Language Operations and a Structure Theory of ω-Languages

February 14, 2014

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   - via $* \to \omega$ language operators: BC lim $\mathcal{L}^*(\mathrm{reg})$

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We have the standard $\mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^\omega)$ language operators:

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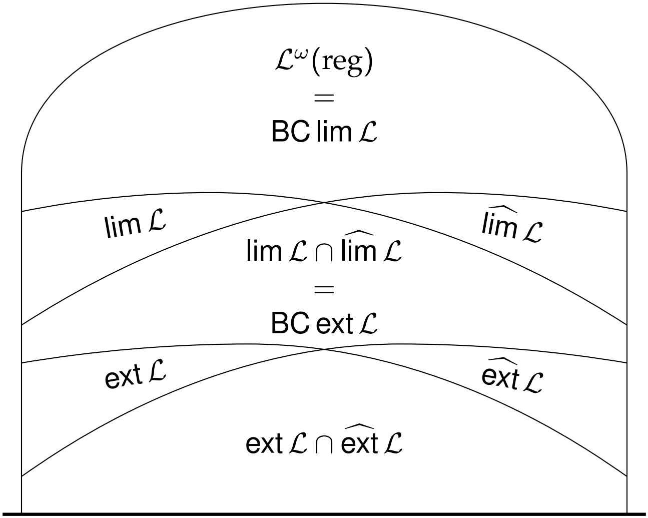
$\mathcal{L} := \mathcal{L}^*(\mathrm{reg})$ inclusion diagram



All inclusions are strict.

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- Chapter 3: general results on arbitrary $\mathcal{L}$, given some introduced properties on $\mathcal{L}$
- Chapter 4: concrete $*$-language classes

Properties on $\mathcal{L}$

1. $\mathcal{L}$ **closed under suffix-independence**: $L \in \mathcal{L} \Rightarrow L \cdot \Sigma^* \in \mathcal{L}$
   Examples: $\mathcal{L}^*(\mathrm{reg})$, $\mathcal{L}(\mathrm{starfree})$, $\mathcal{L}(\mathrm{PT}_n)$ (Lemma 4.10), $\mathcal{L}(\mathrm{PT})$, $\mathcal{L}(\mathrm{LT})$, $\mathcal{L}(\mathrm{LTT})$
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5. $\mathcal{L}$ **closed under alphabet permutation**: For all permutations $\sigma : \Sigma \to \Sigma$ and $L \in \mathcal{L}$, we have $L_\sigma := \{\sigma(w) \mid w \in L\} \in \mathcal{L}$

General results: ext $\subseteq$ lim

- Lemma 3.3: $\mathcal{L}$ closed under suffix-independence $\Rightarrow$

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- Counter examples are $\mathcal{L}(\mathrm{finite})$ or somewhat artificial (Example 3.9)

$\mathrm{ext} \cup \widehat{\mathrm{ext}}\, \mathcal{L}^*(\mathrm{reg}) \subsetneq \mathrm{BC}\,\mathrm{ext}\, \mathcal{L}^*(\mathrm{reg})$

- We have

$$\mathrm{ext} \cup \widehat{\mathrm{ext}}\, \mathcal{L} := (\mathrm{ext}\, \mathcal{L}) \cup (\widehat{\mathrm{ext}}\, \mathcal{L}) \subsetneq \mathrm{BC}\,\mathrm{ext}\, \mathcal{L}$$

  for $\mathcal{L} = \mathcal{L}^*(\mathrm{reg})$.

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- Separating languages: Let $\Sigma := \{a, b, c\}$.

$$L_a := \Sigma^* a \in \mathcal{L}, \quad L_b := \Sigma^* b \in \mathcal{L},$$

$$\tilde{L}_1 := \mathrm{ext}\, L_a \cap - \mathrm{ext}\, L_b, \quad \tilde{L}_2 := \lim L_a \cap - \lim L_b.$$

  Then

$$\tilde{L}_1 \notin \mathrm{ext} \cup \widehat{\mathrm{ext}}\, \mathcal{L} \quad \text{but} \quad \tilde{L}_1 \in \mathrm{BC}\, \mathrm{ext}\, \mathcal{L}$$

$$\Rightarrow \mathrm{ext} \cap \widehat{\mathrm{ext}}\, \mathcal{L} \subsetneqq \mathrm{ext} \cup \widehat{\mathrm{ext}}\, \mathcal{L} \subsetneqq \mathrm{BC}\, \mathrm{ext}\, \mathcal{L},$$

$$\tilde{L}_2 \notin \lim \cup \widehat{\lim}\, \mathcal{L} \quad \text{but} \quad \tilde{L}_2 \in \mathrm{BC} \lim \mathcal{L}$$

$$\Rightarrow \lim \cap \widehat{\lim}\, \mathcal{L} \subsetneqq \lim \cup \widehat{\lim}\, \mathcal{L} \subsetneqq \mathrm{BC} \lim \mathcal{L}.$$

$\mathsf{ext} \cup \widehat{\mathsf{ext}}\, \mathcal{L}^*(\mathrm{reg}) \subsetneqq \mathsf{BC}\,\mathsf{ext}\, \mathcal{L}^*(\mathrm{reg})$

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for $\mathcal{L} = \mathcal{L}^*(\mathrm{reg})$.
- Separating languages: Let $\Sigma := \{a, b, c\}$.
$$L_a := \Sigma^* a \in \mathcal{L}, \quad L_b := \Sigma^* b \in \mathcal{L},$$
$$\tilde{L}_1 := \mathsf{ext}\, L_a \cap -\mathsf{ext}\, L_b, \quad \tilde{L}_2 := \lim L_a \cap -\lim L_b.$$
Then
$$\begin{aligned} &\tilde{L}_1 \notin \mathsf{ext} \cup \widehat{\mathsf{ext}}\, \mathcal{L} \quad \text{but} \quad \tilde{L}_1 \in \mathsf{BC}\,\mathsf{ext}\, \mathcal{L} \\ &\quad \Rightarrow \mathsf{ext} \cap \widehat{\mathsf{ext}}\, \mathcal{L} \subsetneqq \mathsf{ext} \cup \widehat{\mathsf{ext}}\, \mathcal{L} \subsetneqq \mathsf{BC}\,\mathsf{ext}\, \mathcal{L}, \\ &\tilde{L}_2 \notin \lim \cup \widehat{\lim}\, \mathcal{L} \quad \text{but} \quad \tilde{L}_2 \in \mathsf{BC} \lim \mathcal{L} \\ &\quad \Rightarrow \lim \cap \widehat{\lim}\, \mathcal{L} \subsetneqq \lim \cup \widehat{\lim}\, \mathcal{L} \subsetneqq \mathsf{BC} \lim \mathcal{L}. \end{aligned}$$
- $L_a, L_b \in \mathcal{L}(\mathrm{starfree}) \cap \mathcal{L}(\mathrm{LT}) \cap \mathcal{L}(\mathrm{LTT})$

General results: $\mathrm{ext} \cup \widehat{\mathrm{ext}} \subsetneq \mathrm{BC\,ext}$

Definition 3.12. A language $L \subseteq \Sigma^*$ is called M-**invariant** for $M \subseteq \Sigma$ iff for all $w_1, w_2 \in \Sigma^*$, $a \in M$,

$$w_1 a w_2 \in L \quad \Rightarrow \quad w_1 M^* w_2 \subseteq L.$$

A language $L \subseteq \Sigma^*$ is called M-**relevant** iff L is not M-invariant and $\Sigma^* a \Sigma^* \cap L \neq \emptyset$ for every $a \in M$.

Theorem 3.15. Let $\mathcal{L}$ be closed under negation and under alphabet permutation. Let $\{a, b, c\} \subseteq \Sigma$. Let $L_a \in \mathcal{L}$ be $\{a\}$*-relevant* and $\{b, c\}$*-invariant*. Then

$$\mathrm{ext}\, L_a \notin \widehat{\mathrm{ext}}\, \mathcal{L}^*(\mathrm{reg}) \quad \Rightarrow \quad \mathrm{ext} \cup \widehat{\mathrm{ext}}\, \mathcal{L} \subsetneq \mathrm{BC\,ext}\, \mathcal{L}$$

and

$$\lim L_a \notin \widehat{\lim}\, \mathcal{L}^*(\mathrm{reg}) \quad \Rightarrow \quad \lim \cup \widehat{\lim}\, \mathcal{L} \subsetneq \mathrm{BC} \lim \mathcal{L}.$$

General results

- **Theorem 3.19.** (Staiger-Wagner 1) $\mathcal{L}$ closed under change of final states. Then

$$\lim \cap \widehat{\lim}\, \mathcal{L} \subseteq \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}.$$

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- **Theorem 3.19.** (Staiger-Wagner 1) $\mathcal{L}$ closed under change of final states. Then
$$\lim \cap \widehat{\lim}\, \mathcal{L} \subseteq \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}.$$
- **Theorem 3.20.** (Staiger-Wagner 2) $\mathcal{L}$ closed under suffix-independence, negation, union and change of final states. Then
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$$\mathrm{BC}\, \mathrm{ext}\, \mathcal{L} \subseteq \lim \cap \widehat{\lim}\, \mathcal{L}.$$
- **Theorem 3.22.** $\mathcal{L}$ closed under suffix-independence, negation, union, change of final states and alphabet permutation. Then we have
$$\mathrm{ext} \cap \widehat{\mathrm{ext}}\, \mathcal{L} \overset{(1.)}{\subseteq} \mathrm{ext} \cup \widehat{\mathrm{ext}}\, \mathcal{L} \overset{(2.)}{\subseteq} \mathrm{BC}\, \mathrm{ext}\, \mathcal{L} \overset{(3.)}{=}$$
$$\lim \cap \widehat{\lim}\, \mathcal{L} \overset{(4.)}{\subseteq} \lim \cup \widehat{\lim}\, \mathcal{L} \overset{(5.)}{\subseteq} \mathrm{BC}\lim \mathcal{L}.$$
With $L_a \in \mathcal{L}$ and $\mathrm{ext}\, L_a \notin \widehat{\mathrm{ext}}\, \mathcal{L}^*(\mathrm{reg})$, the inclusions in (1) and (2) are strict. With $L'_a \in \mathcal{L}$ and $\lim L'_a \notin \widehat{\lim}\, \mathcal{L}^*(\mathrm{reg})$, the inclusions in (4) and (5) are strict.

Kleene closure

$$\mathrm{Kleene}(\mathcal{L}) := \left\{ \bigcup_{i=1}^{n} U_i \cdot V_i^{\omega} \,\middle|\, U_i, V_i \subseteq \Sigma^*, U_i \cdot V_i^* \in \mathcal{L}, n \in \mathbb{N}_0 \right\}$$

$$\mathcal{L}^{\omega}(\mathrm{reg}) = \mathrm{Kleene}(\mathcal{L}^*(\mathrm{reg})) = \mathrm{BC}\lim \mathcal{L}^*(\mathrm{reg})$$

- **Lemma 3.24.**

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$$\mathcal{L}^\omega(\mathrm{reg}) = \text{Kleene}(\mathcal{L}^*(\mathrm{reg})) = \text{BC}\lim \mathcal{L}^*(\mathrm{reg})$$

- **Lemma 3.24.**
  - Generic power-set construction based on a non-det. UV^* automaton which results in a det. co-Büchi automaton for parts of UV^ω. deterministic *simplified* automata.

Kleene closure

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- **Lemma 3.25.** $\mathcal{L}$ closed under change of final states. Then

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Congruence based classes $\mathcal{L}(R)$

Motivation: $\mathcal{L}(\mathrm{LT}_n)$ or $\mathcal{L}(\mathrm{PT}_n)$

Let $R \subseteq \Sigma^* \times \Sigma^*$ be a congruence relation.

$$\mathcal{L}^*(R) := \{L \subseteq \Sigma^* \mid L \text{ is finite union of } R\text{-equivalence-classes}\}.$$

There is a canonical deterministic automaton with states $S_R := \Sigma^*/R$. We call it the R-automaton.

- Lemma 3.28. $\mathcal{L}(R)$ is *closed under change of final states*.

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General results: $\mathrm{BC}\lim \mathcal{L}(R)$ in $\mathcal{L}^{\omega}(\mathrm{reg})$

- **Lemma 3.33.** $\mathrm{BC}\lim \mathcal{L}(R) \cap \mathrm{ext}\, \mathcal{L}^{*}(\mathrm{reg}) \subseteq \mathrm{ext}\, \mathcal{L}(R)$
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General results: BC lim $\mathcal{L}(R)$ in $\mathcal{L}^\omega(\mathrm{reg})$

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- **Definition 3.41.** If there is a SCC $Q \subseteq S_R$ including two loops $P_1, P_2 \subseteq Q$, $P_1 \neq P_2$ with $P_1 \not\subseteq P_2$, $P_2 \not\subseteq P_1$, then call $\mathcal{L}(R)$ **postfix-loop-deterministic.**
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  Examples:
  - $\mathcal{L}(\mathrm{PT}_n)$ for all n and $\mathcal{L}(\mathrm{LT}_1)$ are not postfix-loop-deterministic
  - $\mathcal{L}(\mathrm{LT}_n)$ for $n \geq 2$ is postfix-loop-deterministic (Lemma 4.14)
- **Theorem 3.44.** $\mathcal{L}(R)$ is not *postfix-loop-deterministic* $\Leftrightarrow$

$$\text{BC} \lim \mathcal{L}(R) \cap \lim \mathcal{L}^*(\mathrm{reg}) = \lim \mathcal{L}(R).$$

General results: $\mathcal{L}(R)$: Staiger-Wagner

- **Theorem 3.47.** (Staiger-Wagner) $\mathcal{L}(R)$ not postfix-loop-deterministic. $\mathrm{BC}\,\mathrm{ext}\,\mathcal{L}(R) \subseteq \mathrm{BC}\,\lim\,\mathcal{L}(R)$. Then
$$\lim \cap \widehat{\lim}\,\mathcal{L}(R) = \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}(R)$$
Example: $\mathcal{L}(\mathrm{PT}_n)$

Concrete results

For $\mathcal{L} := \mathcal{L}(\mathrm{starfree})$, via Theorem 3.22, we get

$$\mathsf{ext} \cap \widehat{\mathsf{ext}}\,\mathcal{L} \subsetneqq \mathsf{ext} \cup \widehat{\mathsf{ext}}\,\mathcal{L} \subsetneqq \mathsf{BC}\,\mathsf{ext}\,\mathcal{L} =$$
$$\lim \cap \widehat{\lim}\,\mathcal{L} \subsetneqq \lim \cup \widehat{\lim}\,\mathcal{L} \subsetneqq \mathsf{BC}\lim\mathcal{L}.$$

For $\mathcal{L} := \mathcal{L}(\mathrm{LT})$ or $\mathcal{L} := \mathcal{L}(\mathrm{LTT})$, via Theorem 3.22, we get

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Conclusion

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Conclusion

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- Another possible generalization: class of $\mathcal{L}$ automata (instead of single fixed R-automata as in $\mathcal{L}(R)$). e.g. $\bigcup_n \mathrm{PT}_n -$ automata.
- More concrete language classes can be studied. Supersets of the class of regular languages weren't studied at all here. Natural generalization would be to use pushdown automata in the proofs for the class of context free languages.