HERMITIAN MODULAR FORMS FOR FIELDS OF LOW DISCRIMINANT

DIPLOMA THESIS
in Mathematics

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Chapter 1

Introduction

We develop an algorithm to compute Fourier expansions of Hermitian modular forms of degree 2 over $\mathrm{Sp}_2(\mathcal{O})$ for $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$, $\Delta \in \{3, 4, 8\}$.

In [PY07], spaces of Siegel modular cusp forms are calculated.

A similar algorithm is also [Rau12, Algorithm 4.3] for Jacobi forms.

We are doing the same for Hermitian modular forms.

Along with the theoretical work, the algorithm has also been implemented. The implementation has been done with the Sage ([S$^+$13]) framework. It is implemented in C++ ([Str83]), Cython ([BBS$^+$13]) and Python ([vR13]). The code can be found on GitHub ([Zey13a]) and another backup might be on [Zey13b].

Chapter 2

Preliminaries

$\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$ are all **integers**. $\mathbb{Q}$ are all the **rational numbers**, $\mathbb{R}$ are the **real numbers** and $\mathbb{C}$ are the **complex numbers**. $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$, $\mathbb{R}^\times$ and $\mathbb{C}^\times$ denotes all non-zero numbers.

Let $\mathrm{Mat}_n(R)$ be the set of all $n \times n$ **matrices** over some commutative ring R. Likewise, $\mathrm{Mat}_n^T(R)$ are the **symmetric** $n \times n$ matrices. X^T is the **transposed** matrix of $X \in \mathrm{Mat}_n(R)$. $\overline{Z}$ is the **conjugated** matrix of $Z \in \mathrm{Mat}_n(\mathbb{C})$. For $R \subseteq \mathbb{C}$, $\overline{R} \subseteq R$, the set of **Hermitian matrices** in R is defined as

$$\mathrm{Her}_n(R) = \left\{ Z \in \mathrm{Mat}_n(R) \,\middle|\, \overline{Z}^T = Z \right\}.$$

A matrix $Y \in \mathrm{Mat}_n(\mathbb{C})$ is greater 0 if and only if $\forall x \in \mathbb{C}^n - \{0\} : Y[x] := \overline{x}^T Y x \in \mathbb{R}^+$. Such matrices are called the **positive definitive matrices**, defined by

$$\mathcal{P}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid X > 0\}$$

for $R \subseteq \mathbb{C}$. Note that $\mathcal{P}_n(R) \subseteq \mathrm{Her}_n(R)$, i.e. all positive definite matrices are Hermitian. For a matrix over $\mathbb{R}$, it means that it is also symmetric.

For $A, X \in \mathrm{Mat}_n(\mathbb{C})$, we define $A[X] := \overline{X}^T A X$. The **denominator** of a matrix $Z \in \mathrm{Mat}_n(\mathbb{Q})$ is the smallest number $x \in \mathbb{N}$ such that $xZ \in \mathrm{Mat}_n(\mathbb{Z})$. We also write $\mathrm{denom}(Z) = x$. $1_n \in \mathrm{Mat}_n(\mathbb{Z})$ denotes the **identity matrix**. We use the **Gauß notation** $[a, b, c] := \left(\begin{smallmatrix} a & b \\ \overline{b} & c \end{smallmatrix}\right) \in \mathrm{Her}_n(\mathbb{C})$.

The **general linear group** is defined by

$$\mathrm{GL}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid \det(X) \text{ is a unit in } R\}$$

and the **special linear group** by

$$\mathrm{SL}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid \det(X) = 1\}.$$

The **orthogonal group** is defined by

$$\mathrm{O}_n(R) = \left\{ X \in \mathrm{GL}_n(R) \,\middle|\, X^T 1_n X = 1_n \right\} \subseteq \mathrm{GL}_n(R).$$

The **symplectic group** is defined by

$$\mathrm{Sp}_n(R) = \left\{ X \in \mathrm{GL}_{2n}(R) \,\middle|\, \overline{X}^T J_n X = J_n \right\} \subseteq \mathrm{GL}_{2n}(R) \subseteq \mathrm{Mat}_{2n}(R)$$

where $J_n := \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right) \in \mathrm{SL}_{2n}(R)$ (as in [Der01]). (Note that some authors (e.g. [PY07]) define J_n negatively.) $\mathrm{Sp}_n(R)$ is also called the **unitary group**. Note that [Der01] uses $\mathrm{U}_n(R) = \mathrm{Sp}_n(R)$. Also note that $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_1(\mathbb{Z}) \Leftrightarrow ad - bc = 1 \Leftrightarrow M \in \mathrm{SL}_2(\mathbb{Z})$. Thus, $\mathrm{Sp}_1(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$.

In addition, for a ring $R \subseteq \mathbb{C}$, define

$$\mathrm{Rot}(U) := \begin{pmatrix} \overline{U}^T & \\ & U^{-1} \end{pmatrix} \in \mathrm{Sp}_2(R), \qquad U \in \mathrm{GL}_2(R)$$

$$\mathrm{Trans}(H) := \begin{pmatrix} 1_2 & H \\ & 1_2 \end{pmatrix} \in \mathrm{Sp}_2(R), \qquad H \in \mathrm{Her}_2(R)$$

and note that we have $J_2 = \left(\begin{smallmatrix} & -1_2 \\ 1_2 & \end{smallmatrix}\right) \in \mathrm{Sp}_2(R)$. Those tree types of matrices form a generator set for the group $\mathrm{Sp}_2(R)$.

For $Z \in \mathrm{Mat}_n(\mathbb{C})$, we call

$$\Re(Z) := \frac{1}{2}\left(Z + \overline{Z}^T\right) \in \mathrm{Mat}_n(\mathbb{C})$$

the **real** part and

$$\Im(Z) := \frac{1}{2i}\left(Z - \overline{Z}^T\right) \in \mathrm{Mat}_n(\mathbb{C})$$

the **imaginary** part of Z and we have $Z = \Re(Z) + i\Im(Z)$. Note that we usually have $\Re(Z), \Im(Z) \notin \mathrm{Mat}_n(\mathbb{R})$ but we have $\Re(Z), \Im(Z) \in \mathrm{Her}_n(\mathbb{C})$.

We say that some function $f\colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{A} \subseteq \mathrm{Mat}_n(R)$, $\mathcal{B} \subseteq R$ is k-**invariant** under some $\mathcal{X} \subseteq \mathrm{Mat}_n(R)$ where $\mathcal{A}[\mathcal{X}] \subseteq \mathcal{A}$ if and only if $\det(U)^k f(T[U]) = f(T)$ for all $T \in \mathcal{A}$, $U \in \mathcal{X}$.

2.1 Elliptic modular forms

Elliptic modular forms are holomorphic functions over the set

$$\mathcal{H}_1 := \{z \in \mathbb{C} \mid \Im(z) > 0\} \subseteq \mathbb{C}$$

which is called the **Poincaré upper half plane**.

Let f be a holomorphic function $\mathcal{H}_1 \to \mathbb{C}$. **Modular forms** are functions which are invariant with regard to a specific **translation**. In this case, the translation is given by some $M \in \mathrm{Sp}_1(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$ and a **weight** $k \in \mathbb{Z}$.

Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_1(\mathbb{Z})$ and $\tau \in \mathcal{H}_1$. We write

$$M\tau := \frac{a\tau + b}{c\tau + d}.$$

Note that we have $\Im(M\tau) = \frac{\Im(\tau)}{(c\Re(\tau)+d)^2+(c\Im(\tau))^2} > 0$ and thus $M\tau \in \mathcal{H}_1$. We define the **translated function** $f|M\colon \mathcal{H}_1 \to \mathbb{C}$ as

$$(f|M)(\tau) := (c\tau + d)^{-k} \cdot f(M\tau).$$

Let Γ be a subgroup of $\mathrm{Sp}_1(\mathbb{Z})$. We also call Γ the **translation group**.

An **Elliptic modular form** with weight $k \in \mathbb{Z}$ over Γ is a holomorphic function

$$f\colon \mathcal{H}_1 \to \mathbb{C}$$

with

$$\begin{aligned} &(1)\ \ f|M = f \quad \forall\, M \in \Gamma, \\ &(2)\ \ f(\tau) = O(1) \quad \text{for } \tau \to i\infty. \end{aligned}$$

Thus, (1) yields the equation

$$f\left(\frac{a\tau + b}{b\tau + c}\right) = (c\tau + d)^k \cdot f(\tau) \quad \forall\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma, \tau \in \mathcal{H}_1.$$

$\mathcal{M}_k(\Gamma)$ denotes the vector space of such Elliptic modular forms.

In this work, we use a specific subgroup of $\mathrm{Sp}_1(\mathbb{Z})$. We define

$$\Gamma_0(l) := \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{Sp}_1(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{l} \right\} \subseteq \mathrm{Sp}_1(\mathbb{Z}) \subseteq \mathrm{Mat}_2(\mathbb{Z})$$

as a subgroup of $\mathrm{Sp}_1(\mathbb{Z})$.

An **Elliptic modular cusp form** is an Elliptic modular form $f\colon \mathcal{H}_1 \to \mathbb{C}$ with

$$\lim_{t\to\infty} f(it) = 0.$$

We can represent the cusps with $\Gamma\backslash\mathbb{Q}$.

2.2 Siegel modular forms

Siegel modular forms are a generalization of Elliptic modular forms for higher dimensions. Let

$$\mathcal{H}_n := \left\{ Z \in \mathrm{Mat}_n^T(\mathbb{C}) \,\middle|\, \Im(Z) > 0 \right\}$$

be the **Siegel upper half space**. We call $\mathrm{Sp}_n(\mathbb{Z})$ the **Siegel modular group**. Siegel modular forms are holomorphic functions $\mathcal{H}_n \to \mathbb{C}$ for a given **degree** $n \in \mathbb{N}$.

The **translation group** Γ is a subgroup of $\mathrm{Sp}_n(\mathbb{Z})$. For $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$ and $Z \in \mathcal{H}_n$, we write

$$M \cdot Z := (AZ + B) \cdot (CZ + D)^{-1}.$$

Again, we can confirm that $M \cdot Z \in \mathcal{H}_n$. Generalizing the Elliptic translation, the Siegel **translated function** $f|M \colon \mathcal{H}_n \to \mathbb{C}$ is defined as

$$(f|M)(Z) := \det(CZ + D)^{-k} \cdot f(M \cdot Z)$$

A **Siegel modular form** of degree $n \in \mathbb{N}$ with weight $k \in \mathbb{Z}$ over Γ is a holomorphic function

$$f \colon \mathcal{H}_n \to \mathbb{C}$$

with

(1) $f|M = f \quad \forall\, M \in \Gamma$,

(2) for $n = 1$: $f(Z) = O(1) \quad$ for $Z \to i\infty$

$\mathcal{M}_k^{\mathcal{H}_n}(\Gamma)$ denotes the vector space of such Siegel modular forms.

Note that Elliptic modular forms are Siegel modular forms of degree $n = 1$. Thus we have $\mathcal{M}_k(\Gamma) = \mathcal{M}_k^{\mathcal{H}_1}(\Gamma)$.

Siegel modular forms aren't directly used in this work. However, the idea of this work is inspired by [PY07] and they are using them.

2.3 Hermitian modular forms

Let

$$\mathbb{H}_n := \{Z \in \mathrm{Mat}_n(\mathbb{C}) \mid \Im(Z) > 0\}$$

be the **Hermitian upper half space**. Note that these matrices are not symmetric as the Siegel upper half space $\mathcal{H}_n$ but we have $\mathcal{H}_n \subseteq \mathbb{H}_n$ and $\mathcal{H}_1 = \mathbb{H}_1 \subseteq \mathbb{C}$.

Hermitian modular forms are holomorphic functions $\mathbb{H}_n \to \mathbb{C}$. They are a generalization of Siegel modular forms where the **translation group** Γ is not a subgroup of $\mathrm{Sp}_n(\mathbb{Z})$ but a subgroup of $\mathrm{Sp}_n(\mathcal{O})$ for some $\mathcal{O} \subseteq \mathbb{C}$.

More specificially, let $\Delta \in \mathbb{N}$ so that we have the imaginary quadratic number field $\mathbb{K} := \mathbb{Q}(\sqrt{-\Delta})$ where $-\Delta$ is the fundamental discriminant. Then, let $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ be the maximum order. We call $\mathrm{Sp}_n(\mathcal{O})$ the **Hermitian modular group**. Let Γ be a subgroup of $\mathrm{Sp}_n(\mathcal{O})$.

Again, with $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathcal{O})$, $Z \in \mathbb{H}_n$, $M \cdot Z := (AZ+B) \cdot (CZ+D)^{-1} \in \mathbb{H}_n$ as for Siegel modular forms and the **weight** $k \in \mathbb{Z}$, we define the **translated function** $f|M \colon \mathbb{H}_n \to \mathbb{C}$ as

$$(f|M)(Z) := \det(CZ+D)^{-k} \cdot f(M \cdot Z).$$

A **Hermitian modular form** of **degree** $n \in \mathbb{N}$ with **weight** $k \in \mathbb{Z}$ over Γ is a holomorphic function

$$f \colon \mathbb{H}_n \to \mathbb{C}$$

with

(1) $f|M = f \quad \forall\, M \in \Gamma, Z \in \mathbb{H}_n$,

(2) for $n = 1$: f is holomorphic in all cusps.

$\mathcal{M}_k^{\mathbb{H}_n}(\Gamma)$ denotes the vector space of such Hermitian modular forms.

As it can be done for Siegel modular forms, we generalize this further by introducing a **Multiplicative character** $\nu \colon \Gamma \to \mathbb{C}^\times$. Thus, for $M_1, M_2 \in \Gamma$, we have $\nu(M_1) \cdot \nu(M_2) = \nu(M_1 \cdot M_2)$.

A **Hermitian modular form** over Γ and ν is a holomorphic function

$$f \colon \mathbb{H}_n \to \mathbb{C}$$

with

(1) $f|M = \nu(M) \cdot f \quad \forall\, M \in \Gamma, Z \in \mathbb{H}_n$,

(2) for $n = 1$: f is holomorphic in all cusps.

$\mathcal{M}_k^{\mathbb{H}_n}(\Gamma, \nu)$ denotes the vector space of such Hermitian modular forms.

For $f \in \mathcal{M}_k^{\mathbb{H}_n}(\Gamma, \nu)$, define the **Siegel Φ-operator** as

$$(f|\Phi)(Z') := \lim_{t \to \infty} f \begin{pmatrix} Z' & 0 \\ 0 & it \end{pmatrix}, \quad Z' \in \mathbb{H}_{n-1}.$$

Then (see [Der01]), $f|\Phi \colon \mathbb{H}_{n-1} \to \mathbb{C}$ is a well-defined Hermitian modular form of degree $n-1$.

A Hermitian modular form $f \in \mathcal{M}_k^{\mathbb{H}_n}(\Gamma, \nu)$ is a **Hermitian modular cusp form**, if and only if for all $R \in \mathrm{Sp}_n(\mathbb{K})$, it holds

$$(f|R)|\Phi \equiv 0.$$

In this work, we will always use Hermitian modular forms of degree $n = 2$.

2.3.1 Properties

Because $-\Delta$ is fundamental, we have two possible cases:

1. $\Delta \equiv 3 \pmod 4$ and Δ is square-free, or
2. $\Delta \equiv 0 \pmod 4$, $\Delta/4 \equiv 1, 2 \pmod 4$ and $\Delta/4$ is square-free.

And for the **maximal order** $\mathcal{O}$, we have (compare [Der01])

$$\begin{aligned} \mathcal{O} &= \mathbb{Z} + \mathbb{Z}\frac{-\Delta + i\sqrt{\Delta}}{2}, \\ \mathcal{O}^{\#} &= \mathbb{Z}\frac{i}{\sqrt{\Delta}} + \mathbb{Z}\frac{1 + i\sqrt{\Delta}}{2}. \end{aligned}$$

From now on, we will always work with Hermitian modular forms of degree $n = 2$. We also use $\Gamma = \mathrm{Sp}_2(\mathcal{O})$ for simplicity.

Chapter 3

Theory

Lemma 3.1. *Let $f\colon \mathbb{H}_2 \to \mathbb{C}$ be a Hermitian modular form of weight k with $\nu \equiv 1$. Let $S \in \mathcal{P}_2(\mathcal{O})$. Then, $\tau \mapsto f(S\tau)\colon \mathbb{H}_1 \subseteq \mathbb{C} \to \mathbb{C}$ is an Elliptic modular form of weight $2k$ to $\Gamma_0(l)$, where l is the denominator of S^{-1}.*

We write

$$f[S]\colon \mathbb{H}_1 \to \mathbb{C}, \quad \tau \mapsto f(S\tau).$$

Proof. Define $\Gamma^H := \mathrm{Sp}_2(\mathcal{O})$ as the translation group for f. Let $\tau \in \mathbb{H}_1$. With $S = [s,t,u] \in \mathcal{P}_2(\mathbb{C})$ we have

$$\begin{aligned}
\Im(S\tau) &= \frac{1}{2i}\left(S\tau - \overline{S}^T\overline{\tau}\right)\\
&= \frac{1}{2i}S(\tau - \overline{\tau})\\
&= \frac{1}{2i}S \cdot 2i\Im(\tau)\\
&= S\Im(\tau) > 0,
\end{aligned}$$

thus $S\tau \in \mathbb{H}_2$. Thus, $\tau \mapsto f(S\tau)$ is a function $\mathbb{H}_1 \to \mathbb{C}$.

Let $l := \det(S)$. That is the denominator of S^{-1}. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(l) \subseteq \mathrm{SL}_2(\mathbb{Z})$. We have

$$\begin{aligned}
& S\frac{a\tau + b}{c\tau + d}\\
= \; & (a(S\tau) + bS) \cdot ((cS^{-1})(S\tau) + d)^{-1}\\
= \; & \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \cdot S\tau.
\end{aligned}$$

Define

$$M := \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \in \mathrm{Mat}_4(\mathbb{C}).$$

With $l|c$, we also have $cS^{-1} = \frac{c}{l}[u, -t, s] \in \mathrm{Mat}_2(\mathcal{O})$, thus we have $M \in \mathrm{Mat}_4(\mathcal{O})$. Recall that we have $S = \overline{S}^T$ and $ad - bc = 1$. Verify that we have $M \in \mathrm{Sp}_2(\mathcal{O}) = \Gamma^H$:

$$\begin{aligned}
& \overline{M}^T J_2 M \\
= & \overline{\begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix}}^T J_2 \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \\
= & \begin{pmatrix} (-acS^{-1} + ac\overline{S^{-1}}^T) & (-ad1_2 + cb\overline{S^{-1}}^T S) \\ (-bc\overline{S}^T S^{-1} + ad1_2) & (-bd\overline{S}^T + bdS) \end{pmatrix} \\
= & J_2.
\end{aligned}$$

Thus, because f is a Hermitian modular form, we have

$$\begin{aligned}
& f[S]\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau\right) \\
= & f\left(S\frac{a\tau + b}{c\tau + d}\right) \\
= & f\left(M \cdot S\tau\right) \\
= & \nu(M) \cdot \det(cS^{-1}S\tau + d1_2)^k \cdot f(S\tau) \\
= & (c\tau + d)^{2k} \cdot f[S](\tau).
\end{aligned}$$

This is the same as

$$(f[S])|_{2k}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = f[S].$$

It follows that $f[S]$ is an Elliptic modular form of weight $2k$ to $\Gamma_0(l)$. $\square$

Remark 3.2. Let us analyze the case $\nu \not\equiv 1$. According to [Der01], only for $\Delta \equiv 0 \pmod 4$, there is a single non-trivial Abel character ν. This ν has the following properties (see [Der01]):

$$\begin{aligned}
\nu(J_2) &= 1, \\
\nu(\mathrm{Trans}(H)) &= (-1)^{h_1 + h_4 + |h_2|^2}, & H &= [h_1, h_2, h_4] \in \mathrm{Her}_2(\mathcal{O}) \\
\nu(\mathrm{Rot}(U)) &= (-1)^{|1+u_1+u_4|^2|1+u_2+u_3|^2 + |u_1u_4|^2} & U &= \left(\begin{smallmatrix} u_1 & u_2 \\ u_3 & u_4 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O})
\end{aligned}$$

Consider the proof of the previous lemma. To calculate $\nu(M)$ with the given equations, we need to represent M in the generating system J_2, $\mathrm{Trans}(H)$ and $\mathrm{Rot}(U)$.

We must consider two different cases. Recall that we have $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, i.e. $ad - bc = 1$, $S = [s, t, u] \in \mathcal{P}_2(\mathcal{O})$ and

$$M = \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \in \mathrm{Sp}_2(\mathcal{O}).$$

Case 1: $c = 0$. Then we have $ad = 1$. Define $T := \frac{b}{d}S$. Then we have

$$\begin{aligned}
& \mathrm{Trans}\left(\frac{b}{d}S\right)\mathrm{Rot}\left(\frac{1}{d}1_2\right) \\
= \; & \begin{pmatrix} 1_2 & \frac{b}{d}S \\ & 1_2 \end{pmatrix} \begin{pmatrix} \frac{1}{d}1_2 & \\ & d1_2 \end{pmatrix} \\
= \; & \begin{pmatrix} \frac{1}{d}1_2 & bS \\ & d1_2 \end{pmatrix} \\
= \; & M.
\end{aligned}$$

And we have

$$\begin{aligned}
\nu\left(\mathrm{Trans}\left(\frac{b}{d}S\right)\right) &= (-1)^{\frac{b}{d}s + \frac{b}{d}u + \left|\frac{b}{d}t\right|^2}, \\
\nu\left(\mathrm{Rot}\left(\frac{1}{d}1_2\right)\right) &= (-1)^{\left|1+\frac{2}{d}\right|^2 + \left|\frac{1}{d^2}\right|^2} = 1.
\end{aligned}$$

Case 2: $c \neq 0$. Then we have

$$\begin{aligned}
& \mathrm{Trans}\left(\frac{a}{c}S\right)\mathrm{Rot}\left(-\frac{1}{c}S\right)(-J_2)\,\mathrm{Trans}\left(-\frac{d}{c}S\right)^{-1} \\
= \; & \begin{pmatrix} 1_2 & \frac{a}{c}S \\ & 1_2 \end{pmatrix} \begin{pmatrix} -\frac{1}{c}\overline{S}^T & \\ & -cS^{-1} \end{pmatrix} (-J_2) \begin{pmatrix} 1_2 & -\frac{d}{c}S \\ & 1_2 \end{pmatrix}^{-1} \\
= \; & \begin{pmatrix} -\frac{1}{c}\overline{S}^T & -a1_2 \\ & -cS^{-1} \end{pmatrix} \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix} \begin{pmatrix} 1_2 & \frac{d}{c}S \\ & 1_2 \end{pmatrix} \\
= \; & \begin{pmatrix} -\frac{1}{c}\overline{S}^T & a1_2 \\ & -cS^{-1} \end{pmatrix} \begin{pmatrix} & 1_2 \\ -1_2 & -\frac{d}{c}S \end{pmatrix} \\
= \; & \begin{pmatrix} a1_2 & -\frac{1}{c}\overline{S}^T + \frac{ad}{c}S \\ cS^{-1} & d1_2 \end{pmatrix} \\
= \; & M.
\end{aligned}$$

And we have

$$\begin{aligned}
\nu\left(\operatorname{Trans}\left(\frac{a}{c}S\right)\right) &= (-1)^{\frac{a}{c}s+\frac{a}{c}u+\left|\frac{a}{c}t\right|^2},\\
\nu\left(\operatorname{Rot}\left(-\frac{1}{c}S\right)\right) &= (-1)^{\left|1-\frac{1}{c}s-\frac{1}{c}u\right|^2\left|1-\frac{2}{c}\Re(t)\right|^2+\left|\frac{su}{c^2}\right|^2},\\
\nu\left(-J_2\right) &= -1,\\
\nu\left(\operatorname{Trans}\left(-\frac{d}{c}S\right)\right)^{-1} &= (-1)^{-\frac{d}{c}s-\frac{d}{c}u+\left|\frac{d}{c}t\right|^2}.
\end{aligned}$$

As a conclusion for now, it looks complicated to restrict $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, i.e. the translation group Γ^E for the Elliptic modular forms, to satisfy $\nu(M) = 1$. For example, for the case $c = 0$, one fulfilling condition would be $2|b$.

To avoid such complications, we will use $\nu \equiv 1$ for the rest of our work. □

Preliminaries 3.3. We want to calculate a generating set for the Fourier expansions of Hermitian modular forms.

We define the index set

$$\Lambda := \left\{ 0 \leq \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} \in \operatorname{Mat}_2(\mathcal{O}^{\#}) \,\middle|\, a, c \in \mathbb{Z} \right\}$$

as the index for the Fourier coefficients of the Fourier expansions of our Hermitian modular forms.

Remark 3.4. For a holomorphic function $f\colon \mathbb{H}_2 \to \mathbb{C}$, we write its Fourier expansion as

$$f(Z) = \sum_{T\in\Lambda} a(T)\cdot e^{2\pi i\cdot \operatorname{tr}(TZ)}$$

with its Fourier coefficients $a\colon \Lambda \to \mathbb{Q}$.

For any $S \in \mathcal{P}_2(\mathcal{O})$, for the restricted function $f[S]\colon \mathbb{H}_1 \to \mathbb{C}$, this gives us

$$f[S](\tau) = \sum_{T\in\Lambda} a(T)\cdot e^{2\pi i\cdot \operatorname{tr}(TS\tau)}.$$

We use $a[S]\colon \mathbb{N}_0 \to \mathbb{Q}$ for the Fourier coefficients of $f[S]$, i.e. we have

$$f[S](\tau) = \sum_{n\in\mathbb{N}_0} a[S](n)\cdot e^{2\pi i n\tau}.$$

This gives us

$$a[S](n) = \sum_{T \in \Lambda, \text{tr}(ST) = n} a(T).$$

For the implementation of the algorithm, we need to define a finite precision of the index set of the Fourier coefficients of the Hermitian modular forms. Fix $B := B_{\mathcal{F}} \in \mathbb{N}$ as a limit. Define the precision of the Fourier coefficient index

$$\mathcal{F} := \mathcal{F}_B := \left\{ \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \Lambda \,\middle|\, 0 \leq a, c < B_{\mathcal{F}} \right\} \subseteq \Lambda.$$

The main algorithm is going to be described in Algorithm 3.9. It will start with the vectorspace of all possible Fourier expansions for the precision index set $\mathcal{F}$ and reduce that vectorspace.

Lemma 3.5. *Given a Hermitian modular form f and its Fourier expansion coefficients $a\colon \mathcal{F}_B \to \mathbb{Q}$ of the precision index set $\mathcal{F}_B$ and a matrix $S = [s, t, u] \in \mathcal{P}_2(\mathcal{O})$, the precision of the Fourier expansion of the Elliptic modular form $f[S]$ is given by*

$$\mathcal{F}(S) = B \cdot (s + u - 2\,|t|),$$

i.e. we can calculate the Fourier expansion coefficients (as described in remark 3.4)

$$a[S]\colon \{k \in N_0 \mid k < \mathcal{F}(S)\} \to \mathbb{Q}.$$

Proof. For a given $S \in \mathcal{S}$ and limit $B \in \mathbb{N}$ which restricts $\mathcal{F} \subset \Lambda$, $\mathcal{F}(S) \in \mathbb{N}_0$ is the limit such that for any $T \in \Lambda - \mathcal{F}$, $\text{tr}(ST) \geq \mathcal{F}(S)$. Thus, for calculating the Fourier coefficients $T \in \Lambda$ with $\text{tr}(ST) \in \{0, \ldots, \mathcal{F}(S) - 1\}$, it is sufficient to enumerate the $T \in \mathcal{F}$.

Let $S = [s, t, u]$ and $T = [a, b, c]$. Recall that $S \in \mathcal{P}_2(\mathcal{O})$. Then we have

$$\text{tr}(ST) = as + \bar{t}b + t\bar{b} + cu = as + cu + 2\Re(\bar{t}b).$$

Because $T \geq 0$, we have $ac \geq |b|^2$ and thus

$$|b| \leq \sqrt{ac} \leq \max(a, c).$$

Thus,

$$2\Re(\bar{t}b) \geq -2\,|t|\,|b| \geq -2\,|t| \max(a, c).$$

We also have $as + cu \geq \max(a,c)(s+u)$. Assuming $T \in \Lambda - \mathcal{F}$, we have $\max(a,c) \geq B$. For such T, we get

$$\text{tr}(ST) \geq B \cdot (s + u - 2\,|t|).$$

Given $S > 0$, we have $su > |t|^2$. Then we have

$$\begin{aligned} & s + u - 2\,|t| > 0 \\ \Leftrightarrow\quad & su + u^2 - 2\,|t|\,u > 0 \\ \Leftrightarrow\quad & \left(|t|^2 + u^2 - 2\,|t|\,u\right) + (su - |t|^2) > 0 \\ \Leftrightarrow\quad & (|t| - u)^2 + (su - |t|^2) > 0. \end{aligned}$$

Thus, for $B > 0$, we have

$$B \cdot (s + u - 2\,|t|) > 0.$$

All inequalities were sharp estimations[1], thus we get

$$\mathcal{F}(S) = B \cdot (s + u - 2\,|t|). \qquad \square$$

Remark 3.6. Let $\mathcal{M}_i$ be a sub vector space of Fourier expansions $a\colon \mathcal{F} \to \mathbb{Q}$, i.e. $\mathcal{M}_i \subset \mathcal{FE}_{\mathcal{F}}(\mathcal{M}_k^{\mathbb{H}_2}(\Gamma))$. Remark 3.4 and lemma 3.5 gives us the tools to reduce $\mathcal{M}_i$ to a sub vector space $\mathcal{M}_{i+1} \subset \mathcal{M}_i$.

For a given $S \in \mathcal{P}_2(\mathcal{O})$, calculating the restrictions $a \mapsto a[S]$ for all $a \in \mathcal{M}_i$, we must only get Fourier expansions of Elliptic modular forms.

Thus,

$$\mathcal{M}_{i+1} := \left\{a \in \mathcal{M}_i \,\middle|\, a[S] \in \mathcal{FE}_{\mathcal{F}(S)}(\mathcal{M}_k(\Gamma_0(l_S)))\right\} \cup \{a \in \mathcal{M}_i \mid a[S] \equiv 0\}.$$

Remark 3.7. With $[a,b,c] \in \mathcal{F}$, we have $0 \leq a, c < B$, thus there are only a finite number of possible $(a,c) \in \mathbb{N}_0^2$. Because $0 \leq [a,b,c]$, we get $ac - |b|^2 \geq 0$ and thus b is also always limited. Thus, $\mathcal{F}$ is finite but it might be huge for even small B. For example[2],

$$\begin{aligned} &\text{for } D = -3, B = 10, \quad \text{we have } \#\mathcal{F} = 21892. \\ &\text{for } D = -3, B = 20, \quad \text{we have } \#\mathcal{F} = 413702. \end{aligned}$$

[1] For example, let $S = [2, -1, 1]$. Then you have $s + u - 2\,|t| = 1$. With $c = B$ and $a = b = 1$, you hit the limit $\text{tr}(ST) = 2 + B - 2 = B = \mathcal{F}(S)$.

[2] This example was calculated with the code at [Zey13a].

Because we want $a \in \mathbb{Q}^{\mathcal{F}}$ to be Fourier expansions of Hermitian modular forms, we can assume that a is invariant under $\mathrm{GL}_2(\mathcal{O})$. This means that we have

$$\det(U)^k a(T[U]) = a(T) \quad \forall\, U \in \mathrm{GL}_2(\mathcal{O}),$$

where k is the weight of the Hermitian modular forms. Restricting the elements in $\mathcal{F}$ by the $\mathrm{GL}_2(\mathcal{O})$-invariation makes the set $\mathcal{F}^{\mathrm{GL}_2(\mathcal{O})} \subseteq \mathcal{F}$ much smaller and better to handle in computer calculations. For example,

$$\text{for } D=-3, B=10, \quad \text{we have } \#\left(\mathcal{F}^{\mathrm{GL}_2(\mathcal{O})}\right) = 420,$$
$$\text{for } D=-3, B=20, \quad \text{we have } \#\left(\mathcal{F}^{\mathrm{GL}_2(\mathcal{O})}\right) = 4840.$$

We use this set to identify a base of the finite dimension vector space $\left(\mathbb{Q}^{\mathcal{F}}\right)^{\mathrm{GL}_2(\mathcal{O})}$. □

Remark 3.8. From remark 3.4 and lemma 3.5, we have

$$a[S](k) = \sum_{T \in \mathcal{F}, \mathrm{tr}(ST)=k} a(T)$$

for $k \in \mathbb{N}_0, k < \mathcal{F}(S)$.

Let $v = (v_1, \ldots, v_n) \in \mathbb{Q}^n$ be a vector in the base of reduced indices of $\mathcal{F}$ as described in remark 3.7 with $n = \#\left(\mathcal{F}^{\mathrm{GL}_2(\mathcal{O})}\right)$. And let

$$\mathcal{F}^{\mathrm{GL}_2(\mathcal{O})} = (T_1, \ldots, T_n).$$

We have

$$\det(U)^k a(T_i[U]) = a(T_i) = v_i$$

for all $i \leq n$, $U \in \mathrm{GL}_2(\mathcal{O})$. For any $T \in \mathcal{F}$, we can uniquely find $i \leq n$ and $U \in \mathrm{GL}_2(\mathcal{O})$ such that

$$T_i[U] = T.$$

Then we have

$$a(T) = a(T_i[U]) = \det(U)^{-k} a(T_i) = \det(U)^{-k} v_i.$$

This gives us the formula to calculate the Fourier expansion $a[S]\colon \{k \in N_0 \mid k < \mathcal{F}(S)\} \to \mathbb{Q}$. □

Algorithm 3.9. We have the Hermitian modular form degree $n = 2$ fixed, as well as some Δ (for now, $\Delta \in \{3, 4, 8\}$). Then we select some form weight $k \in \mathbb{Z}$ ($k \in \{1, \dots, 20\}$ or so), let $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ be the maximal order (see chapter 2.3.1) and some subgroup Γ of $\mathrm{Sp}_2(\mathcal{O})$. Then we select an abel character $\nu\colon \Gamma \to \mathbb{C}^\times$ of $\mathrm{Sp}_2(\mathcal{O})$ (we just use $\nu \equiv 1$, see remark 3.2).

1. Enumerate matrices $S \in \mathcal{P}_2(\mathcal{O})$ and for each matrix perform the following steps.

2. We set

   $$\mathcal{M}^H_{k,\mathcal{S},\mathcal{F}} := \left\{ (a[S])_{S\in\mathcal{S}} \,\middle|\, a \in \mathbb{Q}^{\mathcal{F}} \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant} \right\} \subseteq \bigoplus_{S\in\mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)}.$$

   The elements $a \in \mathbb{Q}^{\mathcal{F}}$ are Fourier expansions of Elliptic modular forms ($\mathbb{H}_1 \to \mathbb{C}$) and $a(T) \in \mathbb{Q}$ for $T \in \mathcal{F} \subseteq \mathrm{Mat}_2(\mathcal{O}^{\#})$ are the Fourier coefficients.

   We identify

   $$\bigoplus_{S\in\mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)} = \mathbb{Q}^N, \ \ N = \sum_{S\in\mathcal{S}} \mathcal{F}(S).$$

   See lemma 3.5.

   We want to calculate the matrix of the linear function

   $$\mathbb{Q}^{\mathcal{F}^{\mathrm{GL}_2(\mathcal{O})}} \to \bigoplus_{S\in\mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)}, \ \ a \mapsto (a[S])_{S\in\mathcal{S}}.$$

   The base of the destination room is canonical. The dimension is N. The base of the source room can be identified by $\mathcal{F}^{\mathrm{GL}_2(\mathcal{O})}$.

   And we set

   $$\mathcal{M}_{k,\mathcal{S},\mathcal{F}} := \bigoplus_{S\in\mathcal{S}} \mathcal{F}\mathcal{E}_{\mathcal{F}(S)}(\mathcal{M}_k(\Gamma_0(l_S)))$$

   where $\mathcal{M}_k(\Gamma_0(l_S))$ is the vectorspace of Elliptic modular forms over $\Gamma_0(l_S)$.

3. If

   $$\dim \mathcal{M}^H_{k,\mathcal{S},\mathcal{F}} \cap \mathcal{M}_{k,\mathcal{S},\mathcal{F}} = \dim \mathcal{M}^{\mathbb{H}_2}_k(\Gamma, \nu),$$

   then we are ready and we can reconstruct the Fourier expansion in the following way:

   ...

   If not, then return to step 1, and enlarge $\mathcal{S}$.

Chapter 4

Implementation

In this chapter, we are describing the implementation.

Chapter 5

Conclusion

Blub

Chapter 6

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