Math 104: Real Analysis

Albert Ye

June 20, 2023

CONTENTS

Chapter 1
Week 1

1.1 Lecture 1

1.1.1 Logic and Sets

For clauses p, q: we have $p \wedge q$, $p \vee q$, $\neg p$. These are *and*, *or*, *not*; respectively.

Moreover, we have $p \implies q$ meaning that q is true if p is true. Moreover, we have $p \iff q$ meaning that p is true if q is true and q is true if p is true.

Other terminology: $:=$ is a definition, $\forall$ is for all, $\exists$ is exists, $a \in A$ means that element a is in the set A, $a \notin A$ means that element a isn't in the set A.

For sets, we have $\subset, =, \subseteq$ to determine subset and equality relations. Moreover, we have $\cap, \cup$ to represent union and intersections of sets. There is also $A \setminus B$ to denote everything in A but not B, and we have A^C to denote every element not in A.

Theorem 1.1 (DeMorgan's Laws)**.** Let A and B be sets.

(a) $(A \cup B)^C = A^C \cap B^C$

(b) $(A \cap B)^C = A^C \cup B^C$

(c) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

(d) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

1.1.2 Indexed Sets

Let Λ be a set and suppose for each $a \in \Lambda$ there is a set A_a. The set $\{A_a : a \in \Lambda\}$ is called a **collection of sets indexed by Λ**. In this case, Λ is called the **indexing set** for this collection.

$$\bigcup_{a \in A} = \{x | x \in A_a \text{ for some } a \in A\}$$

$$\bigcap_{a \in A} = \{x | x \in A_a \text{ for all } a \in A\}.$$

We can generalize DeMorgan's laws to indexed collections:

Theorem 1.2 (Generalized DeMorgan)**.** If $\{B_a : a \in \Lambda\}$ is an indexed collection of sets and A is a set, then

$$A \setminus \bigcup_{a\in\Lambda} B_a = \bigcap_{a\in\Lambda}(A \setminus B_a),$$
$$A \setminus \bigcap_{a\in\Lambda} B_a = \bigcup_{a\in\Lambda}(A \setminus B_a).$$

1.1.3 Set of Natural Numbers

We set $\mathbb{N}$ to be all positive integers, $\mathbb{Z}$ to be all integers, and $\mathbb{N}_0$ to be all nonnegative integers.

Definition 1.3 (Peano Axioms)**.**
1. $1 \in \mathbb{N}$.
2. If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$. We'll call this the **successor**.
3. 1 is not the successor of any element
4. If $n, m \in NN$ have the same successor, then $n = m$.
5. (Induction) If $S \subseteq \mathbb{N}$ with the properties $1 \in S$ and $n \in S \implies n + 1 \in S$, then $S = \mathbb{N}$. This becomes induction when we have S as the set of elements where a certain property holds.

So, for induction, we have a base case where we have P_0 or P_1 or some starting value. And then, we have induction that proves that P_k being true implies P_{k+1} is true. Then it dominoes over.

Remember that we didn't prove that P_{n+1} is true, but rather that it can be implied from P_n.

1.1.4 Set of Rational Numbers

We define $\mathbb{Q}$, the set of rational numbers, by $\mathbb{Q} := \{\frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0\}$.

Remark 1.4. $\mathbb{Q}$ contains all terminating decimals.

Remark 1.5. If $\frac{m}{n} \in \mathbb{Q}$ and $r \in \mathbb{Z} \setminus \{0\}$, then $\frac{m}{n} = \frac{rm}{rn}$, so we assume that m, n are coprime usually.

Definition 1.6 (Field Axioms)**.** Remembering these is now an exercise for the reader.

We see that the set of rational numbers with addition and multiplication is a field. Going through the axioms is left as an exercise to the reader.