Problemset 7

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1 Counting, Counting, and More Counting

a) $\binom{n+k}{n}$

b) $3 \cdot 2^6$

c) (i) $\binom{52}{13}$

(ii) $\binom{48}{13}$

(iii) $\binom{48}{9}$

(iv) $\binom{13}{6}\binom{39}{7}$

d) $\frac{104!}{2^{52}}$

e) 2^{98}

f) (i) $\frac{7!}{4!}$

(ii) $\frac{7!}{2!2!}$

g) (i) $5! = 120$

(ii) $\frac{6!}{2}$

h) 27^9

i) $\binom{36}{8}$

j) $\binom{8}{6}$

k) $\prod_{i=1}^{10} \binom{2i}{2}$, $\frac{20!}{2^{10}}$

l) $\binom{n+k}{k}$

m) $n-1$

n) $\binom{n-1}{k}$

2 Grids and Trees!

a) A shortest path can only go upwards and rightwards from $(0,0)$ to (n,n), so there are n upwards moves and n rightwards moves for $\binom{2n}{n}$ total moves.

b) Using the same logic there are $n+1$ upwards moves and $n-1$ rightwards moves for $\binom{2n}{n-1}$ total moves.

c) Let's say the path crosses at a point (i,i). That means that it ends up reaching $(i,i+1)$. If we were to reflect the path before $(i,i+1)$ across the line $y=x+1$, we find that the path ends at $(-1,1)$ instead of $(0,0)$. Therefore, for each path that reaches $(i,i+1)$ on its way to (n,n), there is an equivalent path of size $(n+1)$. Therefore, the number of paths crossing $y=x$ is $\binom{2n}{n-1}$.

d) We find that the total number of paths from $(0,0)$ to (n,n) that don't cross $y=x$ is $\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$.

e) Assume without loss of generality that the path goes at or below the line $y=x$. If the path intersects $y=x$ for the last time at (i,i), the path from $(0,0)$ to (i,i) must go at or below $y=x$, and the path from (i,i) to (n,n) should go strictly below $y=x$.

The part from $(0,0)$ to (i,i) is clearly equivalent to F_i ways. For the part from (i,i) to (n,n), the first move must be rightwards and the last move must be upwards, and the remaining moves must be at or below the line $y=x-1$. Observe that translation of the entire grid keeps the number of paths the same. Therefore, this problem is equivalent to that going from $(0,0)$ to $(n-i-1, n-i-1)$. therefore, this segment is clearly equivalent to F_{n-i-1} ways. $\square$

f) The last time the path intersects $y=x$ before reaching the end can be at any point between $(0,0)$ and $(n-1,n-1)$. From part (e), the answer for each i equals $F_i F_{n-i-1}$. Therefore, the total number of ways to reach (n,n) from $(0,0$ is the total for all possible i, which is $\sum_{i=0}^{n-1} F_i F_{n-i-1}$.

g) Let T_k be the number of ways to construct a tree of size k. We claim $T_n = F_n$.

To construct each subtree for a tree of size k, we will need to put i nodes into the left subtree for some integer i. Then, the other $k-i-1$ nodes that aren't in the root or the left subtree must necessarily be in the right subtree. But we also have to loop over all possible sizes i from 0 to $k-1$ as the $(0,k)$ and $(k,0)$ cases are equal, so $T_k = \sum_{i=0}^{k-1} T_i T_{k-i-1}$.

This is the same relation as is used to find F_n. Moreover, note that $T_0 = T_1 = 1$, similar to how $F_0 = F_1 = 1$, so the base cases are also identical. Therefore, we must have $T_n = F_n$ for all $n \in \mathbb{Z}$.

3 Fermat's Wristband

a) k^p

b) $k^p - k$

c) Because p is prime, there are p equivalent rotations for each string with two or more different colors, and only 1 equivalent rotation for each string with one color only.

Thus, the number of possible rotations is $\frac{k^p-k}{p} + k$.

d) Because the number of possible rotations is a whole number, $k^p - k$ must be divisible by p, or, in other words, $k^p \equiv k \pmod{p}$.

4 Counting on Graphs + Symmetry

a) There are $6! = 720$ ways to color the six faces. Each face has six possible locations, and if one location is fixed, there are four possible rotations of the cube. Therefore, there must be $6 \times 4 = 24$ rotations of each colors. Therefore, there are $\frac{720}{24} = 30$ colorings.

b) There are $n!$ ways to rearrange the beads and n ways to rotate each arrangement, for a total of $\frac{n!}{n} = (n-1)!$ ways.

c) There are $\frac{n(n-1)}{2} = \binom{n}{2}$ possible edges, and any subset of them will form a valid undirected graph. Therefore, there are $2^{\binom{n}{2}}$ graphs.

d) For every subset of the vertices of size $k \geq 3$, we must have at least one cycle. There are $\binom{n}{k}$ possible unordered subsets of size k. Each subset can be ordered in $k!$ ways, but rotations of the same sequence are considered identical and each sequence has k rototations. Therefore, there are $(k-1)!$ rotations for each subset. Thus, for each k we have $\binom{n}{k}(k-1)!$ ways, so the total is

$$\sum_{k=3}^{n} \binom{n}{k}(k-1)!$$

□

5 Proofs of the Combinatorial Variety

a) Imagine we have n people in a club, we want to choose k officers and one president for any $k \in [0, n]$. There are a total of $k\binom{n}{k}$ ways to do that for each k. Summing for all valid k, we find a sum of $\sum_{k=0}^{n} k\binom{n}{k}$. However, now let's choose a president first and then choose remaining officers. We have n ways to choose the president, and among the $n-1$ other club members, we can choose an arbitrary number of officers. There are $\sum_{k=0}^{n-1} \binom{n-1}{k}$ ways to get the arbitrary number of officers, so there are $n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}$ ways to select the same scheme using this method. Therefore, the left and right hand sides must be equivalent. □

b) Let's have three committees A, B, C, such that there are n members each. We try to form a super-committee with m members, some from each of A, B, or C. However, it does not matter how many are from each. Note that there are $\binom{3n}{m}$ ways to choose such a supercommittee because the number in each committee doesn't matter. However, if we were to go by committee, we can choose a from A, b from B, and c from C. We see that $a + b + c = m$. The total number of ways to choose subcommittees from members of A, B, C is the sum over $\binom{n}{a}\binom{n}{b}\binom{n}{c}$ over all valid (a, b, c). Therefore, the left hand side and right hand side count equivalent values. □

6 Fibonacci Fashion

a) We use induction on t. For $t = 1$, we have that there are $2 = F_{1+2} = F_3$ such sequences. For $t = 2$ we have $3 = F_{2+2} = F_4$ such sequences: all of them except for 00.

Let's claim that for all $i < n$, there are F_{i+2} ways to get a sequence of i bits. Now, let's add a character to an existing bit string to get a sequence of length n. If the last character equals 1, there is no extra restriction on the rest of the sequee, and there are F_{n+1} ways to build it.

However, if the last character equals 0, the second-to-last character must be 1. There are no extra restrictions on the other characters, so there are F_n ways for this case.

Therefore, the total number of sequences for n is $F_n + F_{n+1} = F_{n+2}$, and we are done by induction. □

b) Let each accessory be represented by a bit string of length t, where 1 means the accessory is not used and 0 means the accessory is used. Note that there are F_{t+2} ways to make a sequence of days where no accessory is used in two consecutive days from part (a). Since there are n accessories the total number of ways is $(F_{t+2})^n$.

Now, we count inductively by days. Starting on day 1, we can pick x_1 accessories in $\binom{n}{x_1}$ ways. Assuming we've picked a certain x_i accessories on day i, we cannot pick the same accessories on day $i+1$, so we must choose x_{i+1} accessories from the $n - x_i$ given. Therefore on day 1 we have a total of $\binom{n}{x_1}$ and on a given day i from day 2 to day t, we have a total of $\binom{n-x_{i-1}}{i}$ ways. Now, all that remains is for us to add all possible values of each x_i and multiply all days together to get the total number of ways to choose n accessories over the next t days such that no accessory is worn two days in a row. The result of that equation is

$$\sum_{x_1 \geq 0} \sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n}{x_1} \binom{n - x_1}{x_2} \cdots \binom{n - x_{t-1}}{x_t}.$$

Since the left-hand-side and right-hand-side of this equation end up calculating the same thing with two different approaches, the equation must hold. $\square$