Linear map from higher dim. space to lower dim. space can never be <u>injective</u>

from smaller to larger cannot be surjective

- for injective & surjective linear map, it's necessary & sufficient that dim is equal!
  - inj: null space is trivial
  - surj.
- conversely, suppose dim V = dim $W < \infty$.
  - construct bijective linear map.
  - transform the basis for V & W.
- define a linear map by its action on a basis,
  - $T\vec{v}_j = \vec{w}_j$ ⇒ we can do a linear combination of v-side + get w-side
  - isomorphism btwn. two spaces.

For example: an isomorphism $T: \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^4$.

canonical bases: $\underbrace{(1, x, x^2, x^3)}_{\text{null}=0} \to \underbrace{(e_1, e_2, e_3, e_4)}_{\text{null}=0}$.

$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \to a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$.

- Bijection vs. Isomorphism
  - bijection may not be linear.
- BTW, isomorphism inverse is also necessarily a linear map.
- another classical isomorphism:
  - Suppose dim $V = m$, dim $W = n$. We want to represent an element of $\mathcal{L}(V, W)$ as a <u>matrix</u>.
  - start w/ $\vec{v}_1, \dots, \vec{v}_m$ a basis for V,

$$\boxed{\begin{aligned} T\vec{v}_1 &= \sum_{j=1}^{n} a_{j1} \vec{w}_j \\ T\vec{v}_m &= \sum_{j=1}^{n} a_{jm} \vec{w}_j \end{aligned} \qquad \left[a_{ji} \right]_{j=1,\ i=1}^{n,\ m}} \leftarrow \text{matrix } a.$$

Point is, this is the canonical correspondence b/w matrices & linear maps