Math 110 – Lecture 11

- section 3e – products & quotient spaces is being skipped.
- $M(T)(i,j)$ is the coefficient in front of w_i of the vector $T\vec{v}_j$.
  - $\underbrace{i \in \{1 \cdots n\}}_{\text{target}}$ $\underbrace{j \in \{1 \cdots m\}}_{\text{domain}}$.

- likewise, we can define $M(v)$ or $M(w)$ for $v \in V$ and $w \in W$.
- if $\vec{v} = a_1\vec{v}_1 + \cdots + a_m\vec{v}_m \Rightarrow M(\vec{v}) = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$
  - so our actions can be purely understood by matrix multiplication:
    $M(Tv) = M(T)\,M(v)$.

$$\left[M(T\vec{v}_1), \ldots, M(T\vec{v}_m)\right]\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = a_1 M(T\vec{v}_1) + \cdots + a_m M(T\vec{v}_m).$$
$$= M(T\vec{v}) \text{ where } M(\vec{v}) = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$$

- can split things column by column or row by row.

Ex: $V = W = \mathcal{P}_{\leq 3}(\mathbb{R})$. with the same basis for V and W $(1, x, x^2, x^3)$.

$$T = D \simeq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad p(x) = -\pi + ex + \frac{x^3}{3}.$$
$$Tp(x) = \frac{dp}{dx} = 0 + e + x^2.$$

$$M(p) = \begin{bmatrix} -\pi \\ e \\ 0 \\ 1/3 \end{bmatrix} \qquad M(Tp) = \begin{bmatrix} e \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\pi \\ e \\ 0 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} e \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

- The space $\mathcal{L}(V, \mathbb{F})$ is a.k.a. the dual space to V and is denoted by V'.
  $\dim \mathcal{L}(V, \mathbb{F}) = \dim V$.

Dual Basis: suppose $\dim V < \infty$ and $\vec{v}_1 \ldots \vec{v}_m$ is a basis for V.
we say $\varphi_1, \ldots, \varphi_n \in V'$ is the dual basis if
$$\varphi_i(\vec{v}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

- not clear why φ's ever form a basis, or why it's unique.

- uniqueness
  - $\varphi_1(v_1) = 1$, $\varphi_1(v_{\text{anything else}}) = 0$.
  - LM is defined by its actions and φ_i act a certain way on $\vec{v}_i$ so must be unique.
- linear independence
  - Suppose we have scalars $a_1 \cdots a_n$ s.t. $a_1\varphi_1 + a_2\varphi_2 + \cdots + a_n\varphi_n = 0$.

$a_1\varphi_1(v_1) + a_2\varphi_2(v_1) + \cdots + a_n\varphi_n(v_1) = a_1 + 0 + \cdots = a_1$. So $a_1 = 0$ for $v_1 \mapsto 0$.

$a_1\varphi_1(v_2) + a_2\varphi_2(v_2) + \cdots + a_n\varphi_n(v_2) = a_2 \longrightarrow a_2 = 0$ for $v_2 \mapsto 0$.

$\vdots$

$a_n = 0$ for $v_n \mapsto 0$.

  - So for a linear combination $w = b_1v_1 + \cdots + b_nv_n$, we would need all $a_i = 0$ to guarantee $w = 0$.

- spanning
  - dimension is n, so lin. ind. set of size n is spanning so must be a basis.

Ex: $V = \mathcal{P}_{\leq 3}(\mathbb{R})$. basis $1, x, x^2, x^3$.

- find dual basis.

$\varphi_1(1) = 1$
$\varphi_1(x) = 0$
$\varphi_1(x^2) = 0$
$\varphi_1(x^3) = 0$.

$\longrightarrow$

$\left.\begin{array}{l} \varphi_1 = \delta_0 \\ \varphi_2 = \delta_0' \\ \varphi_3 = \delta_0''/2 \\ \varphi_4 = \delta_0'''/3. \end{array}\right\}$ taylor series, if we continue.

$\psi : p \mapsto \int_0^1 p(t)\, dt.$

$\psi = a_0\delta_0 + a_1\delta_0' + \frac{a_2}{2}\delta_0'' + \frac{a_3}{6}\delta_0'''$ for some a_0, a_1, a_2, a_3.

$\psi(1) = 1 = a_0$

$\psi(x) = \frac{1}{2} = a_1$

$\psi(x^2) = \frac{1}{3} = a_2$

$\psi(x^3) = \frac{1}{4} = a_3$

$= \delta_0 + \frac{1}{2}\delta_0' + \frac{1}{6}\delta_0'' + \frac{1}{24}\delta_0'''.$

The dual map:

Setup: $T \in \mathcal{L}(V, W)$, $\dim V < \infty$, $\dim W < \infty$. The map dual to T (T') is defined as

$$\varphi \in W' \longmapsto \varphi \circ T \in V'$$

Example: $V = \mathcal{P}_{\leq 3}(\mathbb{R})$, $W = \mathcal{P}_{\leq 2}(\mathbb{R})$. $T = D$, $\varphi = (\delta_1$ on $W)$

$T'(\varphi) = \delta_1'$