# Math 115 - Final Review

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All of the material between the second midterm and the final.

## 1 Unit Groups

## 2 Gauss Sums

## 3 Jacobi Sums

Some initial definitions would help provide some context:

**Definition 1** (Multiplicative Character)
We call $\chi$ a **multiplicative character** if $\chi(ab) = \chi(a)\chi(b)$, over all $a, b \in F_p$.

We have a couple other rules for this function:

1. $\chi(1) = 1$
2. $\chi(a)$, over any $a$, is a $p-1$th root of unity
3. $\chi(a^{-1}) = \chi(a)^{-1} = \overline{\chi(a)}$.

We also have the identity character $\varepsilon$, where $\varepsilon(a) = 1$ over all $a \in F_p$.

**Definition 2** (Jacobi Sum)
The **Jacobi sum** of two multiplicative characters $\chi, \lambda$, given $a, b \in F_p$ for some prime $p$, is $J(\chi, \lambda) = \sum_{a+b=1} \chi(a)\lambda(b)$.

There are also a couple theorems you should know, and one you should know the proof of:

**Theorem 3**

1. $J(\epsilon, \epsilon) = p$.
2. $J(\epsilon, \chi) = 0$.
3. $J(\chi, \chi^{-1}) = -\chi(-1)$.
4. If $\chi\lambda \neq \epsilon$, then

$$J(\chi, \lambda) = \frac{g(\chi)g(\lambda)}{g(\chi)(\lambda)}.$$

This one we need to prove: The first two statements can be done just by checking the sum over all $a, b \in F_p$ such that $a + b = 1$.

The third statement is true because $\chi(a)\chi^{-1}(b) = \chi(a)\chi(b^{-1}) = \chi\left(\frac{a}{b}\right)$, so we're summing over $\chi(a(1-a)^{-1})$. We see that this maps to every value but $-1$, so we have that our sum includes everything but $\chi(-1)$. But as the sum of everything is 0, we have $J(\chi, \chi^{-1}) = -\chi(-1)$.

The last statement is true because