Lecture 8: Linear Maps

- $T: V \to V \Rightarrow$ needs to satisfy

$$T(\vec{a}+\vec{b}) = T(a) + T(b)$$
$$T(a\vec{b}) = aT(\vec{b}).$$

- given 2 vec. spaces V, W over same field $\mathbb{F}$, and

$$\mathcal{L}(V,W) = \{T: V \to W \text{ and } T \text{ is linear}\}$$

  - it turns out that in some "natural" sense, $\mathcal{L}(V,W)$ can be viewed as a vector space over $\mathbb{F}$ as well.

find sum map

$$\begin{cases} (T_1+T_2)(\vec{v}) := T_1\vec{v} + T_2\vec{v} & \forall\, \vec{v} \in V. \\ (aT_1)(\vec{v}) := a\cdot(T_1(\vec{v})) & \forall\, a \in \mathbb{F} \end{cases}$$

$$(T_1+T_2)(a_1\vec{v}_1 + a_2\vec{v}_2) := T_1(a_1\vec{v}_1 + a_2\vec{v}_2) + T_2(a_1\vec{v}_1 + a_2\vec{v}_2).$$

$$= (a_1T_1\vec{v}_1 + a_2T_1\vec{v}_2) + (a_1T_2\vec{v}_1 + a_2T_2\vec{v}_2).$$

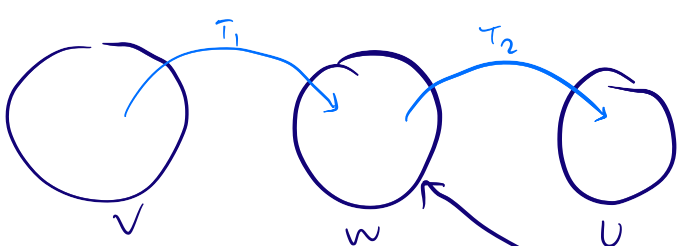
Goal: show that this is a map.

$$= \underbrace{a_1(T_1+T_2)\vec{v}_1 + a_2(T_1+T_2)\vec{v}_2}_{\substack{\text{as } (T_1+T_2) \text{ works,} \\ \text{we can see that } T_1+T_2 \text{ is a linear map.}}}$$

$$=$$

Likewise, we can check aT_1 is also a linear map for scalar a, LM T_1

$\Longrightarrow$ set of linear maps b/w vec. spaces is a vector space.



$T_2 \circ T_1 : V \to U$ is linear?

$$T_2 \circ T_1(a_1\vec{v}_1 + a_2\vec{v}_2) = T_2(a_1T_1\vec{v}_1 + a_2T_1\vec{v}_2) = a_1T_2(T_1\vec{v}_1) + a_2T_2(T_1\vec{v}_2)$$

$$= a_1(T_2\circ T_1)(\vec{v}_1) + a_2(T_2\circ T_1)(\vec{v}_2)$$

T_2 applied to T_1

Remember we always have $T_V : V \to$, $I_W : W \to W$

So for $T \in \mathcal{L}(V, W)$

$I_W \circ T = T$, $T \circ I_V = T$.

$W \to W$ $W \to V$ $W \to V$ $V \to V$

Setting $T : V \to W$, dim $V = n < \infty$.

- T is uniquely determined by n vectors $T\vec{v}_1, \cdots, T\vec{v}_n$ where $\vec{v}_1, \cdots, \vec{v}_n$ is any basis of V.

$$\vec{v} = a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n$$

$$T\vec{v} = T(a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n)$$

$$= \underbrace{a_1 T(\vec{v}_1) + \cdots + a_n T(\vec{v}_n)}_{\vec{v}_i\text{'s must be a basis for this reason.}}$$

choice of $\vec{w}_i$ is completely arbitrary, pick any w and you'll get a linear map.

Sometimes, T needs special properties.

- 1-1 (injective)
- onto (surjective)

Consider $\text{Null}(T) = \{\vec{v} : T\vec{v} = \vec{0}\}$.

Claim: $\text{Null}(T)$ is a subspace of V.

- 0 in $V \to 0$ in W. ✓
- $\vec{v}_1, \vec{v}_2 \in \text{Null}(T)$.

$\vec{v}_1 + \vec{v}_2 \overset{?}{\in} \text{Null}(T)$.

$\Rightarrow T(\vec{v}_1 + \vec{v}_2) \overset{?}{=} 0 \Rightarrow T\vec{v}_1 + T\vec{v}_2 = 0 + 0 = 0$.

example: $P_{\le 3}(\mathbb{R})$.

highest powers diff = linearly indep. for

$3x^2 + x + 1$, $16x$, -1, $x^3 + 3$

$6x^2 + x$ $16x$ 0 $3x^3$

uniquely determines T.

$T : p(x) \mapsto xp'(x)$.

choose the outcomes which uniquely determines the linear map

$\text{Range}(T) := \{T\vec{v} : \vec{v} \in V.\}$

Claim: $\text{range}(T)$ is a subspace of W.

$0 \to 0$.

- $T(\vec{v}_1 + \vec{v}_2) = T\vec{v}_1 + T\vec{v}_2 = \vec{w}_1 + \vec{w}_2$, so $\vec{w}_1 + \vec{w}_2 \in$ Range.
- $\alpha \in \mathbb{F}$, $\vec{w} = \text{Range}(T)$, then there exists a $\vec{v} \in V$ s.t. $T\vec{v} = \vec{w}$, and $T(\alpha\vec{v}) = \alpha T(\vec{v}) = \alpha\vec{w}$. so $\alpha\vec{w} \in$ range.

Incidentally, $T\vec{u} = T\vec{v} \Rightarrow \vec{v} - \vec{u} \in \text{Null}(T)$.

Lemma:

$T \in \mathcal{L}(V,W)$ injective $\iff$ Null space is trivial, $\text{Null}(T) = 0$.

- we cannot have 2 inputs w/ identical output.
  - If T is 1-1, then $\vec{0}_V$ is the only vector mapping to $\vec{0}_W$.
- if $\text{Null}(T) = \{0\}$, then $T\vec{v}_1 = T\vec{v}_2 \to \vec{v}_1 - \vec{v}_2 \in \text{Null}(T)$

$\to \vec{v}_1 - \vec{v}_2 = \vec{0}$

$\to \vec{v}_1 = \vec{v}_2$.