Hermitian Modular Forms for Fields of Low Discriminant

Diploma Thesis
in Mathematics

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Chapter 1

Introduction

We develop an algorithm to compute Fourier expansions of Hermitian modular forms of degree 2 over $\mathrm{Sp}_2(\mathcal{O})$ for $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$, $\Delta \in \{3, 4, 8\}$.

In [**?**], spaces of Siegel modular cusp forms are calculated.

A similar algorithm is also [**?**, Algorithm 4.3] for Jacobi forms.

We are doing the same for Hermitian modular forms.

Chapter 2

Preliminaries

$\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$ are all **integers**. $\mathbb{Q}$ are all the **rational numbers**, $\mathbb{R}$ are the **real numbers** and $\mathbb{C}$ are the **complex numbers**. $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$, $\mathbb{R}^\times$ and $\mathbb{C}^\times$ denotes all non-zero numbers.

Let $\mathrm{Mat}_n(R)$ be the set of all $n \times n$ **matrices** over some commutative ring R. Likewise, $\mathrm{Mat}_n^T(R)$ are the **symmetric** $n \times n$ matrices. X^T is the **transposed** matrix of $X \in \mathrm{Mat}_n(R)$. $\overline{Z}$ is the **conjugated** matrix of $Z \in \mathrm{Mat}_n(\mathbb{C})$. A matrix $Y \in \mathrm{Mat}_n(\mathbb{C})$ is greater 0 if and only if $\forall x \in \mathbb{C}^n - \{0\} : Y[x] := \overline{x}^T Y x \in \mathbb{R}^+$. Such symmetric matrices are called the **positive definitive matrices**, defined by $\mathcal{P}_n(R) = \left\{X \in \mathrm{Mat}_n^T(R) \,\middle|\, X > 0\right\}$. For $A, X \in \mathrm{Mat}_n(\mathbb{C})$, we define $A[X] := \overline{X}^T A X$. For $Z \in \mathrm{Mat}_n(\mathbb{C})$, we call $\Re(Z) = \frac{1}{2}(Z + \overline{Z}^T) \in \mathrm{Mat}_n(\mathbb{R})$ the **real** part and $\Im(Z) = \frac{1}{2i}(Z - \overline{Z}^T) \in \mathrm{Mat}_n(\mathbb{R})$ the **imaginary** part of Z and we have $Z = \Re(Z) + i\Im(Z)$. The **denominator** of a matrix $Z \in \mathrm{Mat}_n(\mathbb{Q})$ is the smallest number $x \in \mathbb{N}$ such that $xZ \in \mathrm{Mat}_n(\mathbb{Z})$.

We say that some function $f\colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{A} \subseteq \mathrm{Mat}_n(R)$, $\mathcal{B} \subseteq R$ is k**-invariant** under some $\mathcal{X} \subseteq \mathrm{Mat}_n(R)$ where $\mathcal{A}[\mathcal{X}] \subseteq \mathcal{A}$ if and only if $\det(U)^k f(T[U]) = f(T)$ for all $T \in \mathcal{A}$, $U \in \mathcal{X}$.

Let S be a set with G-action. Then the set of G-invariants S^G is the set of all $s \in S$ satisfying $gs = s$ for all G. We can equip the set of functions $\mathcal{F} \to \mathbb{C}$ with the action $(gf)(T) = det(g)^k f(T[g])$ and this lead to the definition that we need.

The **general linear group** is defined by $\mathrm{GL}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid \det(X) \text{ is a unit in } R\}$ and the **special linear group** by $\mathrm{SL}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid \det(X) = 1\}$. The **orthogonal group** is defined by $\mathrm{O}_n(R) = \left\{X \in \mathrm{GL}_n(R) \,\middle|\, X^T 1_n X = 1_n\right\}$.

For $R \subseteq \mathbb{C}$, $\overline{R} \subseteq R$, the set of **hermitian matrices** in R is defined as $\mathrm{Her}_n(R) := \left\{Z \in \mathrm{Mat}_n(R) \,\middle|\, \overline{Z}^T = Z\right\}$. The **symplectic group** is defined by $\mathrm{Sp}_n(R) = \left\{X \in \mathrm{GL}_{2n}(R) \,\middle|\, \overline{X}^T J_n X = J_n\right\}$ where $J_n := \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right) \in \mathrm{SL}_{2n}(R)$. $\mathrm{Sp}_n(R)$ is also called the **unitary group**.

2.1 Siegel modular forms

Let $\mathcal{H}_n := \left\{Z \in \mathrm{Mat}_n^T(\mathbb{C}) \,\middle|\, \Im(Z) > 0\right\}$ be the **Siegel upper half space**. Thus, $\mathcal{H}_1$ is the **Poincaré upper half plane**. We call $\mathrm{Sp}_n(\mathbb{Z})$ the **Siegel modular group**.

A **Siegel modular cusp form** of degree $n \in \mathbb{N}$ for some $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{Z})$, Γ subgroup of $\mathrm{Sp}_n(\mathbb{Z})$, is a holomorphic function

$$f\colon \mathcal{H}_n \to \mathbb{C}$$

with

(1) $f|_k y = f \ \ \forall\, y \in \Gamma$

(2) for $n = 1$: $f(Z) = O(1)$ for $Z \to i\infty$

where

$$\left(f|_k \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right)(Z) = f((AZ+B)(CZ+D)^{-1}) \cdot \det(CZ+D)^{-k}$$

with $Z \in \mathcal{H}_n$, $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$.

2.2 Elliptic modular forms

$\Gamma_0(l)$

2.3 Hermitian modular forms

Let $\mathbb{H}_n := \{Z \in \mathrm{Mat}_n(\mathbb{C}) \mid \Im(Z) > 0\}$ be the **Hermitian upper half space**.

Let $\Delta \in \mathbb{N}$ so that we have the field $\mathbb{Q}(\sqrt{-\Delta})$. Then, let $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ be the maximum order. We call $\mathrm{Sp}_n(\mathcal{O})$ the **Hermitian modular group**. Let Γ be a subgroup of $\mathrm{Sp}_n(\mathcal{O})$. Let $\nu\colon \Gamma \to \mathbb{C}^\times$ be an abel character of $\mathrm{Sp}_n(\mathcal{O})$.

A **Hermitian modular form** of degree $n \in \mathbb{N}$ with weight $k \in \mathbb{Z}$ over Γ and ν is a holomorphic function

$$f\colon \mathbb{H}_n \to \mathbb{C}$$

with

(1) $f(M \cdot Z) = \nu(M) \det(CZ+D)^k f(Z), \quad M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma, Z \in \mathbb{H}_n,$

(2) for $n = 1$: f is holomorphic in all cusps.

$[\Gamma, k, \nu]$ denotes the vector space of such Hermitian modular forms.

In this work, we will concentrate on Hermitian modular forms of degree 2. We will start with $\Delta \in \{3, 4, 8\}$.

Note that if Δ is fundamental (see [**?**]), we have

$$\begin{aligned}\mathcal{O} &= \mathbb{Z} + \mathbb{Z}\frac{-\Delta + \sqrt{-\Delta}}{2},\\ \mathcal{O}^{\#} &= \mathbb{Z}\frac{i}{\sqrt{-\Delta}} + \mathbb{Z}\frac{1 + \sqrt{-\Delta}}{2}.\end{aligned}$$

From now on, we will always work with Hermitian modular forms of degree 2, i.e. we will always have $n = 2$, except if otherwise stated.

Chapter 3

Theory

Lemma 3.1. *Let $f\colon \mathbb{H}_2 \to \mathbb{C}$ be a Hermitian modular form of weight k. Let $S \in \mathcal{P}_2(\mathbb{C})$. Then, $f(S\tau)\colon \mathbb{H}_1 \subseteq \mathbb{C} \to \mathbb{C}$ is an elliptic modular form of weight $2k$ to $\Gamma_0(l)$, where l is the denominator of S^{-1}.*

Lemma 3.2. *Prop 7.3. von Poor für herm Modulformen. $\Gamma(\mathcal{L}) \supseteq \Gamma_0(l)$ for $l \in \mathbb{Z}^+, ls^{-1} \in \mathcal{P}_n(\mathcal{O})$.*

We want to calculate a generating set for the Fourier expansions of Hermitian modular forms. Now we will formulate the main algorithm of our work.

Algorithm 3.3. We have the Hermitian modular form degree $n = 2$ fixed, as well as some Δ (for now, $\Delta \in \{3, 4, 8\}$). Then we select some form weight $k \in \mathbb{Z}$ ($k \in \{1, \ldots, 20\}$ or so), some $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ and some subgroup Γ of $\mathrm{Sp}_2(\mathcal{O})$. Then we select an abel character $\nu\colon \Gamma \to \mathbb{C}^\times$ of $\mathrm{Sp}_2(\mathcal{O})$.

We define the index set

$$\Lambda := \left\{ 0 \leq \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \mathrm{Mat}_2(\mathcal{O}^\#) \;\middle|\; a, c \in \mathbb{Z} \right\}.$$

Fix $B \in \mathbb{N}$ as a limit. Select a precision

$$\mathcal{F} := \left\{ \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \;\middle|\; 0 \leq a, c < B, b \in \mathcal{O}^\# \right\} \subseteq \Lambda.$$

1. We start with $l = 1$ and increase it but only use the square-free numbers.

2. Set $\mathcal{S} = \{\}$,

3. Enumerate matrices $S \in \mathrm{Mat}_2^T(\mathbb{Z})$, and set $\mathcal{S} \leftarrow \mathcal{S} \cup \{S\}$ and for each time you add a new matrix perform the following steps.

4. We set

$$\mathcal{M}^H_{k,\mathcal{S},\mathcal{F}} := \left\{ (f[S])_{S\in\mathcal{S}} \,\middle|\, f \in \mathbb{Q}^{\mathcal{F}} \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant} \right\} \subseteq \bigoplus_{S\in\mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)},$$

   where

$$f[S] := \mathbb{H}_1 \to \mathbb{Q}, \tau \mapsto f(S\tau),$$

   and

$$\mathcal{M}_{k,\mathcal{S},\mathcal{F}} := \bigoplus_{S\in\mathcal{S}} \mathcal{FE}_{\mathcal{F}(S)}(M_k(\Gamma(l_S)))$$

   where M_k is the vectorspace of elliptic modular forms.

5. If

$$\dim \mathcal{M}^H_{k,\mathcal{S},\mathcal{F}} \cap \mathcal{M}_{k,\mathcal{S},\mathcal{F}} = \dim[\Gamma, k, \nu],$$

   then we are ready and we can reconstruct the Fourier expansion in the following way:

   ...

   If not, then return to step **??**, and enlarge $\mathcal{S}$.

Chapter 4

Implementation

In this chapter, we are describing the implementation.

Chapter 5

Conclusion

Blub

Chapter 6

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