110 L7

Thm: given finite dimension V, then finite dimension U is ~~subspace~~ of U s.t. $\dim(U) \le \dim(V)$.

↳ any lin. ind. list $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ into a finite-dimensional space V can be expanded into a basis.

↳ any spanning list $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ can be thinned down into a basis of V.

⇒ $|$lin. ind. list$| \le |$spanning list$|$

⇒ proof of $\dim U \le \dim V$ is similar to previous ones, i.e., starting w/ the empty list, we keep adding vectors in the subspace U that are not in the span of the current list.

This terminates because U (and V) are finite-dimensional; Then,

# of these vectors $\underbrace{(\text{lin. ind } \& \text{ spanning } U)}_{\text{spanning happens @ end}} \le$ # vectors in basis of V.

$|$basis of $U| \le |$basis of $V| \Rightarrow \dim U \le \dim V$. □

Consider:

$$V = \mathbb{R}^3, \ U = \{(x, y, 2x) : x \in \mathbb{R}\}$$

$(1,0,2) \in U$. $(0,1,0) \notin \operatorname{Span}((1,0,2))$.

⇒ $[(1,0,2), (0,1,0)]$ does, however, span.

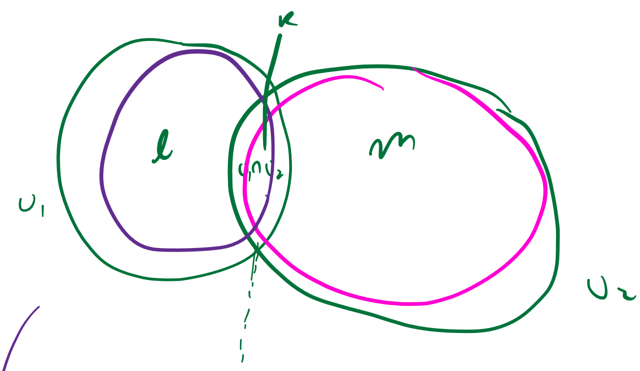
⇒ this means $\operatorname{Span}((1,0,2), (0,1,0)) = U$. $\quad u = x(1,0,2) + y(0,1,0) \ \forall u \in U$.

Say U_1, U_2 are subspaces of V. suppose $\dim U_1, \dim U_2 < \infty$.

$\exists$? formula for $\dim(U_1 + U_2)$, regardless of directness, in terms of $\dim U_1$, $\dim U_2$, $\underbrace{\dim(U_1 \cap U_2)}_{< \infty}$.

$\dim U_1 + \dim U_2 \Rightarrow$ size of two basis, overcounting the overlap?

Claim: $\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$.



take basis of $U_1 \cap U_2$. $\Rightarrow$ $\boxed{\vec{u}_1, \dots, \vec{u}_k}$ $\to$ this can be extended to basis of U_1 using l vecs.

can also extend to U_2, using m vecs.

claim $\longleftrightarrow$ $\dim(U_1 + U_2) \to k + l + m = (k+l) + (k+m) - k$,

To establish this equality, let's establish that $\vec{u}_1, \dots, \vec{u}_k, \vec{v}_1, \dots, \vec{v}_l, \vec{w}_1, \dots, \vec{w}_m$ is a basis for the sum, (linearly indep. & spanning).

$\mathrm{span}(\vec{u}_1, \dots, \vec{u}_k, \vec{v}_1, \dots, \vec{v}_l) = U_1$

$+ \ \mathrm{span}(\vec{u}_1, \dots, \vec{u}_k, \vec{w}_1, \dots, \vec{w}_m) = U_2$.

$\mathrm{span}(\text{union of these lists}) = U_1 + U_2$. } should thus be spanning.

$\vec{u}_1, \dots, \vec{u}_k, \vec{v}_1, \dots, \vec{v}_l, \vec{w}_1, \dots, \vec{w}_m$.

Now to check linear independence for $\vec{u}_1, \dots, \vec{u}_k, \vec{v}_1, \dots, \vec{v}_\ell, \vec{z}_1, \dots, \vec{w}_m$.

↳ say we have $a_1 u_1 + \dots + a_k u_k + \beta_1 v_1 + \dots + \beta_\ell v_\ell + \gamma_1 w_1 + \dots + \gamma_m w_m = \vec{0}$.

$$\underbrace{\alpha_1 u_1 + \dots + \beta_1 v_1 + \dots}_{U_1} = \underbrace{-(\gamma_1 w_1 + \dots +)}_{\in U_2,\ \text{but not in } U_1}$$

if $\vec{x} \in U_1$, $\vec{x} \in U_2$, then we have $\vec{x} \in U_1 \cap U_2$

$\Rightarrow$ $\vec{x}$ should be a lin. comb. of the $a_i u_i$.

$$-(\gamma_1 \vec{w}_1 + \dots + \gamma_m \vec{w}_m) = \delta_1 \vec{u}_1 + \dots + \delta_k \vec{u}_k.$$

$$\Rightarrow \underbrace{\delta_1 \vec{u}_1 + \dots \delta_k \vec{u}_k + \gamma_1 \vec{w}_1 + \dots + \gamma_m \vec{w}_m}_{\text{Basis of } U_2} = 0. \longrightarrow \gamma, \delta \text{ must all be } 0.$$

↓

a, β must also be 0.

$\Longrightarrow$ $\vec{u}_1, \dots, \vec{u}_k, \vec{v}_1, \dots, \vec{v}_\ell, \vec{m}_1, \dots, \vec{w}_m$ is lin. independent.

Linear Maps

- much more abstract?

function sends v/s from $V \to U$.

setup: 2 worlds $V \longrightarrow U$

vector spaces

- Let T be function from $V \to U$. it's a linear map if $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ $\forall \vec{v}_1, \vec{v}_2, \vec{v}$.

$T(\alpha \vec{v}) = \alpha T(\vec{v})$ $\forall \alpha \in \mathbb{F}$

ex: $V = \mathbb{R}^3$, $U = \mathbb{R}^2$.

we would have $T: (x, y, z) \mapsto (x, y)$.

$(x_1, y_1, z_1) \mapsto (x_1, y_1)$ | $(x_2, y_2, z_2) \mapsto (x_2 + y_2)$. } linear.

$(x_1 + x_2, y_1 + y_2, z_1 + z_2) \mapsto (x_1 + x_2, y_1 + y_2)$

test scalar mult. similarly.

non-linear. { $T_1: (x, y, z) \mapsto (x^2, y^2)$

↳ not linear b/c $(x_1^2 + x_2^2, y_1^2 + y_2^2)$

$\neq ((x_1 + x_2)^2, (y_1 + y_2)^2)$, usually

$\vec{v} = (1,1,1) \quad \alpha = 2.$

e.g. $\alpha\vec{v} = (2,2,2) \overset{T_2}{\longmapsto} (4,4) \longrightarrow$ not homogeneous

$T_2(\vec{v}) + T_2(\vec{v}) \Rightarrow (1,1,1) \to (1,1)$

$\times 2\,[(2,2,2) \to (2,2)] \Rightarrow \alpha T_1(b) \neq T_1(\alpha b).$

Consider $S: U \to U$, $p \mapsto P$

$\hookrightarrow P(x) = \int_0^x p(t)\,dt.$

not important, will return to.