Linear Programming

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1 Duality

We want to find $\max 5x_1 + 4x_2$ such that

$$\begin{aligned} 2x_1 + x_2 &\leq 100(\cdot y_1) \\ x_1 \quad\quad &\leq 30(\cdot y_2) \\ x_2 &\leq 60(\cdot y_1) \end{aligned}$$

Then, we have that $(2y_1 + y_2) \cdot x_1 + (y_1 + y_3) \cdot x_2 \leq 100y_1 + 30y_2 + 60y_3$.

Now, we want to find $\min 100y_1 + 30y_2 + 60y_3$ such that

$$\begin{aligned} y_1, y_2, y_3 &\geq 0 \\ 5 &\leq 2y_1 + y_2 \\ 4 &\leq y_1 + y_2 \end{aligned}$$

This results in another linear program: the **dual** linear program. We then have $5x_1+4x_2 \leq 100y_1+30y_2+60y_3$.

In matrix form, we have:

Primal LP: $\max C^T\vec{x}$ s.t. $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq 0$.

Dual LP: $\min \vec{b}^T\vec{y}$ s.t. $A^t\vec{y} \geq c$ and $\vec{y} \geq 0$.

1.1 Weak and Strong Duality

Theorem 1.1 (Weak Duality)**.** The value of the feasible solution $\vec{x}$ to primal linear program **must be** $\leq$ the value of the feasible solution $\vec{y}$ to dual linear program.

Proof. FILL IN LATER ■

Theorem 1.2 (Strong Duality)**.** If the primal opt is bounded, then primal opt = dual opt.

For example, min cut = max flow.

1.2 Zero-sum Games

Input: a payoff matrix M, with the row and column values determining different actions. The row player picks row r and column player picks row c.

The row player wins $+M_{rc}$ and the column player wins $-M_{rc}$. It's called a **zero-sum game** because the scores sum to 0.

Types of Strategies:

1. **Pure Strategy**: a single row / column, e.g. the row player always picks rock (beaten by column player always playing paper)
2. **Mixed Strategy**: probability distribution over pure strategies. For example, $\Pr[\text{Rock}] = \frac{1}{3}, \Pr[\text{Paper}] = \frac{1}{3}, \Pr[\text{Scissors}] = \frac{1}{3}$. The average score is 0 regardless of what the column player does.

Example 1.1 (Game 1).

$$M = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}.$$

Row player goes first, column player goes second.

First, the row player announces a mixed strat $p = (p_1, p_2)$, then the column player announces a mixed strat $q = (q_1, q_2)$.

Definition 1.3 (Average Score). Row player's **average score** is $\text{Score}(p, q) = 3p_1q_1 - 1p_1q_2 - 2p_2q_1 + 1p_2q_2$.

The column player's best strat is to minimize $\text{Score}(p, q)$ over all mixed strategies q, which is equivalent to minimizing $\text{Score}(p, q)$ over all pure strategies q. This is equal to $\min(3p_1 - 2p_2, -p_1 + p_2)$, which is the minimum of the column 1 and column 2 score.

The row player will have to pick (p_1, p_2) that gives them the maximum $\text{Score}(p, q)$. We can compute the $\max_p(\min(3p_1 - 2p_2, -p_1 + p_2))$ with linear programming.

Proof. Maximize z, subject to

$$\begin{aligned} z &\leq 3p_1 - 2p_2 \\ z &\leq -p_1 + p_2 \\ p_1 + p_2 &= 1 \\ p_1 &\geq 0, p_2 \geq 0. \end{aligned}$$

This is a linear program. Note that $z = \min(3p_1 - 2p_2, -p_1 + p - 2)$. ■

Example 1.1 (Game 2). Same as game 1, except the column player goes first and the row player goes second.

Now, the row player does a pure strategy and the col player does a mixed strategy.

Note that the payoff of row 1 is $3q_1 - q_2$ and the payoff of row 2 is $-2q_1 + q_2$. So the **row** player's best strat is $\max(3q_1 - q_2, -2q_1 + q_2)$. As a result, the column player's best strat is $\min_q max(3q_1 - q_2, -2q_1 + q_2)$, where $\min_q(x)$ means the minimum value of x over all strategies q.

Comparing the two games, we have $\max_p(\min_q(score(p, q))) \leq \min_q(\max_p(score(p, q)))$. We can go further with strong duality, though:

Theorem 1.4 (Min-Max Theorem). $\max_p(\min_q(score(p, q))) = \min_q(\max_p(score(p, q)))$

1.3 Experts Problem

There are n experts $E_1, \dots, E_n$. Every day, they make a prediction. On the tth day,

1. Pick and expert E_i
2. Each expert E_j incurs a loss $l_j^{(t)} \in \{0, 1\}$.
3. You incur the loss $l_i^{(t)}$

Allowed to pick a random expert on the tth day.

Remark 1.5. Surprising features of the model:

1. **Not** assuming that past performance predicts future performance
   $\therefore$ all algorithms are useless.
2. Expert losses can be adversarial, that is, losses can depend on the distribution you specify.

Theorem 1.6. There exists an algorithm with regret $R^T \leq 2\sqrt{T \ln n}$.

$\therefore$ after time T, **average regret** $\frac{R^T}{T} \leq \frac{2\sqrt{\ln n}}{\sqrt{T}}$, so as $T \to \infty$, $\frac{R^T}{T} \to 0$.