

# HERMITIAN MODULAR FORMS FOR FIELDS OF LOW DISCRIMINANT

DIPLOMA THESIS  
in Mathematics

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# Chapter 1

## Introduction

We develop an algorithm to compute Fourier expansions of Hermitian modular forms of degree 2 over  $\mathrm{Sp}_2(\mathcal{O})$  for  $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ ,  $\Delta \in \{3, 4, 8\}$ .

In [PY07], spaces of Siegel modular cusp forms are calculated.

A similar algorithm is also [Rau12, Algorithm 4.3] for Jacobi forms.

We are doing the same for Hermitian modular forms.

## Chapter 2

### Background results

#### 2.1 Preliminaries

$\mathbb{N}$  denotes the set  $\{1, 2, 3, \dots\}$ ,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$  and  $\mathbb{Z}$  are all integers.  $\mathbb{Q}$  are all the rational numbers,  $\mathbb{R}$  are the real numbers and  $\mathbb{C}$  are the complex numbers.  $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$ ,  $\mathbb{R}^\times$  and  $\mathbb{C}^\times$  denotes all non-zero numbers.

Let  $M_n(R)$  be the set of all  $n \times n$  matrices over some commutative ring  $R$ . Likewise,  $M_n^T(R)$  are the symmetric  $n \times n$  matrices. A matrix  $Y \in M_n(\mathbb{C})$  is greater 0 iff  $\forall x \in \mathbb{C}^n - \{0\} : Y[x] := x^{\text{tr}} Y x \in \mathbb{R}^+$ . Such symmetric matrices are called the **positive definite matrices**, defined by  $\mathcal{P}_n(R) = \{X \in M_n^T(R) \mid X > 0\}$ . For  $A, X \in M_n(\mathbb{C})$ , we define  $A[X] := \overline{X}^{\text{tr}} A X$ . For  $Z \in M_n(\mathbb{C})$ , we call  $\Re(Z) = \frac{1}{2}(Z + \overline{Z}^{\text{tr}}) \in M_n(\mathbb{R})$  the real part and  $\Im(Z) = \frac{1}{2i}(Z - \overline{Z}^{\text{tr}}) \in M_n(\mathbb{R})$  the imaginary part of  $Z$  and we have  $Z = \Re(Z) + i\Im(Z)$ .

The **general linear group** is defined by  $\text{GL}_n(R) = \{X \in M_n(R) \mid \det(X) \text{ is a unit in } R\}$  and the **special linear group** by  $\text{SL}_n(R) = \{X \in M_n(R) \mid \det(X) = 1\}$ . The **orthogonal group** is defined by  $\text{O}_n(R) = \{X \in \text{GL}_n(R) \mid X^T 1_n X = 1_n\}$ .

The **symplectic group** is defined by  $\text{Sp}_n(R) = \{X \in \text{GL}_{2n}(R) \mid X^T J_n X = J_n\}$  where  $J_n := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \in \text{SL}_{2n}(R)$ .  $\text{Sp}_n(R)$  is also called the **Hermitian modular group** or the **unitary group**.

Let  $\mathcal{H}_n := \{Z \in M_n^T(\mathbb{C}) \mid \Im(Z) > 0\}$  be the **Siegel upper half space**. Thus,  $\mathcal{H}_1$  is the **Poincaré upper half plane**.

A **Siegel modular cusp form** of degree  $n \in \mathbb{N}$  for some  $\Gamma \subseteq \text{Sp}_n(\mathbb{Z})$ ,  $\Gamma$  subgroup of  $\text{Sp}_n(\mathbb{Z})$ , is a holomorphic function

$$f: \mathcal{H}_n \rightarrow \mathbb{C}$$

with

- (1)  $f|_k y = f \quad \forall y \in \Gamma$
- (2) for  $n = 1$ :  $f(Z) = O(1)$  for  $Z \rightarrow i\infty$

where

$$\left(f|_k \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)(Z) = f((AZ + B)(CZ + D)^{-1}) \cdot \det(CZ + D)^{-k}$$

with  $Z \in \mathcal{H}_n$ ,  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ .

Let  $\mathbb{H}_n := \{Z \in \mathbb{M}_n(\mathbb{C}) \mid \Im(Z) > 0\}$  be the **Hermitian upper half space**.

A **Hermitian modular form** of degree  $n \in \mathbb{N}$  is a holomorphic function

$$f: \mathbb{H}_n \rightarrow \mathbb{C}$$

with weight  $k \in \mathbb{Z}$  for some  $\Gamma \subseteq \mathrm{Sp}_n(\mathcal{O})$ ,  $\Gamma$  subgroup of  $\mathrm{Sp}_n(\mathcal{O})$ ,  $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ ,  $\Delta \in \mathbb{N}$ ,  $\nu: \Gamma \rightarrow \mathbb{C}^\times$  is an abel character of  $\mathrm{Sp}_n(\mathcal{O})$ , with

- (1)  $f(M \cdot Z) = \nu(M) \det(CZ + D)^k f(Z)$ ,  $M = \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma$ ,  $Z \in \mathbb{H}_n$ ,
- (2) for  $n = 1$ :  $f$  is holomorphic in all cusps.

$[\Gamma, k, \nu]$  denotes the vector space of such hermitian modular forms.

In this work, we will concentrate on Hermitian Modular forms of degree 2. We will start with  $\Delta \in \{3, 4, 8\}$ .

Note that if  $\Delta$  is fundamental, we have

$$\begin{aligned} \mathcal{O} &= \mathbb{Z} + \mathbb{Z} \frac{-\Delta + \sqrt{-\Delta}}{2}, \\ \mathcal{O}^\# &= \mathbb{Z} \frac{i}{\sqrt{-\Delta}} + \mathbb{Z} \frac{1 + \sqrt{-\Delta}}{2}. \end{aligned}$$

From now on, we will always work with Hermitian modular forms of degree 2, i.e. we will always have  $n = 2$ , except if otherwise stated.

## Chapter 3

### Theory

**Lemma 3.1.** *Let  $f: M_2(\mathbb{C}) \rightarrow \mathbb{C}$  be a Hermitian Modular form of weight  $k$ . Let  $S \in \mathcal{P}_2(\mathbb{C})$ . Then,  $f(S\tau): \mathbb{H}_1 \subseteq \mathbb{C} \rightarrow \mathbb{C}$  is an elliptic modular form of weight  $2k$  to  $\Gamma_0(l)$ , where  $l$  is the denominator of  $S^{-1}$ .*

**Lemma 3.2.** *Prop 7.3. von Poor für herm Modulformen.  $\Gamma(\mathcal{L}) \supseteq \Gamma_0(l)$  for  $l \in \mathbb{Z}^+, ls^{-1} \in \mathcal{P}_n(\mathcal{O})$ .*

We want to calculate a generating set for the Fourier expansions of Hermitian modular forms. Now we will formulate the main algorithm of our work.

**Algorithm 3.3.** We have the Hermitian modular form degree  $n = 2$  fixed, as well as some  $\Delta$  (for now,  $\Delta \in \{3, 4, 8\}$ ). Then we select some form weight  $k \in \mathbb{Z}$  ( $k \in \{1, \dots, 20\}$  or so), some  $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$  and some subgroup  $\Gamma$  of  $\text{Sp}_2(\mathcal{O})$ . Then we select an abel character  $\nu: \Gamma \rightarrow \mathbb{C}^\times$  of  $\text{Sp}_2(\mathcal{O})$ .

We define the index set

$$\Lambda := \left\{ 0 \leq \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in M_2(\mathcal{O}^\#) \mid a, c \in \mathbb{Z} \right\}.$$

1. Select a set of matrices  $\mathcal{S} \subseteq M_2^T(\mathbb{Z})$  with  $0 < S \in \mathcal{S}$ . Make  $\mathcal{S}$  big enough. Now, for some  $S \in \mathcal{S}$ :
2. Fix  $B \in \mathbb{N}$  as a limit. Select a precision

$$\mathcal{F} := \left\{ \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \mid 0 \leq ac < B, b \in \mathcal{O}^\# \right\} \subseteq \Lambda.$$

- 3.

$$\mathcal{M}_{k,S,\mathcal{F}}^H = \{f[S] \mid f \in \mathbb{Q}^{\mathcal{F}}\},$$

$$\mathcal{M}_{k,S} = \mathcal{FE}_{\mathcal{F}(S)}(M_k(\Gamma(S)))$$

4. If

$$\dim \mathcal{M}_{k,S,\mathcal{F}}^H \cap \bigoplus_{S \in \mathcal{S}} \mathcal{M}_{k,S} = \dim M_k^H,$$

then we are ready and we can reconstruct the Fourier expansion in the following way:

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## **Chapter 4**

### **Implementation**

In this chapter, we are describing the implementation.



## **Chapter 5**

## **Conclusion**

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## Chapter 6

### References

- [PY07] C. Poor and D.S. Yuen. Computations of spaces of siegel modular cusp forms. *Journal of the Mathematical Society of Japan*, 59(1):185–222, 2007.
- [Rau12] M. Raum. Computing Jacobi Forms and Linear Equivalences of Special Divisors. *ArXiv e-prints*, December 2012.