Lecture 6

span reprisal

find the basis?

$\cancel{\vec{v}_1 = (0,0,0)}$ $\quad \vec{v}_2 = (2,1,0) \quad \vec{v}_3 = (e, \pi, -1) \quad \vec{v}_4 = (1,1,1)$

note: $1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3 + 0 \cdot \vec{v}_4 = \vec{0}$. so we remove v_1.

$\vec{v}_2$ isn't already in a span. It's not the 0 vector.

$\vec{v}_3$ is not in span of $\vec{v}_2$ because of the nonzero 3rd element in $\vec{v}_2$.

$\vec{v}_4 \to a_2 \vec{v}_2 \stackrel{?}{=} (e+1, \pi+1, 0)$. There is no a_2 solving this.

example 2:

$(1,0,1)$ $\cancel{(0,0,0)}$, $\cancel{(-\pi, 0, \pi)}$, $(1,2,3)$, $(4,5,6)$

$\to (1, 0, 1) \quad (1,2,3) \quad (4,5,6)$

is 0 $\quad$ $\times \frac{5}{2}$ $\to$ doesn't match.

$\boxed{\text{basis: } \{(1,0,1) \;\; (1,2,3) \;\; (4,5,6)\}}$

Basis

Claim: length of any linearly ind. list $\leq$ length of any spanning list.

Proof. Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ linearly independent and $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$ is the span of V. Assume FTSOC $m \leq n$.

Step 1: Consider the list

$\vec{v}_1, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$.

$\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$ is the span, so $\vec{v}_1$ must be in this span and it's linearly dependent.

But $\vec{v}_1$ not in span of previous.

So there exists $j \in \{1, \dots, n\}$ s.t. $\vec{w}_j$ in span of $\{\vec{v}_1, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_{j-1}\}$. So we remove $\vec{w}_j$ and continue in the same way.

$\Rightarrow$ add $\vec{v}_2 \Rightarrow \vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$. so we can find & remove another $\vec{w}_k$.

So some $k \in \{1, \dots, j-1, j+1, \dots\}$ s.t. $\vec{w}_k$ can be removed without changing the span.

As a result, we can perform the swap m times, since none of the $\vec{v}_i$ are in the span of the preceding ones.

↳ so we end up w/ $\{\underbrace{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m}_{}, \underbrace{\vec{w}_{??} \dots}_{\text{possibly some } w}\}$

we always need to remove w_j.

↳ however, if $m > n$, we will run out of $\vec{w}$'s to remove. ⟶ the spanning set will be at most size n. otherwise, we will have to remove something in the set to add v_{n+1}, but we don't have any w_j left. So we have to remove some v_i, contradiction.

↳ we say that a list of vectors is a $\boxed{\text{basis}}$ for U if it's simultaneously linearly independent & spanning set (for U).

Thm: every vector space that's finite dimensional has a finite basis.

↳ finite-dimensional : finite list of vectors spans the space.

↳ needa start reading Axler before class.

↳ remove some vecs. from span to create basis

⟹ finitely sized basis. ∎

↳ basis needs to be linearly-independent & spanning. so $|\text{one basis}| \le |\text{another basis}|$. But swapping "one basis" w/ "another basis" ⟹ $|\text{one basis}| \ge |\text{another basis}|$

⟹ $|\text{one basis}| = |\text{another basis}|$

⟹ all bases of V are same sized.

$\dim \mathbb{R}^n = n$, classical basis $(1,0,\ldots,0)$, $(0,1,0,\ldots,0)$ $\cdots$ $(0,0,\ldots,1)$

How can you restate the lin. ind. of $v_1, \ldots, v_k$ using the notion of span & direct sum?

$(\vec{v}_1, \ldots \vec{v}_k)$ is lin. indep. $\iff$ $\text{span}(\vec{v}_1) \oplus \text{span}(\vec{v}_2) \oplus \cdots \oplus \text{span}(\vec{v}_k)$ is direct.

$W \subseteq V$, V is finite-dimensional. is there a basis for W?

$\hookrightarrow$ $V = \mathbb{R}^3$, $W := \{(x, x, 0) : x \in \mathbb{R}\}$

$\hookrightarrow$ $(1,0,0)$ $(0,1,0)$ $(0,0,1)$

If W is nontrivial, then $\exists$ $w_1 \in W$, and w_1 is not the 0 vector. If $\text{span}(w_1) = W$, then done.

If not then take another vector $w_2 \notin \text{span}(w_1)$. Now if $\text{span}(w_1, w_2)$, then we're done. Otherwise $\exists w_3 \ldots$

I see where this is going. But it can't go more than $\dim V$ steps, because the # vectors we're gathering now can't exceed the size of the spanning set of V, b/c a subset of that spans W.

$\rightarrow$ Moreover, $W \subseteq V$ & V finite dimensional, W has direct complement U s.t. $W \oplus U = V$.

We already know $\exists$ a basis $w_1 \ldots w_k$ for W. Likewise we have $v_1 \ldots v_n$ for V.

List the w's first, then v's:

$$\underbrace{w_1, w_2, \ldots, w_k}_{\text{lin. independent}},\ \underbrace{v_1, \ldots v_n}_{\text{could } \exists \text{ dependencies, that we throw out.}}$$

anything else we add is the direct complement!