Language Operations and a Structure Theory of ω-Languages

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Contents

1 Introduction

Language theory is strongly connected to the theory of automata. With some interpretation of run-acceptance in an automaton, we canonically get a language.

We call languages over infinite words the $*$-languages and often use $\mathcal{L}^*$ or some variant for such language class. Likewise, ω-languages ($\mathcal{L}^\omega$) are over infinite words. The acceptance-condition in automata for $*$-languages is straight-forward. If we look at ω-languages, several different types of automata and their acceptance have been thought of. For the class of regular languages, we see that many of them are equivalent.

For all types, we can also argue with equivalent language-theoretical operators which operate on a $*$-language. We will study the equivalences in more detail.

Depending on the $* \to \omega$ language operator (or the ω-automaton acceptance condition), we get different ω-language classes. This was studied earlier already in detail for the class of regular $*$-languages.

When we look at other $*$-language classes, we might get different results. This study is the main topic of this thesis.

2 Background results on regular ω-languages

2.1 The class of regular ω-languages

The class of regular ω-languages can be defined in many different ways. We will use one common definition and show some equivalent descriptions.

$$\mathcal{L}^\omega(reg) := \{\cup_i \, U_i \cdot V_i^\omega \mid U_i, V_i \in \mathcal{L}^*(reg)\}$$

A different, very common description is in terms of automata.

An automaton $\mathcal{A} = (Q, \Sigma, E, I, F)$ **Büchi-accepts** a word $\alpha = (a_0, a_1, a_2, ...) \in \Sigma^\omega$ iff there is an infinite run $q_0 \to^{a_0} q_1 \to^{a_1} q_2 \to^{a_2} q_3 ...$ with $q_0 \in I$ and $\{q_i | q_i \in F\}$ infinite, i.e. which reaches a state in F infinitely often.

The language $L^\omega(\mathcal{A})$ is defined as the set of all infinite words which are Büchi-accepted by $\mathcal{A}$.

An automaton $\mathcal{A}$ is a Büchi automaton iff we use the Büchi-acceptence.

The set of all languages accepted by a non-deterministic Büchi automaton is exactly $\mathcal{L}^\omega(reg)$. (S218,R101) Deterministic Büchi automata are less powerful, e.g. they cannot recognise $(a + b)^* b^\omega$.

There are some different forms of ω-automata, e.g. the Rabin automata and the Muller automata. We see that the class of languages accepted by non-deterministic Büchi automata is equal to deterministic Rabin automata and deterministic Muller automata. (S407)

We also see that this is equal to boolean combinations of languages accepted by deterministic Büchi automata. Under this regard, an operator of interest is $\lim(L) := \{\alpha \in \Sigma^\omega \mid \exists^\omega n \colon \alpha[0, n] \in L\}$. We see that $\lim(\mathcal{L}^\omega)$ is equal to the languages accepted by deterministic Büchi automata. (S407) Thus:

$$\mathrm{BC} \lim \mathcal{L}^*(reg) = \mathcal{L}^\omega(reg)$$

Some other descriptions: $\mathcal{L}^\omega(reg) = \{\cup_i U_i \cdot \lim V_i \mid U_i, V_i \in \mathcal{L}^*(reg)\}$ (S218,S411,R107) $\mathcal{L}^\omega(reg) = \{A \subset \Sigma^\omega \mid A \text{definable in} L_2(\Sigma)\}$

We will formulate some properties of interest in a general form for a $*$-language class $\mathcal{L}$ which all hold for $\mathcal{L}^*(reg)$. We get some general results based on these properties in chapter 3.

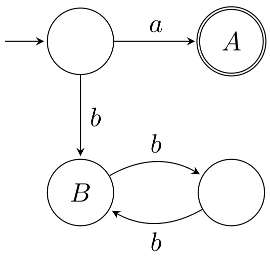
Let $L, A, B \in \mathcal{L}$.

- E1: $L \cdot \Sigma^* \in \mathcal{L}$ (not suffix sensitive)
- E2a: $A \cup B \in \mathcal{L}$
- E2b: $A \cap B \in \mathcal{L}$
- E3: $-L \in \mathcal{L}$ (closed under complementation) (S303.E3, S218, R101)
- In some proofs, e.g. in 3.10 or 3.11, we have an automaton based on some language of the language class and we do some modifications on it, e.g. we modify the final state set. If we stay in the language class, we call this the E4 property. Formally:

  E4: $\forall$ deterministic automaton $\mathcal{A} = (Q, q_0, \Delta, F), L^*(\mathcal{A}) = L \colon \forall F' \subseteq Q : L^*((Q, q_0, \Delta, F')) \in \mathcal{L}$

  For $\mathcal{L}^*(reg)$, this property holds obviously.

  For $\mathcal{L}^*(\mathrm{FO}[<])$, it does not hold:



This is a deterministic automaton for the language $\{a\} \in \mathcal{L}^*(\mathrm{FO}[<])$. If you make B also a final state, we get the language $a + b(bb)^* \notin \mathcal{L}^*(\mathrm{FO}[<])$.

So, E4 seems too restricted.

2.2 Language Operators: Transformation of $*$-languages to ω-languages

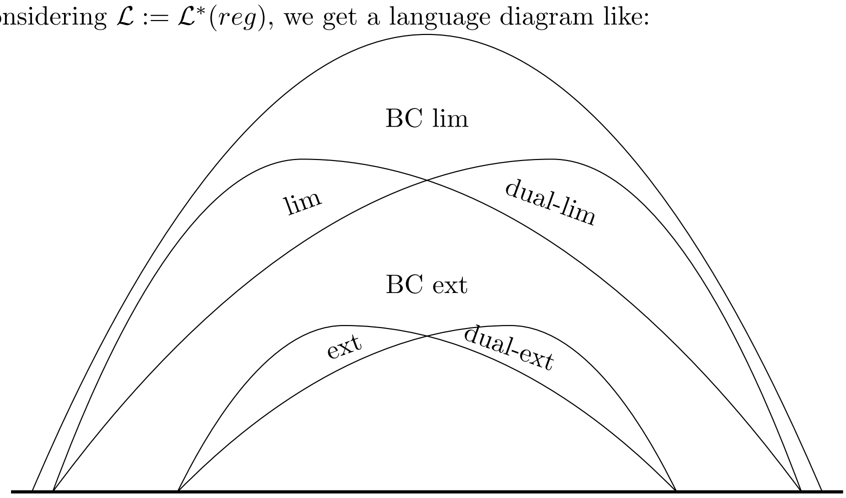
We already introduced lim. We can define a family of language operators, partly also derived from the study of $\mathcal{L}^\omega(reg)$. Some of these operators operate on a single language and not on the class. Let $\mathcal{L}$ be a $*$-language class. Let $L \in \mathcal{L}$.

1. $\mathrm{ext}(L) := \{\alpha \in \Sigma^\omega \mid \exists n\colon \alpha[0,n] \in L\} = L \cdot \Sigma^\omega$
2. $\overline{\mathrm{ext}}(L) := \{\alpha \in \Sigma^\omega \mid \forall n\colon \alpha[0,n] \in L\} = L \cdot \Sigma^\omega$
3. $\mathrm{BC}\,\mathrm{ext}$
4. $\lim(L) := \{\alpha \in \Sigma^\omega \mid \forall N\colon \exists n > N\colon \alpha[0,n] \in L\} = \{\alpha \in \Sigma^\omega \mid \exists^\omega n\colon \alpha[0,n] \in L\}$
5. $\overline{\lim}(L) := \{\alpha \in \Sigma^\omega \mid \exists N\colon \forall n > N\colon \alpha[0,n] \in L\}$
6. $\mathrm{BC}\lim$
7. Kleene-Closure of $\mathcal{L}$: $\mathrm{Kleene}(\mathcal{L}) := \{\bigcup_{i=1}^n U_i \cdot V_i^\omega \mid U_i, V_i \in \mathcal{L}\}$
8. $\{\bigcup_{i=1}^n U_i \cdot \lim V_i \mid U_i, V_i \in \mathcal{L}\}$

From language operators, we get language class operators in a canonical way, e.g. $\lim(\mathcal{L}) := \{\lim L \mid L \in \mathcal{L}\}$.

2.3 Classification of regular ω-languages

Considering $\mathcal{L} := \mathcal{L}^*(reg)$, we get a language diagram like:



where all inclusions are strict. In more detail:

- P1: $\mathrm{ext}\,\mathcal{L} \cap \overline{\mathrm{ext}}\,\mathcal{L} \neq \emptyset$
  Proof: $\tilde{L}_1 := a\Sigma^\omega \in \mathrm{ext} \cap \overline{\mathrm{ext}}\,\mathcal{L}$ with $\tilde{L}_1 = \mathrm{ext}(a)$ and $\tilde{L}_1 = \overline{\mathrm{ext}}(a\Sigma^*)$. (R101, prop, p.38)
- P2a: $\mathrm{ext}\,\mathcal{L} \cap \overline{\mathrm{ext}}\,\mathcal{L} \subsetneq \mathrm{ext}\,\mathcal{L}$
  Proof: $\tilde{L}_{2a} := \mathrm{ext}(a^*b) = a^*b\Sigma^\omega \in \mathrm{ext}\,\mathcal{L}$. Assume some A-automaton $\mathcal{A}$ with n states accepts $\tilde{L}_{2a}$. $\mathcal{A}$ would also accept $a^n b^\omega$. I.e. the $(n+1)$th state after the run of a^n would also accept a, i.e. $\mathcal{A}$ would accept a^{n+1}. By inclusion, $\mathcal{A}$ would accept a^ω. That is a contradiction. Thus, there is no such A-automat. Thus, $\tilde{L}_{2a} \notin \overline{\mathrm{ext}}\,\mathcal{L}$.

- P2b: $\operatorname{ext}\mathcal{L} \cap \overline{\operatorname{ext}}\,\mathcal{L} \subsetneqq \overline{\operatorname{ext}}\,\mathcal{L}$
  Proof: $\tilde{L}_{2b} := -\tilde{L}_{2a} \in \overline{\operatorname{ext}}\,\mathcal{L}$, $\tilde{L}_{2b} \notin \operatorname{ext}\mathcal{L}$.

- P3: $\operatorname{ext}\mathcal{L} \neq \overline{\operatorname{ext}}\,\mathcal{L}$
  Proof: Follows directly from P2a and P2b.

- P4: $\operatorname{ext}\mathcal{L} \cup \overline{\operatorname{ext}}\,\mathcal{L} \subsetneqq \operatorname{BC}\operatorname{ext}\mathcal{L}$
  Proof: $\tilde{L}_4 := \Sigma^* a \Sigma^\omega \cap -(\Sigma^* b \Sigma^\omega)$, $\Sigma = \{a, b, c\}$. Then we have $\tilde{L}_4 \notin \operatorname{ext} \cup \overline{\operatorname{ext}}\,\mathcal{L}$, $\tilde{L}_4 \in \operatorname{BC}\operatorname{ext}\mathcal{L}$. (R101, p.38)

- P5: $\operatorname{BC}\operatorname{ext}\mathcal{L} = \lim\mathcal{L} \cap \overline{\lim}\,\mathcal{L}$
  Proof: 3.10 (Staiger-Wagner-recognizable)

- P6a: $\lim\mathcal{L} \cap \overline{\lim}\,\mathcal{L} \subsetneqq \lim\mathcal{L}$
  Proof: $\tilde{L}_{6a} := \lim(\Sigma^* a) = (\Sigma^* a)^\omega$. Assume there is $L \subseteq \Sigma^*$ with $\lim(L) = -\tilde{L}_{6a}$. Let $(w_0, w_1, w_2, \dots) \in (\Sigma^*)^{\mathbb{N}}$ so that $w_0 \in L, w_0 a w_1 \in L, \dots, w_0 \prod_{i=0}^{n} a w_i \in L \ \forall n \in \mathbb{N}$. Thus, $\alpha := w_0 \prod_{i \in \mathbb{N}} a w_i \in \lim L$. But $\alpha \notin -\tilde{L}_{6a}$. That is a contradiction. Thus, $-\tilde{L}_{6a} \notin \lim\mathcal{L}$. With E3, we get $\tilde{L}_{6a} \notin \overline{\lim}\,\mathcal{L}$.

- P6b: $\lim\mathcal{L} \cap \overline{\lim}\,\mathcal{L} \subsetneqq \overline{\lim}\,\mathcal{L}$
  Proof: Analog to P6a with $\tilde{L}_{6b} := -\tilde{L}_{6a}$.

- P7: $\lim\mathcal{L} \neq \overline{\lim}\,\mathcal{L}$
  Proof: Follows directly from P6a and P6b.

- P8: $\lim\mathcal{L} \cup \overline{\lim}\,\mathcal{L} \subsetneqq \operatorname{BC}\lim\mathcal{L}$
  Proof: $\tilde{L}_8 := (\Sigma^* a)^\omega \cap -(\Sigma * b)^\omega$. Then $\tilde{L}_8 \notin \lim \cup \overline{\lim}\,\mathcal{L}$ but $\tilde{L}_8 \in \operatorname{BC}\lim\mathcal{L}$. (R101, prop, p.38)

- P9: $\operatorname{BC}\lim\mathcal{L} = \{\bigcup_{i=1}^{n} U_i \cdot V_i^\omega \mid U_i, V_i \in \mathcal{L}\}$
  Proof: This is explained already in chapter 2.1.

- P10: $\operatorname{BC}\lim\mathcal{L} = \{\bigcup_{i=1}^{n} U_i \cdot \lim V_i \mid U_i, V_i \in \mathcal{L}\}$
  Proof: This is explained already in chapter 2.1.

- P11: $\operatorname{BC}\lim\mathcal{L} = \{L^\omega_{\text{B"uchi}}(\mathcal{A}) \mid \mathcal{A} \text{ automaton so that } L^*(\mathcal{A}) \in \mathcal{L}\}$
  Proof: (R101,Th.12,p.9) says $\{L^\omega_{\text{B"uchi}}(\mathcal{A}) \mid \dots\} = \{\bigcup_{i=1}^{n} U_i \cdot V_i^\omega \mid U_i, V_i \in \mathcal{L}\}$. The rest follows with P9.

2.4 Towards a theory for subclasses of the regular language class

This was studied in detail for $\mathcal{L}^*(reg)$. We are now studing relations of resulting ω-language classes for different $*$-language classes.

Esp.:

- $\operatorname{BC}\operatorname{ext}\mathcal{L} \stackrel{?}{=} \operatorname{BC}\lim\mathcal{L}$

3 General results

Let $\mathcal{L}$ be a $*$-language class.

3.1 Background

Lemma 3.1. *Let $L, A, B \in \mathcal{L}$.*

1. $\operatorname{ext} L = L \cdot \Sigma^\omega$
2. $\operatorname{ext} L = \lim L \cdot \Sigma^*$
3. $\operatorname{ext} L = \overline{\lim} L \cdot \Sigma^*$
4. $-\lim(-L) = \overline{\lim} L$
5. $\overline{\lim} L \subseteq \lim L$
6. $\lim A \cup \lim B = \lim A \cup B$
   Proof:
   $$\begin{aligned} & \alpha \in \lim A \cup \lim B \\ \Leftrightarrow\ & \exists^\omega n\colon \alpha[0,n] \in A \ \vee\ \exists^\omega n\colon \alpha[0,n] \in B \\ \Leftrightarrow\ & \exists^\omega n\colon \alpha[0,n] \in A \cup B \\ \Leftrightarrow\ & \alpha \in \lim A \cup B \end{aligned}$$
7. $\overline{\lim} A \cup \overline{\lim} B \subseteq \overline{\lim} A \cup B$
   Proof:
   $$\begin{aligned} & \alpha \in \overline{\lim} A \cup \overline{\lim} B \\ \Leftrightarrow\ & \exists N\colon \forall n \geq N\colon \alpha[0,n] \in A \ \vee\ \exists N\colon \forall n \geq N\colon \alpha[0,n] \in B \\ \Rightarrow\ & \exists N\colon \forall n \geq N\colon \alpha[0,n] \in A \cup B \end{aligned}$$
   There is no equality in general: $A = (00)^*$, $B = (00)^*0$.

Lemma 3.2. *The ω-language-class accepted by deterministic E-automata is equal to non-deterministic E-automata. I.e., for every non-deterministic E-automaton, we can construct an equivalent deterministic E-automaton.*

Proof. Let $\mathcal{A}^N$ be any non-deterministic automaton and $\mathcal{A}^D$ an ($*$-)equivalent deterministic automaton. Then:

$$\begin{aligned} & \alpha \in L^{E\omega}(\mathcal{A}^N) \\ \Leftrightarrow\ & \exists n\colon \alpha[0,n] \in L(\mathcal{A}^N) \\ \Leftrightarrow\ & \exists n\colon \alpha[0,n] \in L(\mathcal{A}^D) \\ \Leftrightarrow\ & \alpha \in L^{E\omega}(\mathcal{A}^D) \end{aligned}$$

where $L^{E\omega}(\mathcal{A})$ denotes the set of ω-words which are E-accepted by $\mathcal{A}$. $\mathcal{A}^N$ can be interpreted as an arbitary E-automata and we have shown that we get an equivalent deterministic E-automata. $\square$

We are interested in relations like $\mathrm{BC}\,\mathrm{ext}\,\mathcal{L} \overset{?}{\subsetneq} \mathrm{BC}\lim\mathcal{L}$ or $\mathrm{ext}\,\mathcal{L} \overset{?}{\subsetneq} \lim\mathcal{L}$. With $\mathcal{L} = \{\{a\}\}$, we realize that even $\mathrm{ext}\,\mathcal{L} \subseteq \lim\mathcal{L}$ is not true in general ($\mathrm{ext}\,\{\{a\}\} = \{a\Sigma^\omega\} \neq \emptyset = \lim\{\{a\}\}$). In 3.3, we see a sufficient condition for this property, though.

3.2 Classification for arbitrary language classes

Lemma 3.3. *If E1 ($\forall L \in \mathcal{L}\colon L \cdot \Sigma^* \in \mathcal{L}$, i.e. $\mathcal{L}$ is non suffix sensitive) holds for $\mathcal{L}$: For $L \in \mathcal{L}$, we have* $\mathrm{ext}\,L = \lim L\Sigma^* = \overline{\lim}\,L\Sigma^*$ *and thus*

$$\mathrm{ext}\,\mathcal{L} \subseteq \lim\mathcal{L} \cup \overline{\lim}\,\mathcal{L}.$$

Lemma 3.4. *From* $\mathrm{ext}\,\mathcal{L} \subseteq \lim\mathcal{L}$, *it directly follows* $\{-\,\mathrm{ext}\,L \mid L \in \mathcal{L}\} \subseteq \{-\lim L \mid L \in \mathcal{L}\}$. *Thus, it also follows*

$$\mathrm{BC}\,\mathrm{ext}\,\mathcal{L} \subseteq \mathrm{BC}\lim\mathcal{L}.$$

Lemma 3.5. *From* $\mathrm{ext}\,\mathcal{L} \subseteq \lim\mathcal{L}$, *we need E3 ($\mathcal{L}$ closed under negation) to get* $\overline{\mathrm{ext}}\,\mathcal{L} \subseteq \overline{\lim}\,\mathcal{L}$.

This is in contrast to 3.4, where it directly follows. We have to be careful about the difference $-\,\mathrm{ext}\,\mathcal{L} \neq \overline{\mathrm{ext}}\,\mathcal{L}$ (in general, if E3 does not hold).

Lemma 3.6.

- *E2a (closed under union)* $\Rightarrow \bigcup \mathrm{ext}\,\mathcal{L} \subseteq \mathrm{ext}\,\mathcal{L}$.
- *E2b (closed under intersection)* $\Rightarrow \bigcap \mathrm{ext}\,\mathcal{L} \subseteq \mathrm{ext}\,\mathcal{L}$.

Lemma 3.7. *If there is* $L_\Sigma \in \mathcal{L}_\Sigma$ *with* $L_\Sigma \in \mathrm{ext}\,\mathcal{L}_\Sigma, L_\Sigma \notin \overline{\mathrm{ext}}\,\mathcal{L}_\Sigma$ *and E3 (closed under negation) holds for $\mathcal{L}$:*

$$\Rightarrow\ -L_\Sigma \in \overline{\mathrm{ext}}\,\mathcal{L}_\Sigma, -L_\Sigma \notin \mathrm{ext}\,\mathcal{L}_\Sigma$$

$$\Rightarrow\ L_{\Sigma_1} \cup -L_{\Sigma_2} \in \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}_{\Sigma_1 \dot\cup \Sigma_2}$$
$$L_{\Sigma_1} \cup -L_{\Sigma_2} \notin \mathrm{ext} \cup \overline{\mathrm{ext}}\,\mathcal{L}_{\Sigma_1 \dot\cup \Sigma_2}$$

Thus,

$$\mathrm{ext} \cup \overline{\mathrm{ext}}\,\mathcal{L} \subsetneq \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}.$$

Lemma 3.8. *Similarily, if there is* $L_\Sigma \in \lim\mathcal{L}_\Sigma, L_\Sigma \notin \overline{\lim}\,\mathcal{L}_\Sigma$ *and E3 holds for $\mathcal{L}$:*

$$\Rightarrow\ -L_\Sigma \in \overline{\lim}\,\mathcal{L}_\Sigma, -L_\Sigma \notin \lim\mathcal{L}_\Sigma$$

$$\Rightarrow\ L_{\Sigma_1} \cup -L_{\Sigma_2} \in \mathrm{BC}\lim\mathcal{L}_{\Sigma_1 \dot\cup \Sigma_2}$$
$$L_{\Sigma_1} \cup -L_{\Sigma_2} \notin \lim \cup \overline{\lim}\,\mathcal{L}_{\Sigma_1 \dot\cup \Sigma_2}$$

Thus,

$$\lim \cup \overline{\lim}\,\mathcal{L} \subsetneq \mathrm{BC}\lim\mathcal{L}.$$

Definition 3.1. A Staiger-Wagner automaton (weak Muller automaton) is of the same form $\mathcal{A} = (Q, \Sigma, q_0, \delta, \mathcal{F})$ as a Muller automaton with the acceptance condition that a run ρ is accepting if and only if $\mathrm{Occ}(\rho) := \{q \in Q\colon \text{q occurs in } p\} \in \mathcal{F}$. (R101,Def.61,p.43)

Lemma 3.9. *We see (R101,Th.63+64,p.44) that the class of Staigner-Wagner-recognized languages is exactly the class* $\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(reg)$ *and also* $\lim\cap\overline{\lim}\,\mathcal{L}^*(reg)$*.*

We are now formulating a more general and direct proof for the $\mathrm{BC}\,\mathrm{ext}\,\mathcal{L} = \lim\cap\overline{\lim}\,\mathcal{L}$ equality without Staiger-Wagner-automata (where some of the ideas are loosely based on (R101,Th.63+64,p.44)).

Lemma 3.10.

$$\mathrm{BC}\,\mathrm{ext}\,\mathcal{L} = \lim\cap\overline{\lim}\,\mathcal{L}$$

Proof. First, we show $\lim\cap\overline{\lim}\,\mathcal{L} \subseteq \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}$.

Let $\tilde{L} \in \lim\cap\overline{\lim}\,\mathcal{L}$, i.e. there are deterministic automaton $\mathcal{A}$ and $\overline{\mathcal{A}}$ so that $L^\omega_{\text{Büchi}}(\mathcal{A}) = L^\omega_{\text{co-Büchi}}(\overline{\mathcal{A}}) = \tilde{L}$. Let $Q, \overline{Q}$ be the states of $\mathcal{A}, \overline{\mathcal{A}}$. Now look at the product automaton $\mathcal{A} \times \overline{\mathcal{A}} =: \overset{\times}{\mathcal{A}}$ with states $Q \times \overline{Q}$ and final states $F \times \overline{F} \subseteq Q \times \overline{Q}$. $\overset{\times}{\mathcal{A}}$ is also deterministic.

In $\overset{\times}{\mathcal{A}}$, we have

$$\begin{aligned} &\alpha \in \tilde{L} \\ \Leftrightarrow\ &\forall N\colon \exists n \geq N\colon \overset{\times}{\rho}(\alpha)[n] \in F \times \overline{Q} \\ \Leftrightarrow\ &\exists N\colon \forall n \geq N\colon \overset{\times}{\rho}(\alpha)[n] \in Q \times \overline{F} \end{aligned}$$

Look at strongly connected component (SCC) S in $\overset{\times}{\mathcal{A}}$. We have $S \cap F \times \overline{Q} \neq \emptyset$, iff S accepts. It follows that all states in S are finite states in $\overline{\mathcal{A}}$, i.e. $S \cap Q \times \overline{F} = S$.

Single $\overset{\times}{q} \in \overset{\times}{Q}$ which are not part of a SCC can be ignored. For the acceptance of infinte words, only SCCs are relevant. For S, define $S_+ := \left\{ \overset{\times}{q} \in \overset{\times}{Q} - S \,\middle|\, \overset{\times}{q} \text{ can be visited after } S \right\}$.

Then we have

$$\begin{aligned} \tilde{L} = \bigcup_{\text{SCC } S} \; & S \text{ will be visited} \wedge \\ & \text{all states of } S \text{ will be visited forever after some step} \wedge \\ & S_+ \text{ will not be visited.} \end{aligned}$$

S will be visited: Let S exactly be the finite states. This interpreted as an E-automaton $\mathcal{A}^E_S$ is exactly the condition.

Only the allowed states will be visited but nothing followed after S: Mark S and all states on all paths to S as finite states. This as an A-automaton $\mathcal{A}^A_S$ is exactly the condition.

A similar negated condition might be simpler: Let S_+ be exactly the finite states. Interpret this as an E-automaton $\mathcal{A}^E_{S_+}$.

Then we have

$$\begin{aligned} \tilde{L} &= \bigcup_{\text{SCC } S} L^\omega_E(\mathcal{A}^E_S) \cap L^\omega_A(\mathcal{A}^A_S) \\ &= \bigcup_{\text{SCC } S} L^\omega_E(\mathcal{A}^E_S) \cap -L^\omega_E(\mathcal{A}^E_{S_+}). \end{aligned}$$

Thus,

$$\tilde{L} \in \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(reg).$$

Open question at this point: Is $L^*(\mathcal{A}_S^E), L^*(\mathcal{A}_{S_+}^E) \in \mathcal{L}$? With E4, this is obviously the case. However, E4 is too strong.

Now let us show $\mathrm{BC}\,\mathrm{ext}\,\mathcal{L} \subseteq \lim \mathcal{L}$.

With E1, we get $\mathrm{ext}\,\mathcal{L} \subseteq \lim \mathcal{L}$ and $\mathrm{ext}\,\mathcal{L} \subseteq \overline{\lim}\,\mathcal{L}$. I.e. $\mathrm{ext}\,\mathcal{L} \subseteq \lim \cap \overline{\lim}\,\mathcal{L}$. Let us show that $\lim \cap \overline{\lim}\,\mathcal{L}$ is closed under BC.

Let $\tilde{L}_a, L_b \in \lim \cap \overline{\lim}\,\mathcal{L}$, i.e. $\exists L_{a1}, L_{a2}, L_{b1}, L_{b2} \in \mathcal{L}\colon \tilde{L}_a = \lim L_{a1} = \overline{\lim}\, L_{a2}, \tilde{L}_b = \lim L_{b1} = \overline{\lim}\, L_{b2}$. Let us show 1. $-\tilde{L}_a \in \lim \cap \overline{\lim}\,\mathcal{L}$, 2. $\tilde{L}_a \cup \tilde{L}_b \in \lim \cap \overline{\lim}\,\mathcal{L}$.

1. $-\tilde{L}_a = -\lim L_{a1} = \overline{\lim} -L_{a1}$, $-\tilde{L}_b = -\overline{\lim}\, L_{a2} = \lim -L_{a2}$. With E3 (closed under negation), we get

   $$-\tilde{L}_a \in \lim \cap \overline{\lim}\,\mathcal{L}.$$

2. $\tilde{L}_a \cup \tilde{L}_b = \lim L_{a1} \cup \lim L_{b1} = \lim L_{a1} \cup L_{b1}$ (3.1). Thus, with E2a, we have

   $$\tilde{L}_a \cup \tilde{L}_b \in \lim \mathcal{L}.$$

   The $\overline{\lim}\,\mathcal{L}$ case is harder. Let $\mathcal{A}_a$, $\mathcal{A}_b$ be deterministic automaton, so that $L^\omega_{\text{Büchi}}(\mathcal{A}_a) = L^\omega_{\text{co-Büchi}}(\mathcal{A}_a) = \tilde{L}_a$, $L^\omega_{\text{Büchi}}(\mathcal{A}_b) = L^\omega_{\text{co-Büchi}}(\mathcal{A}_b) = \tilde{L}_b$. Look at the product automaton $\mathcal{A}_a \times \mathcal{A}_b =: \overset{\times}{\mathcal{A}}$. Then we have $L^\omega_{\text{Büchi}}(\overset{\times}{\mathcal{A}}) = L^\omega_{\text{co-Büchi}}(\overset{\times}{\mathcal{A}}) = \tilde{L}_a \cup \tilde{L}_b$.

   Thus,

   $$\tilde{L}_a \cup \tilde{L}_b \in \overline{\lim}\,\mathcal{L}^*(reg).$$

   Open question: Is $L^*(\overset{\times}{\mathcal{A}}) \in \mathcal{L}$?

□

Lemma 3.11.

$$\mathrm{BC} \lim \mathcal{L} = \mathrm{Kleene}\,\mathcal{L}$$

Proof. Let $U, V \in \mathcal{L}$. Look at the non-deterministic automaton $\mathcal{A}$ defined as:

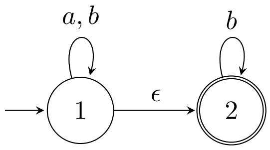
$$\begin{tikzcd} {} \arrow[r] & U \arrow[r, "\epsilon"] & V\odot \arrow[loop above] \end{tikzcd}$$

Then we have $L^\omega_{\text{Büchi}}(\mathcal{A}) = U \cdot V^\omega$.

Let us construct deterministic automata for $\mathcal{A}$ so that we can formulate 'V will be visited and not be left anymore' and 'finite states of the V-related automaton will be visited infinitely often' (or 'UV^* will be visited infinitely often').

In a constructed automaton, we must be able to tell wether we are in U or we deterministically have been in U the previous state. In a state power set construction, we can tell wether we are deterministically in U or not. If we are non-determinstic and we may be in both U or V and we get an input symbol which determines that we have been in U, we might not be able to tell from the following power set. Example:

Let $U = (a+b)^*$, $V = \{b\}$. I.e. $UV^\omega = \{\alpha \in \{a,b\}^\omega \mid \text{at one point in } \alpha, \text{ there are only } b\text{s}\}$. The non-deterministic automaton is:



Powerset construction: The initial state is $\{1,2\}$. Then we have:

- $\{1,2\} \overset{a}{\to} \{1,2\}$
- $\{1,2\} \overset{b}{\to} \{1,2\}$

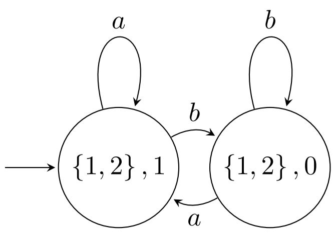
This gives the $*$-language $\{a,b\}^*$ and we cannot formulate UV^ω in any way from there.

In the construction, when we got the a from $\{1,2\}$, we knew that we have been deterministically in 1, i.e. in U. We loose this information. To keep it, we introduce another state flag which exactly says wether we have determined that we have been in U. Thus, we construct an automaton with the states $\mathcal{P}(Q) \times \mathbb{B}_{\text{det. been in } U}$, where Q are the states from $\mathcal{A}$.

For the example, we get the initial state $(\{1,2\}, 1)$. Then we have:

- $(\{1,2\}, 1) \overset{a}{\to} (\{1,2\}, 1)$
- $(\{1,2\}, 1) \overset{b}{\to} (\{1,2\}, 0)$
- $(\{1,2\}, 0) \overset{a}{\to} (\{1,2\}, 1)$
- $(\{1,2\}, 0) \overset{b}{\to} (\{1,2\}, 0)$

This is the automaton



When we mark all states from V and where we have not been deterministically in U as final, this as a co-Büchi automaton gives exactly the condition 'V will be visited and not be left anymore'. Let L_E be the $*$-language of this automata. Note that $L_E \neq UV^*$ in general and esp. in the example.

When we mark the final states as in the original non-deterministic automata, no matter about $\mathbb{B}_{\text{det. been in } U}$, with Büchi-acceptance, we get the condition 'UV^* will be visited infinitely often'. This is just $\lim UV^*$.

Together, we get UV^ω, i.e.:

$$\lim UV^* \cap \overline{\lim} L_E = UV^\omega$$

If we have $L_E \in \mathcal{L}$, it follows

$$\left\{ \bigcup_{i=1}^{n} U_i \cdot V_i^\omega \;\middle|\; U_i, V_i \in \mathcal{L} \right\} = \text{Kleene}\,\mathcal{L} \subseteq \text{BC}\lim \mathcal{L}.$$

Open question: Is $L_E \in \mathcal{L}$?

We also need to show $\text{BC}\lim \mathcal{L} \subseteq \text{Kleene}\,\mathcal{L}$.
Show: $\lim \mathcal{L} \subseteq \text{Kleene}\,\mathcal{L}$.
Proof: Let $\mathcal{A}$ be a deterministic Büchi automaton for some language $\tilde{L} = L^\omega_{\text{Büchi}}(\mathcal{A}) \in \mathcal{L}$ with final states F.

For all finite states $q \in F$: If q is not part of a strongly connected component (SCC), we can ignore it. Let S be the SCC where $q \in S$. Then the set of all $\alpha \in \Sigma^\omega$ which are infinitely often in q can be described as $U_q \cdot V_q^\omega$, where U_q is the set of words so that we arrive in q and V_q is the set of words so that we get from q to q. Both sets are regular.

Thus,

$$\tilde{L} = L^\omega_{\text{Büchi}}(\mathcal{A}) = \bigcup_{q \in F} U_q V_q^\omega.$$

Obviously, the Kleene-Closure is closed under union.

TODO: Show that Kleene-Closure is closed under negation. (S306.5) (Follows with non-det Büchi complementation but a more generic proof might be useful.) □

3.3 Congruenced based language classes

3.3.1 Introduction

We define $\mathcal{L}(R)$ for an equivalence relation $R \subseteq \Sigma^* \times \Sigma^*$.

Definition 3.2. If a language class $\mathcal{L}(R)$ is defined as finite union of equivalation classes of a relation $R \subseteq \Sigma^* \times \Sigma^*$ and

- the set of equivalent classes of R is finite,
- $(v, w) \in R \;\Leftrightarrow\; (va, wa) \in R \;\forall a \in \Sigma$

(this is the case for LT, LTT, PT), then we can construct a canonical deterministic automaton $\mathcal{A}_R$ which has $S_R := \Sigma^*/R$ as states, $\langle \epsilon \rangle_R$ is the initial state and the transitions are according to concatenation. Call this an R-automaton.

Definition 3.3. The set of all such R-automata, varying in the final state set, is isomprophic to $\mathcal{L}(R)$. We have

$$\mathcal{L}(R) = \{L(\mathcal{A}_R(F)) \mid F \subseteq S_R\} =: \mathcal{L}^*(\mathcal{A}_R).$$

Definition 3.4. Analogously for ω, we get the set of R-E-automata with the ω-language-class

$$\mathcal{L}_E^\omega(\mathcal{A}_R) := \left\{L^\omega(\mathcal{A}_R^E(F)) \,\middle|\, F \subseteq S_R\right\},$$

R-Büchi-automata and

$$\mathcal{L}_{\text{Büchi}}^\omega(\mathcal{A}_R) := \left\{L^\omega(\mathcal{A}_R^{\text{Büchi}}(F)) \,\middle|\, F \subseteq S_R\right\},$$

R-Muller-automata and

$$\mathcal{L}_{\text{Muller}}^\omega(\mathcal{A}_R) := \left\{L^\omega(\mathcal{A}_R^{\text{Muller}}(\mathcal{F})) \,\middle|\, \mathcal{F} \subseteq 2^{S_R}\right\}.$$

Definition 3.5. For a relation R on Σ^*, there are various ways to construct a relation on Σ^ω. For now, we mainly study $R^\omega := \overline{\text{ext}}\, R$, i.e.

$$(\alpha, \beta) \in R^\omega \;\; :\Leftrightarrow \;\forall n \colon (\alpha[0,n], \beta[0,n]) \in R.$$

Analogously to $\mathcal{L}(R)$, define the ω-language-class

$$\mathcal{L}^\omega(R^\omega) := \{L^\omega \mid L^\omega \text{ is finite union of } R^\omega\text{-equivalence-classes}\}.$$

With this preparation, we show for some R the equalities:

- $\mathcal{L}_E^\omega(\mathcal{A}_R) = \text{ext}\,\mathcal{L}(R)$ (3.12)
- $\mathcal{L}_{\text{Büchi}}^\omega(\mathcal{A}_R) = \lim \mathcal{L}(R)$ (3.13)
- $\mathcal{L}_{\text{Muller}}^\omega(\mathcal{A}_R) = \text{BC}\lim \mathcal{L}(R)$ (3.14)
- $\mathcal{L}^\omega(R^\omega) = \text{BC}\,\text{ext}\,\mathcal{L}(R)$
- $\text{BC}\lim \mathcal{L}(R) \cap \text{ext}\,\mathcal{L}(reg) = \text{ext}\,\mathcal{L}(R)$
- $\text{BC}\lim \mathcal{L}(R) \cap \lim \mathcal{L}(reg) = \lim \mathcal{L}(R)$
- $\lim \mathcal{L}(R) \cap \overline{\lim}\, \mathcal{L}(R) = \text{BC}\,\text{ext}\,\mathcal{L}(R)$

We will see that all those equations hold for $\mathcal{L}(LT)$, $\mathcal{L}(LTT)$ and $\mathcal{L}(PT)$.

Lemma 3.12.

$$\mathcal{L}_E^\omega(\mathcal{A}_R) = \text{ext}\,\mathcal{L}(R)$$

Proof. Let $L = \bigcup_i \langle w_i \rangle_R, L \in \mathcal{L}(R)$. Then

$$\begin{aligned}
&L^\omega = \text{ext}\, L \\
\Leftrightarrow\ &L^\omega = \left\{\alpha \in \Sigma^\omega \,\middle|\, \exists n \colon \alpha[0,n] \in \bigcup_i \langle w_i \rangle_R \right\} \\
\Leftrightarrow\ &L^\omega = \{\alpha \in \Sigma^\omega \mid \exists n \colon \delta_{\mathcal{A}_R}(\alpha[0,n]) \in \{\langle w_i \rangle_R \subseteq S_R \mid i\}\} \\
\Leftrightarrow\ &L^\omega = L^\omega(\mathcal{A}_R^E(\{\langle w_i \rangle_R \subseteq S_R \mid i\}))
\end{aligned}$$

□

Lemma 3.13.

$$\mathcal{L}^{\omega}_{B\ddot{u}chi}(\mathcal{A}_R) = \lim \mathcal{L}(R)$$

Proof. Let $L = \bigcup_i \langle w_i \rangle_R$, $L \in \mathcal{L}(R)$. Then

$$\begin{aligned}
& L^{\omega} = \lim L \\
\Leftrightarrow\ & L^{\omega} = \left\{ \alpha \in \Sigma^{\omega} \;\middle|\; \exists^{\infty} n \colon \alpha[0,n] \in \bigcup_i \langle w_i \rangle_R \right\} \\
\Leftrightarrow\ & L^{\omega} = \{ \alpha \in \Sigma^{\omega} \mid \exists^{\infty} n \colon \delta_{\mathcal{A}_R}(\alpha[0,n]) \in \{ \langle w_i \rangle_R \subseteq S_R \mid i \} \} \\
\Leftrightarrow\ & L^{\omega} = L^{\omega}(\mathcal{A}_R^{\text{Büchi}}(\{ \langle w_i \rangle_R \subseteq S_R \mid i \}))
\end{aligned}$$

□

Lemma 3.14.

$$\mathcal{L}^{\omega}_{Muller}(\mathcal{A}_R) = \mathrm{BC} \lim \mathcal{L}(R)$$

Proof. Any $L^{\omega} \in \mathrm{BC} \lim \mathcal{L}(R)$ can be described by $\mathrm{BC}\, 2^{S_R}$. $2^{2^{S_R}}$ is also finite. Thus, any $A \in \mathrm{BC}\, 2^{S_R}$ can be represented in $2^{2^{S_R}}$. This is exactly an acceptance condition in Muller. □

Lemma 3.15.

$$\mathcal{L}^{\omega}(R^{\omega}) = \mathrm{BC}\, \mathrm{ext}\, \mathcal{L}(R)$$

Proof. TODO... □

Lemma 3.16.

$$\mathrm{BC} \lim \mathcal{L}(R) \cap \mathrm{ext}\, \mathcal{L}(reg) = \mathrm{ext}\, \mathcal{L}(R)$$

Proof. We have $\mathrm{ext}\, \mathcal{L}(R) \subseteq \mathrm{ext}\, \mathcal{L}(reg)$ and $\mathrm{ext}\, \mathcal{L}(R) \subseteq \mathrm{BC} \lim \mathcal{L}(R)$. Thus, "$\supseteq$" is shown.

Now, we show "$\subseteq$". Let $L^{\omega} \in \mathrm{BC} \lim \mathcal{L}(R) \cap \mathrm{ext}\, \mathcal{L}(reg)$. Because $L^{\omega} \in \mathrm{ext}\, \mathcal{L}(reg)$, there is an E-automaton $\mathcal{A}^E$ which accepts L^{ω}. We can assume that $\mathcal{A}^E$ is deterministic (with 3.2).

We must find an R-E-automaton which accepts L^{ω}. We will call it the $\overline{\mathcal{A}}^M$ E-automaton and will construct it in the following.

Let $\mathcal{A}^M$ be the deterministic R-Muller-automaton for L^{ω} (according to 3.3.1 and 3.14). Without restriction, there are no final state sets in $\mathcal{A}^M$ which are not loops. Then, $\overline{\mathcal{A}}^M$ has the same states and transitions as $\mathcal{A}^M$.

Look at a final state q^E of $\mathcal{A}^E$. Without restriction, we can assume that there is no path that we can reach multiple final states at once. Let L_{q^E} be all words which reach q^E exactly once at the end.

Let $w \in L_{q^E}$. Let q be the state in $\mathcal{A}^M$ which is reached after w. Let S be the set of states in $\mathcal{A}^M$ which can be reached from q.

Then, $\mathcal{A}^M$ accepts all words in $L_q \cdot L^{\omega}_{q,S}$, where L_q is the set of words to q and $L^{\omega}_{q,S}$ is the set of words of possible infinite postfixes after q in S so that they are accepted. Any word with a prefix in L_q, which is not in $L_q \cdot L^{\omega}_{q,S}$, will not be accepted by $\mathcal{A}^M$ because $\mathcal{A}^M$

is deterministic. Also, because $L_{q^E} \cap L_q \neq \emptyset$ and $L_{q^E} \cdot \Sigma^\omega \subseteq L^\omega$ and $L_q \cdot L^\omega_{q,S} \subseteq L^\omega$, we get $L^\omega_{q,S} \neq \emptyset$.

Assuming $L^\omega_{q,S} \neq \Sigma^\omega$. Then we would have $L^\omega \notin \operatorname{ext} \mathcal{L}(reg)$, which is a contradiction. I.e. $L^\omega_{q,S} = \Sigma^\omega$.

Thus, $\mathcal{A}^M$ accepts all words in $L_q \cdot \Sigma^\omega$. Mark q as a final state in $\overline{\mathcal{A}}^M$. Thus, $\overline{\mathcal{A}}^M$ E-accepts all words in $L_q \cdot \Sigma^\omega \subseteq L^\omega$.

Because we did this for all final states in $\mathcal{A}^E$, there is no $\alpha \in L^\omega$ which is not accepted by $\overline{\mathcal{A}}^M$. I.e., the R-E-automata $\overline{\mathcal{A}}^M$ accepts exactly L^ω. I.e. $L^\omega \in \operatorname{ext} \mathcal{L}(R)$.

3.4 $\operatorname{BC} \lim \mathcal{L}(R) \cap \lim \mathcal{L}(reg) = \lim \mathcal{L}(R)$

This proof is loosely analogue to the proof in 3.16.

We have $\lim \mathcal{L}(R) \subseteq \lim \mathcal{L}(reg)$ and $\lim \mathcal{L}(R) \subseteq \operatorname{BC} \lim \mathcal{L}(R)$. Thus, "$\supseteq$" is shown.

Now, we show "$\subseteq$". Let $L^\omega \in \operatorname{BC} \lim \mathcal{L}(R) \cap \lim \mathcal{L}(reg)$. Because $L^\omega \in \lim \mathcal{L}(reg)$, there is an Büchi-automaton $\mathcal{A}^B$ which accepts L^ω. We can assume that $\mathcal{A}^B$ is deterministic (with 3.2).

We must find an R-Büchi-automaton which accepts L^ω. We will call it the $\overline{\mathcal{A}}^M$ Büchi-automaton and will construct it in the following.

Let $\mathcal{A}^M$ be the deterministic R-Muller-automaton for L^ω (according to 3.3.1 and 3.14). Without restriction, there are no final state sets in $\mathcal{A}^M$ which are not loops. Then, $\overline{\mathcal{A}}^M$ has the same states and transitions as $\mathcal{A}^M$.

Look at the SCC S in $\mathcal{A}^M$. Let $q \in S$. Let $\mathcal{F}_q \subseteq 2^S$ be the set of final states in $\mathcal{A}^M$ with $q \in F$ for all $F \in \mathcal{F}_q$. Let $\mathcal{S}_q \subseteq 2^S$ be the set of loops in S which include q.

Case 1: $\mathcal{F}_q \neq \mathcal{S}_q$.

Case 2: $\mathcal{F}_q = \mathcal{S}_q$. In that case, mark q as a final state in $\overline{\mathcal{A}}^M$.

For the constructed Büchi-automaton $\overline{\mathcal{A}}^M$, we show that it accepts exactly L^ω.

Let $\alpha \in L^\omega(\overline{\mathcal{A}}^M)$. Let q be some final state in $\overline{\mathcal{A}}^M$ which is infinitely often visited by α. Then, $\mathcal{F}_q = \mathcal{S}_q$ from the construction. I.e., no matter what loops through q of the related SCC are visited infinitely often by α, it will be accepted by $\mathcal{A}^M$. Thus, $\alpha \in L^\omega$.

Let $\alpha \in L^\omega$. Then, the set of states F infinitely often visited by α in $\mathcal{A}^M$ is some final state set of the Muller-automaton $\mathcal{A}^M$. In $\mathcal{A}^B$, there is a final state $\tilde{q}$ infinitely often visited by α. Let $\alpha =: \prod_{i=1}^\infty w_i$ so that $\prod_{i=1}^n w_i$ ends up in $\tilde{q}$ in $\mathcal{A}^B$ for all $n \in \mathbb{N}$ for shortest possible w_i (i.e. we don't miss any $\tilde{q}$). Let S be the SCC in $\mathcal{A}^M$ where we finally end up with α. Then, $F \subseteq S$.

There must be a $q \in F$ so that $\mathcal{F}_q = \mathcal{S}_q$. Then, by construction of $\overline{\mathcal{A}}^M$, q is a final state in $\overline{\mathcal{A}}^M$ and thus, $\alpha \in L^\omega(\overline{\mathcal{A}}^M)$.

Let us show that there is such $q \in F$ by contradiction. I.e. assume there is no such $q \in F$. I.e. for all $q \in F$, $\mathcal{F}_q \neq \mathcal{S}_q$. Of course we have $F \in \mathcal{F}_q$ for all $q \in F$.

Let $\mathcal{P}_{\tilde{q}}$ be the set of loops in $\mathcal{A}^M$ so that all words which end up looping there infinitely would also visit $\tilde{q}$ infinitely often in $\mathcal{A}^B$. Of course, all $P \in \mathcal{P}_{\tilde{q}}$ will be final state sets in $\mathcal{A}^M$ because $\mathcal{A}^B$ would accept. Define $\mathcal{P}_{\tilde{q},S} := \{P \in \mathcal{P}_{\tilde{q}} \mid P \subseteq S\}$. No matter how much other infinte loops in S we add to α so that we still visit some loops from $\mathcal{P}_{\tilde{q},S}$ infinitely often, $\mathcal{A}^M$ and $\mathcal{A}^B$ will keep accepting. Thus, for $P \in \mathcal{P}_{\tilde{q},S}$, every $P' \supseteq P$, $P' \in \mathcal{S}_S$, we have $P' \in \mathcal{P}_{\tilde{q},S}$.

...

□

Lemma 3.17.

$$\underline{\lim}\,\mathcal{L}(R) \cap \overline{\lim}\,\mathcal{L}(R) = \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}(R)$$

Proof. □

4 $*$-language classes

4.1 Overview

We already showed many results for $\mathcal{L}^*(reg)$.

4.2 FO[<] / starfree

Theorem 4.1.

$$\mathcal{L}^\omega(\mathrm{FO}[<]) = \mathrm{BC}\lim \mathcal{L}^*(\mathrm{FO}[<])$$

Proof. Let $\varphi \in \mathrm{FO}[<]$. By the [Tho81, Normal Form Theorem (4.4)] there are bounded formulas $\varphi_1(y), \cdots, \varphi_r(y), \psi_1(y), \cdots, \psi_r(y)$ such that for all $\alpha \in \Sigma^\omega$:

$$\alpha \models \varphi \Leftrightarrow \alpha \models \bigvee_{i=1}^{r} (\forall x \exists y > x \colon \varphi_i(y)) \wedge \neg (\forall x \exists y > x \colon \psi_i(y))$$

Thus:

$$\begin{aligned}\alpha \models \varphi \Leftrightarrow \bigvee_{i=1}^{r} &\underbrace{(\alpha \models \forall x \exists y > x \colon \varphi_i(y))} \wedge \neg (\alpha \models \forall x \exists y > x \colon \psi_i(y)) \\ &\Leftrightarrow \forall x \exists y > x \colon \alpha[0,n] \models \varphi_i(\omega) \\ &\Leftrightarrow \exists^\omega n \colon \alpha[0,n] \models \varphi_i(\omega) \\ &\Leftrightarrow \alpha \in \lim L^*(\varphi_i(\omega))\end{aligned}$$

where $\varphi_i(\omega)$ stands for φ_i with all bounds removed. I.e. we have

$$L^\omega(\varphi) = \bigcup_{i=1}^{r} \lim(L^*(\varphi_i(\omega)) \cap \neg \lim(L^*(\psi_i(\omega))),$$

and thus

$$L^\omega(\varphi) \in \mathrm{BC}\lim \mathcal{L}^*(\mathrm{FO}[<]).$$

We have prooved the $\subseteq$-direction. For $\supseteq$:

$$\begin{aligned}&\alpha \in \lim(L^*(\varphi)) \\ \Leftrightarrow\ &\exists^\omega n \colon \alpha[0,n] \models \varphi \\ \Leftrightarrow\ &\alpha \models \forall x \exists y > x \colon \varphi(y) \\ \Leftrightarrow\ &\alpha \in L^\omega(\forall \exists y > x \colon \varphi(y))\end{aligned}$$

where $\varphi(y)$ stands for φ with all variables bounded by y. I.e.

$$\lim \mathcal{L}^*(\mathrm{FO}[<]) \subseteq \mathcal{L}^\omega(\mathrm{FO}[<]),$$

and thus also

$$\mathrm{BC}\lim \mathcal{L}^*(\mathrm{FO}[<]) \subseteq \mathcal{L}^\omega(\mathrm{FO}[<]).$$

Thus we have prooved the equality. $\square$

Theorem 4.2.

$$\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\mathrm{FO}[<]) \subsetneqq \mathrm{BC}\lim\mathcal{L}^*(\mathrm{FO}[<])$$

Proof. $\subseteq$: $L \subset \Sigma^\omega$ starfree $\Rightarrow$ $L\Sigma^\omega \in \lim(\mathcal{L}^*(\mathrm{FO}[<]))$ □

Proof. $\neq$:

$$L := (\Sigma^* a)^\omega$$
$$\Rightarrow L = \lim((\Sigma^* a)^*)$$
$$\Rightarrow L = L^\omega(\exists^\omega x : Q_a x)$$

And we have $L \notin \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\mathrm{FO}[<])$. □

With 3.3, we get $\mathrm{ext}\,\mathcal{L} \subseteq \lim\mathcal{L}$.
$\tilde{L} := \lim(\Sigma^* a) = (\Sigma^* a)^\omega \in \lim\mathcal{L}$ but $\tilde{L} \notin \mathrm{ext}\,\mathcal{L}$ as shown in chapter 2.3.

- P1: $\{a\} \in \mathcal{L}$. $a\Sigma^* \in \mathcal{L}$, thus $a\Sigma^\omega = \mathrm{ext}(\{a\}) = \overline{\mathrm{ext}}\,a\Sigma^*$.
- P2a: $\tilde{L}_{2a} := \mathrm{ext}(a^*b) = a^*b\Sigma^\omega$, $a^*b \in \mathcal{L}$. Then $\tilde{L}_{2a} \notin \mathrm{ext}\,\mathcal{L}^*(reg) \supseteq \mathcal{L}^*(\mathrm{FO}[<])$.
- P2b: $-\tilde{L}_{2a} := \overline{\mathrm{ext}}(-a^*b)$, $-a^*b \in \mathcal{L}$. Then $-\tilde{L}_{2a} \notin \mathrm{ext}\,\mathcal{L}$.
- P3: Follows directly from P2a and P2b.
- P4: $\tilde{L}_4 := \mathrm{ext}(\Sigma^* a) \cap \overline{\mathrm{ext}}(-\Sigma^* b) = \Sigma^* a\Sigma^\omega \cap -(\Sigma^* b\Sigma^\omega)$, whereby $\Sigma^* a \in \mathcal{L}$, $-\Sigma^* b \in \mathcal{L}$. $\tilde{L}_4 \notin \mathrm{ext} \cup \overline{\mathrm{ext}}\,\mathcal{L}^*(reg) \supseteq \mathcal{L}^*(\mathrm{FO}[<])$ but $\tilde{L}_4 \in \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}$.
- P5: TODO
- P6a/P6b/P7/P8: $\Sigma^* a \in \mathcal{L}$. We can use the same arguments as for $\mathcal{L}^*(reg)$.
- P9: TODO
- P10: TODO

4.3 FO[+1]

Theorem 4.3.

$$\mathcal{L}^\omega(\mathrm{FO}[+1]) = \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\mathrm{FO}[+1])$$

Proof. From [Tho96, Theorem 4.8], we know that each formular in FO[+1] is equivalent (for both finite and infinite words) to a boolean combination of statements "sphere $\sigma \in \Sigma^+$ occurs $\geq n$ times". That statement can be expressed by a sentence of the form

$$\psi := \exists\overline{x_1} \cdots \exists\overline{x_n}\varphi(\overline{x_1}, \cdots, \overline{x_n})$$

where each $\overline{x_i}$ is a $|\sigma|$-tuple of variables and the formula φ states:

$$\bigwedge_{\substack{i,j \in \underline{n}, \\ i \neq j, \\ k,l \in \underline{|\sigma|}}} x_{i,k} \neq x_{j,l} \;\wedge\; \bigwedge_{\substack{i \in \underline{n}, \\ k \in \underline{|\sigma|-1}}} x_{i,k+1} = x_{i,k} + 1 \;\wedge\; \bigwedge_{\substack{i \in \underline{n}, \\ k \in \underline{|\sigma|}}} Q_{\sigma_k} x_{i,k}$$

For ψ, we have:

$$\alpha \models \psi \Leftrightarrow \exists n\colon \alpha[0,n] \models \psi \quad \text{for all } \alpha \in \Sigma^\omega,$$

i.e.

$$L^\omega(\psi) = \operatorname{ext} L^*(\psi).$$

Any formular in FO[+1] can be expressed as a boolean combination of ψ-like formular. With

$$\begin{aligned}
&L^\omega(\neg\psi) = \neg L^\omega(\psi)\\
&L^\omega(\psi_1 \wedge \psi_2) = L^\omega(\psi_1) \cap L^\omega(\psi_2)\\
&L^\omega(\psi_1 \vee \psi_2) = L^\omega(\psi_1) \cup L^\omega(\psi_2)
\end{aligned}$$

we get:

$$\mathcal{L}^\omega(\mathrm{FO}[+1]) = \mathrm{BC}\operatorname{ext}\mathcal{L}^*(\mathrm{FO}[+1]).$$

□

4.4 FO[]

4.5 piece-wise testable

Theorem 4.4.

$$\mathrm{BC}\operatorname{ext}\mathcal{L}^*(\text{piece-wise testable}) = \mathrm{BC}\lim\mathcal{L}^*(\text{piece-wise testable})$$

Proof. L piece-wise testable $\Leftrightarrow$ L is a boolean algebra of $\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*$

$\subseteq$: It is sufficient to show $\operatorname{ext}(\mathcal{L}^*(\text{piece-wise testable})) \subseteq \mathrm{BC}\lim\mathcal{L}^*(\text{piece-wise testable})$. By complete induction:

$$\begin{aligned}
&\operatorname{ext}(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*) = \Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^\omega = \lim(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*)\\
&\operatorname{ext}(\neg(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*)) = \Sigma^\omega = \lim(\Sigma^*)\\
&\operatorname{ext}(\emptyset) = \emptyset = \lim(\emptyset)
\end{aligned}$$

It is sufficient to show negation only for such ground terms because we can always push the negation down.

$$\begin{aligned}
&\operatorname{ext}(A \cup B) = \operatorname{ext}(A) \cup \operatorname{ext}(B)\\
&\operatorname{ext}(A \cap B) = \operatorname{ext}(A) \cap \operatorname{ext}(B)
\end{aligned}$$

This makes the induction complete.

$\supseteq$: It is sufficient to show $\lim(\mathcal{L}^*(\text{piece-wise testable})) \subseteq \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{piece-wise testable})$.

$$\begin{aligned}
&\lim(\emptyset) = \mathrm{ext}(\emptyset),\ \lim(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*) = \mathrm{ext}(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*) \quad \text{(see above)}\\
&\lim(\neg(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*)) = \{\alpha \in \Sigma^\omega \mid \exists^\omega n\colon \alpha[0,n] \notin \Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*\}\\
&\qquad = \{\alpha \in \Sigma^\omega \mid \forall n\colon \alpha[0,n] \notin \Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*\}\\
&\qquad = \neg\, \mathrm{ext}(\Sigma^* a_1 \Sigma^* a_2 \cdots a_n \Sigma^*)\\
&\lim(A \cup B) = \{\alpha \in \Sigma^\omega \mid \exists^\omega n\colon \alpha[0,n] \in A \cup B\} = \lim(A) \cup \lim(B)\\
&\lim(A \cap B) = \{\alpha \in \Sigma^\omega \mid \exists^\omega n\colon \alpha[0,n] \in A \cap B\}
\end{aligned}$$

and because A, B are piece-wise testable

$$= \{\alpha \in \Sigma^\omega \mid \exists n : \forall m > n\colon \alpha[0,m] \in A \cap B\} = \lim(A) \cap \lim(B)$$

□

4.6 positive piece-wise testable

Theorem 4.5.

$$\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{pos-PT}) = \mathrm{BC}\,\lim\,\mathcal{L}^*(\text{pos-PT})$$

Proof. $\subseteq$: Exactly like the proof for PT except that we leave out the negated part. $\supseteq$: Also like the proof for PT. □

4.7 locally testable

Theorem 4.6.

$$\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{locally testable}) \subsetneq \mathrm{BC}\,\lim\,\mathcal{L}^*(\text{locally testable})$$

Proof. Let $w \in \Sigma^+$.

$$\begin{aligned}
&\mathrm{ext}(w\Sigma^*) = \lim(w\Sigma^*)\\
&\mathrm{ext}(\Sigma^* w) = \Sigma^* w \Sigma^\omega = \lim(\Sigma^* w \Sigma^*)\\
&\mathrm{ext}(\Sigma^* w \Sigma^*) = \Sigma^* w \Sigma^\omega = \lim(\Sigma^* w \Sigma^*)
\end{aligned}$$

Thus we have

$$\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{locally testable}) \subseteq \mathrm{BC}\,\lim\,\mathcal{L}^*(\text{locally testable}).$$

But we also have

$$\lim(\Sigma^*) = (\Sigma^* w)^\omega \notin \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{locally testable}).$$

□

4.8 endwise testable

- $\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\mathit{endwise}) \neq \mathrm{BC}\,\lim\,\mathcal{L}^*(\mathit{endwise})$ because $\Sigma^* a \in \mathcal{L}^*(\mathit{endwise})$.
- $\mathrm{ext}(a\Sigma^* a) = a\Sigma^* a \Sigma^\omega \notin \mathrm{BC}\,\lim\,\mathcal{L}^*(\mathit{endwise})$

4.9 local

4.10 finite / co-finite

- $\lim \mathcal{L}^*(finite) = \{\emptyset\}$
- $\text{ext}\, \mathcal{L}^*(finite) = \mathcal{L}^*(finite) \cdot \Sigma^\omega$
- $\lim \mathcal{L}^*(co - finite) = \{\Sigma^\omega\}$
- $\text{ext}\, \mathcal{L}^*(co - finite) = \{\Sigma^\omega\}$

4.11 dot-depth-n

4.12 L-trivial

4.13 R-trivial

4.14 locally modulo testable

4.15 context free

5 Lemmas

5.1 pos-PT and PT

Theorem 5.1.

$$\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{pos-PT}) = \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\mathrm{PT})$$

Proof. In the proof of $\lim \mathcal{L}^*(\mathrm{PT}) \subseteq \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\mathrm{PT})$ we actually proved $\mathrm{BC}\lim \mathcal{L}^*(\mathrm{PT}) \subseteq \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\text{pos-PT})$. Similiarly we also proved $\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}^*(\mathrm{PT}) \subseteq \mathrm{BC}\lim \mathcal{L}^*(\text{pos-PT})$.

With 4.6 and 4.5 we get the claimed equality. $\square$

References

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