Problemset 1

Albert Ye

September 3, 2022

1 Solving a System of Equations

$$\begin{aligned} a+b+c &= 16 \\ 2a+3b &= 23 \\ a+2b+3c &= 35 \end{aligned}$$

We can reduce this into a matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 16 \\ 23 \\ 35 \end{bmatrix}.$$

We can use augmented matrix form to reduce the matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 2 & 3 & 0 & 23 \\ 1 & 2 & 3 & 35 \end{array}\right]$$

$$\implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & -2 & -9 \\ 0 & 1 & 2 & 19 \end{array}\right]$$

$$\implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & -2 & -9 \\ 0 & 0 & 4 & 28 \end{array}\right]$$

$$\implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$$\implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{array}\right].$$

Therefore, we have $a = 4, b = 5, c = 7$. $\square$

2 Integration Purgatory

a.

$$\begin{aligned}
\int_0^\infty \sin(t)e^{-t}dt &= -\int_0^\infty \sin(t)d(e^{-t}). \\
\int \sin(t)d(e^{-t}) &= \sin(t)e^{-t} - \int e^{-t}d(\sin t) \\
-\int \sin(t)e^{-t}dt &= \sin(t)e^{-t} - \int e^{-t}\cos t dt. \\
\int \cos te^{-t}dt &= -\int \cos td(e^{-t}) \\
&= -(\cos te^{-t} - \int e^{-t}d(\cos t)) \\
&= -(\cos te^{-t} + \int \sin(t)e^{-t}dt) \\
-\int \sin(t)e^{-t}dt &= \sin(t)e^{-t} + \cos(t)e^{-t} + \int \sin(t)e^{-t}dt \\
-2\int \sin(t)e^{-t}dt &= (\sin t + \cos t)e^{-t} \\
\int \sin(t)e^{-t}dt &= -e^{-t}\left(\frac{\sin t + \cos t}{2}\right).
\end{aligned}$$

Therefore, we have

$$\int_0^\infty \sin(t)e^{-t}dt = \lim_{t\to\infty} -e^{-t}\left(\frac{\sin t + \cos t}{2}\right) + e^0\left(\frac{\sin 0 + \cos 0}{2}\right) = 0 + \frac{1}{2} = \boxed{\frac{1}{2}}.$$

□

b. We want to find values of x such that

$$\frac{d}{dx}\int_0^{x^2} e^{t^2}dt = 0.$$

Let $F = \int e^{t^2}dx$. Then, the integral we want to find becomes $F(x^2) - F(0)$. We know that $\frac{d}{dx}F(x) = e^{x^2}$, so applying the chain rule, we have $F(x^2) = e^{x^4}(2x)$ and $F(0) = e^0 \cdot 0$. Therefore,

$$\frac{d}{dx}\int_0^{x^2} e^{t^2}dt = e^{x^4}(2x).$$

This equals 0 either when e^{x^4} reaches zero (which is never), or $2x$ reaches zero (which happens at $x = 0$). We know that $\int_c^c f(x)dx = 0$ for any $c \in \mathbb{R}$ and function f in terms of x, so the minimum value is $\int_0^0 e^{t^2}dt = \boxed{0}$. □

c.

$$\begin{aligned}
\int\int_R 2x + ydA &= \int_0^1\int_0^x 2x + ydydx \\
&= \int_0^1 \left(2xy + \frac{y^2}{2}\right|_0^x dx \\
&= \int_0^1 2x^2 + \frac{x^2}{2}dx \\
&= \int_0^1 2.5x^2dx = \left(\frac{2.5x^3}{3}\right|_0^1 = \boxed{\frac{5}{6}}.
\end{aligned}$$

□

3 Implication

a. True, because the two statements basically say the same thing. $\exists x \exists y$ means there exists both an x and a y that satisfy $Q(x, y)$, while $\exists y \exists x$ means the exact same thing.

b. False. Consider $y = x$. For all x, there exists a y such that $y = x$; but there doesn't exist a y such that $y = x$ for all x.

c. True, because if there exists an x that satisfies $Q(x, y)$ for all y, then for all y there must exist an x that satisfies $Q(x, y)$ because the original x works.

d. False. Take $x = y = 1$. There exists x, y that satisfy this but this does not hold for all y (obviously).

4 Logical Equivalence?

a. True, if $P(x) \wedge Q(x)$ is true for all x, then $P(x)$ must be true for all x and $Q(x)$ must be true for all x necessarily. And if $P(x)$ and $Q(X)$ are both true for all x, then $P(x) \wedge Q(x)$ must also be true for all x because there are no x such that either component of the AND relation is false.

b. False, $P(x) \vee Q(x)$ being true for all x does not mean that either $P(x)$ is true for all x or $Q(x)$ is true for all x.

c. True, if there exists an x such that $P(x) \vee Q(x)$ holds then there must exist an x such that either $P(x)$ holds or $Q(x)$ holds. These basically mean the same thing, since if $P(x)$ or $Q(x)$ hold for x, then $P(x) \vee Q(x)$ must hold for x.

d. False, if $P(x) \wedge Q(x)$ is true, there must be one x that satisfies both; but the right hand side can have different x's for P, Q.

5 Preserving Set Operations

a. Imagine we have a set P such that $f^{-1}(A) = P$, and another set Q such that $f^{-1}(B) = Q$. We know that if $x \in P$ and $x \in Q$, then $f(x)$ must be in both A and B by definition of image. So if there is an x such that $x \in P \cap Q$, then $f(x) \in A \cap B$ must follow.

   If x is not in P, then by definition of preimage, $f(x)$ cannot be in A, and the same logic applies for Q and B. Therefore, if there is an element in both A and B, it must be in both P and Q. This, along with the previous paragraph, confirms that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. $\square$

b. Define P, Q similarly. Then $f^{-1}(A \setminus B) = f^{-1}(A \setminus (A \cap B))$. We know from part (a) that $f^{-1}(A \cap B) = P \cap Q$. The set of inputs in A that also are in B is $P \cap Q$, so we must exclude $P \cap Q$ from the preimage of $A \setminus B$.

   Since all other elements of P do not map to elements of B by definition of preimage, we can conclude that the preimage of $A \setminus B$ is $P \setminus (P \cap Q) = P \setminus Q$. Substituting in the original terms, we have $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$. $\square$

c. $A \cap B$ is the set of values that are elements of both A and B. It follows that every element of $f(A \cap B)$ must be an element of $f(A)$ because $A \cap B \subset A$, and $f(A \cap B)$ must be an element of $f(B)$ becaues $A \cap B \subset B$ as well. Thus, $f(A \cap B) \in f(A) \cap f(B)$. $\square$

   Consider the line $y = 3$. Let A be the negative numbers and B be the positive numbers. Then $A \cap B =$ but $f(A) \cap f(B)$ is not null.

d. $A \setminus B$ is everything in A excluding $A \cap B$. Since $f(A \cap B) \subset f(A) \cap f(B)$, we have that $f(A) \setminus (f(A) \cap f(B))$ is a superset of $f(A \setminus (A \cap B))$. $\square$

   The same example from before still applies to this situation Let A be $[0, 4]$ and B be $[3, 8]$. Then $f(A \setminus B) = \{3\}$ but $f(A) \setminus f(B) =$.

6 Prove or Disprove

a. If n is odd, it immediately follows that $n^2 = n \cdot n$ is odd. Moreover, $4n$ is guaranteed to be even. So $n^2 + 4n \equiv 1 + 0 \pmod 2 \equiv 1 \pmod 2$, so $n^2 + 4n$ is odd. $\square$

b. Assume for the sake of contradiction that $a > 11$, and $b > 4$. Then, we would have $a + b > 11 + 4 = 15$, meaning that $a + b > 15$, which contradicts the fact $a + b \leq 15$. Therefore, for $a + b \leq 15$ we must have $a \leq 11$ or $b \leq 4$. $\square$

c. Assume for the sake of contraduction that r is rational. Then, we can express r as $\frac{p}{q}$ for $p, q \in \mathbb{Z}$. Then, $r^2 = \frac{p^2}{q^2}$, and clearly if $p, q \in \mathbb{Z}$ then $p^2, q^2 \in \mathbb{Z}$. Thus, r^2 must also be rational. Therefore, the contrapositive must also be true: if r^2 is not rational, then r cannot be positive. $\square$

d. False. Consider $n = 10$. Then $5n^3 = 5000$ while $n! = 3628800$. Clearly, in this case, $n! > 5n^3$. $\square$

7 Rationals and Irrationals

Let the rational number be expressed as $\frac{p}{q}$ for $p, q \in \mathbb{Z}$, and let the irrational number be r. We seek to prove that $\frac{pr}{q}$ cannot be rational. Assume for the sake of contradiction that $\frac{pr}{q}$ is a rational number, which we can call $\frac{a}{b}$ for $a, b \in \mathbb{Z}$.

We find the quotient of $\frac{a}{b}$ and $\frac{p}{q}$, which is $\frac{\frac{a}{b}}{\frac{p}{q}} = \frac{aq}{bp}$. As a, b, p, q are all integers, $\frac{aq}{bp}$ must be rational, contradicting the claim that r is irrational. Therefore, we know the contrapositive is true: if r is irrational, then $\frac{pr}{q}$ must also be irrational. $\square$

8 Twin Primes

a. If p is not of the form $3k + 1$ or $3k - 1$, that means that $p \equiv 0 \pmod 3$ because the numbers of the form $3k + 1$ are the ones with residue 1 modulo 3, and the numbers of the form $3k - 1$ are the ones with residue 2 modulo 3. Hence, p cannot be prime if it isn't among one of these forms. $\square$

b. Assume we have another two twin prime pairs surrounding a value k. Then, we need $k - 2$ and $k + 2$ to both be prime. However, none of $k, k - 2, k + 2$ are equivalent modulo 3, meaning that each of them will have to take a different residue modulo 3, so one of the values must be divisible by 3. This means the only valid set of $k, k - 2, k + 2$ must contain 3. Going through all triplets of $k - 2, k, k + 2$, the only two pairs that work would be $(3, 5)$ and $(5, 7)$. Thus, 5 is the only possible number that is part of two twin prime pairs. $\square$

9 Sundry

I worked on this alone, but I discussed the approaches towards some problems with my roommate Saathvik Selvan.