Math 110: Linear Algebra Done Wrong

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Contents

CHAPTER 1
WEEK 1

1.1 Lecture 1

1.1.1 Logistics

- 30% HW, 30% MT, 40% final
- or if two midterms, 20% HW, 20+20% MT, 40% final
- OH Saturday 2-3 @ Free Speech Cafe, Friday 6-7PM online
- If in doubt, refer to the book

1.1.2 Complex Numbers

Definition 1.1 (Complex Number). A **complex number** is an ordered pair of real numbers (a, b), which can be then represented as $a + bi$ where $i = \sqrt{-1}$.

A complex number can also be represented in exponential form, so $z = pe^{i\varphi}$, where φ is the angle / **argument** of z.

Theorem 1.2 (DeMoivre's). Complex numbers can be written both in the form of $pe^{i\varphi}$ or $p(\cos\varphi + i \sin\varphi)$.

The proof for this uses Taylor expansion but I forgot / I am too lazy to actually write it down and it doesn't matter much for this course regardless.

1.2 Lecture 2

1.2.1 Fields

Definition 1.3 (Field). A **field** is a set and two operations of addition and multiplication that also satisfies a number of properties.

Addition must be **closed**, be **commutative**, have an **additive identity**, be **invertible**, and be **associative**.

Multiplication excludes the additive identity, but it must also be **closed**, be **commutative**, have a **multiplicative identity**, be **associative**, be **invertible** (once again excluding the additive identity), and be **distributive**.

Common examples of fields include $\mathbb{R}$ and $\mathbb{C}$, but we can also prove that $\mathbb{Z}/p\mathbb{Z}$ is a field for prime p under modular addition and multiplication.

Lemma 1.4. $\mathbb{Z}/2\mathbb{Z}$ is a field.

Proof. fill in later lol ■

However, for non-primes, $\mathbb{Z}/p\mathbb{Z}$ is not a field. Consider the case for $\mathbb{Z}/6\mathbb{Z}$. $3x = 1 \implies 2(3 \cdot x) = 2 \cdot 1 = 2$, but $2 \cdot (3 \cdot x) = (2 \cdot 3) \cdot x = 0 \cdot x = 0$. This means that $\mathbb{Z}/6\mathbb{Z}$ isn't a field. More generally, for pq, we can use the case of $qx = 1$ and then try associativity on $p \cdot q \cdot x$.

1.2.2 Vector Spaces

Definition 1.5 (Vector Space). A **vector space** can be written as $(V, \mathbb{F}, +, \cdot)$ where V is a set, $\mathbb{F}$ is the field over which the space is defined, $+$ is a function $+ : V \to V$ (in other wodes, V should be closed under vector addition), and $\cdot$ is a function $\cdot : \mathbb{F} \to V$ which represents scalar multiplication.

From this definition, we gather that V is an abelian group under $+$, and $\cdot$ is associative with regards to scalar multiplication in $\mathbb{F}$. Also, $+$ and $\cdot$ must be distributive, or $(a + b) \cdot x = ax + bx$ and $a \cdot (x + y)$ = $ax + ay$. Be careful as to which operations are numerical multiplication and which ones are scalar multiplication.

1.3 Discussion 1

$\mathbb{F}^n$ is the coordinate space. The reason $\mathbb{F}$ must be a field is because it would be much more difficult otherwise because both working addition and working multiplication are needed in a vector space.

An example of a coordinate space is $\mathbb{R}^3 = \{(a, b, c) : a, b, c \in \mathbb{R}\}$.

1.3.1 Complex Numbers

Complex numbers exist because some equations are not closed in the real numbers (i.e. $x^2 + 1$ not closed for $x \in \mathbb{R}$).

Problem 1.1. Show that $\mathbb{C}$ is an $\mathbb{R}$-vector space with usual definitions of $+$ and $\cdot$.

We need to check all conditions.

- Commutativity, associativity, and invertibility in $+$ is pretty much given by the definition of the operation.

- Commutativity and invertibility for $\cdot$ are als o given by the operation. We also know from definition that $\cdot$ is distributive in the form of $\lambda(u+v) = \lambda u + \lambda v$, $(\lambda_1 + \lambda_2)u = \lambda_1 u + \lambda_2 u$, and that $\cdot$ is associative in $(\lambda_1 \lambda_2)v = \lambda_1(\lambda_2 v)$.
- Remember which multiplication operations are scalar and which multiplication operations are vector.
- Closure: Note that $(a+bi)+(c+di) = (a+c)+(b+d)i \in \mathbb{C}$, and that $a+c$ and $b+d$ are reals by definition. Moreover, note that $\lambda(a+bi) = (\lambda a) + (\lambda b)i \in \mathbb{C}$, as λa and λb are in $\mathbb{R}$ by definition as well.
- Additive identity: the value $0+0i$ represents the $\vec{0}$ vector in this case. If $\vec{v} = a+bi$, then $\vec{0}+\vec{v} = (0+0i)+(a+bi) = (0+a)+(0+b)i = a+bi = \vec{v}$.
- "Multiplicative identity": $1 \cdot (a+bi) = a+bi$. This follows because $1 \cdot (a+bi) = 1 \cdot a + 1 \cdot bi = a+bi$.

Problem 1.2. If V is a $\mathbb{F}$-vector space and $\mathbb{F}' \subset \mathbb{F}$, then V is an $\mathbb{F}'$-vector space.

Proof. Addition, and its properties, make no reference to the field.

All other axioms are carried over from V being an $\mathbb{F}$-vector space, because $\mathbb{F}'$ is a subfield of $\mathbb{F}$. We know that invertibility, associativity, and distributivity are guaranteed within $\mathbb{F}$ by the property of being a field. Closure of multiplication is also given in V because $\mathbb{F}'$ is a subset of $\mathbb{F}$ so $f \in \mathbb{F}'$ must also be an element of $\mathbb{F}$. ∎

Chapter 2
Week 2

2.1 Lecture 3

Given a field $\mathbb{F}$ and $n \in \mathbb{N}$, consider $\mathbb{F}^n = \{(f_1, f_2, \ldots, f_n)\}$ with componentwise addition and multiplication.

2.1.1 Vector Spaces with Infinite Vectors

Countable

If we take $n \to \infty$, then we have an infinitely long list $\mathbb{F}^\infty : \{(x_1, x_2, \ldots) : x_j \in \mathbb{F}\}$ for a countably infinite number of elements in the list.

> **Problem 2.1.** Consider this with componentwise addition and componentwise scalar multiplication. Is this a space?

Solution. We would need to prove five axioms given componentwise addition and multiplication, as the rest follow from the nature of the operation.

There is guaranteed to be a zero in the list $(0, 0, \ldots)$.

The additive inverse of $(x_1, x_2, \ldots)$ is $(-x_1, -x_2, \ldots)$.

The multiplicative inverse of $c(x_1, x_2, \ldots)$ is $\frac{1}{c}(x_1, x_2, \ldots)$.

Associativity and distributivity are extrapolatable from shorter sequences because the addition and multiplication schemes are the same. ■

Uncountable

If we now consider the case of $\mathbb{F}^\mathbb{R}$, we see that traditional list format breaks. Thus, we can try defining by function. $\mathbb{F}^S$ is defined as the set of functions $f : \mathbb{F} \to S$. Notice that $\mathbb{F}^\infty$ is equivalent to $\mathbb{F}^\mathbb{N}$ in this format.

If addition and multiplication are component-wise for squares, we can think of them as point-wise for functions. Thus, we have $h = f : g \implies h(a) = f(a) + g(a)$ for $a \in S$. For $a \in \mathbb{F}$ and $f \in \mathbb{F}^S$, we have that $\lambda f \in \mathbb{F}^S$ is define by $(\lambda f)(a) = \lambda \cdot f(a)$ for $a \in S$.

2.1.2 Structure of a Vector Space

Ignoring all tangible formations of a vector space as a set of lists or functions, what information do we have?

- The 0 element/vector of a vector space is unique. Suppose there are two zeroes, 0 and $0'$. Then, $0' = 0 + 0' = 0$.
- Additive inverses must thus be unique. If $a + b = 0$ and $a + c = 0$, then $b = b + 0 \implies b = b + a + c = 0 + c = c$.
- Also notice that additive inverse is actually always $-1 \cdot v$.

2.1.3 Subspaces

A subspace is a vector space defined inside another vector space, with the same rules for addition and multiplication applied over a subset of the values in the space.

Theorem 2.1. $W \subseteq V$ is a vector subspace of V wrt $+, \cdot$ iff

a) $0 \in W$

b) $\lambda v \in W \forall v \in W, \lambda \in \mathbb{F}$

c) $w_1 + w_2 \in W \forall w_1, w_2 \in W$

2.2 Lecture 4

Continuing from previuos lecture...

Example 2.1. If W_1, W_2 are subspaces of a vector space V, is $W_1 \cap W_2$ a subspace?

Solution. Yes. For zero, we have that $0 \in W_1, W_2$ by definition. Now, all there is to check is if $W_1 \cap W_2$ is additively and multiplicatively closed.

Additive Closure. If we have $u, v \in W_1 \cap W_2$, this means that $u, v \in W_1$ must hold AND $u, v \in W_2$ must hold. Therefore, we have that W_1 is a subspace, so $u + v \in W_1$; and W_2 is a subspace, so $u + v \in W_2$. Therefore, $u + v \in W_1 \cap W_2$ as well, so $W_1 \cap W_2$ is additively closed.

Multiplicative Closure. Similarly, note that if $u \in W_1 \cap W_2$, then $u \in W_1$ and $u \in W_2$. Therefore, $\lambda u \in W_1$ and $\lambda u \in W_2$, so $\lambda u \in W_1 \cap W_2$. ∎

Example 2.2. If W_1, W_2 are subspaces of a vector space V, is $W_1 \cup W_2$ a subspace?

Solution. No. Consider $a \in W_1$ but $a \notin W_2$, and $b \in W_2, b \notin W_1$. Then, $a + b$ is not closed because it takes an element solely of W_1 and an element solely of W_2, so the result cannot be of W_1 because one of the summands isn't in W_1 and the result cannot be of W_2 because one of the summands isn't in W_2. Therefore, $W_1 \cup W_2$ is not additively closed, so it cannot be a subspace. ∎

So how are we going to restrict W_1, W_2? The only case is if one of the spaces is a subspace of the other.

2.2.1 Sums (possibly direct)

How are we going to replace the union? With a sum!

Definition 2.2. The **sum** $W_1 + W_2$ is defined as $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$.

Theorem 2.3. The sum of two vector spaces W_1 and W_2 is itself a vector space.

Proof. Zero: $0_{\in W_1} + 0_{\in W_2} = 0$ must hold by definition.

Additive closurse: Take two already-summed vectors $u = w_1 + w_2, v = y_1 + y_2$. Therefore, our total is $u + v = (w_1 + y_1) + (w_2 + y_2)$. Our first term must be in W_1, and our second term must be in W_2, so u must be in $W_1 + W_2$.

Multiplicative closer: Given a scalar λ and a vector $u = w_1 + w_2$, we have that the product is $\lambda u = \lambda(w_1 + w_2) = \lambda w_1 + \lambda w_2$, where $w_1 \in W_1, w_2 \in W_2$. Note that the first term is once again in W_1 and the second term in W_2, so u must be in $W_1 + W_2$. ■

A special case is the **direct sum**, which is if any vector $u \in W_1 \oplus W_2$ can be split uniquely as $u = w_1 + w_2$, where the summands belong to their corresponding spaces.

How do we determine if $W_1 + W_2$ is direct?

1. Our sum is direct iff $0 = 0_{\in W_1} + 0_{\in W_2}$ is the **only** way to get 0 in $W_1 + W_2$.
2. Our sum is direct iff $W_1 \cap W_2 = \{0\}$.

Notice that Criterion 1 generalizes to more than 2 subspaces, but Criterion 2 doesn't. Consider the example of $(x, y, 0), (0, 0, z), (0, y, y)$. The pairwise intersections are trivial, but it's still not a direct sum.

CHAPTER 3

WEEK 3

3.1 Lecture 5

3.1.1 Linear Independence