Hermitian Modular Forms for Fields of Low Discriminant

Diploma Thesis
in Mathematics

by
Albert Zeyer

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Supervisor: Prof. Dr. Aloys Krieg
Second examiner: Dr. Martin Raum

written at the
Lehrstuhl A für Mathematik
Prof. Dr. A. Krieg

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Chapter 1

Introduction

We develop an algorithm to compute Fourier expansions of Hermitian Modular Forms of degree 2 over $\mathrm{Sp}_2(\mathcal{O})$ for $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$, $\Delta \in \{3, 4, 8\}$.

In [PY07], spaces of Siegel Modular Cusp forms are calculated.

A similar algorithm is also [Rau12, Algorithm 4.3] for Jacobi forms.

We are doing the same for Hermitian Modular forms.

Chapter 2

Background results

2.1 Preliminaries

$\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$ are all integers. $\mathbb{Q}$ are all the rational numbers, $\mathbb{R}$ are the real numbers and $\mathbb{C}$ are the complex numbers. $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$, $\mathbb{R}^\times$ and $\mathbb{C}^\times$ denotes all non-zero numbers.

Let $\mathrm{M}_n(R)$ be the set of all $n \times n$ matrices over some commutative ring R. Likewise, $\mathrm{M}_n^T(R)$ are the symmetric $n \times n$ matrices. A matrix $Y \in \mathrm{M}_n(\mathbb{C})$ is greater 0 iff $\forall x \in \mathbb{C}^n - \{0\} : Y[x] := \overline{x}^{\mathrm{tr}} Y x \in \mathbb{R}^+$. Such matrices are called the **positive definitive matrices**, defined by $\mathcal{P}_n(R) = \{X \in \mathrm{M}_n(R) \mid X > 0\}$. For $M, X \in \mathrm{M}_n(\mathbb{C})$, we define $M[X] := \overline{X}^{\mathrm{tr}} M X$. For $Z \in \mathrm{M}_n(\mathbb{C})$, we call $\Re(Z) = \frac{1}{2}(Z + \overline{Z}^{\mathrm{tr}}) \in \mathrm{M}_n(\mathbb{R})$ the real part and $\Im(Z) = \frac{1}{2i}(Z - \overline{Z}^{\mathrm{tr}}) \in \mathrm{M}_n(\mathbb{R})$ the imaginary part of Z and we have $Z = \Re(Z) + i\Im(Z)$.

The **general linear group** is defined by $\mathrm{GL}_n(R) = \{X \in \mathrm{M}_n(R) \mid \det(X) \text{ is a unit in } R\}$ and the **special linear group** by $\mathrm{SL}_n(R) = \{X \in \mathrm{M}_n(R) \mid \det(X) = 1\}$. The **orthogonal group** is defined by $\mathrm{O}_n(R) = \left\{X \in \mathrm{GL}_n(R) \,\middle|\, X^T 1_n X = 1_n\right\}$.

The **symplectic group** is defined by $\mathrm{Sp}_n(R) = \left\{X \in \mathrm{GL}_{2n}(R) \,\middle|\, X^T J_n X = J_n\right\}$ where $J_n := \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right) \in \mathrm{SL}_{2n}(R)$. $\mathrm{Sp}_n(R)$ is also called the **Hermitian modular group** or the **unitary group**.

Let $\mathcal{H}_n := \left\{Z \in \mathrm{M}_n^T(\mathbb{C}) \,\middle|\, \Im(Z) > 0\right\}$ be the **Siegel upper half space**. Thus, $\mathcal{H}_1$ is the **Poincaré upper half plane**.

A **Siegel modular cusp form** of degree $n \in \mathbb{N}$ for some $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{Z})$, Γ subgroup of $\mathrm{Sp}_n(\mathbb{Z})$, is a holomorphic function

$$f \colon \mathcal{H}_n \to \mathbb{C}$$

with

(1) $f|_k y = f \;\; \forall\, y \in \Gamma$

(2) for $n = 1$: $f(Z) = O(1)$ for $Z \to i\infty$

where

$$\left(f|_k \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)(Z) = f((AZ+B)(CZ+D)^{-1}) \cdot \det(CZ+D)^{-k}$$

with $Z \in \mathcal{H}_n$, $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$.

Let $\mathbb{H}_n := \{Z \in \mathrm{M}_n(\mathbb{C}) \mid \Im(Z) > 0\}$ be the **Hermitian upper half space**.

A **Hermitian modular form** of degree $n \in \mathbb{N}$ is a holomorphic function

$$f \colon \mathbb{H}_n \to \mathbb{C}$$

with weight $k \in \mathbb{Z}$ for some $\Gamma \subseteq \mathrm{Sp}_n(\mathcal{O})$, Γ subgroup of $\mathrm{Sp}_n(\mathcal{O})$, $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$, $\Delta \in \mathbb{N}$, $\nu \colon \Gamma \to \mathbb{C}^\times$ is an abel character of $\mathrm{Sp}_n(\mathcal{O})$, with

(1) $f(M \cdot Z) = \nu(M) \det(CZ + D)^k f(Z), \quad M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma, Z \in \mathbb{H}_n,$

(2) for $n = 1$: f is holomorphic in all cusps.

$[\Gamma, k, \nu]$ denotes the vector space of such hermitian modular forms.

In this work, we will concentrate on Hermitian Modular forms of degree 2. We will start with $\Delta \in \{3, 4, 8\}$.

Note that if Δ is fundamental, we have

$$\begin{aligned} \mathcal{O} &= \mathbb{Z} + \mathbb{Z}\frac{-\Delta + \sqrt{-\Delta}}{2}, \\ \mathcal{O}^\# &= \mathbb{Z}\frac{i}{\sqrt{-\Delta}} + \mathbb{Z}\frac{1 + \sqrt{-\Delta}}{2}. \end{aligned}$$

From now on, we will always work with Hermitian modular forms of degree 2, i.e. we will always have $n = 2$, except if otherwise stated.

Chapter 3

Theory

Lemma 3.1. *Let* $f\colon \mathrm{M}_2(\mathbb{C}) \to \mathbb{C}$ *be a Hermitian Modular form of weight* k*. Let* $S \in \mathcal{P}_2(\mathbb{C})$*. Then,* $f(S\tau)\colon \mathbb{H}_1 \subseteq \mathbb{C} \to \mathbb{C}$ *is an elliptic modular form of weight* $2k$ *to* $\Gamma_0(l)$*, where* l *is the denominator of* S^{-1}*.*

Lemma 3.2. *Prop 7.3. von Poor für herm Modulformen.* $\Gamma(\mathcal{L}) \supseteq \Gamma_0(l)$ *for* $l \in \mathbb{Z}^+, ls^{-1} \in \mathcal{P}_n(\mathcal{O})$*.*

Now we will formulate the main algorithm of our work.

Algorithm 3.3.

1. Select a set of matrices $\mathcal{S} \subseteq \mathrm{M}_2^T(\mathbb{Z})$ with $0 < S \in \mathcal{S}$. Make $\mathcal{S}$ big enough. Now, for some $S \in \mathcal{S}$:

2. Fix $B \in \mathbb{N}$ as a limit. Or select a precision

$$\mathcal{F} := \left\{ \left(\begin{array}{cc} a & b \\ \bar{b} & c \end{array} \right) \middle| \, 0 \le ac < B, b \in \mathcal{O}^{\#} \right\} \subseteq \Lambda,$$

where

$$\Lambda := \left\{ 0 \le \left(\begin{array}{cc} a & b \\ \bar{b} & c \end{array} \right) \in \mathrm{M}_2(\mathcal{O}^{\#}) \, \middle| \, a, c \in \mathbb{Z} \right\}.$$

3.

$$\mathcal{M}^H_{k,\mathcal{S},\mathcal{F}} = \left\{ f[S] \,\middle|\, f \in \mathbb{Q}^{\mathcal{F}} \right\},$$

$$\mathcal{M}_{k,S} = \mathcal{FE}_{\mathcal{F}(S)}(\mathrm{M}_k(\Gamma(S)))$$

4. If

$$\dim \mathcal{M}^H_{k,\mathcal{S},\mathcal{F}} \cap \bigoplus_{S \in \mathcal{S}} \mathcal{M}_{k,S} = \dim M^H_k,$$

then we are ready and we can reconstruct the Fourier expansion in the following way:

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Chapter 4

Implementation

In this chapter, we are describing the implementation.

Chapter 5

Conclusion

Blub

Chapter 6

References

[PY07] C. Poor and D.S. Yuen. Computations of spaces of siegel modular cusp forms. *Journal of the Mathematical Society of Japan*, 59(1):185–222, 2007.

[Rau12] M. Raum. Computing Jacobi Forms and Linear Equivalences of Special Divisors. *ArXiv e-prints*, December 2012.