Letting $T \in \mathcal{L}(V, W)$

Theorem: V, W finite dimensional, $T \in \mathcal{L}(V, W)$.

Then, a) $\text{Null } T' = (\text{Range } T)^0$.

b) $\dim \text{Null } T' = \dim \text{Null } T + \dim W - \dim V$.

a) $\varphi \in W'$ belongs to $\text{Null } T'$

$$\Updownarrow$$

$$\varphi \cdot T = 0$$

$$\Updownarrow$$

$$(\varphi \cdot T)\vec{v} = 0 \qquad \forall \vec{v} \in V.$$

$$\Updownarrow$$

$$\varphi(T\vec{v}) = 0 \qquad \forall \vec{v} \in V.$$

$\Leftrightarrow \varphi(\vec{w}) = 0 \quad \forall \vec{w} \in \text{Range } T. \Leftrightarrow \varphi \in \text{Range}(T^0)$.

So, $\text{Null } T' = (\text{Range } T)^0$.

b) $\underline{\dim \text{Null } T'} = \dim \text{Null } T + \dim W - \dim V$

$$\begin{aligned}
\dim \text{Null } T' &= \dim (\text{Range } T)^0 \\
&= \dim W - \dim \text{Range } T \\
&= \dim W - \dim \text{Range } T. \\
&= \dim W - (\dim V - \dim \text{Null } T) \\
&= \dim \text{Null } T + \dim W - \dim V.
\end{aligned}$$

Corollary: $\text{Null } T' = \{0\} \Leftrightarrow T'$ is injective $\Leftrightarrow (\text{Range } T')^0 = \{0\}$
$\Leftrightarrow \text{Range } T = W \Leftrightarrow T$ is surjective.

$\hookrightarrow$ Therefore, if T' is injective, then T is surjective.

Theorem: V, W finite dimensional, $T \in \mathcal{L}(V, W)$. Then,

a) $\dim \text{Range } T' = \dim \text{Range } T$.

b) $\text{Range } T' = (\text{Null } T)^0$.

Proof: a) $\dim \text{Range } T' = \dim W' - \dim \text{Null } T' = \dim W - (\dim \text{Null } T + \dim W - \dim V)$

$= \dim V - \dim \text{Null } T. = \dim \text{Range } T.$ ∎

b) $\text{Range } T' = (\text{Null } T)^0$

$\psi \in \text{Range } T' \Leftrightarrow \psi \in \varphi \circ T$

$\forall \vec{v} \in \text{Null } T, \ \psi(\vec{v}) = (\varphi \circ T)(\vec{v}) = \varphi(\underbrace{T\vec{v}}_{\vec{0}}) = \varphi(\vec{0}) = 0.$

If $\psi \in \text{Range } T'$, then $\psi \in (\text{Null } T)^0$. (the annihilator is a vector space)

since we already know from (a) that $\dim \text{Range } T' = \dim (\text{Null } T)^0$,

we can conclude $\text{Range } T' = (\text{Null } T)^0$.

Question: given $T \in \mathcal{L}(V, W)$, $\dim V = m$, $\dim W = n$. Choose two bases

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ of V and $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ of W to represent T

as $\mathcal{M}(T)$. Can we represent T' in an organic way?

$\varphi \in W'$ $\mathcal{M}(\varphi)$ is a matrix of size $1 \times n$.

$\varphi \circ T$ represented by $\mathcal{M}(\varphi) \cdot \mathcal{M}(T)$ of size $1 \times m$.

↳ how to compute functional by map.

↳ $T' \in \mathcal{L}(W', V')$.

↳ need a basis for dual space of W & V.

What is the connection between $\mathcal{M}(T)$ wrt $\vec{v}_1, \dots, \vec{v}_m$ and $\vec{w}_1, \dots, \vec{w}_n$

&

$\mathcal{M}(T')$ wrt $\varphi_1, \dots, \varphi_m$ and $\psi_1, \dots, \psi_m$.

$\mathcal{M}(T)(i,j) = i^{th}$ coeff. of $T(v_j)$

$\mathcal{M}(T)(i,j) = i^{th}$ coeff of $T\vec{v}_j = \varphi_i(T\vec{v}_j)$

$\mathcal{M}(T')(j,i) = j^{th}$ coeff of $T'(\varphi_i) = (T'(\varphi_i))(\vec{v}_j) = (\varphi_i \circ T)(v'_j)$.

$\Rightarrow \mathcal{M}(T)^T = \mathcal{M}(T')$.

Rank of Matrices

- # of pivots
- dim of column space.

$$(AB)^T = B^T \cdot A^T$$