Lecture 5

Linear independence

$\vec{u} = (2,3,6)$ in $\vec{v}_1 = (1,0,0)$ $\vec{v}_2 = (0,1,0)$ $\vec{v}_3 = (0,0,1)$

- unique,
- $\vec{u} = 2\vec{v}_1 + 3\vec{v}_2 - 6\vec{v}_3$.
- is linearly indep.

$\vec{u} = (2,3,-6)$ in $\vec{v}_1 = (1,0,0)$ $\vec{v}_2\ (0,1,0)$ $\vec{v}_3\ (0,0,1)$ $\vec{v}_4\ (1,-1,3)$

$\Rightarrow \vec{v}_1 + 4\vec{v}_2 - 9\vec{v}_3 + \vec{v}_4$ (not unique)

$= 2\vec{v}_1 + 3\vec{v}_2 - 6\vec{v}_3 + 0\vec{v}_4$

infinite # of choices, b/c not linearly independent

Given $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ we say this list is lin. indep. if the eqn.

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k = \vec{0}$$

has a unique solution $a_1 = a_2 = \dots = a_k = 0$ (in $\mathbb{F}$)

i.e. only linear combinations of $v_1, \dots v_k$ has all 0 coef.

---

why 0?

return to direct sums.

- only needed to test the 0 vector (will prove later)
- not dissimilar to linear independence.

Checking Lin. ind.

- $\vec{v}_1 = (1,0,0)$ $\vec{v}_2 = (0,1,0)$ $\vec{v}_3 = (0,0,1)$

1) add coefficients, then add together

$\exists\, a_1, a_2, a_3$ s.t. $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1) = (0,0,0)$

$\Rightarrow (a_1, a_2, a_3) = (0,0,0)$

$\Rightarrow$ literally this has to be 0...

2) then check the SOE to see what needs to be 0.

- $\vec{w_1} = (1,2,3)$ $\vec{w_2} = (0,1,-1)$ $\vec{w_3} = (0,0,5)$
  - 1^{st} row determined by $\vec{w_1}$
  - 2^{nd} row determined by $\vec{w_2}$ (partly)
  - 3^{rd} row " " $\vec{w_3}$

$a_1(1,2,3) + a_2(0,1,-1) + a_3(0,0,5) = 0$

$\Rightarrow a_1 = 0,\ \cancel{2a_1} + a_2 = 0,\ \cancel{3a_1} - \cancel{a_2} + 5a_3 = 0.$

$\Rightarrow a_1 = a_2 = a_3 = 0.$

- list of vectors is either indep. or dependent.
  - if dependent, $\nexists$ no unique soln., $\exists$ a nontrivial lin. comb. $= 0$.
  - "nontrivial" → not all coeff. are 0.

Span

- span $(\vec{v}_1 \dots \vec{v}_k)$: set of LC of $\{\vec{v}_1, \dots, \vec{v}_k\}$.
- Definition of finite-dimensional VS:
  A vector space U is "finite dimensional" if $\exists$ a finite list of vectors $\vec{v}_1, \dots \vec{v}_k \in U$ s.t. $U = \text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k)$.
  - per this defn, $\mathbb{R}^3$ is finite dimensional b/c of $(1,0,0)$ $(0,1,0)$ $(0,0,1)$
- example of infinite-dimension VS:
  - polynomials (in 1 variable)
  - def. function $f: \mathbb{F} \to \mathbb{F}$ is a polynomial if $f(z) = a_0 + a_1 z + \dots + a_m z^m$ for some $a_0, \dots, a_m \in \mathbb{F}$ and $m \in \{0, 1, 2, \dots\}$
- If $a_m \neq 0$, then degree of $f = m$. So degree must be nonnegative int. OR $-\infty$ for the 0 function.

consider the set of polys $\mathcal{P}(\mathbb{F})$

1) is this a vector space over $\mathbb{F}$?
   - zero poly. included
   - scalar mult. = multiplying p by constant leads to other polynomial
   - vector add. = adding coeffs from 1 poly to next ⇒ result = polynomial.
   - everything else is done by coeff. wise addition
   - also this is a subspace of $\mathbb{F}^{\mathbb{F}}$ so we barely even need to check!

2) if $\mathcal{P}(\mathbb{F})$ finite-dimensional? No!!

- in any finite set of polynomials $f_1, \cdots f_k$, their span $\{f_1, \cdots, f_k\}$ consists of polys. of degree $\le \max(\deg(f_1), \cdots, \deg(f_k)) = x$.
- no terms available in higher degrees.
- now consider polynomials of degree $x+1$.
  - we will need to expand set to account for size $x+1$.
- so $\mathcal{P}(f) \supsetneq \text{span}(f_1, f_2, \cdots, f_k)$.

$\Rightarrow$ $\mathcal{P}(\mathbb{F})$ must be infinite-dimensional. SAD. $\blacksquare$

<u>Claim:</u> in a finite dimensional VS V the size of any lin. ind. list doesn't exceed size of spanning list.

- first explore a dependent list $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$.
  - remove vectors, keeping same span.
  - redundancy in the representation in span.
- there coeffs. $a_1, \cdots, a_k$ s.t. $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}$
  - remove any given j s.t. $a_j \neq 0$ & rewrite this as
    $-a_j \vec{v}_j = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$.

$$\vec{v}_j = -\frac{1}{a_j}\Big(\underbrace{a_1\vec{v}_1 + \cdots + a_k\vec{v}_k}_{\text{sum w/o } a_j\vec{v}_j}\Big) \text{ so } \vec{v}_j = \text{span}(\vec{v}_1, \cdots, \vec{v}_{j-1}, \vec{v}_{j+1}, \cdots, \vec{v}_k).$$

- how to tell if old span is in here? any vector in span$(\vec{v}_1, \cdots, \vec{v}_k)$ can be written w/o $\vec{v}_j$ by using v_j i.t.o all other vectors.
- notice: can choose j s.t. $a_j \neq 0$ is the last nonzero coeff.
  Then $\vec{v}_j \in \text{span}(\vec{v}_1, \cdots \vec{v}_{j-1})$.
- empty sum $= \vec{0}$, empty prod $= 1$