Language Operations and a Structure Theory of ω-Languages

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Introduction: $\mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*)$

We have the common $\mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*)$ language operators:

1. $\mathrm{ext}(L) := \{\alpha \in \Sigma^\omega \mid \exists n\colon \alpha[0,n] \in L\} = L \cdot \Sigma^\omega$
2. $\widehat{\mathrm{ext}}(L) := \{\alpha \in \Sigma^\omega \mid \forall n\colon \alpha[0,n] \in L\}$
3. $\lim(L) := \{\alpha \in \Sigma^\omega \mid \forall N\colon \exists n > N\colon \alpha[0,n] \in L\} =$
   $\{\alpha \in \Sigma^\omega \mid \exists^\omega n\colon \alpha[0,n] \in L\}$
4. $\widehat{\lim}(L) := \{\alpha \in \Sigma^\omega \mid \exists N\colon \forall n > N\colon \alpha[0,n] \in L\}$

Introduction: $\mathcal{P}(\mathcal{P}(\Sigma^*)) \to \mathcal{P}(\mathcal{P}(\Sigma^*))$

From these, define language class operators:

1. $\mathrm{ext}(\mathcal{L}) := \{\lim L \mid L \in \mathcal{L}\}$
2. $\widehat{\mathrm{ext}}(\mathcal{L}) := \left\{\widehat{\mathrm{ext}}\, L \,\middle|\, L \in \mathcal{L}\right\}$
3. $\lim(\mathcal{L}) := \{\lim L \mid L \in \mathcal{L}\}$
4. $\widehat{\lim}(\mathcal{L}) := \left\{\widehat{\lim}\, L \,\middle|\, L \in \mathcal{L}\right\}$

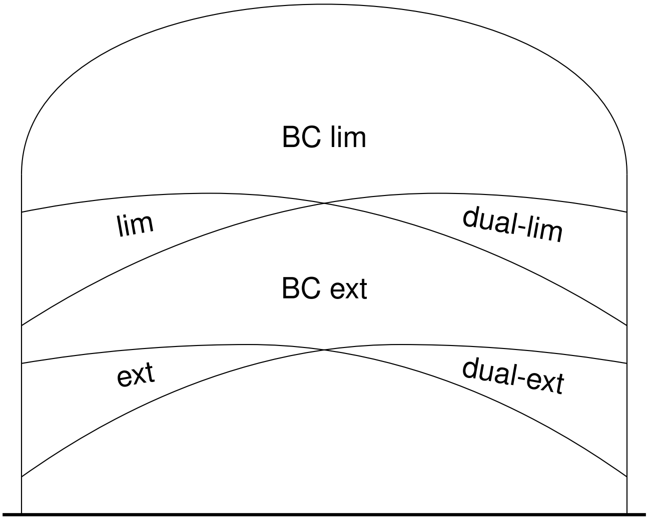
We combine these operators via union or intersection, e.g.

$$\mathrm{ext} \cup \widehat{\mathrm{ext}}\, \mathcal{L} := \mathrm{ext}\, \mathcal{L} \cup \widehat{\mathrm{ext}}\, \mathcal{L}.$$

Or boolean combinations:

1. $\mathrm{BC}\, \mathrm{ext}\, \mathcal{L} = \mathrm{BC}(\mathrm{ext}(\mathcal{L}))$
2. $\mathrm{BC} \lim \mathcal{L} = \mathrm{BC}(\lim(\mathcal{L}))$

Diagram



Questions

- instead of regular $*$-languages, look at other $*$-language classes, e.g. FO[$<$] (starfree), FO[+1], PT, positive PT, ...
- does it result in the same relations as in the diagram? are the enclosures strict?

Some results

1. $\mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\mathrm{PT}) = \mathsf{BC}\,\mathsf{lim}\,\mathcal{L}^*(\mathrm{PT})$
2. $\mathcal{L}^\omega(\mathrm{FO}[+1]) = \mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\mathrm{FO}[+1])$
3. $\mathcal{L}^\omega(\mathrm{FO}[<]) = \mathsf{BC}\,\mathsf{lim}\,\mathcal{L}^*(\mathrm{FO}[<])$
4. $\mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\mathrm{FO}[<]) \subsetneq \mathsf{BC}\,\mathsf{lim}\,\mathcal{L}^*(\mathrm{FO}[<])$
5. $\mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\mathrm{LT}) \subsetneq \mathsf{BC}\,\mathsf{lim}\,\mathcal{L}^*(\mathrm{LT})$
6. $\mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\text{pos-PT}) = \mathsf{BC}\,\mathsf{lim}\,\mathcal{L}^*(\text{pos-PT})$
7. $\mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\text{pos-PT}) = \mathsf{BC}\,\mathsf{ext}\,\mathcal{L}^*(\mathrm{PT})$

More general results

1. $\forall L \in \mathcal{L} \colon L \cdot \Sigma^* \in \mathcal{L}$
   $\Rightarrow \mathrm{ext}(\mathcal{L}^*) \subset \lim(\mathcal{L}^*)$
2. $\exists \tilde{L} \in \lim \mathcal{L}, \tilde{L} \neq \Sigma^\omega, \forall n \colon \tilde{L}[0, n] = \Sigma^n$
   $\Rightarrow \tilde{L} \notin \mathrm{BC}\,\mathrm{ext}(\mathcal{L})$
   $\Rightarrow \mathrm{BC}\,\mathrm{ext}(\mathcal{L}) \neq \mathrm{BC}\,\lim(\mathcal{L})$