Language Operations and a Structure Theory of ω-Languages

July 30, 2012

Introduction: $\mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*)$

We have the common $\mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*)$ language operators:

1. $\mathrm{ext}(L) := \{\alpha \in \Sigma^\omega \mid \exists n \colon \alpha[0, n] \in L\} = L \cdot \Sigma^\omega$
2. $\widehat{\mathrm{ext}}(L) := \{\alpha \in \Sigma^\omega \mid \forall n \colon \alpha[0, n] \in L\}$
3. $\lim(L) := \{\alpha \in \Sigma^\omega \mid \forall N \colon \exists n > N \colon \alpha[0, n] \in L\} =$ $\{\alpha \in \Sigma^\omega \mid \exists^\omega n \colon \alpha[0, n] \in L\}$
4. $\widehat{\lim}(L) := \{\alpha \in \Sigma^\omega \mid \exists N \colon \forall n > N \colon \alpha[0, n] \in L\}$

Introduction: $\mathcal{P}(\mathcal{P}(\Sigma^*)) \to \mathcal{P}(\mathcal{P}(\Sigma^*))$

From these, define language class operators:

1. $\mathrm{ext}(\mathcal{L}) := \{\lim L \mid L \in \mathcal{L}\}$
2. $\widehat{\mathrm{ext}}(\mathcal{L}) := \left\{\widehat{\mathrm{ext}}\, L \,\middle|\, L \in \mathcal{L}\right\}$
3. $\lim(\mathcal{L}) := \{\lim L \mid L \in \mathcal{L}\}$
4. $\widehat{\lim}(\mathcal{L}) := \left\{\widehat{\lim}\, L \,\middle|\, L \in \mathcal{L}\right\}$

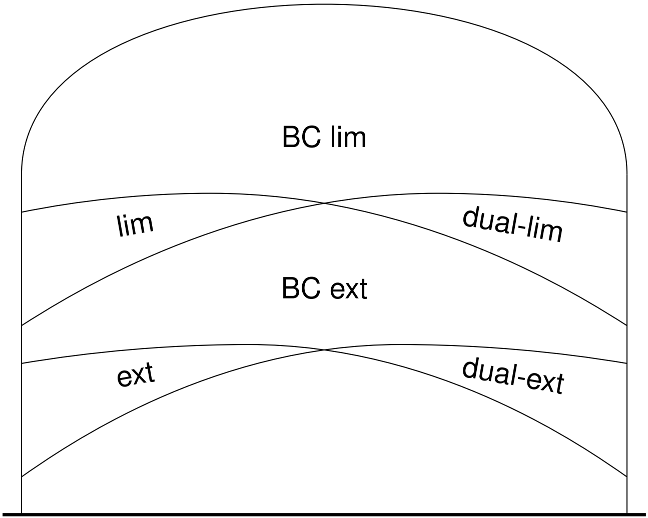
We combine these operators via union or intersection, e.g.

$$\mathrm{ext} \cup \widehat{\mathrm{ext}}\, \mathcal{L} := \mathrm{ext}\, \mathcal{L} \cup \widehat{\mathrm{ext}}\, \mathcal{L}.$$

Or boolean combinations:

1. $\mathrm{BC}\,\mathrm{ext}\,\mathcal{L} = \mathrm{BC}(\mathrm{ext}(\mathcal{L}))$
2. $\mathrm{BC}\lim\mathcal{L} = \mathrm{BC}(\lim(\mathcal{L}))$

$\mathcal{L}^*(\mathrm{reg})$ inclusion diagram



Questions

- instead of the class of regular $*$-languages, look at other $*$-language classes, e.g. starfree, LT, PT, or any arbitrary $*$-language class $\mathcal{L}$
- does it result in the same relations as in the diagram? are the enclosures strict?

My Diplom thesis:

- Chapter 3: general results on arbitrary $\mathcal{L}$, given some introduced properties on $\mathcal{L}$
- Chapter 4: concrete $*$-language classes

Properties on $\mathcal{L}$

Let $L, A, B \in \mathcal{L}$.

1. **Closure under suffix-independence**: $L \cdot \Sigma^* \in \mathcal{L}$
2. **Closure under union, intersection**: $A \cup B \in \mathcal{L}$, $A \cap B \in \mathcal{L}$
3. **Closure under negation**: $-L \in \mathcal{L}$
4. **Closure under change of final states**: Let $\mathcal{A}_L = (Q, \Sigma, q_0, \delta, F_L)$ be the minimal deterministic automaton for L, i.e. with $L^*(\mathcal{A}_L) = L$. Then, for all $F' \subseteq Q$, we have $L^*((Q, \Sigma, q_0, \delta, F')) \in \mathcal{L}$.
5. **Closure under alphabet permutation**: For all permutations $\sigma : \Sigma \to \Sigma$, we have $L_\sigma := \{\sigma(w) \mid w \in L\} \in \mathcal{L}$

General results

- Lemma 3.3: Closure under suffix-independence $\Rightarrow$ $\mathrm{ext}\,\mathcal{L} \subseteq \lim \cap \widehat{\lim}\,\mathcal{L}$ (but $\not\Leftarrow$)
- Lemma 3.8: Closure under suffix-independence and negation $\Rightarrow$ $\mathrm{ext} \cup \widehat{\mathrm{ext}}\,\mathcal{L} \subseteq \lim \cap \widehat{\lim}\,\mathcal{L}$

Separating language for $\mathrm{ext} \cup \widehat{\mathrm{ext}} \subsetneq \mathrm{BC}\,\mathrm{ext}$, $\lim \cup \widehat{\lim} \subsetneq \mathrm{BC}\lim$:
$\Sigma := \{a, b, c\}$, $L_a := \Sigma^* a$, $L_b := \Sigma^* b$.
$\tilde{L}_1 := \mathrm{ext}\,L_a \cap -\mathrm{ext}\,L_b$, $\tilde{L}_2 := \lim L_a \cap -\lim L_b$.
$\tilde{L}_1 \notin \mathrm{ext} \cup \widehat{\mathrm{ext}}\,\mathcal{L}$ but $\tilde{L}_1 \in \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}$.
$\tilde{L}_2 \notin \lim \cup \widehat{\lim}\,\mathcal{L}$ but $\tilde{L}_2 \in \mathrm{BC}\lim\mathcal{L}$.
More general:

General results

Definition 3.12. A language $L \subseteq \Sigma^* \cup \Sigma^\omega$ is called ***M*-invariant** for $M \subseteq \Sigma$ iff for all $w \in \Sigma^* \cup \Sigma^\omega$,

$$w \in L \quad \Leftrightarrow \quad w|_M \in L,$$

where $w|_M$ is the word w with all letters from M removed. There is always exactly one **maximum invariant alphabet set** $M_m \subseteq \Sigma$ **of** L such that L is M_m-invariant. Then call $\Sigma - M_m$ the **non-invariant alphabet set of** L.

Theorem 3.15. Let $\mathcal{L}$ be closed under negation and under alphabet permutation. Let $\{a, b, c\} \subseteq \Sigma$. Let there be $L_a \in \mathcal{L}$. Let $\{a\}$ be the *non-invariant alphabet set of* L_a and let L_a be $\{b, c\}$*-invariant*. Then

$$\operatorname{ext} L_a \notin \widehat{\operatorname{ext}}\, \mathcal{L}^*(\mathrm{reg}) \quad \Rightarrow \quad \operatorname{ext} \cup \widehat{\operatorname{ext}}\, \mathcal{L} \subsetneqq \operatorname{BC} \operatorname{ext} \mathcal{L}$$

and

$$\lim L_a \notin \widehat{\lim}\, \mathcal{L}^*(\mathrm{reg}) \quad \Rightarrow \quad \lim \cup \widehat{\lim}\, \mathcal{L} \subsetneqq \operatorname{BC} \lim \mathcal{L}.$$

General results

- **Theorem 3.19.** (Staiger-Wagner 1) $\mathcal{L}$ closed under change of final states. Then
$$\lim \cap \widehat{\lim}\, \mathcal{L} \subseteq \mathrm{BC}\,\mathrm{ext}\,\mathcal{L}.$$
- **Theorem 3.20.** (Staiger-Wagner 2) $\mathcal{L}$ closed under suffix-independence, negation, union and change of final states. Then
$$\mathrm{BC}\,\mathrm{ext}\,\mathcal{L} \subseteq \lim \cap \widehat{\lim}\, \mathcal{L}.$$
- **Theorem 3.22.** $\mathcal{L}$ closed under suffix-independence, negation, union, change of final states and alphabet permutation. Then we have
$$\mathrm{ext} \cap \widehat{\mathrm{ext}}\, \mathcal{L} \overset{(1.)}{\subseteq} \mathrm{ext} \cup \widehat{\mathrm{ext}}\, \mathcal{L} \overset{(2.)}{\subseteq} \mathrm{BC}\,\mathrm{ext}\,\mathcal{L} \overset{(3.)}{=}$$
$$\lim \cap \widehat{\lim}\, \mathcal{L} \overset{(4.)}{\subseteq} \lim \cup \widehat{\lim}\, \mathcal{L} \overset{(5.)}{\subseteq} \mathrm{BC}\lim \mathcal{L}.$$
With $\mathcal{L}$-ext-$\widehat{\mathrm{ext}}$-separating language L_a, the inclusions in (1) and (2) are strict. With $\mathcal{L}$-lim-$\widehat{\lim}$-separating language L'_a, the inclusions in (4) and (5) are strict.

General results: Kleene closure

$$\text{Kleene}(\mathcal{L}) := \left\{ \bigcup_{i=1}^{n} U_i \cdot V_i^\omega \;\middle|\; U_i, V_i \subseteq \Sigma^*, U_i \cdot V_i^* \in \mathcal{L}, n \in \mathbb{N}_0 \right\}$$

- **Lemma 3.24.** $\mathcal{L}$ closed under change of final states for all deterministic simplified automata. Then

  $$\text{Kleene}\,\mathcal{L} \subseteq \text{BC}\lim \mathcal{L}.$$

  (The closure of final states here is stronger.) (The idea in the proof can probably be generalized into a general non-deterministic Büchi to deterministic Muller automaton conversion.)
- **Lemma 3.25.** $\mathcal{L}$ closed under change of final states. Then

  $$\lim \mathcal{L} \subseteq \text{Kleene}\,\mathcal{L}.$$

General results: congruence based classes

Let $R \subseteq \Sigma^* \times \Sigma^*$ be a congruence relation.

$$\mathcal{L}^*(R) := \{L \subseteq \Sigma^* \mid L \text{ is finite union of } R\text{-equivalence-classes}\}\,.$$

There is a canonical deterministic automaton with states $S_R := \Sigma^*/R$. We call it the R-automaton.

- Lemma 3.28. $\mathcal{L}(R)$ is *closed under change of final states*.
- Lemma 3.28. $\mathcal{L}(R)$ is *closed under negation, union and intersection*.
- Example 3.29. *Closure under suffix-independence* doesn't directly follow from this.
- Lemma 3.30. $\mathcal{L}^\omega_E(\mathcal{A}_R) = \operatorname{ext} \mathcal{L}(R)$
- Lemma 3.31. $\mathcal{L}^\omega_{\text{Büchi}}(\mathcal{A}_R) = \lim \mathcal{L}(R)$
- Lemma 3.32. $\mathcal{L}^\omega_{\text{Muller}}(\mathcal{A}_R) = \operatorname{BC} \lim \mathcal{L}(R)$
- Lemma 3.33. $\operatorname{BC} \lim \mathcal{L}(R) \cap \operatorname{ext} \mathcal{L}^*(\mathrm{reg}) = \operatorname{ext} \mathcal{L}(R)$

General results: $\mathrm{BC}\lim \mathcal{L}(R) \cap \lim \mathcal{L}^*(\mathrm{reg}) = \lim \mathcal{L}(R)$

- **Definition 3.35.** $\mathcal{L}$ is **infinity-postfix-independent.**
  Lemma 3.36. $\mathcal{L}(R)$ is *infinity-postfix-independent* $\Leftrightarrow$ every SCC Q in the R-automata has exactly one looping subset, i.e. Q itself is the only loop in Q.
- **Lemma 3.39** $\mathcal{L}(R)$ *infinity-postfix-independent*. Then

  $$\mathrm{BC}\lim \mathcal{L}(R) \cap \lim \mathcal{L}^*(\mathrm{reg}) = \lim \mathcal{L}(R).$$

  (But $\not\Leftarrow$. Example 3.37 and 3.40.)
- **Definition 3.41.** If there is a SCC $Q \subseteq S_R$ including two loops $P_1, P_2 \subseteq Q$, $P_1 \neq P_2$ with $P_1 \not\subseteq P_2$, $P_2 \not\subseteq P_1$, then call $\mathcal{L}(R)$ **postfix-loop-deterministic.**
- **Theorem 3.44.** $\mathcal{L}(R)$ is not *postfix-loop-deterministic* $\Leftrightarrow$

  $$\mathrm{BC}\lim \mathcal{L}(R) \cap \lim \mathcal{L}^*(\mathrm{reg}) = \lim \mathcal{L}(R).$$

General results: $\mathcal{L}(R)$: Staiger-Wagner

- **Example 3.46.** There is $\mathcal{L}(R)$ infinity-postfix-independent and not postfix-loop-deterministic and

$$\operatorname{ext}\mathcal{L}(R) \not\subseteq \lim\mathcal{L}(R).$$

- **Theorem 3.47.** (Staiger-Wagner) $\mathcal{L}(R)$ not postfix-loop-deterministic. $\mathrm{BC}\operatorname{ext}\mathcal{L}(R) \subseteq \mathrm{BC}\lim\mathcal{L}(R)$. Then

$$\lim \cap \widehat{\lim}\,\mathcal{L}(R) = \mathrm{BC}\operatorname{ext}\mathcal{L}(R)$$

Concrete results

$\mathcal{L}(\text{starfree})$:

- Theorem 4.3. $\mathcal{L}^\omega(\text{FO}[<]) = \mathsf{BC}\,\mathsf{lim}\,\mathcal{L}^*(\text{FO}[<])$
- Theorem 4.4. $\mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\text{FO}[<]) \subsetneq \mathsf{BC}\,\mathsf{lim}\,\mathcal{L}^*(\text{FO}[<])$
- Lemma 4.5. $\mathcal{L}^*(\text{starfree})$ closed under change of final states.

$\mathcal{L}(\text{dot-depth-}n)$:

- $\mathsf{ext} \cap \widehat{\mathsf{ext}}\,\mathcal{L} \subsetneq \mathsf{ext} \cup \widehat{\mathsf{ext}}\,\mathcal{L} \subsetneq \mathsf{BC}\,\mathsf{ext}\,\mathcal{L}$,
  $\mathsf{lim} \cap \widehat{\mathsf{lim}}\,\mathcal{L} \subsetneq \mathsf{lim} \cup \widehat{\mathsf{lim}}\,\mathcal{L} \subsetneq \mathsf{BC}\,\mathsf{lim}\,\mathcal{L}$.
- Lemma 4.6. $\mathcal{L}(\text{dot-depth-0})$ closed under change of final states and we have
  $\mathsf{ext}\,\mathcal{L}(\text{dot-depth-0}) = \widehat{\mathsf{ext}}\,\mathcal{L}(\text{dot-depth-0})$,
  $\mathsf{lim}\,\mathcal{L}(\text{dot-depth-0}) = \widehat{\mathsf{lim}}\,\mathcal{L}(\text{dot-depth-0})$.
- Lemma 4.7.
  $\mathsf{BC}\,\mathsf{ext}\,\mathcal{L}(\text{dot-depth-0}) = \mathsf{lim} \cap \widehat{\mathsf{lim}}\,\mathcal{L}(\text{dot-depth-0})$

Concrete results

$\mathcal{L}(\mathrm{PT})$:

- Theorem 4.9. $\mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\mathrm{PT}) = \mathsf{BC}\,\mathsf{lim}\,\mathcal{L}^*(\mathrm{PT})$
- Lemma 4.10. $\mathcal{L}(\mathrm{PT}_n)$ closed under suffix-independence.
- For $\mathcal{L} = \mathcal{L}(\mathrm{PT}_n)$ and $\mathcal{L}(\mathrm{PT})$: $\mathsf{ext} \cap \widehat{\mathsf{ext}}\,\mathcal{L} \subsetneqq \mathsf{ext} \cup \widehat{\mathsf{ext}}\,\mathcal{L} \subsetneqq \mathsf{BC}\,\mathsf{ext}\,\mathcal{L} = \mathsf{lim} \cap \widehat{\mathsf{lim}}\,\mathcal{L} = \mathsf{lim} \cup \widehat{\mathsf{lim}}\,\mathcal{L} = \mathsf{BC}\,\mathsf{lim}\,\mathcal{L}$
- Theorem 4.11. $\mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\mathrm{pos\text{-}PT}) = \mathsf{BC}\,\mathsf{lim}\,\mathcal{L}^*(\mathrm{pos\text{-}PT})$
- Lemma 4.12. $\mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\mathrm{pos\text{-}PT}) = \mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\mathrm{PT})$

$\mathcal{L}(\mathrm{LT})$:

- Theorem 4.13. $\mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\mathrm{LT}) \subsetneqq \mathsf{BC}\,\mathsf{lim}\,\mathcal{L}^*(\mathrm{LT})$
- Lemma 4.14. $\mathcal{L}(\mathrm{LT}_n)$ is *postfix-loop-deterministic* and not *infinity-postfix-independent* for $n \geq 2$.
- $\mathsf{BC}\,\mathsf{lim}\,\mathcal{L}(\mathrm{LT}_n) \cap \mathsf{lim}\,\mathcal{L}^*(\mathrm{reg}) \supsetneqq \mathsf{lim}\,\mathcal{L}(\mathrm{LT}_n)$ for $n \geq 2$
- $\mathsf{BC}\,\mathsf{lim}\,\mathcal{L}(\mathrm{LT}_1) \cap \mathsf{lim}\,\mathcal{L}^*(\mathrm{reg}) = \mathsf{lim}\,\mathcal{L}(\mathrm{LT}_1)$

$\mathcal{L}(\mathrm{LTT})$:

- Theorem 4.15. $\mathcal{L}^\omega(\mathrm{FO}[+1]) = \mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\mathrm{FO}[+1])$

Conclusion

- Closure under change of final state or variants of this closure was important in some proofs, e.g. Staiger-Wagner or Kleene closure.
- Another possible generalization: class of $\mathcal{L}$ automata (instead of single fixed R-automata as in $\mathcal{L}(R)$). e.g. $\bigcup_n \mathrm{PT}_n -$ automata.
- More concrete language classes can be studied. Supersets of the class of regular languages weren't studied at all here. Natural generalization would be to use pushdown automata in the proofs for the class of context free languages.