

# Language Operations and a Structure Theory of $\omega$ -Languages

August 3, 2012

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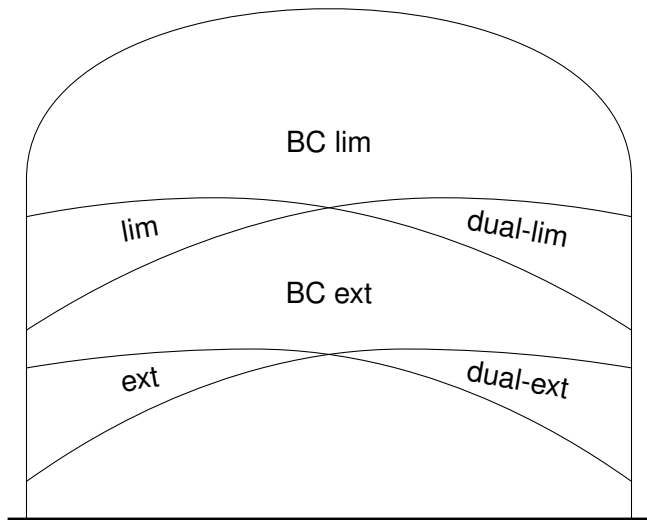
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# $\mathcal{L}^*(\text{reg})$ inclusion diagram



# Questions

- ▶ Instead of the class of regular  $\ast$ -languages, look at other  $\ast$ -language classes, e.g. starfree, LT, PT, or any arbitrary  $\ast$ -language class  $\mathcal{L}$ .

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- ▶ Chapter 4: concrete  $*$ -language classes



# Properties on $\mathcal{L}$

1.  $\mathcal{L}$  **closed under suffix-independence**:  $L \in \mathcal{L} \Rightarrow L \cdot \Sigma^* \in \mathcal{L}$

Examples:  $\mathcal{L}^*(\text{reg})$ ,  $\mathcal{L}(\text{starfree})$ ,  $\mathcal{L}(\text{PT}_n)$  (Lemma 4.10),  
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$\mathcal{A} = (Q, \Sigma, q_0, \delta, F)$  be a minimal deterministic automaton with  $L^*(\mathcal{A}) \in \mathcal{L}$ . Then, for all  $F' \subseteq Q$ , we have

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5.  **$\mathcal{L}$  closed under alphabet permutation:** For all permutations  $\sigma: \Sigma \rightarrow \Sigma$  and  $L \in \mathcal{L}$ , we have

$L_\sigma := \{\sigma(w) \mid w \in L\} \in \mathcal{L}$

## General results: $\text{ext} \subseteq \lim$

- ▶ Lemma 3.3:  $\mathcal{L}$  closed under suffix-independence  $\Rightarrow$

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- ▶ Counter examples are  $\mathcal{L}(\text{finite})$  or somewhat artificial (Example 3.9)

$$\text{ext} \cup \widehat{\text{ext}} \mathcal{L}^*(\text{reg}) \subsetneq \text{BC ext } \mathcal{L}^*(\text{reg})$$

► We have

$$\text{ext} \cup \widehat{\text{ext}} \mathcal{L} := (\text{ext } \mathcal{L}) \cup (\widehat{\text{ext}} \mathcal{L}) \subsetneq \text{BC ext } \mathcal{L}$$

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- Separating languages: Let  $\Sigma := \{a, b, c\}$ .

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$$\tilde{L}_1 := \text{ext } L_a \cap -\text{ext } L_b, \quad \tilde{L}_2 := \lim L_a \cap -\lim L_b.$$

Then

$$\tilde{L}_1 \notin \text{ext} \cup \widehat{\text{ext}} \mathcal{L} \quad \text{but} \quad \tilde{L}_1 \in \text{BC ext } \mathcal{L}$$

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- $L_a, L_b \in \mathcal{L}(\text{starfree}) \cap \mathcal{L}(\text{LT}) \cap \mathcal{L}(\text{LTT})$

## General results: $\text{ext} \cup \widehat{\text{ext}} \subsetneq \text{BC ext}$

**Definition 3.12.** A language  $L \subseteq \Sigma^*$  is called  **$M$ -invariant** for  $M \subseteq \Sigma$  iff for all  $w_1, w_2 \in \Sigma^*$ ,  $a \in M$ ,

$$w_1 a w_2 \in L \Rightarrow w_1 M^* w_2 \subseteq L.$$

A language  $L \subseteq \Sigma^*$  is called  **$M$ -relevant** iff  $L$  is not  $M$ -invariant and  $\Sigma^* a \Sigma^* \cap L \neq \emptyset$  for every  $a \in M$ .

**Theorem 3.15.** Let  $\mathcal{L}$  be closed under negation and under alphabet permutation. Let  $\{a, b, c\} \subseteq \Sigma$ . Let  $L_a \in \mathcal{L}$  be  $\{a\}$ -relevant and  $\{b, c\}$ -invariant. Then

$$\text{ext } L_a \notin \widehat{\text{ext}} \mathcal{L}^*(\text{reg}) \Rightarrow \text{ext} \cup \widehat{\text{ext}} \mathcal{L} \subsetneq \text{BC ext } \mathcal{L}$$

and

$$\lim L_a \notin \widehat{\lim} \mathcal{L}^*(\text{reg}) \Rightarrow \lim \cup \widehat{\lim} \mathcal{L} \subsetneq \text{BC } \lim \mathcal{L}.$$

# General results

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- **Theorem 3.20.** (Staiger-Wagner 2)  $\mathcal{L}$  closed under suffix-independence, negation, union and change of final states. Then

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- **Theorem 3.22.**  $\mathcal{L}$  closed under suffix-independence, negation, union, change of final states and alphabet permutation. Then we have

$$\begin{aligned} \text{ext} \cap \widehat{\text{ext}} \mathcal{L} &\stackrel{(1.)}{\subseteq} \text{ext} \cup \widehat{\text{ext}} \mathcal{L} \stackrel{(2.)}{\subseteq} \text{BC ext } \mathcal{L} \stackrel{(3.)}{=} \\ \lim \cap \widehat{\lim} \mathcal{L} &\stackrel{(4.)}{\subseteq} \lim \cup \widehat{\lim} \mathcal{L} \stackrel{(5.)}{\subseteq} \text{BC lim } \mathcal{L}. \end{aligned}$$

With  $L_a \in \mathcal{L}$  and  $\text{ext } L_a \notin \widehat{\text{ext}} \mathcal{L}^*(\text{reg})$ , the inclusions in (1) and (2) are strict. With  $L'_a \in \mathcal{L}$  and  $\lim L'_a \notin \widehat{\lim} \mathcal{L}^*(\text{reg})$ , the inclusions in (4) and (5) are strict.



# Kleene closure

$$\text{Kleene}'(\mathcal{L}) := \left\{ \bigcup_{i=1}^n U_i \cdot V_i^\omega \mid U_i, V_i \subseteq \Sigma^*, U_i \cdot V_i^* \in \mathcal{L}, n \in \mathbb{N}_0 \right\}$$

► **Lemma 3.24.**

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► **Lemma 3.24.**

- Generic power-set construction based on a non-det.  $UV^*$  automaton which results in a det. co-Büchi automaton for parts of  $UV^\omega$ .

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- **Lemma 3.25.**  $\mathcal{L}$  closed under change of final states. Then

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# Congruence based classes $\mathcal{L}(R)$

Motivation:  $\mathcal{L}(\text{LT}_n)$  or  $\mathcal{L}(\text{PT}_n)$

Let  $R \subseteq \Sigma^* \times \Sigma^*$  be a congruence relation.

$\mathcal{L}^*(R) := \{L \subseteq \Sigma^* \mid L \text{ is finite union of } R\text{-equivalence-classes}\}.$

There is a canonical deterministic automaton with states  $S_R := \Sigma^*/R$ . We call it the  $R$ -automaton.

► Lemma 3.28.  $\mathcal{L}(R)$  is *closed under change of final states*.

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- ▶ Lemma 3.28.  $\mathcal{L}(R)$  is *closed under change of final states*.
- ▶ Lemma 3.28.  $\mathcal{L}(R)$  is *closed under negation, union and intersection*.



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- ▶ **Lemma 3.33.**  $\text{BC lim } \mathcal{L}(R) \cap \text{ext } \mathcal{L}^*(\text{reg}) \subseteq \text{ext } \mathcal{L}(R)$   
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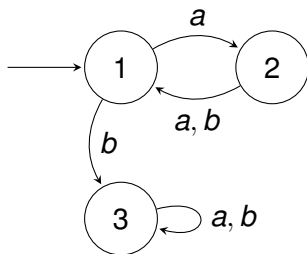
## General results: $\text{BC lim } \mathcal{L}(R)$ in $\mathcal{L}^\omega(\text{reg})$

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  - ▶  $\mathcal{L}(\text{LT}_n)$  for  $n \geq 2$  is postfix-loop-deterministic (Lemma 4.14)
- ▶ **Theorem 3.44.**  $\mathcal{L}(R)$  is not *postfix-loop-deterministic*  $\Leftrightarrow$

$$\text{BC lim } \mathcal{L}(R) \cap \text{lim } \mathcal{L}^*(\text{reg}) = \text{lim } \mathcal{L}(R).$$

## General results: $\mathcal{L}(R)$ : Staiger-Wagner

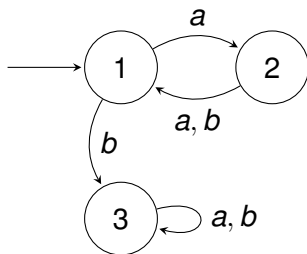
- **Example 3.46.** There is  $\mathcal{L}(R)$  infinity-postfix-independent and not postfix-loop-deterministic and  $\text{ext } \mathcal{L}(R) \not\subseteq \lim \mathcal{L}(R)$ .



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We have  $\text{ext } L_2 = a\Sigma^\omega \notin \text{BC } \lim \mathcal{L}(R)$ .

- **Theorem 3.47.** (Staiger-Wagner)  $\mathcal{L}(R)$  not postfix-loop-deterministic.  $\text{BC } \text{ext } \mathcal{L}(R) \subseteq \text{BC } \lim \mathcal{L}(R)$ .  
Then

$$\lim \cap \widehat{\lim} \mathcal{L}(R) = \text{BC } \text{ext } \mathcal{L}(R)$$

Example:  $\mathcal{L}(\text{PT}_n)$

# Concrete results

For  $\mathcal{L} := \mathcal{L}(\text{starfree})$ , via Theorem 3.22, we get

$$\begin{aligned}\text{ext} \cap \widehat{\text{ext}} \mathcal{L} &\subsetneq \text{ext} \cup \widehat{\text{ext}} \mathcal{L} \subsetneq \text{BC ext } \mathcal{L} = \\ \lim \cap \widehat{\lim} \mathcal{L} &\subsetneq \lim \cup \widehat{\lim} \mathcal{L} \subsetneq \text{BC lim } \mathcal{L}.\end{aligned}$$

For  $\mathcal{L} := \mathcal{L}(\text{LT})$  or  $\mathcal{L} := \mathcal{L}(\text{LTT})$ , via Theorem 3.22, we get

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- ▶ Another possible generalization: class of  $\mathcal{L}$  automata (instead of single fixed  $R$ -automata as in  $\mathcal{L}(R)$ ). e.g.  $\bigcup_n \text{PT}_n$  – automata.
- ▶ More concrete language classes can be studied. Supersets of the class of regular languages weren't studied at all here. Natural generalization would be to use pushdown automata in the proofs for the class of context free languages.