Chapter 1. Probability

1. Probabilities and Events

1.1. Definitions

- Sample Space (*S*): the set of all possible outcomes of an experiment
- Event: any set of possible outcomes of the an experiment

1.2. Probabilities

- All Outcomes

$$p_i \geq 0, i = 1, 2, ..., m.$$

$$\sum_{i=1}^{m} = 1$$

- Event A

$$P\left(A\right) = \sum_{i \in A} p_i$$

- The Complement of Event A

$$P(A^c) = 1 - P(A)$$

- The Union of Event A and B: Addition Theorem of Probabilities

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

- Conditional Probability

$$P\left(A|B\right) = P\left(AB\right)/P\left(B\right)$$

- The Intersection of Event A and B: Multiplication Theorem of Probabilities

$$P(AB) = P(A|B)P(B)$$

1.3. Relations between Events

- Mutually Exclusive / Disjoint

$$A \cup B = \varnothing$$

$$P(AB) = 0$$

- Independent

$$P(A|B) = P(A)$$

2. Random Variables

2.1. Definitions

- Random Variable: numerical quantities whose values are determined by the outcome of an experiment
- Probability Distribution: the set of the probabilities of assigning different values to an random variable

$$P\left\{X=x_j\right\}, j=1,2,\ldots,n$$

$$\sum_{j=1}^{n} P\left\{X=x_j\right\}=1$$

- Bernoulli Random Variable: A random variable *X*, which is equal to 1 with probability *p* and to 0 with probability 1-*p*, is said to be a *Bernoulli* random variable with parameter *p*
- Binomial Random Variable: Considered *n* independent trials, each of which is a success with probability *p*. The random variable X, equal to the total number of successes that occur, is called a *binomial* random variable with parameters *n* and *p*

2.2. Expected Value

- Expected Value

$$E\left[X\right]=\sum_{j=1}^{n} x_j P\left\{X=x_j\right\}$$

- Fair Bet: *E[X]=0*
- **Formula 1**: *Y=aX+b*

$$E\left[Y\right]=aE\left[X\right]+b$$

- **Formula 2**: the expected value of a sum of random variables is equal to the sum of their expected values.

- Expected Value of a Bernoulli Random Variable: *p*
- Expected Value of a Binomial Random Variable: *np*

2.3. Variance

- Variance

$$Var\left(X\right)=E\left[\left(X-E\left[X\right]\right)^2\right]$$

- Standard Deviation: the square root of the variance
- **Formula 1**: *Y=aX+b*

$$Var\left(Y\right)=a^2Var\left(X\right)$$

- **Formula 2**: the variance of a sum of random variables if the random variables are independent

$$Var\left(\sum_{j=1}^{k}X_j\right)=\sum_{j=1}^{k}Var(X_j)$$

- Variance of a Bernoulli Random Variable: *p-p^2*
- Variance of a Binomial Random Variable: *np(1-p)*

2.4. Covariance and Correlation

- The covariance of X and Y

$$Cov\left(X,Y\right)=E\left[\left(X-E\left[X\right]\right)\left(Y-E\left[Y\right]\right)\right]$$

$$Cov\left(X,Y\right)=E\left[XY\right]-E\left[X\right]E\left[Y\right]$$

- The covariance of independent random variables is equal to 0.
- **Formulas**

$$Cov\left(X,Y\right)=Cov\left(Y,X\right)$$

$$Cov\left(X,X\right)=Var\left(X\right)$$

$$Cov\left(cX,Y\right)=c\cdot Cov\left(X,Y\right)$$

$$Cov\left(c,Y\right)=0$$

- **Linear Property**

$$Cov\left(X_1+X_2,Y\right)=Cov\left(X_1,Y\right)+Cov\left(X_2,Y\right)$$

- **Generalized Linear Property**

$$Cov\left(\sum_{i=1}^{n}X_i,\sum_{j=1}^{m}Y_j\right)=\sum_{i=1}^{n}\sum_{j=1}^{m}Cov\left(X_i,Y_j\right)$$

- **The variance of the sum of random variables**

$$Var\left(\sum_{i=1}^{n}X_i\right)=\sum_{i=1}^{n}Var\left(X_i\right)+\sum_{i=1}^{n}\sum_{j\neq 1}Cov\left(X_i,X_j\right)$$

- The correlation of *X* and *Y*: in range [-1,1]

$$\rho\left(X,Y\right)=\frac{Cov\left(X,Y\right)}{\sqrt{Var\left(X\right)Var\left(Y\right)}}$$

- Given *Y=a+bX*, *\rho=1* if *b>0*, *\rho=-1* if *b<0*

2.5. Conditional Expectation

- The conditional expectation of *X* given that *Y=y*

$$E\left[X|Y=y\right]=\sum_{x}xP\left(X=x|Y=y\right)$$

- **The Relation between Expectation and Conditional Expectation**

$$E\left[X\right]=\sum_{y}E\left[X|Y=y\right]P\left(Y=y\right)$$

$$E\left[X\right]=E\left[E\left[X|Y\right]\right]$$