Math 104: Real Analysis

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Contents

Chapter 1
Week 1

1.1 Lecture 1

1.1.1 Logic and Sets

For clauses p, q: we have $p \wedge q$, $p \vee q$, $\neg p$. These are *and*, *or*, *not*; respectively.

Moreover, we have $p \implies q$ meaning that q is true if p is true. Moreover, we have $p \iff q$ meaning that p is true if q is true and q is true if p is true.

Other terminology: $:=$ is a definition, $\forall$ is for all, $\exists$ is exists, $a \in A$ means that element a is in the set A, $a \notin A$ means that element a isn't in the set A.

For sets, we have $\subset, =, \subseteq$ to determine subset and equality relations. Moreover, we have $\cap, \cup$ to represent union and intersections of sets. There is also $A \setminus B$ to denote everything in A but not B, and we have A^C to denote every element not in A.

Theorem 1.1 (DeMorgan's Laws)**.** Let A and B be sets.

(a) $(A \cup B)^C = A^C \cap B^C$

(b) $(A \cap B)^C = A^C \cup B^C$

(c) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

(d) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

1.1.2 Indexed Sets

Let Λ be a set and suppose for each $a \in \Lambda$ there is a set A_a. The set $\{A_a : a \in \Lambda\}$ is called a **collection of sets indexed by Λ**. In this case, Λ is called the **indexing set** for this collection.

$$\bigcup_{a \in A} = \{x | x \in A_a \text{ for some } a \in A\}$$

$$\bigcap_{a \in A} = \{x | x \in A_a \text{ for all } a \in A\}.$$

We can generalize DeMorgan's laws to indexed collections:

Theorem 1.2 (Generalized DeMorgan). If $\{B_a : a \in \Lambda\}$ is an indexed collection of sets and A is a set, then

$$A \setminus \bigcup_{a\in\Lambda} B_a = \bigcap_{a\in\Lambda} (A \setminus B_a),$$
$$A \setminus \bigcap_{a\in\Lambda} B_a = \bigcup_{a\in\Lambda} (A \setminus B_a).$$

1.1.3 Set of Natural Numbers

We set $\mathbb{N}$ to be all positive integers, $\mathbb{Z}$ to be all integers, and $\mathbb{N}_0$ to be all nonnegative integers.

Definition 1.3 (Peano Axioms).

1. $1 \in \mathbb{N}$.
2. If $n \in \mathbb{N}$, then $n+1 \in \mathbb{N}$. We'll call this the **successor**.
3. 1 is not the successor of any element
4. If $n, m \in NN$ have the same successor, then $n = m$.
5. (Induction) If $S \subseteq \mathbb{N}$ with the properties $1 \in S$ and $n \in S \implies n+1 \in S$, then $S = \mathbb{N}$. This becomes induction when we have S as the set of elements where a certain property holds.

So, for induction, we have a base case where we have P_0 or P_1 or some starting value. And then, we have induction that proves that P_k being true implies P_{k+1} is true. Then it dominoes over.

Remember that we didn't prove that P_{n+1} is true, but rather that it can be implied from P_n.

1.1.4 Set of Rational Numbers

We define $\mathbb{Q}$, the set of rational numbers, by $\mathbb{Q} := \{\frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0\}$.

Remark 1.4. $\mathbb{Q}$ contains all terminating decimals.

Remark 1.5. If $\frac{m}{n} \in \mathbb{Q}$ and $r \in \mathbb{Z} \setminus \{0\}$, then $\frac{m}{n} = \frac{rm}{rn}$, so we assume that m, n are coprime usually.

Definition 1.6 (Field Axioms). Remembering these is now an exercise for the reader.

We see that the set of rational numbers with addition and multiplication is a field. Going through the axioms is left as an exercise to the reader.

1.2 Lecture 2

1.2.1 Ordered Sets

Definition 1.7 (Ordered Set)**.** We define an **ordered set** to be a set S with an order satisfying the following criteria:

1. $\forall \alpha, \beta \in S$, either $\alpha < \beta, \alpha = \beta, \alpha > \beta$.
2. $\alpha < \beta, \beta < \gamma \implies \alpha < \gamma$.
3. $\alpha \leq \beta \implies \alpha + \gamma \leq \beta + \gamma$
4. $\alpha \leq \beta, \gamma \geq 0 \implies \alpha\gamma \leq \beta\gamma$
5. $\alpha \leq \beta, \beta \leq \alpha \implies \alpha = \beta$

A set that is a field and an ordered set can be called an **ordered field**.

1.2.2 Defects of $\mathbb{Q}$

Theorem 1.8 (Irrationality of $\sqrt{2}$)**.** There is no α such that $\alpha^2 = 2$.

Proof. Suppose for the sake of contradiction that there is $\alpha \in \mathbb{Q}$ such that $\alpha^2 = 2$. We see that $\alpha = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ such that $\gcd(m, n) = 1$.

Since $a^2 = 2$, we have $\frac{m^2}{n^2} = 2$, implying that $m^2 = 2n^2$, or $2|m^2$. As 2 is prime, we see that $2|m \implies m = 2p$ for some integer p. Then, $(2p)^2 = 2n^2 \implies 4p^2 = 2n^2 \implies n^2 = 2p^2$. Once again, we have that $2|n$ from above, so m, n are both even. This contradicts the claim that m, n are coprime, so $\sqrt{2}$ cannot be expressed in a rational form. ∎

This motivates the concept of incompleteness.

Definition 1.9 (Incompleteness)**.** Let S be an ordered set, and let $A \subseteq S$.

1. An element $\beta \in S$ is an **upper bound** for A if $\alpha \leq \beta, \forall \alpha \in A$.

   Then, we say that A is **bounded above**.
2. An element $\beta \in S$ is a **lower bound** for A if $\alpha \geq \beta, \forall A \in A$.

   Then, we say that A is **bounded below**.

Definition 1.10 (Supremum)**.** Suppose S is an ordered set and $A \in S$ is bounded above. Suppose $\exists B \in S$ such that:

1. β is an upper bound for A.
2. If r is another upper bound, then $r \geq \beta$.

Then, we will call β the least upper bound, or the **supremum** ($\sup$) of A.

The greatest lower bound is then called the **infimum** ($\inf$) of A.

Remark 1.11. Supremum and infimum may not exist or belong to A.

Definition 1.12 (Completeness)**.** An ordered set S is said to have the least upper bound property, or **completeness**, if every upper bounded set has a supremum in S.

1.2.3 Real Numbers

Theorem 1.13 (Real Numbers)**.** There is a unique ordered field $(\mathbb{R}, +, \cdot, \leq)$ that has the following properties:

1. Completeness.
2. $\mathbb{Q} \subseteq \mathbb{R}$ is an ordered subfield; i.e., $(+, \cdot, \leq)$ restricted to $\mathbb{Q}$ are the usual $(+, \cdot, \leq)$ on $\mathbb{Q}$.

Lecturer says we will be using the result, and not the proof.

There's another theorem with properties of real number arithmetic but honestly I'm too lazy to write it as of now so you'll see it later.

1.2.4 Consequences of the Completeness Axiom

Existence of Infimum

Theorem 1.14. Let $E \subset \mathbb{R}$ be a set bounded below. Then $\inf E$ exists in $\mathbb{R}$.

Proof. Define $-E$ to be $\{-x|x \in E\}$. Then, we see that $-E$ must be bounded above, implying that it must have a supremum by definition of completeness. Then, we let $\sup -E = \beta$.

Then, for any α such that $x \geq \alpha \forall x \in E$, then we have $y \leq -\alpha \forall y \in E$. We see that $-\alpha \geq \beta$ by definition of supremum, so $\alpha \leq -\beta$. As a result, $\inf E = -\beta$. ■

1.3 Lecture 3

1.3.1 Archimedean Property

Theorem 1.15 (Archimedean Property). If $a > 0$ and $b > 0$, then for some positive integer n, we have $na > b$.

Proof. Assume that the Archimedean Property fails. Then, for all positive integers n, we have $na < b$ for some positive a and b.

Now, let's observe $S = \{na | n \in \mathbb{N}\}$. Then, let $b = \sup S$, which must exist by completeness.

Consider $b - a$. Since b is a supremum and a is positive, then $\exists s \in S$ such that $s > b - a$. However, $a + s$ must also be in S by definition, and $a + s > a + (b - a) = b$, contradicting the claim that $b = \sup S$. ■

Corollary 1.16.
1. If $a > 0$, then $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < a$.
2. If $b > 0$, then $\exists n \in \mathbb{N}$ such that $b < n$.

Theorem 1.17 (Density of $\mathbb{Q}$). If $a, b \in \mathbb{R}$ and $a < b$, then $\exists r \in \mathbb{Q}$ such that $a < r < b$.

Proof. This is an exercise for the reader. Till I fill the proof in. ■

Theorem 1.18 (Existence of nth roots). Given any $\alpha \in \mathbb{R}$, $\alpha > 0$, and any $n \in \mathbb{N}$, there's a $\beta \in \mathbb{R}$ s.t. $\beta^n = \alpha$.

Proof. This is an exercise for the reader until I feel like writing more about this. ■

Corollary 1.19.
1. $b_1, b_2 > 0$ s.t. $b_1^n = b_2^n$. Then, $b_1 = b_2$.
2. If $a, b > 0$, then $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.

1.3.2 (Gates to) Infinity

We define the set of **extended reals** to be $\mathbb{R}^* = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, with the extended order $-\infty < \alpha < \infty$, $\forall \alpha \in \mathbb{R}$.

Then, ∞ is an upper bound for any $E \subset \mathbb{R}$ and $-\infty$ is a lower bound for any $E \subset \mathbb{R}$.

We can extend the definition of $\sup$ and $\inf$ such that

- $\sup E = \inf$ if E is not bounded above, and
- $\inf E = -\inf$ if E is not bounded below.

Note that $\mathbb{R}^*$ does not form a field. As a result, we cannot apply a theorem or exercise stated for real numbers to ∞, $-\infty$. This set doesn't have an algebraic structure.

We also denote unbounded intervals using $-\infty, \infty$ instead of real numbers.

Remark 1.20. Let S be any nonempty subset of $\mathbb{R}$. The symbols $\sup S$ and $\inf S$ always make sense. If S is bounded above, then $\sup S$ is a real; otherwise, it is $+\infty$. Same logic applies to lower bounds and $-\infty$.

Moreover, the statement $\inf S \leq \sup S$ also always makes sense.

Chapter 2
Week 2

2.1 Lecture 4

2.1.1 Limits of Sequences

Definition 2.1 (Sequence). A **sequence** is a function S whose domain is a set of the form $\{n \in \mathbb{Z} : n \geq m\}$; m is usually 1 or 0.

Or, a sequence is an infinite list of real numbers.

Note that we must be careful in ensuring that we have $\ldots$ at the end of our list for repeating sequences, to make clear that our sequence goes to infinity.

Given a sequence $S_1, S_2, \ldots$, we want to figure out what happens to S_n as $n \to \infty$.

Example 2.1. $S_n = \frac{1}{\sqrt{n}}, n \in \mathbb{N}$. The terms seem to "approach" zero.

Example 2.2. $S_n = (-1)^n, n \geq 0$. The sequence jumps around, and it appears to not approach any single value.

Intuitively: $\lim_{n\to\infty} S_n = S$ means that as S gets large, then S_n goes to S.

Epsilon-Delta

Definition 2.2 (Formal Definition of Convergence). A sequence S_n of real numbers is said to **converge** to the real number S if:

$$\forall \epsilon > 0, \exists N = N(\epsilon) \text{s. t. } n > N \implies |S_n - S| \leq \epsilon.$$

So, no matter how small ϵ is, there is a threshold N s.t. once you have $n > N$, then you can guarantee that S_n is at most ϵ away from S.

Definition 2.3 (Limits)**.** If S_n converges to S, we will write that

$$\lim_{n\to\infty} S_n = S, \text{ or } S_n \to S.$$

The number S is called the **limit** of the sequence (S_n).

A sequence that doesn't converge to any set number is said to **diverge**.

Remark 2.4.

1. The threshold N in the first definition can be treated as a positive integer by the Archimedean Property.
2. It's traditional to use ϵ and δ in situations where the interesting values are small positive values.
3. The first definition is an infinite number of statements, one for each ϵ.

Also, usually N depends on ϵ, usually with an inverse relationship.

2.1.2 Proving Limits

Example 2.3. $\lim \frac{1}{\sqrt{n}} = 0$.

We see that $|S_n - S| = |\frac{1}{\sqrt{n}} - 0| = |\frac{1}{\sqrt{n}}| = \frac{1}{\sqrt{n}}$. We claim that this is less than ϵ for some n.

We have that $\frac{1}{n} < \epsilon^2 \implies n > \frac{1}{\epsilon^2}$. However, as n is unbounded, we see that there is some $n \in \mathbb{N}$ such that $n > \frac{1}{\epsilon^2}$.
But this isn't a rigorous mathematical proof.

Proof. Let's set $N = \frac{1}{\epsilon^2}$. We claim that $|\frac{1}{\sqrt{n}} - 0| \leq \epsilon$. Setting $n > \frac{1}{\epsilon^2}$ implies that $\frac{1}{n} < \epsilon^2$. Therefore, we have $|]frac1\sqrt{n} - 0| = |\frac{1}{\sqrt{n}}| = \frac{1}{\sqrt{n}} < \epsilon$. Therefore, our sequence $S_n = \frac{1}{\sqrt{n}}$ must converge to 0. ∎

Example 2.4. $\lim_{n\to\infty} \frac{2n+4}{5n+2} = \frac{2}{5}$.

We see that

$$\begin{aligned}|S_n - S| &= |\frac{2n+4}{5n+2} - \frac{2}{5}| \\ &= |\frac{10n + 20 - (10n+4)}{5(5n+2)}| \\ &= |\frac{16}{25n+10}| \\ &= \frac{16}{25n+10}.\end{aligned}$$

We want this value to be less than ϵ, so we have $\frac{16}{25n+10} < \epsilon \implies 25n + 10 > \frac{16}{\epsilon} \implies n > \frac{16-10\epsilon}{25\epsilon}$.

Proof. We set N to be $\frac{16-10\epsilon}{25\epsilon}$ for all $\epsilon > 0$. Then, we claim that for all $n > N$, $|\frac{2n+4}{5n+2} - \frac{2}{5}| < \epsilon$.

We see from above that this is equivalent to $\frac{16}{25n+10} < \epsilon$. However, we know that $n > N \implies n > \frac{16-10\epsilon}{25\epsilon}$. Then, we have

$$\frac{16}{25n+10} < \frac{16}{25\left(\frac{16-10\epsilon}{25\epsilon}\right)+10} = \frac{16}{\frac{16-10\epsilon}{\epsilon}+10} = \frac{16}{\frac{16}{\epsilon}} = \epsilon.$$

Therefore, our sequence $S_n = \frac{2n+4}{5n+2}$ does indeed converge to $\frac{2}{5}$. ■

We *must* write the formal proof, because all we do in the first part is find a potential threshold N, and not prove that it is valid.

2.2 Lecture 5

2.2.1 Limits Theorem for Sequences

We start by showing that limits are unique.

Theorem 2.5. If $\lim_{n\to\infty} S_n = s$ and $\lim_{n\to\infty} S_n = t$, then $s = t$.

Proof. We first reframe this in terms of ϵ. Then, we see that for each $\epsilon > 0$, there exists an N_1, N_2 such that $n > N_1 \implies |S_n - s| < \frac{\epsilon}{2}$. Also, $n > N_2 \implies |S_n - t| < \frac{\epsilon}{2}$. Now, we consider $N = \max(N_1, N_2)$.

From the Triangle Inequality, we see that for all $n > N$, $|s - t| \leq |S_n - s| + |S_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. As a result, $\forall n > N, |s - t| < \epsilon$. Therefore, $|s - t| < \epsilon$ for all $\epsilon > 0$, implying that $|s - t| = 0$, or $s = t$. ∎

Definition 2.6 (Bounded Sequence)**.** A sequence S_n is said to be **bounded** if its set of values $\{S_n : n \in \mathbb{N}\}$ is a bounded set.

Theorem 2.7. Convergent sequences are bounded.

Proof. Let S_n be a convergent sequence, and let $S = \lim_{n\to\infty} S_n$. Take $\epsilon = 1$, then we obtain $N \in \mathbb{N}$ such that

$$\forall n > N, |S_n - S| < 1.$$

By the triangle inequality, for all $n > N$, we have $|S_n| = |S_n - s + S| \leq |S_n - S| + |S| < 1 + |S|$.

Now, let $M = \max\{|S_1|, |S_2|, \ldots, |S_N|, |S| + 1$. Then, we see that $|S_n| \leq M, \forall n \in \mathbb{N}$. Therefore, (S_n) is a bounded sequence. ∎

Remark 2.8. The converse of the above theorem is not true. Consider $(-1, 1, \ldots)$. Some condition must be added.

For cases where we need to prove a sequence is bounded, we should set ϵ to 1 and try to find a resulting upper bound.

2.2.2 Limit Laws

Theorem 2.9. If $S_n \to S$ and $k \in \mathbb{R}$, then $kS_n \to kS$.

That is, $\lim kS_n = k \lim S_n$.

Proof. Let $k \neq 0$, and let $\epsilon > 0$.

For all $n > N$, we have $|S_n - S| < \frac{\epsilon}{|k|}$. We want to show that $|kS_n - kS| < \epsilon$ for $n > N$ as well.

Then,

$$\begin{aligned} n > N &\implies |k| \cdot |S_n - S| < \epsilon \\ n > N &\implies |kS_n - kS| < \epsilon, \end{aligned}$$

and we are done. ∎

Theorem 2.10. If $s_n \to s$ and $t_n \to t$, then $(s_n + t_n) \to s + t$.

Proof. We let $\epsilon > 0$.

We know that there exists $N_1, N_2 \in \mathbb{N}$ such that for $n > N_1$, we have $|s_n - s| < \epsilon/2$, and for $n > N_2$ we have $|t_n - t| < \epsilon/2$.

Then, let $N = \max(N_1, N_2)$. then, $\forall n > N$ we have $|s_n - s| + |t_n - t| < \epsilon$. We have that that $|s_n - s| + |t_n - t| \geq |s_n + t_n - s - t| = |(s_n + t_n) - (s + t)|$, so it follows that $|(s_n + t_n) - (s + t)| < \epsilon/2 + \epsilon/2 = \epsilon$. ∎

Theorem 2.11. If $s_n \to s$ and $t_n \to t$, then $s_n t_n \to st$.

Proof. I'm gettin lazy ∎

Lemma 2.12. If $s_n \neq 0$ for all n and $s_n \to s \neq 0$, then $\frac{1}{s_n} \to \frac{1}{s}$.

Theorem 2.13. $\lim \frac{t_n}{s_n} = \frac{\lim t_n}{\lim s_n}$.

Theorem 2.14. We have the following limit laws:

(a) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$.

(b) $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$

(c) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(d) $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ if $a > 0$.

The proof uses both the binomial and squeeze theorems. Proof happens in the next lecture.

Theorem 2.15 (Squeze Theorem)**.** Let $(a_n), (b_n), (c_n)$ be sequences such that $a_n \leq b_n \leq c_n$. Then,

1. If $a_n \to A$, $b_n \to B$, $c_n \to C$, then $A \leq B \leq C$.
2. If $\lim a_n = \lim c_n = L$, then $\lim b_n$ exists and is equal to L.

2.3 Lecture 6

2.3.1 Proofs of Limit Laws

(a) fill out later

(b) fill out later

(c) fill out later

(d) fill out later

2.3.2 Infinite Limits

Definition 2.16 (Divergence)**.** There are two cases for divergence.

$\lim_{n\to\infty} s_n = \infty$ if for each $M > 0$, there exists an N such that $n > N \implies s_n > M$.

$\lim_{n\to\infty} s_n = -\infty$ if for each $M < 0$, there exists an N such that $n > N \implies s_n < M$.

From now on, we'll say that a limit **converges**, **diverges to** $+\infty$, or **diverges to** $-\infty$.

We now say that (s_n) **has a limit**, or the limit **exists**, if it is convergent or divergent.

Remark 2.17. Many sequences don't have limits of $+\infty$, $-\infty$, even when unbounded.

Example 2.5. Show that $\lim_{n\to\infty} \sqrt{n-5}+3 = +\infty$.

Proof. Our first step is to find N. We have

$$\begin{aligned}\sqrt{n-5}+3 > M &\implies \sqrt{n-5} > M-3\\ &\implies n-5 > (M-3)^2\\ &\implies n > (M-3)^2+5.\end{aligned}$$

Thus, we let $N = (M-3)^2+5$.

Next, we let $M > 0$ be given, and let $N = (M-3)^2+5$ s.t. $n > N \implies \sqrt{n-5}+3 > \sqrt{(M-3)^2+5-5}+3 = M-3+3 = M$. ■

Theorem 2.18. Suppose $s_n \to +\infty$, $t_n \to t$. Then, $s_n t_n \to +\infty$. This is true for any $t > 0$ or $t = +\infty$.

Proof. Let $M > 0$. Select a real number m such that $0 < m < \lim t_n$. Regardless of if $t_n \to \infty$, there exist an N_1 such that $n > N_1 \implies t_n > m$.

$S_n \to +\infty$ implies that there exists an N_2 such that $n > N_2 \implies s_n > \frac{M}{m}$. Now, let $N = \max(N_1, N_2)$. We see that $n > N \implies s_n t_n > \frac{M}{m} \cdot m = M$. ■

Theorem 2.19 (Duality)**.** If $s_n > 0 \forall n$, then

$$s_n \to +\infty \iff \frac{1}{s_n} \to 0.$$

Proof. For the forward direction, let $\epsilon > 0$ be given. Let $M = \frac{1}{\epsilon}$. Since $s_n \to \infty$, then there must exist an N such that $\forall n > N \implies s_n > M = \frac{1}{\epsilon}$.

But then, for that same N, if $n > N$, we have $|\frac{1}{s_n} - 0| = \frac{1}{s_n} < \frac{1}{M} = \epsilon$ from the previous paragraph. This implies that $|\frac{1}{s_n} - 0| = 0$, or the limit is 0. ■

For the backwards direction, we let $M > 0$ be given. Let $\epsilon = \frac{1}{M}$. Then, since $\frac{1}{s_n} \to 0$, there exists an N such that $\forall n > N$, $|\frac{1}{s_n}| = \frac{1}{s_n} < \epsilon = \frac{1}{M}$.

But for the same N, we have $n > N \implies S_n > \frac{1}{\epsilon} = \frac{1}{\frac{1}{M}} = M$, so the limit must tend to infinity. ■

2.4 Lecture 7

2.4.1 Monotonic Sequences

Definition 2.20 (Monotonicity)**.** A sequence (s_n) is increasing if $s_{n+1} \geq s_n$ for all n, and decreasing of $s_{n+1} \leq s_n$ for all n. If either of these describes a sequence, that sequence is **monotonic**.

The following theorem shows why we care about monotonic sequences:

Theorem 2.21. Let (s_n) be a sequence.

1. If (s_n) is increasing and bounded above then (s_n) converges.
2. If (s_n) is decreasing and bounded below then (s_n) converges.

Corollary 2.22. All monotonic sequences that are bounded converge.

Theorem 2.23.
1. An increasing unbounded sequence diverges to $+\infty$.
2. A decreasing unbounded sequence diverges to $-\infty$.

Proof.

1. We claim for the sake of contradiction that (s_n) is unbounded but does not diverge. Then, for some $M > 0$, there exists s_n such that s_n
2. Similar argument.

■

2.4.2 Cauchy Sequences

Definition 2.24. A sequence (s_n) of real numbers is called a **Cauchy sequence** if

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } n, m > N \implies |s_n - s_m| < \epsilon.$$

In other words, the terms get closer and closer to each other.

Lemma 2.25. If a sequence is convergent, it's a Cauchy sequence.

Proof. Suppose (s_n) is convergent, say $s_n \to s$. Then, let $\epsilon > 0$ be given. Then, there must exist an N such that $n > N \implies |s_n - s| < \epsilon/2$. However, for any $m > N$ we also have $s_m - s < \epsilon/2$. Then, it follows that $|s_n - s| + |s_m - s| < \epsilon$. Moreover, from the Triangle Inequality, $|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s_m - s|$, so we have $|s_n - s_m| < \epsilon$. ∎

And what about the other direction?

Lemma 2.26. Cauchy sequences are bounded.

The proof is similar for that of the proof that convergent sequences are bounded.

Proof. Let (s_n) be a Cauchy sequence. Then, taking $\epsilon = 1$, $\exists N \in \mathbb{N}$ such that $m, n > N \implies |s_n - s_m| = 1$. ∎

We see that for bounded sequences that don't converge, taking the supremum of all elements $n > N$ for each N leads to a convergent sequence. So even when there's no limit, we can still find a way to get a sequence with a limit.

Definition 2.27 ($\limsup$)**.** Let (s_n) be a sequence in $\mathbb{R}$. We define,

$$\limsup s_n = \lim_{n \to \infty} \sup\{s_n : n > N\}.$$

$\liminf$ can be defined similarly.

Chapter 3
Week 3

Chapter 4
Week 4

Chapter 5
Week 5

5.1 Lecture 14

5.2 Lecture 15

5.3 Lecture 16

Theorem 5.1 (Mean Value Theorem).

Theorem 5.2 (Cauchy's MVT).

5.4 Lecture 17

5.4.1 Riemann Integral

Definition 5.3. Let $[a,b]$ be a given interval. By a **partition** P of $[a,b]$ we mean a finite set of points $x_0, x_1, \ldots, x_n$, where $a = x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n = b$.

For any $i = 1, \ldots, n$, we define $\Delta x_i = x_i - x_{i-1}$.

Then, we take Riemann sums.

Suppose $f : [a,b] \to \mathbb{R}$ is bounded. Let P be a partition of $[a,b]$. We put $M_i = \sup f(x), x_{i-1} \leq x \leq x_i$, and $m_i = inf f(x) x_{i-1} \leq x \leq x_i$.

Then, M_i gets too much area and m_i loses area. That is $U(P,f) = \sum\limits_{i=1}^{n} M_i \Delta x_i$, and $L(P,f) = \sum\limits_{i=1}^{n} m_i \Delta x_i$.

The upper Riemann integrals of f are $\int_a^{\bar{b}} f dx = \inf U(P,f)$. The lower Riemann integrals of f are $\int_{\underline{a}}^{b} f dx = \sup L(P,f)$.

There must exist $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$. Thus, $m(b-a) \leq f(x)(b-a) \leq M(b-a)$. Thus, for every P, $m(b-a) \leq \sum\limits_{i=1}^{n} m_i \Delta x \leq \sum\limits_{i=1}^{n} M_i \Delta x \leq M(b-a)$.

If lower and upper integrals are equal, then we just have integral. Another name for lower/upper integrals are Darboux integrals.

Definition 5.4. Given a partition P, a partition P^* is called a **refinement** of P if $P \subseteq P^*$ (that is, if every point of P is a point of P^*).

Given two partitions P_1, P_2, we call $P*$ the **common refinement** if $P^* = P_1 \cup P_2$.

Theorem 5.5. Let $f : [a, b] \to \mathbb{R}$ be bounded and monotonic. Then, $f \in R[a, b]$.

This means f is a Riemann integrable function. This means it must have upper and lower Riemann integrals the same.

CHAPTER 6
WEEK 6

6.1 Lecture 18 - Metric Spaces

Definition 6.1. Let X be a set. A **metric** (distance function) d on X is a function $d : X \times X \to \mathbb{R}, (x, y) \to d(x, y)$.

Definition 6.2 (Metric Space)**.** A metric space is comprised of a set X, distance function $d : X \times X \to \mathbb{R}$. We have multiple properties to make (X, d) a metric space:

1. **Positivity**. $d(x, y) > 0$ if $x \neq y$, $x, y \in X$ or $d(x, x) = 0$.
2. **Symmetry**. $d(x, y) = d(y, x)$, for all $x, y \in X$.
3. **Triangle Inequality**.

Example 6.1. $(\mathbb{R}, d)$ is a metric space with $d(x, y) := |x - y|$.

Clearly, d is a metric on $\mathbb{R}$, as $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. This satisfies:

1. $d(x, y) = |x - y| > 0, d(x, x) = |x - x| = 0$ for all $x \neq y, x, y \in \mathbb{R}$.
2. $d(y, x) = |y - x| = |-(x - y)| = |x - y| = d(x, y)$ for all $x, y \in \mathbb{R}$.
3. $d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$ for all $x, y, z \in \mathbb{R}$.

Definition 6.3 (Euclidean Space)**.** For $n \geq 1$, define n-dimensional Euclidean space:

$$\mathbb{R}^n = \{\vec{x} = (x_1, \dots, x_n) | x_j \in \mathbb{R}, 1 \leq j \leq n\}.$$

Example 6.2 (Discrete Metric)**.** Let x be any set. Define $d_d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$.

1. Positivity holds by definition.
2. $d_d(x, y) = d_d(y, x) = 1$ for $x \neq y$.
3. $d_d(x, y) \leq d_d(x, z) + d_d(z, y)$

Definition 6.4 (balls)**.** Let (X, d) be a metric space.

For any $x \in X$, $r > 0$,

1. The subset $B_r(x) := \{y \in X | d(x, y) < r\}$ is called the **open ball** centered at x with radius r.
2. The subset $\overline{B_r(x)} := \{y \in X | d(x, y) \leq r\}$ is called the **closed ball** centered at x with radius r.
3. An open ball centered at x is also a **neighborhood** of x.

In $\mathbb{R}$, each open/closed ball is equivalent to a finite open/closed interval.

Definition 6.5 (Openness)**.** Let (X, d) be a metric space. A subset $A \subset X$ is called **open** if $A = \phi$ or if for every $x \in A$, there exists some open ball $B_r(x) \subset A$ for some $r > 0$.

Theorem 6.6. Any open ball is open.