent of the representation of the gamma matrices [25], so it can't be linked to a physically meaningful property such as being measurable. But it is interesting to see that it can be directly derived from considerations about measurements. Again, our argument relies on the widespread identification between the notions of mathematical and physical observable. Here, Φ may indeed be hermitian, but is it a physically measurable quantity subject to the condition (MC)?

Selective measurement is a non-linear probabilistic operation on ρ, whereas non-selective measurement is a linear deterministic one, since it merely consists of extracting the diagonal part of the density matrix in the eigenbasis of the measured observable. The theory of decoherence explains how non-selective measurements (6) arise in QM, due to the entanglement between the system and its measurement apparatus [37] [19]. What about the selective measurement (5)?

A.2 The status of the collapse

In any probabilistic model, an update of the probabilities has to be performed when one obtains new empirical information about the system considered.[10]. Since QM is a probabilistic theory, it is very natural that the wavefunction needs to be updated after a gain of information on the system, namely after a selective measurement. It is therefore tempting to say that the physical process affecting the wavefunction during a measurement (ontic side) is the one entailed by decoherence (6), whereas the collapse (5) is not a physical process (epistemic side). Further arguments in favor of the non-physicality of the collapse is that objective collapse models still suffer from issues in their compatibility with special relativity despite numerous attempts [14] [32] [5], and that large portions of the set of possible parameters have already been ruled out experimentally [10]. They are also at odds with the phenomenon of coherence revival [11]. Besides, what can be the status of decoherence in a collapse model, apart from a curious redundant phenomenon that happens to also destroy very efficiently the quantum interferences?

For these reasons, and also because it will appear particularly consistent in the relativistic context (see §2.3), we will adopt in this paper the stance that the collapse is not a physical process, but an epistemic operation. This view, so natural in classical probabilities, obviously comes with its own lot of interpretational issues. This is in particular because quantum probabilities display interferences between all the possible histories before a measurement, so that every potential outcome (*e.g.* left slit or right slit ?) seems to have had an influence on the result of the experiment, hence a sort of 'reality', whereas a probabilistic update is simply supposed to suppress the weights of all the truly unreal histories. One direct consequence of an epistemic collapse is that there are as many wavefunctions of a system as there are observers having different knowledge about it. In any case, note that embracing or rejecting this interpretational stance would not affect the validity of most of the results obtained in this paper.

B Condition (C) ensures covariance

In §2.1, we have motivated the importance of (C) by the fact that two kinds of inconsistencies appear if it were not satisfied. The computation given in 2.2 suffices to exclude the second kind (faster-than-light communication), but lacks an additional argument to get rid of the first kind (non-covariance). Indeed, it has assumed the choice of a fixed reference frame, in which the time evolutions of the systems between the instants t_{1i}, t_{1f} and t_2 (corresponding to the beginning and the end of the evolution undergone by $\mathcal{S}_1$ and the measurement performed on $\mathcal{S}_2$ respectively) are given by unitary operators $V_{t_{1i},t_{1f}}$ and U_{t_{1f},t_2} that we have simply denoted V and U. In another reference frame, however, neither the temporal axis nor the unitary operators are conserved. Even the possible outcomes of the measurements may undergo a Lorenz transformation, if they correspond to the position observable for example. The invariance of the statistics between two reference frames $\mathcal{R}$ and $\mathcal{R}'$ may be written more precisely as follows (one assumes for simplicity $t_{1i} = t_{1f} = t_1 > t_2$, with $t_2 < t_1$ in $\mathcal{R}$ and denote t_0 an earlier time at which the two entangled systems were separated):

$$\begin{cases} \bullet \ \operatorname{tr}\left(U^{(\mathcal{R})}_{t_0,t_2}\rho U^{(\mathcal{R})\dagger}_{t_0,t_2}(\mathbb{1}_1 \otimes \Pi^{(2)}_{y_0})\right) = \operatorname{tr}\left(U^{(\mathcal{R}')}_{t'_0,t'_2}\rho U^{(\mathcal{R}')\dagger}_{t'_0,t'_2}(\mathbb{1}_1 \otimes \Pi^{(2)}_{y'_0})\right) & \text{if } t'_2 < t'_1. \\ \bullet \ \operatorname{tr}\left(U^{(\mathcal{R})}_{t_0,t_2}\rho U^{(\mathcal{R})\dagger}_{t_0,t_2}(\mathbb{1}_1 \otimes \Pi^{(2)}_{y_0})\right) & \\ \qquad = \operatorname{tr}\left(U^{(\mathcal{R}')}_{t'_1,t'_2}\operatorname{tr}_{\mathcal{E}}\left[(V^{(\mathcal{R}')}_{t'_1} \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes U^{(\mathcal{R}')}_{t'_0,t'_1}\rho U^{(\mathcal{R}')\dagger}_{t'_0,t'_1})(V^{(\mathcal{R}')\dagger}_{t'_1} \otimes \mathbb{1}_2)\right]U^{(\mathcal{R}')\dagger}_{t'_1,t'_2}(\mathbb{1}_1 \otimes \Pi^{(2)}_{y'_0})\right) & \text{if } t'_2 > t'_1 \end{cases}$$

[10]As a trivial example, suppose for instance that you ask your computer to choose uniformly a number x_1 in $\{0;1\}$, and then to choose uniformly a second number x_2 in $\{0;1\}$ if $x_1 = 0$, or to choose uniformly x_2 in $\{2;3\}$ if $x_1 = 0$. If you don't look at the result for x_1, you predict x_2 to follow a uniform law in $\{0;1;2;3\}$, but if you look at the outcome for x_1, you obviously update your probabilities.

The first line is simply the covariance of the theory in the absence of measurements: we suppose it already established. The only case to examine involves the changes of reference frames that reverse the temporal order of t_1 and t_2. In particular, since the only potential discontinuity occur when the sign of $t_2 - t_1$ flips, it suffices to write the second line above in the limit $\varepsilon, \varepsilon' \to 0$ of an infinitesimal change of frame with $t_2 = t_1 - \varepsilon$ and $t_2' = t_1' + \varepsilon'$. In that case, $U^{(\mathcal{R}')}_{t_a',t_b'} \to U^{(\mathcal{R})}_{t_a,t_b}$ for all instants t_a and t_b, and $y_0' \to y_0$ so that the condition to verify reads:

$$\mathrm{tr}\left(U^{(\mathcal{R})}_{t_0,t_2}\rho U^{(\mathcal{R})\dagger}_{t_0,t_2}(\mathbb{1}_1\otimes\Pi^{(2)}_{y_0})\right) = \mathrm{tr}\left(U^{(\mathcal{R})}_{t_1,t_2}\,\mathrm{tr}_{\mathcal{E}}\left[(V^{(\mathcal{R})}_{t_1}\otimes\mathbb{1}_2)(\rho_{\mathcal{E}}\otimes U^{(\mathcal{R})}_{t_0,t_1}\rho U^{(\mathcal{R})\dagger}_{t_0,t_1})(V^{(\mathcal{R})\dagger}_{t_1}\otimes\mathbb{1}_2)\right]U^{(\mathcal{R})\dagger}_{t_1,t_2}(\mathbb{1}_1\otimes\Pi^{(2)}_{y_0})\right)$$

but since $U^{(\mathcal{R})}_{t_0,t_2} = U^{(\mathcal{R})}_{t_1,t_2}U^{(\mathcal{R})}_{t_0,t_1}$, this is nothing but the condition (C) that has been proved applied to $U^{(\mathcal{R})}_{t_0,t_1}\rho U^{(\mathcal{R})\dagger}_{t_0,t_1}$ instead of ρ.

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