Problemset 3

Albert Ye

September 17, 2022

1 Build-Up Error?

A counterexample to this would be a graph with 4 vertices and 2 edges such that $1, 2$ and $3, 4$ both have one edge between them. All vertices have a degree of at least one, but there is no connection between the two components so the graph is disconnected.

This proof is incorrect because it relies on a specific construction for a graph of $n + 1$ vertices. It relies on there being $n + 1$ vertices such that the last vertex is linked to another vertex, and the other n nodes being connected. However, the logic of the proof does not apply for all constructions, and there are cases where the induction would fail. For example, consider the case where instead of going to another one of the n nodes, the $n + 1$th vertex had an edge connecting to itself. Then, we would not be able to assume that the inductive step holds.

Generally in an induction, the inductive step must cover all possible cases for the $k + 1$ case, as opposed to just a specific construction that works.

2 Tournament

For a (really boring) two-player tournament, we would have 2 vertices connected by one edge, Regardless of who wins, we can traverse the graph by going from the losing vertex to the winning vertex.

Assume that the case holds for n players. Then, we can add the $n + 1$th player as a vertex that has edges leading to or from all n existing players. Assume WLOG that the Hamiltonian path is $1 \to 2 \to \ldots \to n$, implying that player 1 loses to player 2 and so on to player n.

Clearly, if player $n + 1$ beats player n, then we have another Hamiltonian path. If $n + 1$ loses to n, then we can run the same comparison but with $n - 1$. We will be back to the same two conditions: either $n + 1$ beats $n - 1$ and goes between $n - 1$ and n in the Hamiltonian path, or it loses and gets handed down to $n - 2$. We can repeat this all the way down to player 1. If player $n + 1$ loses to player 1, it is placed at the beginning of the Hamiltonian path. Regardless of how player $n + 1$ plays, it will always have a place in our Hamiltonian path. Hence, we are done by induction. $\square$

3 Proofs in Graphs

a. For the base case $n = 2$, we have one vertex leading to another vertex. The destination vertex will be reachable from the other vertex in distance 1.

   Assuming that we have n vertices such that all vertices reach vertex k in at most two moves. Equivalently, all vertices are a distance 2 from k. Then, for vertex $n + 1$, we have a few possible constructions:

   The first would be if $n + 1$ led to k. Then, we would be done because $n + 1$ would lead to k in just one move.

   The second would be if k led to $n + 1$, but $n + 1$ led to a vertex j that led to k. The claim would still hold because $n + 1$ is a distance of 2 away from k.

Finally, if $n+1$ does not lead to any vertices a distance of 0 or 1 from k, that means it only leads to vertices a distance 2 from k. Therefore, k and all vertices leading to k must have a direct connection to $n+1$. All vertices of distance 2 from k must directly connect to a vertex of distance 1 from k by definition, so they must also be a distance of 2 from $n+1$.

All possible constructions of $n+1$ nodes lead to some vertex being a distance of at most 2 from every other vertex, so the claim is proven by induction. $\square$

b. Denote the odd-degree vertices as $o_1, o_2, \ldots, o_{2m}$. Then, we add m "phantom" edges $(o_1, o_2), (o_3, o_4), \ldots, (o_{2m-1}, o_{2m})$, making the degree of o_i even for all $i \in [1, 2m]$.

Now, we will have an Eulerian tour containing all of the real and phantom edges. Without loss of generality, assume that this Euler tour ends with a phantom edge. Then, when we remove the m phantom edges from the graph we have m walks that all collectively traverse the whole graph.

c. If G is bipartite, then assume it has two sides V_1 and V_2. A move from a node in V_1 would necessarily end up in V_2 by definition, and a move from V_2 would necessarily end up in V_1, so an even number of moves is needed to go between two nodes in V_1. Similarly, an even number of moves is needed to traverse a tour in G.

Now, assume there are no tours of odd length in G. Fix an arbitrary vertex v in a connected component C. Then, let's color the nodes in C that are an odd distance from C black, and color the nodes in C that are an even distance from C white. Because the cycle length is even, we see that black nodes necessarily link only with white nodes, and vice versa. Therefore, the graph that results by iterating over all vertices v and components C must be bipartite. $\square$

4 Planarity and Graph Complements

a. $\overline{G}$ must have all $\frac{v(v-1)}{2}$ edges of K_v except for the e edges in G, so there are a total of $\boxed{\frac{v^2 - v - 2e}{2}}$ edges.

b. Using Euler's theorem, we have $e \leq 3v - 6$ for a planar graph with e edges and v vertices. For a graph with 13 vertices, a planar graph can have at most $3 \cdot 13 - 6 = 33$ edges. However, there are a total of $\frac{13 \cdot 12}{2} = 78$ edges. Regardless of how many edges we give G, we cannot keep G as a planar graph while also having a planar graph $\overline{G}$ because $\overline{G}$ would need at least $78 - 33 = 45$ edges.

For $13 + k$ edges, we can have $p = 2(33 + 3k)$ total edges to keep $G, \overline{G}$ planar and need $q = \frac{k^2 - k}{2}$ edges total between G and $\overline{G}$. As p grows linearly with k and q grows quadratically with k, we can ensure that $q > p$ must hold. Therefore, we cannot have both $G, \overline{G}$ be planar for $v > 13$.

c. Imagine $\overline{G} = K_5$. We know that $\overline{G}$ cannot be planar by Kuratowski's theorem. Then, G would have $p \geq 8$ remaining vertices all completely connected. This means that we can pick 5 vertices in G such that they are all interconnected with each other, implying that G also has a subgraph of K_5. This construction leads to a case where neither G nor $\overline{G}$ are planar, contradicting the claim. $\square$

5 Touring Hypercube

a. For a hypercube in n dimensions, each vertex connects to exactly n other vertices, one vertex per bit. Therefore, if n is even, all vertices have even degree so an Eulerian tour is possible. Otherwise, no vertices will have even degree so an Eulerian tour is not possible. Therefore, there exists an Eulerian tour in an n-D hypercube if and only if n is even. $\square$

b. Let's once again define each vertex of the cube by a number from $00\ldots0_2 = 0$ to $11\ldots1_2 = 2^n - 1$ with n zeroes and n ones, respectively. Now, for a 0-dimensional hypercube we just have a single point (which automatically has a Hamiltonian path).

We can use induction to prove a Hamiltonian path exists for $n+1$ dimensions given the Hamiltonian path for n dimensions. WLOG, assume that the Hamiltonian path starts at vertex 0 and ends at vertex $2^n - 1$ for the n- dimensional hypercube.

Now, for each vertex of the $n+1$-dimensional hypercube, consider the last n vertices. We find that we have two copies of each number from $00\ldots0$ to $11\ldots1$ such that one copy is a vertex $v < 2^n$ (or first digit 0), and one copy is a vertex $v \geq 2^n$ (first digit 1). We must then have two copies of the Hamiltonian path from vertex 0 to vertex $2^n - 1$, one for the vertices with first digit 0 and one for those with first digit 1.

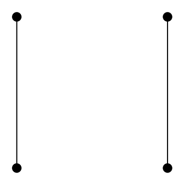
We can use this to construct a Hamiltonian path for an $n+1$-dimensional hypercube: start at vertex 0 and use the Hamiltonian path for an n-dimensional hypercube to get to vertex $2^n - 1 = 011\ldots1$. At this point, we can go from vertex $2^n - 1 = 011\dots1$ to vertex $2^{n+1} - 1 = 111\dots1$. However, we also know there exists a Hamiltonian path from vertex $2^n + 0 = 2^n$ to vertex $2^n + 2^n - 1 = 2^{n+1} - 1$. We can reverse every edge along this path to get a Hamiltonian path from $2^{n+1} - 1$ to 2^n. Since all vertices are covered by this traversal, we have a Hamiltonian path for an $n+1$-dimensional hypercube given an n-dimensional one. $\square$

6 Connectivity

a. We prove the stricter claim that for any two non-adjacent vertices u, v, there must be a third vertex w such that u, v are both adjacent to w. Let the set of adjacent nodes to u be U_a, and the set of adjacent nodes to v be V_a. Then, $|U_a| + |V_a| \geq n - 1 \implies |U_a| + |V_a| > n - 2 = |U_a \cup V_a|$, so $|U_a \cap V_A| \geq 1$ must hold. Thus, if all non-adjacent vertices u, v satisfy $\deg u + \deg v \geq n - 1$, then there must be some vertex w that is adjacent to both u and v.

   If u can reach v for all non-adjacent pairs of vertices (u, v), then the graph must be connected by definition. $\square$

b. In the following graph with 4 vertices, all vertices have degree 1 so all pairs of vertices have a total degree of $4 - 2 = 2$. However, this graph is evidently not connected:

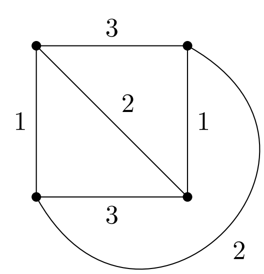


c. If all vertices have degree at least $\frac{n}{2}$, then for every non-adjacent pair of vertices u, v, we would have $\deg u + \deg v \geq \frac{n}{2} + \frac{n}{2} > n - 1$. Therefore, by part (a) the claim must hold.

d. By the Handshake Lemma, the number of vertices with odd degree. Thus, it is impossible for the sum of degrees in a connected component to be odd. Therefore, if there are only 2 nodes with odd degree in a graph, then both will need to be in the same connected component to ensure that the sum of degrees in said component remains even.

7 Edge Colorings

a. One possible coloring is as follows:



b. Given an edge e, no other edges adjacent to e can have the same color. Each end of the edge e can have at most $d - 1$ other edges in order to have maximum degree of d. Therefore, the maximum number of edges adjacent to any current edge is $2(d - 1) = 2d - 2$. If $2d - 1$ colors were used, then there will always exist an open color for every edge.

c. In a tree, each edge e between vertices p, v (where p is closer to the root than v) can still have $d - 1$ adjacent edges on each end, but the nature of the connected vertices differ on each end. Assume that both sides of e have $d - 1$ adjacent edges. Then the $d - 1$ adjacent nodes connected to v are child nodes and the $d - 1$ adjacent nodes connected to p are sibling nodes.

   By the definition of a tree, sibling nodes and child nodes cannot have edges between them, since that would add a cycle to the graph. Therefore, the $d - 1$ edges leading from p and the $d - 1$ edges leading from v do not affect each other, so we can make a stricter bound of d colors for trees.