Lecture 4

- $W_1 \cap W_2$ subspace? $\boxed{\text{Yes}}$
  - zero: $0 \in W_1, W_2$ by defn.
  - add. closure: $u, v \in W_1, W_2 \iff u, v \in W_1, W_2$.

    W_1 is a subspace $\Rightarrow \vec{u} + \vec{v} \in W_1$

    W_2 is a subspace $\rightarrow \vec{u} + \vec{v} \in W_2$

    $\Rightarrow \boxed{\vec{u} + \vec{v} \in W_1, W_2}$
  - now let $\vec{u} \in W_1 \cap W_2$, $a \in \mathbb{F}$.

    $W_1 \ni \vec{u} \longrightarrow W_1 \ni a\vec{u}$

    $W_2 \ni \vec{u} \longrightarrow W_2 \ni a\vec{u}$

    $\Rightarrow W_1 \cap W_2 \ni a\vec{u}$.

Example: $V = \mathbb{R}^3$, $W_1 = \{(x_1, y_1, 0) : x, y \in \mathbb{R}\}$ $\Rightarrow$ can sanity check that these are vec. spaces.

$W_2 = \{(x_2, 0, x_2) : x \in \mathbb{R}\}$

$W_1 \cap W_2 = \{0\}$? Yes

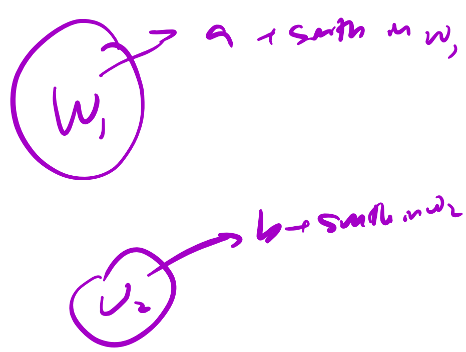
- $y_1 = 0$, $x_2 = x_1$, $x_2 = 0$. $\Rightarrow$ all values must be 0.
- any vec in $W_1 \cap W_2$ must be of form $(x, y, 0)$ and $(z, 0, z)$

- $W_1 \cup W_2$ subspace? $\boxed{\text{No}}$
  - check conditions:

✓ • zero - $0 \in W_1 \cup W_2$ as well.

X • add. closure - $a \in W_1$, $b \in W_2$

$a \notin W_2$ $b \notin W_1$



$(x, 0, 0)$

$(0, y, 0)$ - $a + b$ is not in W_1 b/c a + smth. in W_1, b isn't in W_1

$\Rightarrow (x, y, 0)$ is not in. - $a + b$ is not in W_2 b/c b + smth. in W_2, a isn't in W_2.

- what condition to restrict W_1, W_2 s.t. $W_1 \cup W_2$ is SS?
  - one is a subspace of the other?
- can we replace union w/ smth. else?
  - span
  - "sum" of W_1, W_2 defined as follows:

    $W_1 + W_2$ defined as follows: $W_1 + W_2 = \{\vec{w}_1 + \vec{w}_2, \vec{w}_1 \in W_1, \vec{w}_2 \in W_2\}$.

  - for example, $(x,0,0) + (0,y,0) \to (x,y,0)$.
    - zero: $\vec{0}_{\in W_1} + \vec{0}_{\in W_2} = \vec{0}$. ✓
    - add. closure: take two already-summed vector $\vec{u} = \underset{\in W_1}{\vec{w}_1} + \underset{\in W_2}{\vec{w}_2}$, $\vec{v} = \underset{\in W_1}{\vec{y}_1} + \underset{\in W_2}{\vec{y}_2}$

      $$\text{total} = \vec{u} + \vec{v} = \underbrace{(\vec{w}_1 + \vec{y}_1)}_{\text{in } W_1} + \underbrace{(\vec{w}_2 + \vec{y}_2)}_{\text{in } W_2}$$

      $$\Rightarrow \vec{u} + \vec{v} = [\vec{w}_1^{\,3} + \vec{w}_2^{\,3}] \in W_1 + W_2.$$

    - mult. closure

      scalar λ, vector $\vec{u} = \vec{w}_1 + \vec{w}_2$

      $$\text{total} = \lambda\vec{u} = \lambda(\vec{w}_1 + \vec{w}_2) = \underbrace{\lambda\vec{w}_1}_{\in W_1} + \underbrace{\lambda\vec{w}_2}_{\in W_2}$$

      So $\lambda\vec{u} \in W_1 + W_2$.

- therefore, the span $\boxed{\text{is}}$ a vector space.

- special case is the "direct sum".
  - we say $W_1 + W_2$ is a direct sum if any vector $u \in W_1 + W_2$ can be split uniquely as $u = w_1 + w_2$ where summands belong to corr. spaces.

- take $(1,0,0) + (0,2,0) = (1,2,0)$. only one way to write this $(1,0,0) + (0,2,0)$

  so this is direct sum.

- $(x,y,0)$ splits uniquely into 2 vectors into $(x,0,0) + (0,y,0)$.

Now consider $W_1 = \{(x,0,0) : x \in \mathbb{R}\}$
$W_2 = \{(x,y,0) ; x,y \in \mathbb{R}\}$

$$(1,2,0) = \underbrace{(0,0,0)}_{W_1} + \underbrace{(1,2,0)}_{W_2} = \underbrace{(-1,0,0)}_{W_1} + \underbrace{(2,2,0)}_{W_2}$$

$\Rightarrow$ no uniqueness $\Rightarrow$ not a direct sum.

Q : how to determine if $W_1 + W_2$ is direct?

1) sum direct iff $\vec{0} = \underset{\in W_1}{\vec{0}} + \underset{\in W_2}{\vec{0}}$ is the only composition of 0 of the form $\vec{w}_1 + \vec{w}_2$

- Proof : uniqueness of this split of $\vec{0}$ is a special case of the condition in the defn.

- converse: take $\vec{u} \in W_1 + W_2$, assume there are two ways.

$\vec{u} = \underset{\in W_1}{\vec{w}_1} + \underset{\in W_2}{\vec{w}_2}$, $\vec{u} = \underset{\in W_1}{\vec{y}_1} + \underset{\in W_2}{\vec{y}_2}$

$$\vec{u} - \vec{u} = \left(\vec{w}_1 - \vec{y}_1\right) + \left(\vec{w}_2 - \vec{y}_2\right)$$

$$\Rightarrow \vec{0} = \underbrace{(\vec{w}_1 - \vec{y}_1)}_{\vec{0}} + \underbrace{(\vec{w}_2 - \vec{y}_2)}_{\vec{0}} \Rightarrow w_1 = y_1,\ w_2 = y_2.$$

- Criterion 2 : $W_1 + W_2$ is direct $\Leftrightarrow W_1 \cap W_2 = \{\vec{0}\}$

$W_1 + W_2$ direct $\rightarrow W_1 \cap W_2 = \{\vec{0}\}$.

let's say $a \in W_1, a \in W_2$. by def. of vector space $-a \in W_2$

$\Rightarrow \vec{0} = \vec{a} + (-\vec{a})$ is a possible sum.

$\Rightarrow$ fails criterion 1

$W_1 \cap W_2 = \{\vec{0}\} \rightarrow W_1 + W_2$ direct

$0 = \vec{w}_1 + \vec{w}_2 \Rightarrow \vec{w}_2 = -\vec{w}_1 \Rightarrow \vec{w}_2 \in W_1$ because by definition $-\vec{w}_1 \in W_1$, so $w_2 \in W_1 \cap W_2 \Rightarrow w_2 = 0 \Rightarrow w_1 = 0$.

So... criterion #1 generalizes to > 2 subspaces, criterion #2 doesn't.

example: $\mathbb{R}^3 \in V$

$$\left.\begin{aligned} W_1 &= \{(x,y,0) : x, y \in \mathbb{R}\} \\ W_2 &= \{(0,0,z) : z \in \mathbb{R}\} \\ W_3 &= \{(0,y,y) : y \in \mathbb{R}\} \end{aligned}\right\} \text{pairwise intersections trivial, but still not direct sum.}$$

but $(0,1,0) + (0,0,1) + (0,-1,-1) = 0$ violates C1.