Problemset 10

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1 Cookie Jars

If there are x cookies in jar 2, there is a $\frac{1}{2^{2n-x}}\binom{2n-x}{n}$ probability that x cookies are taken. Therefore, the distribution of X, the number of cookies left in the other jar, is $\mathbb{P}[X = a] = \boxed{\frac{1}{2^{2n-a}}\binom{2n-a}{n}}$.

2 Maybe Lossy Maybe Not

(a) At most one packet can be lost without losing the message, so the probability of the message staying intact is $\mathbb{P}[0] + \mathbb{P}[1]$, where $\mathbb{P}[x]$ is the probability of x packets being dropped. This equals $(1-p)^7 + p(1-p)^6\binom{7}{1} = \boxed{(1-p)^7 + 7p(1-p)^6}$.

(b) One packet may be corrupted after the deletions, so we need at least $6 + 2 = 8$ packets to remain intact after packets are lost. We can thus drop at most 2 packets, so the probability is now $\mathbb{P}[0] + \mathbb{P}[1] + \mathbb{P}[2]$. This is found to be $(1-p)^{10} + p(1-p)^9\binom{10}{1} + p^2(1-p)^8\binom{10}{2} = \boxed{(1-p)^{10} + 10p(1-p)^9 + 45p^2(1-p)^8}$.

(c) There is a probability of p that a packet is dropped. After this, There is a probability of q that a packet is corruptted. We can afford one drop and no subsequent corruption in either case.

There is a probability of p that a packet is dropped, and if not there is a probability of q that it will be corrupted. There is a $(1-p)(1-q)$ chance that a package is neither dropped nor corrupted. Thus, our desired probability is $\mathbb{P}[0,0]+\mathbb{P}[1,0]$ where $\mathbb{P}[x, y]$ is x drops and y corruptions. This equals $[(1-p)(1-q)]^7 + \binom{7}{6}p(1-p)^6(1-q)^7 = \boxed{(1-q)^7[(1-p)^7 + 7(1-p)^6]}$.

3 Class Enrollment

(a) The probability of Lydia getting the geography class on day g is the chance that she does not get the class on all days 1 to $g-1$ and does get it on day g. This has a probability of $\mu(1-\mu)^{g-1}$, so the distribution of G is $\boxed{\mathbb{P}[G=g]=\mu(1-\mu)^{g-1}}$.

(b) $\mathbb{P}[G=i|G>7]$ is the sum of all probabilities $\mathbb{P}[G=i]$ for $i>7$, or $\sum_{i=8}^{\infty}\mathbb{P}[G=i]=\sum_{i=8}^{\infty}\mu(1-\mu)^{i-1}$ which can be evaluated with the geometric series formula to get

$$\mathbb{P}[G=i|G>7]=\frac{\mu(1-\mu)^7}{1-(1-\mu)}=\boxed{\frac{\mu(1-\mu)^7}{\mu}}.$$

(c) $\mathbb{E}[H]$ is $\mathbb{E}[G]+\mathbb{E}[H-G]$, where the former is the expected number of days to get geography and the latter is the expected number of days to get history. Then $\mathbb{E}[G]=\sum_{i=1}^{\infty}i\mu(1-\mu)^{i-1}$. This sum equals

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{i=j}^{\infty}\mu(1-\mu)^{i-1} &= \sum_{j=1}^{\infty}\frac{\mu(1-\mu)^{j-1}}{\mu}\\ &= \sum_{j=1}^{\infty}(1-\mu)^{j-1}\\ &= \frac{1}{\mu}.\end{aligned}$$

The expected number of days to get the history class alone is the same, $\frac{1}{\lambda}$, using the same calculations. So the total $\mathbb{E}[H]=\boxed{\frac{1}{\mu}+\frac{1}{\lambda}}$.

(d) The distribution of G is still $\mathbb{P}[G=g]=\mu(1-\mu)^{g-1}$, but the distribution of H is now completely **independent** of G, being $\boxed{\mathbb{P}[H=h]=\lambda(1-\lambda)^{h-1}}$.

(e) The probability that Lydia gets either a geography or a history class on day i is equal to $\mathbb{P}[G\cup H]=\mathbb{P}[G]+\mathbb{P}[H]-\mathbb{P}[G\cap H]$, which is equal to $\mu(1-\mu)^{i-1}+\lambda(1-\lambda)^{i-1}-(\lambda\mu)(1-\lambda\mu)^{i-1}$ because G,H are independent. Thus, the distribution for A equals $\mathbb{P}[A=i]=\boxed{\mu(1-\mu)^{i-1}+\lambda(1-\lambda)^{i-1}-(\lambda\mu)(1-\lambda\mu)^{i-1}}$.

(f) Instead of placing the two actions of signing up for geography and signing up for history in sequential order, we instead do the two actions simultaneously. This leads to $\mathbb{E}[B]$ being the maximum of $\mathbb{E}[G]$ and $\mathbb{E}[H]$, which equals $\frac{1}{\mu}+\frac{1}{\lambda}-\mathbb{E}[A]$. We find that $\mathbb{E}[A]=\mathbb{E}[G]+\mathbb{E}[H]-\mathbb{E}[G\cap H]$. From part (b), this equals $\frac{1}{\lambda}+\frac{1}{\mu}-\frac{1}{\lambda\mu}$. The former two terms cancel, so $\mathbb{E}[B]=\boxed{\frac{1}{\lambda\mu}}$.

(g) Let I_1 be 1 if Lydia gets geography by day 30, and I_2 be 1 if Lydia gets history by day 30. From linearity of expectation, we find that the expected number of classes Lydia gets by day 30 is $\mathbb{E}[I_1]+\mathbb{E}[I_2]$. Furthermore, we know that $\mathbb{E}[I]=\mathbb{P}[I=1]$, so this is just the sum of the probability that we get either geography or history by day 30. For the case of geography, this equals $\mathbb{P}[I_1=1]=\sum_{i=0}^{29}\mu(1-\mu)^i=\frac{\mu(1-(1-\mu)^{30})}{\mu}=(1-(1-\mu)^{30})$. Similarly, we get that $\mathbb{P}[I_2=1]=(1-(1-\lambda)^{30})$. Therefore, $\mathbb{E}[I_1]+\mathbb{E}[I_2]=\boxed{2-(1-\mu)^{30}-(1-\lambda)^{30}}$.

4 Two Sides of a Coin

(a) We want the expected value n such that the nth toss is the first one that differs from toss 1. This has a value of $\sum_{i=2}^{\infty} \frac{i}{2^i}$, which equals

$$\sum_{i=2}^{\infty} \frac{1}{2^i} + \sum_{j=2}^{\infty}\sum_{i=j}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \sum_{i=2}^{\infty} \frac{1}{2^{i-1}} = \boxed{\frac{3}{2}}.$$

(b) It's expected that there are $\frac{3}{2}$ flips for both sides of the first coin to be seen, and then there are $\frac{3}{2}$ flips for both sides of the second coin to be seen, so the total expected number of flips for both sides of both coins to be seen is $\boxed{3}$ from linearity of expectation.

(c) Let the two random variables for seeing heads and tails of each coin be X and Y. Then both X and Y are identically distributed, and as such the answer will be the same as in part (a), which is $\boxed{\frac{3}{2}}$.

5 Throwing Frisbees

(a) The probability that the frisbee is returned after x turns is $\frac{1}{n-1}\left(\frac{n}{n-1}\right)^{x-2}$, so the expected value is

$$\sum_{x=2}^{\infty} \frac{x}{n-1}\left(\frac{n-2}{n-1}\right)^{x-2} = \sum_{x=2}^{\infty} \frac{1}{n-1}\left(\frac{n-2}{n-1}\right)^{x-2} + \sum_{i=2}^{\infty}\sum_{x=i}^{\infty} \frac{1}{n-1}\left(\frac{n-2}{n-1}\right)^{x-2} = 1 + \sum_{i=2}^{\infty}\left(\frac{n-2}{n-1}\right)^{i-2} = \boxed{1 + \frac{(n-2)^{i-2}}{(n-1)^{i-3}}}.$$

(b) Let the indicator variable I_k be 1 if player k gets the frisbee, and 0 otherwise. Furthermore, let Shahzar be player n. Then, the expected number of people who never feel the loving grasp of a frisbee equals $\sum_{k=1}^{n-1}(1 - \mathbb{E}[I_k])$ by linearity of expectation.

We first calculate the individual $\mathbb{E}[I_k]$. There is a $\frac{1}{n-1}$ chance that k gets the frisbee on the first turn. Then, if we were to add another turn, we should not throw to either Shahzar or player k on the first turn, so of the $n-1$ players we could throw to $n-3$ players. So the probability of getting a frisbee to k in 2 turns is $\frac{n-3}{n-1} \cdot \frac{1}{n-1}$. Similarly, we find that in 3 turns we have a probability of $\frac{1}{n-1}\left(\frac{n-3}{n-1}\right)^2$, and for n turns we have a probability of $\frac{1}{n-1}\left(\frac{n-3}{n-1}\right)^{n-1}$.

Therefore, $\mathbb{E}[I_k]$ can be evaluated with a geometric series to be $\frac{\frac{1}{n-1}}{1-\frac{n-3}{n-1}} = \frac{\frac{1}{n-1}}{\frac{2}{n-1}} = \frac{1}{2}$ for all $k \in [1, n-1]$. Therefore, our answer is

$$\sum_{k=1}^{n-1}(1 - \mathbb{E}[I_k]) = \boxed{\frac{n-1}{2}}.$$

6 Swaps and Cycles

(a) Let an indicator variable I_k equal 1 if k is being switched with some other index, and 0 otherwise. Then, the number of switches n is equal to $\frac{1}{2}\sum_{k=1}^{n} I_k$, so by linearity of expectation $\mathbb{E}[n] = \frac{1}{2}\sum_{k=1}^{n}\mathbb{E}[I_k]$.

We know that $\mathbb{E}[I_k]$ is equal to the probability of putting I_k in a switch pair, and there are $n-1$ possible elements for the other pair. The probability that a fixed k_1, k_2 are in a pair is $\frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$, so the probability that k is in a switch pair with anything else is $\frac{1}{n}$.

Thus,

$$\mathbb{E}[n] = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{n} = \boxed{\frac{1}{2}}.$$

(b) We claim that the answer is $\boxed{\frac{1}{k}}$.

Let I_j be an indicator variable that equals 1 if index j is in a k-cycle and 0 otherwise. Similarly to part (a), the expected number of k-cycles in $\pi(n)$ is the sum of $\frac{1}{k}\sum_{j=1}^{n} 1 - \mathbb{E}[I_j]$.

For a given index j, there are $\frac{(n-1)!}{(n-k)!}$ ways to pick and order $k-1$ other elements to put in the cycle, and then a probability of $\frac{(n-k)!}{n!}$ that a cycle with fixed elements $j_1, j_2, \ldots, j_k$ exists. Therefore, the expected value of I_j is still $\frac{1}{n}$, so

$$\mathbb{E}[n] = \frac{1}{k}\sum_{k=1}^{n}\frac{1}{n} = \frac{1}{k}.$$

$\square$