Language Operations and a Structure Theory of ω-Languages

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1 Introduction

Language theory is strongly connected to the theory of automata. With some interpretation of run-acceptance in an automaton, we canonically get a language.

We call languages over infinite words the $*$-languages and often use $\mathcal{L}^*$ or some variant for such language class. Likewise, ω-languages ($\mathcal{L}^\omega$) are over infinite words. The acceptance-condition in automata for $*$-languages is straight-forward. If we look at ω-languages, several different types of automata and their acceptance have been thought of. For the class of regular languages, we see that many of them are equivalent.

For all types, we can also argue with equivalent language-theoretical operators which operate on a $*$-language. We will study the equivalences in more detail.

Depending on the $* \to \omega$ language operator (or the ω-automaton acceptance condition), we get different ω-language classes. This was studied earlier already in detail for the class of regular $*$-languages.

When we look at other $*$-language classes, we might get different results. This study is the main topic of this thesis.

2 regular ω-languages

The class of regular ω-languages can be defined in many different ways. We will use one common definition and show some equivalent descriptions.

$$\mathcal{L}^\omega(reg) := \left\{\cup_i\, U_i \cdot V_i^\omega \;\middle|\; U_i, V_i \in \mathcal{L}^*(reg)\right\}$$

A different, very common description is in terms of automata.

An automaton $\mathcal{A} = (Q, \Sigma, E, I, F)$ **Büchi-accepts** a word $\alpha = (a_0, a_1, a_2, ...) \in \Sigma^\omega$ iff there is an infinite run $q_0 \to^{a_0} q_1 \to^{a_1} q_2 \to^{a_2} q_3 ...$ with $q_0 \in I$ and $\{q_i | q_i \in F\}$ infinite, i.e. which reaches a state in F infinitely often.

The language $L^\omega(\mathcal{A})$ is defined as the set of all infinite words which are Büchi-accepted by $\mathcal{A}$.

An automaton $\mathcal{A}$ is a Büchi automaton iff we use the Büchi-acceptence.

The set of all languages accepted by a non-deterministic Büchi automaton is exactly $\mathcal{L}^\omega(reg)$. (S218,R101) Deterministic Büchi automata are less powerful, e.g. they cannot recognise $(a + b)^* b^\omega$.

There are some different forms of ω-automata, e.g. the Rabin automata and the Muller automata. We see that the class of languages accepted by non-deterministic Büchi automata is equal to deterministic Rabin automata and deterministic Muller automata. (S407)

We also see that this is equal to boolean combinations of languages accepted by deterministic Büchi automata. Under this regard, an operator of interest is $\lim(L) := \left\{\alpha \in \Sigma^\omega \;\middle|\; \exists^\omega n \colon \alpha[0, n] \in L\right\}$. We see that $\lim(\mathcal{L}^\omega)$ is equal to the languages accepted by deterministic Büchi automata. (S407) Thus:

$$\mathrm{BC}\lim \mathcal{L}^*(reg) = \mathcal{L}^\omega(reg)$$

Some other descriptions: $\mathcal{L}^\omega(reg) = \left\{\cup_i U_i \cdot \lim V_i \;\middle|\; U_i, V_i \in \mathcal{L}^*(reg)\right\}$ (S218,S411,R107) $\mathcal{L}^\omega(reg) = \left\{A \subset \Sigma^\omega \;\middle|\; A \text{definable in} L_2(\Sigma)\right\}$

We will formulate some properties of interest in a general form for a $*$-language class $\mathcal{L}$ which all hold for $\mathcal{L}^*(reg)$. We get some general results based on these properties in chapter 4.

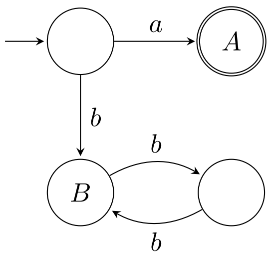
Let $L, A, B \in \mathcal{L}$.

- E1: $L \cdot \Sigma^* \in \mathcal{L}$ (not suffix sensitive)
- E2a: $A \cup B \in \mathcal{L}$
- E2b: $A \cap B \in \mathcal{L}$
- E3: $-L \in \mathcal{L}$ (closed under complementation) (S303.E3, S218, R101)
- In some proofs, e.g. in 4.7 or 4.8, we have an automaton based on some language of the language class and we do some modifications on it, e.g. we modify the final state set. If we stay in the language class, we call this the E4 property. Formally:

  E4: $\forall$ deterministic automaton $\mathcal{A} = (Q, q_0, \Delta, F), L^*(\mathcal{A}) = L \colon \forall F' \subseteq Q : L^*((Q, q_0, \Delta, F')) \in \mathcal{L}$

  For $\mathcal{L}^*(reg)$, this property holds obviously.

  For $\mathcal{L}^*(\mathrm{FO}[<])$, it does not hold:



This is a deterministic automaton for the language $\{a\} \in \mathcal{L}^*(\mathrm{FO}[<])$. If you make B also a final state, we get the language $a + b(bb)^* \notin \mathcal{L}^*(\mathrm{FO}[<])$.

So, E4 seems too restricted.

3 $* \to \omega$

3.1 language operators

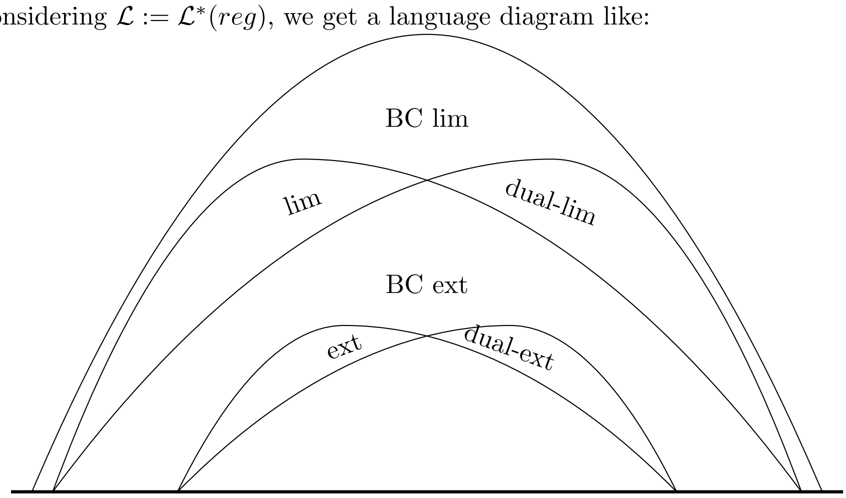
We already introduced lim. We can define a family of language operators, partly also derived from the study of $\mathcal{L}^\omega(reg)$. Some of these operators operate on a single language and not on the class. Let $\mathcal{L}$ be a $*$-language class. Let $L \in \mathcal{L}$.

1. $\mathrm{ext}(L) := \left\{\alpha \in \Sigma^\omega \;\middle|\; \exists n\colon \alpha[0,n] \in L\right\} = L \cdot \Sigma^\omega$
2. $\overline{\mathrm{ext}}(L) := \left\{\alpha \in \Sigma^\omega \;\middle|\; \forall n\colon \alpha[0,n] \in L\right\} = L \cdot \Sigma^\omega$
3. $\mathrm{BC}\,\mathrm{ext}$
4. $\lim(L) := \left\{\alpha \in \Sigma^\omega \;\middle|\; \forall N\colon \exists n > N\colon \alpha[0,n] \in L\right\} = \left\{\alpha \in \Sigma^\omega \;\middle|\; \exists^\omega n\colon \alpha[0,n] \in L\right\}$
5. $\overline{\lim}(L) := \left\{\alpha \in \Sigma^\omega \;\middle|\; \exists N\colon \forall n > N\colon \alpha[0,n] \in L\right\}$
6. $\mathrm{BC}\lim$
7. Kleene-Closure of $\mathcal{L}$: $\left\{\bigcup_{i=1}^n U_i \cdot V_i^\omega \;\middle|\; U_i, V_i \in \mathcal{L}\right\}$
8. $\left\{\bigcup_{i=1}^n U_i \cdot \lim V_i \;\middle|\; U_i, V_i \in \mathcal{L}\right\}$

From language operators, we get language class operators in a canonical way, e.g. $\lim(\mathcal{L}) := \left\{\lim L \;\middle|\; L \in \mathcal{L}\right\}$.

3.2 $\mathcal{L}^*(reg)$

Considering $\mathcal{L} := \mathcal{L}^*(reg)$, we get a language diagram like:



where all inclusions are strict. In more detail:

- P1: $\mathrm{ext}\,\mathcal{L} \cap \overline{\mathrm{ext}}\,\mathcal{L} \neq \emptyset$
  Proof: $\tilde{L}_1 := a\Sigma^\omega \in \mathrm{ext} \cap \overline{\mathrm{ext}}\,\mathcal{L}$ with $\tilde{L}_1 = \mathrm{ext}(a)$ and $\tilde{L}_1 = \overline{\mathrm{ext}}(a\Sigma^*)$. (R101, prop, p.38)

- P2a: $\operatorname{ext}\mathcal{L} \cap \overline{\operatorname{ext}}\,\mathcal{L} \subsetneqq \operatorname{ext}\mathcal{L}$
Proof: $\tilde{L}_{2a} := \operatorname{ext}(a^*b) = a^*b\Sigma^\omega \in \operatorname{ext}\mathcal{L}$. Assume some A-automaton $\mathcal{A}$ with n states accepts $\tilde{L}_{2a}$. $\mathcal{A}$ would also accept $a^n b^\omega$. I.e. the $(n+1)$th state after the run of a^n would also accept a, i.e. $\mathcal{A}$ would accept a^{n+1}. By inclusion, $\mathcal{A}$ would accept a^ω. That is a contradiction. Thus, there is no such A-automat. Thus, $\tilde{L}_{2a} \notin \overline{\operatorname{ext}}\,\mathcal{L}$.

- P2b: $\operatorname{ext}\mathcal{L} \cap \overline{\operatorname{ext}}\,\mathcal{L} \subsetneqq \overline{\operatorname{ext}}\,\mathcal{L}$
Proof: $\tilde{L}_{2b} := -\tilde{L}_{2a} \in \overline{\operatorname{ext}}\,\mathcal{L}$, $\tilde{L}_{2b} \notin \operatorname{ext}\mathcal{L}$.

- P3: $\operatorname{ext}\mathcal{L} \neq \overline{\operatorname{ext}}\,\mathcal{L}$
Proof: Follows directly from P2a and P2b.

- P4: $\operatorname{ext}\mathcal{L} \cup \overline{\operatorname{ext}}\,\mathcal{L} \subsetneqq \operatorname{BC}\operatorname{ext}\mathcal{L}$
Proof: $\tilde{L}_4 := \Sigma^* a \Sigma^\omega \cap -(\Sigma^* b \Sigma^\omega)$, $\Sigma = \{a, b, c\}$. Then we have $\tilde{L}_4 \notin \operatorname{ext} \cup \overline{\operatorname{ext}}\,\mathcal{L}$, $\tilde{L}_4 \in \operatorname{BC}\operatorname{ext}\mathcal{L}$. (R101, p.38)

- P5: $\operatorname{BC}\operatorname{ext}\mathcal{L} = \lim\mathcal{L} \cap \overline{\lim}\,\mathcal{L}$
Proof: S405 / Staigner-Wagner-recognizable

- P6a: $\lim\mathcal{L} \cap \overline{\lim}\,\mathcal{L} \subsetneqq \lim\mathcal{L}$
Proof: $\tilde{L}_{6a} := \lim(\Sigma^* a) = (\Sigma^* a)^\omega$. Assume there is $L \subseteq \Sigma^*$ with $\lim(L) = -\tilde{L}_{6a}$. Let $(w_0, w_1, w_2, \dots) \in (\Sigma^*)^{\mathbb{N}}$ so that $w_0 \in L, w_0 a w_1 \in L, \dots, w_0 \prod_{i=0}^{n} a w_i \in L \ \forall n \in \mathbb{N}$. Thus, $\alpha := w_0 \prod_{i \in \mathbb{N}} a w_i \in \lim L$. But $\alpha \notin -\tilde{L}_{6a}$. That is a contradiction. Thus, $-\tilde{L}_{6a} \notin \lim\mathcal{L}$. With E3, we get $\tilde{L}_{6a} \notin \overline{\lim}\,\mathcal{L}$.

- P6b: $\lim\mathcal{L} \cap \overline{\lim}\,\mathcal{L} \subsetneqq \overline{\lim}\,\mathcal{L}$
Proof: Analog to P6a with $\tilde{L}_{6b} := -\tilde{L}_{6a}$.

- P7: $\lim\mathcal{L} \neq \overline{\lim}\,\mathcal{L}$
Proof: Follows directly from P6a and P6b.

- P8: $\lim\mathcal{L} \cup \overline{\lim}\,\mathcal{L} \subsetneqq \operatorname{BC}\lim\mathcal{L}$
Proof: $\tilde{L}_8 := (\Sigma^* a)^\omega \cap -(\Sigma * b)^\omega$. Then $\tilde{L}_8 \notin \lim \cup \overline{\lim}\,\mathcal{L}$ but $\tilde{L}_8 \in \operatorname{BC}\lim\mathcal{L}$. (R101, prop, p.38)

- P9: $\operatorname{BC}\lim\mathcal{L} = \left\{\bigcup_{i=1}^{n} U_i \cdot V_i^\omega \;\middle|\; U_i, V_i \in \mathcal{L}\right\}$
Proof: This is explained already in chapter 2.

- P10: $\operatorname{BC}\lim\mathcal{L} = \left\{\bigcup_{i=1}^{n} U_i \cdot \lim V_i \;\middle|\; U_i, V_i \in \mathcal{L}\right\}$
Proof: This is explained already in chapter 2.

- P11: $\operatorname{BC}\lim\mathcal{L} = \left\{L^\omega_{\text{Büchi}}(\mathcal{A}) \;\middle|\; \mathcal{A} \text{ automaton so that } L^*(\mathcal{A}) \in \mathcal{L}\right\}$
Proof: (R101,Th.12,p.9) says $\left\{L^\omega_{\text{Büchi}}(\mathcal{A}) \;\middle|\; \dots\right\} = \left\{\bigcup_{i=1}^{n} U_i \cdot V_i^\omega \;\middle|\; U_i, V_i \in \mathcal{L}\right\}$. The rest follows with P9.

3.3 Questions

This was studied in detail for $\mathcal{L}^*(reg)$. We are now studing relations of resulting ω-language classes for different $*$-language classes.

Esp.:

- $\operatorname{BC}\operatorname{ext}\mathcal{L} \stackrel{?}{=} \operatorname{BC}\lim\mathcal{L}$

4 General results

Let $\mathcal{L}$ be a $*$-language class.

4.1 general

Let $L, A, B \in \mathcal{L}$.

1. $\operatorname{ext} L = L \cdot \Sigma^\omega$
2. $\operatorname{ext} L = \lim L \cdot \Sigma^*$
3. $\operatorname{ext} L = \overline{\lim} L \cdot \Sigma^*$
4. $-\lim(-L) = \overline{\lim} L$
5. $\overline{\lim} L \subseteq \lim L$
6. $\lim A \cup \lim B = \lim A \cup B$
   Proof:
   $$\begin{aligned} & \alpha \in \lim A \cup \lim B \\ \Leftrightarrow\ & \exists^\omega n \colon \alpha[0,n] \in A \ \vee\ \exists^\omega n \colon \alpha[0,n] \in B \\ \Leftrightarrow\ & \exists^\omega n \colon \alpha[0,n] \in A \cup B \\ \Leftrightarrow\ & \alpha \in \lim A \cup B \end{aligned}$$
7. $\overline{\lim} A \cup \overline{\lim} B \subseteq \overline{\lim} A \cup B$
   Proof:
   $$\begin{aligned} & \alpha \in \overline{\lim} A \cup \overline{\lim} B \\ \Leftrightarrow\ & \exists N \colon \forall n \geq N \colon \alpha[0,n] \in A \ \vee\ \exists N \colon \forall n \geq N \colon \alpha[0,n] \in B \\ \Rightarrow\ & \exists N \colon \forall n \geq N \colon \alpha[0,n] \in A \cup B \end{aligned}$$
   There is no equality in general: $A = (00)^*$, $B = (00)^*0$.

We are interested in relations like $\mathrm{BC} \operatorname{ext} \mathcal{L} \overset{?}{\subsetneq} \mathrm{BC} \lim \mathcal{L}$ or $\operatorname{ext} \mathcal{L} \overset{?}{\subsetneq} \lim \mathcal{L}$. With $\mathcal{L} = \{\{a\}\}$, we realize that even $\operatorname{ext} \mathcal{L} \subseteq \lim \mathcal{L}$ is not true in general ($\operatorname{ext}\{\{a\}\} = \{a\Sigma^\omega\} \neq \emptyset = \lim\{\{a\}\}$). In 4.2, we see a sufficient condition for this property, though.

4.2 non suffix sensitive

If E1 ($\forall L \in \mathcal{L} \colon L \cdot \Sigma^* \in \mathcal{L}$, i.e. $\mathcal{L}$ is non suffix sensitive) holds for $\mathcal{L}$: For $L \in \mathcal{L}$, we have $\operatorname{ext} L = \lim L\Sigma^* = \overline{\lim} L\Sigma^*$ and thus

$$\operatorname{ext} \mathcal{L} \subseteq \lim \mathcal{L} \cup \overline{\lim} \mathcal{L}.$$

4.3 $\mathrm{BC} \operatorname{ext} \subseteq \mathrm{BC} \lim$

From $\operatorname{ext} \mathcal{L} \subseteq \lim \mathcal{L}$, it directly follows $\{-\operatorname{ext} L \mid L \in \mathcal{L}\} \subseteq \{-\lim L \mid L \in \mathcal{L}\}$. Thus, it also follows

$$\mathrm{BC} \operatorname{ext} \mathcal{L} \subseteq \mathrm{BC} \lim \mathcal{L}.$$

4.4 $\overline{\text{ext}} \subset \overline{\lim}$

From $\text{ext}\,\mathcal{L} \subseteq \lim \mathcal{L}$, we need E3 ($\mathcal{L}$ closed under negation) to get $\overline{\text{ext}}\,\mathcal{L} \subseteq \overline{\lim}\,\mathcal{L}$. This is in contrast to 4.3, where it directly follows. We have to be careful about the difference $-\text{ext}\,\mathcal{L} \neq \overline{\text{ext}}\,\mathcal{L}$ (in general, if E3 does not hold).

4.5 union, intersection

- E2a (closed under union) $\Rightarrow \bigcup \text{ext}\,\mathcal{L} \subseteq \text{ext}\,\mathcal{L}$.
- E2b (closed under intersection) $\Rightarrow \bigcap \text{ext}\,\mathcal{L} \subseteq \text{ext}\,\mathcal{L}$.

4.6 $op \cup \overline{op} \subsetneqq \text{BC}\,op$

If there is $L_\Sigma \in \mathcal{L}_\Sigma$ with $L_\Sigma \in \text{ext}\,\mathcal{L}_\Sigma, L_\Sigma \notin \overline{\text{ext}}\,\mathcal{L}_\Sigma$ and E3 (closed under negation) holds for $\mathcal{L}$:

$$\Rightarrow\ -L_\Sigma \in \overline{\text{ext}}\,\mathcal{L}_\Sigma, -L_\Sigma \notin \text{ext}\,\mathcal{L}_\Sigma$$

$$\begin{aligned}\Rightarrow\ & L_{\Sigma_1} \cup -L_{\Sigma_2} \in \text{BC}\,\text{ext}\,\mathcal{L}_{\Sigma_1 \dot\cup \Sigma_2} \\ & L_{\Sigma_1} \cup -L_{\Sigma_2} \notin \text{ext} \cup \overline{\text{ext}}\,\mathcal{L}_{\Sigma_1 \dot\cup \Sigma_2}\end{aligned}$$

Thus, $\text{ext} \cup \overline{\text{ext}}\,\mathcal{L} \subsetneqq \text{BC}\,\text{ext}\,\mathcal{L}$.

Similarily, if there is $L_\Sigma \in \lim \mathcal{L}_\Sigma, L_\Sigma \notin \overline{\lim}\,\mathcal{L}_\Sigma$ and E3 holds for $\mathcal{L}$:

$$\Rightarrow\ -L_\Sigma \in \overline{\lim}\,\mathcal{L}_\Sigma, -L_\Sigma \notin \lim \mathcal{L}_\Sigma$$

$$\begin{aligned}\Rightarrow\ & L_{\Sigma_1} \cup -L_{\Sigma_2} \in \text{BC}\lim \mathcal{L}_{\Sigma_1 \dot\cup \Sigma_2} \\ & L_{\Sigma_1} \cup -L_{\Sigma_2} \notin \lim \cup \overline{\lim}\,\mathcal{L}_{\Sigma_1 \dot\cup \Sigma_2}\end{aligned}$$

Thus, $\lim \cup \overline{\lim}\,\mathcal{L} \subsetneqq \text{BC}\lim \mathcal{L}$.

4.7 $\text{BC}\,\text{ext} = \lim \cap \overline{\lim}$

A Staiger-Wagner automaton (weak Muller automaton) is of the same form $\mathcal{A} = (Q, \Sigma, q_0, \delta, \mathcal{F})$ as a Muller automaton with the acceptance condition that a run ρ is accepting if and only if $\text{Occ}(\rho) := \{q \in Q \colon \text{q occurs in } p\} \in \mathcal{F}$. (R101,Def.61,p.43)

We see (R101,Th.63+64,p.44) that the class of Staigner-Wagner-recognized languages is exactly the class $\text{BC}\,\text{ext}\,\mathcal{L}^*(reg)$ and also $\lim \cap \overline{\lim}\,\mathcal{L}^*(reg)$.

We are now formulating a more general and direct proof for the $\text{BC}\,\text{ext}\,\mathcal{L} = \lim \cap \overline{\lim}\,\mathcal{L}$ equality without Staiger-Wagner-automata.

First, we show $\lim \cap \overline{\lim}\,\mathcal{L} \subseteq \text{BC}\,\text{ext}\,\mathcal{L}$.

Let $\tilde{L} \in \lim \cap \overline{\lim}\,\mathcal{L}$, i.e. there are deterministic automaton $\mathcal{A}$ and $\overline{\mathcal{A}}$ so that $L^\omega_{\text{Büchi}}(\mathcal{A}) = L^\omega_{\text{co-Büchi}}(\overline{\mathcal{A}}) = \tilde{L}$. Let $Q, \overline{Q}$ be the states of $\mathcal{A}, \overline{\mathcal{A}}$. Now look at the product automaton $\mathcal{A} \times \overline{\mathcal{A}} =: \overset{\times}{\mathcal{A}}$ with states $Q \times \overline{Q}$ and final states $F \times \overline{F} \subseteq Q \times \overline{Q}$. $\overset{\times}{\mathcal{A}}$ is also deterministic.

In $\overset{\times}{\mathcal{A}}$, we have

$$\begin{aligned}& \alpha \in \tilde{L} \\ \Leftrightarrow\ & \forall N \colon \exists n \geq N \colon \overset{\times}{\rho}(\alpha)[n] \in F \times \overline{Q} \\ \Leftrightarrow\ & \exists N \colon \forall n \geq N \colon \overset{\times}{\rho}(\alpha)[n] \in Q \times \overline{F}\end{aligned}$$

Look at strongly connected component (SCC) S in $\overset{\times}{\mathcal{A}}$. We have $S \cap F \times \overline{Q} \neq \emptyset$, iff S accepts. It follows that all states in S are finite states in $\overline{\mathcal{A}}$, i.e. $S \cap Q \times \overline{F} = S$.

Single $\overset{\times}{q} \in \overset{\times}{Q}$ which are not part of a SCC can be ignored. For the acceptance of infinte words, only SCCs are relevant. For S, define $S_+ := \left\{ \overset{\times}{q} \in \overset{\times}{Q} - S \mid \overset{\times}{q} \text{ can be visited after } S \right\}$.

Then we have

$$\begin{aligned}\tilde{L} = \bigcup_{\text{SCC } S} & S \text{ will be visited } \wedge \\ & \text{all states of } S \text{ will be visited forever after some step } \wedge \\ & S_+ \text{ will not be visited.}\end{aligned}$$

S will be visited: Let S exactly be the finite states. This interpreted as an E-automaton $\mathcal{A}^E$ is exactly the condition.

Only the allowed states will be visited but nothing followed after S: Mark S and all states on all paths to S as finite states. This as an A-automaton $\mathcal{A}_S^A$ is exactly the condition.

A similar negated condition might be simpler: Let S_+ be exactly the finite states. Interpret this as an E-automaton $\mathcal{A}_{S_+}^E$.

Then we have

$$\begin{aligned}\tilde{L} &= \bigcup_{\text{SCC } S} L_E^\omega(\mathcal{A}_S^E) \cap L_A^\omega(\mathcal{A}_S^A) \\ &= \bigcup_{\text{SCC } S} L_E^\omega(\mathcal{A}_S^E) \cap -L_E^\omega(\mathcal{A}_{S_+}^E).\end{aligned}$$

Thus, $\tilde{L} \in \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(reg)$.

4.8 Kleene-star $= \mathrm{BC}\lim$

5 $*$-language classes

5.1 Overview

We already showed many results for $\mathcal{L}^*(reg)$.

5.2 FO[<] / starfree

Theorem 5.1.

$$\mathcal{L}^\omega(\mathrm{FO}[<]) = \mathrm{BC}\lim \mathcal{L}^*(\mathrm{FO}[<])$$

Proof. Let $\varphi \in \mathrm{FO}[<]$. By the [Tho81, Normal Form Theorem (4.4)] there are bounded formulas $\varphi_1(y), \cdots, \varphi_r(y), \psi_1(y), \cdots, \psi_r(y)$ such that for all $\alpha \in \Sigma^\omega$:

$$\alpha \models \varphi \Leftrightarrow \alpha \models \bigvee_{i=1}^{r} (\forall x \exists y > x \colon \varphi_i(y)) \wedge \neg (\forall x \exists y > x \colon \psi_i(y))$$

Thus:

$$\begin{aligned} \alpha \models \varphi \Leftrightarrow \bigvee_{i=1}^{r} & \underbrace{(\alpha \models \forall x \exists y > x \colon \varphi_i(y))} \wedge \neg (\alpha \models \forall x \exists y > x \colon \psi_i(y)) \\ & \Leftrightarrow \forall x \exists y > x \colon \alpha[0,n] \models \varphi_i(\omega) \\ & \Leftrightarrow \exists^\omega n \colon \alpha[0,n] \models \varphi_i(\omega) \\ & \Leftrightarrow \alpha \in \lim L^*(\varphi_i(\omega)) \end{aligned}$$

where $\varphi_i(\omega)$ stands for φ_i with all bounds removed. I.e. we have

$$L^\omega(\varphi) = \bigcup_{i=1}^{r} \lim(L^*(\varphi_i(\omega)) \cap \neg \lim(L^*(\psi_i(\omega))),$$

and thus

$$L^\omega(\varphi) \in \mathrm{BC}\lim \mathcal{L}^*(\mathrm{FO}[<]).$$

We have prooved the $\subseteq$-direction. For $\supseteq$:

$$\begin{aligned} & \alpha \in \lim(L^*(\varphi)) \\ \Leftrightarrow\ & \exists^\omega n \colon \alpha[0,n] \models \varphi \\ \Leftrightarrow\ & \alpha \models \forall x \exists y > x \colon \varphi(y) \\ \Leftrightarrow\ & \alpha \in L^\omega(\forall \exists y > x \colon \varphi(y)) \end{aligned}$$

where $\varphi(y)$ stands for φ with all variables bounded by y. I.e.

$$\lim \mathcal{L}^*(\mathrm{FO}[<]) \subseteq \mathcal{L}^\omega(\mathrm{FO}[<]),$$

and thus also

$$\mathrm{BC}\lim \mathcal{L}^*(\mathrm{FO}[<]) \subseteq \mathcal{L}^\omega(\mathrm{FO}[<]).$$

Thus we have prooved the equality. $\square$

Theorem 5.2.

$$\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\mathrm{FO}[<]) \subsetneq \mathrm{BC}\,\lim\mathcal{L}^*(\mathrm{FO}[<])$$

Proof. $\subseteq$: $L \subset \Sigma^\omega$ starfree $\Rightarrow$ $L\Sigma^\omega \in \lim(\mathcal{L}^*(\mathrm{FO}[<]))$ □

Proof. $\neq$:

$$L := (\Sigma^* a)^\omega$$
$$\Rightarrow L = \lim((\Sigma^* a)^*)$$
$$\Rightarrow L = L^\omega(\exists^\omega x : Q_a x)$$

And we have $L \notin \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\mathrm{FO}[<])$. □

With 4.2, we get $\mathrm{ext}\,\mathcal{L} \subseteq \lim\mathcal{L}$.
$\tilde{L} := \lim(\Sigma^* a) = (\Sigma^* a)^\omega \in \lim\mathcal{L}$ but $\tilde{L} \notin \mathrm{ext}\,\mathcal{L}$ as shown in chapter 3.2.

- P1: $\{a\} \in \mathcal{L}$. $a\Sigma^* \in \mathcal{L}$, thus $a\Sigma^\omega = \mathrm{ext}(\{a\}) = \overline{\mathrm{ext}}\, a\Sigma^*$.
- P2a: $\tilde{L}_{2a} := \mathrm{ext}(a^* b) = a^* b\Sigma^\omega$, $a^* b \in \mathcal{L}$. Then $\tilde{L}_{2a} \notin \mathrm{ext}\,\mathcal{L}^*(reg) \supseteq \mathcal{L}^*(\mathrm{FO}[<])$.
- P2b: $-\tilde{L}_{2a} := \overline{\mathrm{ext}}(-a^* b)$, $-a^* b \in \mathcal{L}$. Then $-\tilde{L}_{2a} \notin \mathrm{ext}\,\mathcal{L}$.
- P3: Follows directly from P2a and P2b.
- P4: $\tilde{L}_4 := \mathrm{ext}(\Sigma^* a) \cap \overline{\mathrm{ext}}(-\Sigma^* b) = \Sigma^* a\Sigma^\omega \cap -(\Sigma^* b\Sigma^\omega)$, whereby $\Sigma^* a \in \mathcal{L}$, $-\Sigma^* b \in \mathcal{L}$. $\tilde{L}_4 \notin \mathrm{ext} \cup \overline{\mathrm{ext}}\,\mathcal{L}^*(reg) \supseteq \mathcal{L}^*(\mathrm{FO}[<])$ but $\tilde{L}_4 \in \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}$.
- P5: TODO
- P6a/P6b/P7/P8: $\Sigma^* a \in \mathcal{L}$. We can use the same arguments as for $\mathcal{L}^*(reg)$.
- P9: TODO
- P10: TODO

5.3 FO[+1]

Theorem 5.3.

$$\mathcal{L}^\omega(\mathrm{FO}[+1]) = \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\mathrm{FO}[+1])$$

Proof. From [Tho96, Theorem 4.8], we know that each formular in FO[+1] is equivalent (for both finite and infinite words) to a boolean combination of statements "sphere $\sigma \in \Sigma^+$ occurs $\geq n$ times". That statement can be expressed by a sentence of the form

$$\psi := \exists\overline{x_1}\cdots\exists\overline{x_n}\varphi(\overline{x_1},\cdots,\overline{x_n})$$

where each $\overline{x_i}$ is a $|\sigma|$-tuple of variables and the formula φ states:

$$\bigwedge_{\substack{i,j\in\underline{n},\\ i\neq j,\\ k,l\in\underline{|\sigma|}}} x_{i,k} \neq x_{j,l} \;\wedge\; \bigwedge_{\substack{i\in\underline{n},\\ k\in\underline{|\sigma|-1}}} x_{i,k+1} = x_{i,k}+1 \;\wedge\; \bigwedge_{\substack{i\in\underline{n},\\ k\in\underline{|\sigma|}}} Q_{\sigma_k} x_{i,k}$$

For ψ, we have:

$$\alpha \models \psi \Leftrightarrow \exists n\colon \alpha[0,n] \models \psi \quad \text{for all } \alpha \in \Sigma^\omega,$$

i.e.

$$L^\omega(\psi) = \operatorname{ext} L^*(\psi).$$

Any formular in FO[+1] can be expressed as a boolean combination of ψ-like formular. With

$$L^\omega(\neg\psi) = \neg L^\omega(\psi)$$
$$L^\omega(\psi_1 \wedge \psi_2) = L^\omega(\psi_1) \cap L^\omega(\psi_2)$$
$$L^\omega(\psi_1 \vee \psi_2) = L^\omega(\psi_1) \cup L^\omega(\psi_2)$$

we get:

$$\mathcal{L}^\omega(\mathrm{FO}[+1]) = \mathrm{BC}\operatorname{ext}\mathcal{L}^*(\mathrm{FO}[+1]).$$

□

5.4 FO[]

5.5 piece-wise testable

Theorem 5.4.

$$\mathrm{BC}\operatorname{ext}\mathcal{L}^*(\text{piece-wise testable}) = \mathrm{BC}\lim\mathcal{L}^*(\text{piece-wise testable})$$

Proof. L piece-wise testable $\Leftrightarrow$ L is a boolean algebra of $\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*$

$\subseteq$: It is sufficient to show $\operatorname{ext}(\mathcal{L}^*(\text{piece-wise testable})) \subseteq \mathrm{BC}\lim\mathcal{L}^*(\text{piece-wise testable})$. By complete induction:

$$\operatorname{ext}(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*) = \Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^\omega = \lim(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*)$$
$$\operatorname{ext}(\neg(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*)) = \Sigma^\omega = \lim(\Sigma^*)$$
$$\operatorname{ext}(\emptyset) = \emptyset = \lim(\emptyset)$$

It is sufficient to show negation only for such ground terms because we can always push the negation down.

$$\operatorname{ext}(A \cup B) = \operatorname{ext}(A) \cup \operatorname{ext}(B)$$
$$\operatorname{ext}(A \cap B) = \operatorname{ext}(A) \cap \operatorname{ext}(B)$$

This makes the induction complete.

$\supseteq$: It is sufficient to show $\lim(\mathcal{L}^*(\text{piece-wise testable})) \subseteq \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{piece-wise testable})$.

$$\begin{aligned}
&\lim(\emptyset) = \mathrm{ext}(\emptyset),\ \lim(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*) = \mathrm{ext}(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*) \quad \text{(see above)}\\
&\lim(\neg(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*)) = \{\alpha \in \Sigma^\omega \mid \exists^\omega n\colon \alpha[0,n] \notin \Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*\}\\
&\qquad = \{\alpha \in \Sigma^\omega \mid \forall n\colon \alpha[0,n] \notin \Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*\}\\
&\qquad = \neg\,\mathrm{ext}(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*)\\
&\lim(A \cup B) = \{\alpha \in \Sigma^\omega \mid \exists^\omega n\colon \alpha[0,n] \in A \cup B\} = \lim(A) \cup \lim(B)\\
&\lim(A \cap B) = \{\alpha \in \Sigma^\omega \mid \exists^\omega n\colon \alpha[0,n] \in A \cap B\}
\end{aligned}$$

and because A, B are piece-wise testable

$$= \{\alpha \in \Sigma^\omega \mid \exists n : \forall m > n\colon \alpha[0,m] \in A \cap B\} = \lim(A) \cap \lim(B)$$

□

5.6 positive piece-wise testable

Theorem 5.5.

$$\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{pos-PT}) = \mathrm{BC}\lim\mathcal{L}^*(\text{pos-PT})$$

Proof. $\subseteq$: Exactly like the proof for PT except that we leave out the negated part. $\supseteq$: Also like the proof for PT. □

5.7 locally testable

Theorem 5.6.

$$\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{locally testable}) \subsetneq \mathrm{BC}\lim\mathcal{L}^*(\text{locally testable})$$

Proof. Let $w \in \Sigma^+$.

$$\begin{aligned}
&\mathrm{ext}(w\Sigma^*) = \lim(w\Sigma^*)\\
&\mathrm{ext}(\Sigma^* w) = \Sigma^* w \Sigma^\omega = \lim(\Sigma^* w \Sigma^*)\\
&\mathrm{ext}(\Sigma^* w \Sigma^*) = \Sigma^* w \Sigma^\omega = \lim(\Sigma^* w \Sigma^*)
\end{aligned}$$

Thus we have

$$\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{locally testable}) \subseteq \mathrm{BC}\lim\mathcal{L}^*(\text{locally testable}).$$

But we also have

$$\lim(\Sigma^*) = (\Sigma^* w)^\omega \notin \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{locally testable}).$$

□

5.8 endwise testable

- $\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(endwise) \neq \mathrm{BC}\lim\mathcal{L}^*(endwise)$ because $\Sigma^* a \in \mathcal{L}^*(endwise)$.
- $\mathrm{ext}(a\Sigma^* a) = a\Sigma^* a \Sigma^\omega \notin \mathrm{BC}\lim\mathcal{L}^*(endwise)$

5.9 local

5.10 finite / co-finite

- $\lim \mathcal{L}^*(finite) = \{\emptyset\}$
- $\operatorname{ext} \mathcal{L}^*(finite) = \mathcal{L}^*(finite) \cdot \Sigma^\omega$
- $\lim \mathcal{L}^*(co - finite) = \{\Sigma^\omega\}$
- $\operatorname{ext} \mathcal{L}^*(co - finite) = \{\Sigma^\omega\}$

5.11 dot-depth-n

5.12 L-trivial

5.13 R-trivial

5.14 locally modulo testable

5.15 context free

6 Lemmas

6.1 pos-PT and PT

Theorem 6.1.

$$\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{pos-PT}) = \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\mathrm{PT})$$

Proof. In the proof of $\lim \mathcal{L}^*(\mathrm{PT}) \subseteq \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\mathrm{PT})$ we actually proved $\mathrm{BC}\lim \mathcal{L}^*(\mathrm{PT}) \subseteq \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{pos-PT})$. Similiarly we also proved $\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\mathrm{PT}) \subseteq \mathrm{BC}\lim \mathcal{L}^*(\text{pos-PT})$.

With 5.6 and 5.5 we get the claimed equality. □

Literatur

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