MATH 110 – LECTURE 3

$ \hat z \geq 3 \cdot \hat{} \{ \leq \} \sim $

- Last week:

  vector space $\langle V, \mathbb{F}, +, \cdot \rangle$

  $+ : V \times V \to V$

  vectors scalars

  $\cdot : \mathbb{F} \times V \to V$

GENERALIZING F^n.

- given field $\mathbb{F}$ and $n \in \mathbb{N}$ consider $\mathbb{F}^n = \{(f_1, f_2, \dots, f_n)\}$ w/ componentwise addition & multiplication,

- taking $n \to \infty$.
  - ↳ infinite sequences / list $\mathbb{F}^\infty : \{\underbrace{(x_1, x_2, \dots)}_{\text{countably infinite}} : x_j \in \mathbb{F}\}$.
  - ↳ consider w/ component-wise addition & multiplication.
    is this a space?
    - zero: $(0, 0, 0, \dots, 0)$
    - add. inverse: $(x_1, x_2, \dots) \to (-x_1, -x_2, \dots)$
    - mult. inverse: $c(x_1, x_2, \dots) \to \frac{1}{c}(x_1, x_2, \dots)$
    - associative: can be extrapolated from shorter sequence, b/c same $\cdot$ & scheme
    - distributive: can be extrapolated from shorter sequence, b/c same $\cdot$ and $+$ scheme.

- uncountably infinite components?
  - ↳ $\mathbb{F}^{\mathbb{R}}$?
  - ↳ try $\mathbb{F}^S$ where S is nonempty set. → doesn't need any structure

    $\mathbb{F}^S := \{f : S \to \mathbb{F}\}$

    $\mathbb{F}^n : \mathbb{F}^S$ for $S = \{1, 2, \dots, n\}$

    $\mathbb{F}^{\mathbb{Z}}$ is map of funcs. $\mathbb{Z} \to \mathbb{F}$.

    - ↳ relates to HW #2 ?!?! man was right!
    - ↳ notice that $\mathbb{F}^\infty = \mathbb{F}^{\mathbb{N}}$, which assigns a value in $\mathbb{F}$ assigned to each position
    - ↳ componentwise for seq → "pointwise" for functions
      - $h : f + g \Rightarrow h(a) = f(a) + g(a)$ for $a \in S$.

        for $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, $\lambda f \in \mathbb{F}^S$ is defined via $(\lambda f)(a) = \lambda \cdot f(a)$ for $a \in S$.

Structure of a vector space

- 0 element/vectors of VS is unique:

  suppose there are two zeroes, 0, 0':

  $$0' = 0 + 0' = 0$$

  neutral neutral.

- likewise additive inverses unique.

  $$a+b=0, \quad a+c=0$$

  $$\Rightarrow b = b+\underline{0} = (b+a)+c = \underline{0}+c = c.$$

  associativity

- also notice that add. inverse is actually always $-1(v)$

  $$[v = 1\cdot v] \rightarrow v+(-v) = 1\cdot v + (-1)\cdot v = (1+(-1))\cdot v = 0\cdot v$$

- Lemma (1): $0\vec{v} = 0$

  Lemma (2): $\vec{v}+\vec{w} = \vec{v} \Leftrightarrow \vec{w} = 0.$

  $\Rightarrow u = v+u$ where u = add. inv

  $$\vec{u}+(\vec{v}+\vec{w}) = \vec{u}+\vec{v} = \vec{0}$$

  $$(\vec{u}+\vec{v})+\vec{w} = \vec{0}+\vec{w} \quad \Rightarrow \vec{w} = 0. \quad \square$$

  Now $0\cdot\vec{v} + 0\cdot\vec{v} = (0+0)\cdot\vec{v} = 0\cdot\vec{v} \Rightarrow 0\cdot\vec{v} = 0$ from (2). $\square$

- additive inverse $= -1(\vec{v})$. ■

Subspaces

- setup: $\langle V, \mathbb{F}, +, \cdot \rangle$ is a vector space. Additionally, $W \subseteq V$.
- under which conditions is $\langle W, \mathbb{F}, +, \cdot \rangle$ also VS?

example: $V = \mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$, $+$, $\cdot$

$$W = \{(t, t^2, 0) \text{ for } t \in \mathbb{R}\}.$$

sol: No. just consider $(1,1,0)$ and $(1,1,0)$. This adds to $(2,2,0)$ which breaks the form. Therefore, it's not closed.

- most important to check for 0, check for add, check for sc. mult.

<u>Theorem</u>: $W \subseteq V$ is a vector space wrt $+, \cdot$ iff

(a) $0 \in W$

(b) $\lambda v \in W \;\forall\, v \in W, \lambda \in \mathbb{F}$

(c) $w_1 + w_2 \in W \;\forall\, w_1, w_2 \in W$.

- don't need to worry about associative / distributive / inverse b/c those things are set in stone.

- Lemma: unique $\vec{0}$'s.

- $\vec{0}' \in W$ neutral wrt $+$ in W has to be nec. $= \vec{0} \in V$

$0 \cdot \vec{0} = \vec{0}$, hence $\vec{0} \in W$. Then both $\vec{0}, \vec{0}' \in W$ and W must have unique 0 so $0 = 0'$.

- This means that (a) must hold both ways, because R direction is easy & only if requires uniqueness of $\vec{0}$'s.

- Addition and scalar multiplication are carried over from being a subspace.

Examples:

$W = \{(x,y,0) : x,y \in \mathbb{R}\}$ is a subspace

$W = \{(x,1,0) : x \in \mathbb{R}\}$ is not a subspace; no zero

$W = \{(\cos x, \sin x, 0) : x \in \mathbb{R}\}$ is not a subspace : no zero

of $\mathbb{R}^3$, w/ default $+$ and $\cdot$.

Anything w/ domain $\mathbb{R}^{[0,1]}$?