$T\vec{v}_k = a_{1,k}\vec{w}_1 + a_{2,k}\vec{w}_2 + \cdots + a_{n,k}\vec{w}_n$, for k from 1 to m.

domain

$$\begin{array}{c} \vec{w}_1 \rightsquigarrow \\ \vec{w}_2 \rightsquigarrow \\ \\ \vec{w}_n \rightsquigarrow \end{array} \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n,1} & \cdots & & a_{n,m} \end{bmatrix}$$

codomain

example: $D: P_{\leq 3}(\mathbb{R}) \to P_{\leq 2}(\mathbb{R})$

derivative

$\to P_{\mathbb{R}}(\mathbb{R})$

$\to$ codomain is dictated by specifics of the problem, but you can choose.

Basis of $P_{\leq 3}(\mathbb{R}) = \{1, x, x^2, x^3\}$ of $P_{\leq 2}(\mathbb{R}) = \{1, x, x^2\}$.

$$M(D) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$1 \mapsto 0$

$x \mapsto 1$

$x^2 \mapsto 2x$

$x^3 \mapsto 3x^2$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- e.g. if we keep same basis of V and use it as basis for W, we will add a row of 0's.

- take the original setup: $D: P_{\leq 3}(\mathbb{R}) \to P_{\leq 2}(\mathbb{R})$, and the following bases: $1,\ x-1,\ (x-1)(x-2),\ x(x-1)(x-2)$

  $1,\ x-1,\ (x-1)(x-2)$.

$x(x-1) = (x-2)(x-1) + 2(x-1)$

$(x-1)(x-2) + (x-2)$

$= (x-1)(x-2) + (x-1) - 1$

$1 \mapsto 0$

$x-1 \mapsto 1$

$(x-1)(x-2) \mapsto (x-1) + (x-2) = 2(x-1) - 1$

$3(x-1)(x-2) + 3(x-1) - 1$

$x(x-1)(x-2) \mapsto x(x-1) + x(x-2) + (x-1)(x-2) = (x-1)(x-2) + (x-1)(x-2) + 2(x-1)$

$+ (x-1)(x-2) + (x-1) - 1$

So, in sum,

$1 \mapsto 0$

$x-1 \mapsto 1$

$(x-1)(x-2) \mapsto 2(x-1)-1$

$x(x-1)(x-2) \mapsto 3(x-1)(x-2)+3(x-1)-1.$

WRT these alt. bases,

$$\tilde{M}(D) = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$M(T+S) = M(T)+M(S) \to$ add coeffs

$M(\alpha T) = \alpha M(T) \to$ multi. coeffs

$M: \mathcal{L}(V,W) \to \mathbb{F}^{\dim W \times \dim V}$

is also a linear map!

And also an isomorphism (or bijective linear map)

- M injective: null space is trivial.

$\text{Null}(M) = \{T \in \mathcal{L}(V,W) : M(T) = 0\}$

↳ That means for any $T \in \text{Null}(M)$, $T\vec{v}_k = \vec{0}\ \forall k.$

↳ So, T sends all bases to 0, which means sum of all bases to $0 \to$ all V sent to 0.

↳ Therefore, T = zero map $\to M(T) = [0]$.

⇒ null space is trivial.

- M surjective: Take an arbitrary element in $\mathbb{F}^{\dim W \times \dim V}$

$A = [a_{ij}]_{i=1, j=1}^{\dim W\, \dim V}$

define T by $T\vec{v}_j := \sum_{i=1}^{\dim W} a_{ij} \vec{w}_i.$

This determines a linear map $T: V \to W$, and its matrix $M(T) = A$

So, M is an isomorphism b/w $\mathcal{L}(V,W)$ and $\mathbb{F}^{\dim W \times \dim V}$.

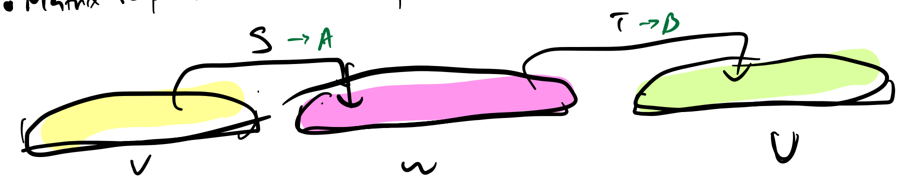
Claim: $\dim \mathbb{F}^{n,m} = nm$.

- canonical basis of $(E_{i,j}, i=1 \cdots n, j=1 \cdots m)$.

for $\mathbb{F}^{2,3}$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = E_{1,3}$

Corollary: $\dim(\mathcal{L}(V,W)) = \dim V \cdot \dim W$. Because M is an isomorphism, and isomorphisms preserve dimensions.

- Matrix respects the composition of linear maps



$\dim V = m$ $\quad \dim W = n$ $\quad \dim U = k$

$M(S \circ T) = ?$

$$(S \circ T)v_j = S(Tv_j) = S\left(\sum_{i=1}^{n} a_{ij} w_i\right)$$

$$= \sum_{i=1}^{n} a_{ij} S w_i = \sum_{i=1}^{n} a_{ij} \sum_{l=1}^{k} b_{li} \vec{u}_l = \left(\sum_{l=1}^{k}\sum_{i=1}^{n} a_{ij} b_{li}\right) \vec{w}_l$$

$b_{li}\, a_{ij}$

$\sum_i a_{ij} b_{li} = (BA)_{l,j}$ usual matrix products

So comp. of lin. maps is represented as a matrix product.