Hermitian Modular Forms for Fields of Low Discriminant

Diploma Thesis
in Mathematics

by
Albert Zeyer

submitted to the
Faculty of Mathematics, Computer Science and Natural Science of
RWTH Aachen University

October 2012
revised version from April 5, 2013

Supervisor: Prof. Dr. Aloys Krieg
Second examiner: Dr. Martin Raum

written at the
Lehrstuhl A für Mathematik
Prof. Dr. A. Krieg

Contents

Chapter 1

Introduction

We develop an algorithm to compute Fourier expansions of Hermitian modular forms of degree 2 over $\mathrm{Sp}_2(\mathcal{O})$ for $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$, $\Delta \in \{3, 4, 8\}$.

In [PY07], spaces of Siegel modular cusp forms are calculated.

A similar algorithm is also [Rau12, Algorithm 4.3] for Jacobi forms.

We are doing the same for Hermitian modular forms.

Chapter 2

Preliminaries

$\mathbb{N}$ denotes the set $\{1,2,3,\ldots\}$, $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and $\mathbb{Z}$ are all **integers**. $\mathbb{Q}$ are all the **rational numbers**, $\mathbb{R}$ are the **real numbers** and $\mathbb{C}$ are the **complex numbers**. $\mathbb{R}^+ := \{x\in\mathbb{R} \mid x>0\}$, $\mathbb{R}^\times$ and $\mathbb{C}^\times$ denotes all non-zero numbers.

Let $\mathrm{Mat}_n(R)$ be the set of all $n\times n$ **matrices** over some commutative ring R. Likewise, $\mathrm{Mat}_n^T(R)$ are the **symmetric** $n\times n$ matrices. X^T is the **transposed** matrix of $X\in\mathrm{Mat}_n(R)$. $\overline{Z}$ is the **conjugated** matrix of $Z\in\mathrm{Mat}_n(\mathbb{C})$. For $R\subseteq\mathbb{C}$, $\overline{R}\subseteq R$, the set of **Hermitian matrices** in R is defined as

$$\mathrm{Her}_n(R) = \left\{ Z\in\mathrm{Mat}_n(R) \,\middle|\, \overline{Z}^T = Z \right\}.$$

A matrix $Y\in\mathrm{Mat}_n(\mathbb{C})$ is greater 0 if and only if $\forall x\in\mathbb{C}^n-\{0\} : Y[x] := \overline{x}^T Y x \in\mathbb{R}^+$. Such symmetric matrices are called the **positive definitive matrices**, defined by

$$\mathcal{P}_n(R) = \{X\in\mathrm{Mat}_n(R) \mid X>0\}$$

for $R\subseteq\mathbb{C}$. Note that $\mathcal{P}_n(\mathbb{C})\subseteq\mathrm{Her}_n(\mathbb{C})$.

For $A,X\in\mathrm{Mat}_n(\mathbb{C})$, we define $A[X] := \overline{X}^T AX$. The **denominator** of a matrix $Z\in\mathrm{Mat}_n(\mathbb{Q})$ is the smallest number $x\in\mathbb{N}$ such that $xZ\in\mathrm{Mat}_n(\mathbb{Z})$. We also write $\mathrm{denom}(Z)=x$. $1_n\in\mathrm{Mat}_n(\mathbb{Z})$ denotes the **identity matrix**. We use the **Gauß notation** $[a,b,c] := \left(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right)\in\mathrm{Mat}_n(\mathbb{C})$.

The **general linear group** is defined by

$$\mathrm{GL}_n(R) = \{X\in\mathrm{Mat}_n(R) \mid \det(X) \text{ is a unit in } R\}$$

and the **special linear group** by

$$\mathrm{SL}_n(R) = \{X\in\mathrm{Mat}_n(R) \mid \det(X)=1\}.$$

The **orthogonal group** is defined by

$$\mathrm{O}_n(R) = \left\{X\in\mathrm{GL}_n(R) \,\middle|\, X^T 1_n X = 1_n\right\} \subseteq \mathrm{GL}_n(R).$$

The **symplectic group** is defined by

$$\mathrm{Sp}_n(R) = \left\{X\in\mathrm{GL}_{2n}(R) \,\middle|\, \overline{X}^T J_n X = J_n\right\} \subseteq \mathrm{GL}_{2n}(R)\subseteq\mathrm{Mat}_{2n}(R)$$

where $J_n := \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right) \in \mathrm{SL}_{2n}(R)$ (as in [Der01]). (Note that some authors (e.g. [PY07]) define J_n negatively.) $\mathrm{Sp}_n(R)$ is also called the **unitary group**. Note that [Der01] uses $\mathrm{U}_n(R) = \mathrm{Sp}_n(R)$. Also note that $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_1(\mathbb{Z}) \Leftrightarrow ad - bc = 1 \Leftrightarrow M \in \mathrm{SL}_2(\mathbb{Z})$. Thus, $\mathrm{Sp}_1(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$.

For $Z \in \mathrm{Mat}_n(\mathbb{C})$, we call

$$\Re(Z) = \frac{1}{2}\left(Z + \overline{Z}^T\right) \in \mathrm{Mat}_n(\mathbb{C})$$

the **real** part and

$$\Im(Z) = \frac{1}{2i}\left(Z - \overline{Z}^T\right) \in \mathrm{Mat}_n(\mathbb{C})$$

the **imaginary** part of Z and we have $Z = \Re(Z) + i\Im(Z)$. Note that we usually have $\Re(Z), \Im(Z) \notin \mathrm{Mat}_n(\mathbb{R})$ but we have $\Re(Z), \Im(Z) \in \mathrm{Her}_n(\mathbb{C})$.

We say that some function $f\colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{A} \subseteq \mathrm{Mat}_n(R)$, $\mathcal{B} \subseteq R$ is k-**invariant** under some $\mathcal{X} \subseteq \mathrm{Mat}_n(R)$ where $\mathcal{A}[\mathcal{X}] \subseteq \mathcal{A}$ if and only if $\det(U)^k f(T[U]) = f(T)$ for all $T \in \mathcal{A}$, $U \in \mathcal{X}$.

2.1 Siegel modular forms

Siegel modular forms aren't directly used in this work. However, the idea of this work is inspired by [PY07] and they are using them. Also, they are a generalization of Elliptic modular forms.

Let $\mathcal{H}_n := \left\{Z \in \mathrm{Mat}_n^T(\mathbb{C}) \,\middle|\, \Im(Z) > 0\right\}$ be the **Siegel upper half space**. We call $\mathrm{Sp}_n(\mathbb{Z})$ the **Siegel modular group**.

A **Siegel modular form** of degree $n \in \mathbb{N}$ with weight $k \in \mathbb{Z}$ for some $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{Z})$, Γ subgroup of $\mathrm{Sp}_n(\mathbb{Z})$, is a holomorphic function

$$f\colon \mathcal{H}_n \to \mathbb{C}$$

with

(1) $f\left((AZ+B)\cdot(CZ+D)^{-1}\right) = \det(CZ+D)^k \cdot f(Z) \quad \forall \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma, Z \in \mathcal{H}_n$

(2) for $n = 1\colon f(Z) = O(1) \quad$ for $Z \to i\infty$

Note that many authors define the transformed function $f|M$ for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ with

$$(f|M)(Z) := f(M \cdot Z) \cdot \det(CZ+D)^{-k}$$

with $Z \in \mathcal{H}_n$, where $M \cdot Z := (AZ+B)\cdot(CZ+D)^{-1}$. Then the first property of Siegel modular forms can be written as

$$f|M = f \quad \forall\, M \in \Gamma.$$

$\mathcal{M}_k^{\mathcal{H}_n}(\Gamma)$ denotes the vector space of such Siegel modular forms.

2.2 Elliptic modular forms

These are functions over $\mathcal{H}_1 = \{z \in \mathbb{C} \mid \Im(z) > 0\} \subseteq \mathbb{C}$ which is called the **Poincaré upper half plane**.

We have Γ as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A **Elliptic modular form** with weight $k \in \mathbb{Z}$ over Γ is a holomorphic function

$$f\colon \mathcal{H}_1 \to \mathbb{C}$$

with

$$\begin{aligned} &(1)\ \ f\left(\frac{a\tau + b}{b\tau + c}\right) = (c\tau + d)^k \cdot f(\tau) \quad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma, \tau \in \mathcal{H}_1 \\ &(2)\ \ f(\tau) = O(1) \quad \text{for } \tau \to i\infty \end{aligned}$$

Note that most authors write $M\tau := \frac{a\tau+b}{b\tau+c}$.

$\mathcal{M}_k(\Gamma)$ denotes the vector space of such Elliptic modular forms.

Note that we have $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Sp}_1(\mathbb{Z})$. We can see that Elliptic modular forms are Siegel modular forms of degree $n = 1$. Thus we have $\mathcal{M}_k(\Gamma) = \mathcal{M}_k^{\mathcal{H}_1}(\Gamma)$.

In this work, we use a specific subgroup of $\mathrm{Sp}_1(\mathbb{Z})$. We define

$$\Gamma_0(l) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_1(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{l} \right\} \subseteq \mathrm{Sp}_1(\mathbb{Z}) \subseteq \mathrm{Mat}_2(\mathbb{Z})$$

as a subgroup of $\mathrm{Sp}_1(\mathbb{Z})$.

2.3 Hermitian modular forms

Let $\mathbb{H}_n := \{Z \in \mathrm{Mat}_n(\mathbb{C}) \mid \Im(Z) > 0\}$ be the **Hermitian upper half space**. Note that these matrices are not symmetric as the Siegel upper half space $\mathcal{H}_n$ but we have $\mathcal{H}_n \subseteq \mathbb{H}_n$ and $\mathcal{H}_1 = \mathbb{H}_1 \subseteq \mathbb{C}$.

Let $\Delta \in \mathbb{N}$ so that we have the imaginary quadratic number field $\mathbb{Q}(\sqrt{-\Delta})$ where $-\Delta$ is the fundamental discriminant. Then, let $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ be the maximum order. We call $\mathrm{Sp}_n(\mathcal{O})$ the **Hermitian modular group**. Let Γ be a subgroup of $\mathrm{Sp}_n(\mathcal{O})$. Let $\nu\colon \Gamma \to \mathbb{C}^\times$ be an abel character of $\mathrm{Sp}_n(\mathcal{O})$.

A **Hermitian modular form** of degree $n \in \mathbb{N}$ with weight $k \in \mathbb{Z}$ over Γ and ν is a holomorphic function

$$f\colon \mathbb{H}_n \to \mathbb{C}$$

with

(1) $f((AZ+B)\cdot(CZ+D)^{-1}) = \nu(M)\det(CZ+D)^k f(Z), \quad M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma, Z \in \mathbb{H}_n,$

(2) for $n = 1$: f is holomorphic in all cusps.

Again as for Siegel modular forms, most authors write $M \cdot Z := (AZ+B)\cdot(CZ+D)^{-1}$.

$\mathcal{M}_k^{\mathbb{H}_n}(\Gamma, \nu)$ denotes the vector space of such Hermitian modular forms.

In this work, we will concentrate on Hermitian modular forms of degree $n = 2$. We will start with $\Delta \in \{3, 4, 8\}$.

2.3.1 Properties

Because $-\Delta$ is fundamental, we have two possible cases:

1. $\Delta \equiv 3 \pmod 4$ and Δ is square-free, or
2. $\Delta \equiv 0 \pmod 4$, $\Delta/4 \equiv 1, 2 \pmod 4$ and $\Delta/4$ is square-free.

And for the **maximum order** $\mathcal{O}$, we have (compare [Der01])

$$\begin{aligned} \mathcal{O} &= \mathbb{Z} + \mathbb{Z}\frac{-\Delta + i\sqrt{\Delta}}{2}, \\ \mathcal{O}^{\#} &= \mathbb{Z}\frac{i}{\sqrt{\Delta}} + \mathbb{Z}\frac{1 + i\sqrt{\Delta}}{2}. \end{aligned}$$

From now on, we will always work with Hermitian modular forms of degree $n = 2$. We also use $\Gamma = \mathrm{Sp}_2(\mathcal{O})$ for simplicity.

Chapter 3

Theory

Lemma 3.1. *Let $f\colon \mathbb{H}_2 \to \mathbb{C}$ be a Hermitian modular form of weight k. Let $S \in \mathcal{P}_2(\mathbb{C})$. Then, $f(S\tau)\colon \mathbb{H}_1 \subseteq \mathbb{C} \to \mathbb{C}$ is an Elliptic modular form of weight $2k$ to $\Gamma_0(l)$, where l is the denominator of S^{-1}.*

Proof. Define $\Gamma^H := \mathrm{Sp}_2(\mathcal{O})$ as the translation group for f. Then, we can verify that

$$M := \begin{pmatrix} 1_2 & B \\ & 1_2 \end{pmatrix} \in \Gamma^H, \quad B \in \mathrm{Her}_2(\mathcal{O}).$$

Let $\tau \in \mathbb{H}_1$. Then we have $S\tau \in \mathbb{H}_2$. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We have

$$\begin{aligned} & S\frac{a\tau + b}{c\tau + d} \\ = \;& (a(S\tau) + bS) \cdot ((cS^{-1})(S\tau) + d)^{-1} \\ = \;& \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \cdot S\tau. \end{aligned}$$

And with $S = \overline{S}^T$ and $ad - bc = 1$ we have

$$\begin{aligned} & \overline{\begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix}}^T J_2 \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \\ = \;& \begin{pmatrix} (-acS^{-1} + ac\overline{S^{-1}}^T) & (-ad1_2 + cb\overline{S^{-1}}^T S) \\ (-bc\overline{S}^T S^{-1} + ad1_2) & (-bd\overline{S}^T + bdS) \end{pmatrix} \\ = \;& J_2, \end{aligned}$$

thus we have

$$M' := \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{C}).$$

Thus, because f is a Hermitian modular form, we have

$$\begin{aligned} & f\left(S\frac{a\tau+b}{c\tau+d}\right) \\ = \quad & f\left(M'\cdot S\tau\right) \\ = \quad & \nu(M')\cdot\det(cS^{-1}S\tau+d1_2)^k\cdot f(S\tau) \\ = \quad & (c\tau+d)^{2k}\cdot f(S\tau). \end{aligned}$$

□

Lemma 3.2. *Prop 7.3. von Poor für herm Modulformen.* $\Gamma(\mathcal{L})\supseteq\Gamma_0(l)$ *for* $l\in\mathbb{Z}^+, ls^{-1}\in\mathcal{P}_n(\mathcal{O})$.

We want to calculate a generating set for the Fourier expansions of Hermitian modular forms. Now we will formulate the main algorithm of our work.

Algorithm 3.3. We have the Hermitian modular form degree $n=2$ fixed, as well as some Δ (for now, $\Delta\in\{3,4,8\}$). Then we select some form weight $k\in\mathbb{Z}$ ($k\in\{1,\dots,20\}$ or so), let $\mathcal{O}\subseteq\mathbb{Q}(\sqrt{-\Delta})$ be the maximum order (see chapter 2.3.1) and some subgroup Γ of $\mathrm{Sp}_2(\mathcal{O})$. Then we select an abel character $\nu\colon\Gamma\to\mathbb{C}^\times$ of $\mathrm{Sp}_2(\mathcal{O})$.

We define the index set

$$\Lambda:=\left\{0\leq\begin{pmatrix} a & b \\ \bar{b} & c\end{pmatrix}\in\mathrm{Mat}_2(\mathcal{O}^{\#})\,\middle|\,a,c\in\mathbb{Z}\right\}.$$

Fix $B\in\mathbb{N}$ as a limit. Select a precision

$$\mathcal{F}:=\left\{\begin{pmatrix} a & b \\ \bar{b} & c\end{pmatrix}\in\Lambda\,\middle|\,0\leq a,c<B\right\}\subseteq\Lambda.$$

1. We start with $l=1$ and increase it but only use the square-free numbers.

2. Set $\mathcal{S}=\{\}$,

3. Enumerate matrices $S\in\mathrm{Mat}_2^T(\mathbb{Z})$, and set $\mathcal{S}\leftarrow\mathcal{S}\cup\{S\}$ and for each time you add a new matrix perform the following steps.

4. We set

$$\mathcal{M}^H_{k,\mathcal{S},\mathcal{F}}:=\left\{(a[S])_{S\in\mathcal{S}}\,\middle|\,a\in\mathbb{Q}^{\mathcal{F}}\text{ is }\mathrm{GL}_2(\mathcal{O})\text{ invariant}\right\}\subseteq\bigoplus_{S\in\mathcal{S}}\mathbb{Q}^{\mathcal{F}(S)},$$

where

$$a[S] := \mathbb{N}_0 \to \mathbb{Q}, n \mapsto \sum_{T \in \Lambda, \mathrm{tr}(ST)=n} a(T),$$

The elements $a \in \mathbb{Q}^{\mathcal{F}}$ are Fourier expansions of Elliptic modular forms ($\mathbb{H}_1 \to \mathbb{C}$) and $a(T) \in \mathbb{Q}$ for $T \in \mathcal{F} \subseteq \mathrm{Mat}_2(\mathcal{O}^{\#})$ are the Fourier coefficients. Recall that a being invariant under $\mathrm{GL}_2(\mathcal{O})$ means that we have

$$\det(U)^k a(T[U]) = a(T) \; \forall \, U \in \mathrm{GL}_2(\mathcal{O}).$$

With $[a, b, c] \in \mathcal{F}$, we have $0 \le a, c < B$, thus there are only a finite number of possibilities. Because $0 \le [a, b, c]$, we get $ac - |b|^2 \ge 0$ and thus b is also always limited. Thus, $\mathcal{F}$ is finite but it might be huge for even small B. Restricting the elements in $\mathcal{F}$ by the $\mathrm{GL}_2(\mathcal{O})$-invariation makes the set $\{x \in \mathcal{F} \mid x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\} \subseteq \mathcal{F}$ much smaller and better to handle in computer calculations. We use this set to identify a base of the finite dimension vector space $\left\{x \in \mathbb{Q}^{\mathcal{F}} \mid x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\right\}$.

We identify

$$\bigoplus_{S \in \mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)} = \mathbb{Q}^N, \;\; N = \sum_{S \in \mathcal{S}} \mathcal{F}(S).$$

For a given $S \in \mathcal{S}$ and limit $B \in \mathbb{N}$ which restricts $\mathcal{F} \subset \Lambda$, $\mathcal{F}(S) \in \mathbb{N}_0$ is the limit such that for any $T \in \Lambda - \mathcal{F}$, $\mathrm{tr}(ST) \ge \mathcal{F}(S)$. Thus, for calculating the Fourier coefficients $T \in \Lambda$ with $\mathrm{tr}(ST) \in \{0, \ldots, \mathcal{F}(S) - 1\}$, it is sufficient to enumerate the $T \in \mathcal{F}$.

Let $S = [s, t, u]$ and $T = [a, b, c]$. Recall that $S \in \mathrm{Mat}_2^T(\mathbb{Z})$. Then we have

$$\mathrm{tr}(ST) = as + \bar{t}b + t\bar{b} + cu = as + cu + 2t\Re(b).$$

Because $T \ge 0$, we have $ac \ge |b|^2$ and thus $\Re(b) \le \sqrt{ac} \le \max(a, c)$. Thus, $2t\Re(b) \ge -2\,|t| \max(a, c)$. We also have $as + cu \ge \max(a, c)(s + u)$. Assuming $T \in \Lambda - \mathcal{F}$, we have $\max(a, c) \ge B$. For such T, we get

$$\mathrm{tr}(ST) \ge B \cdot (s + u - 2\,|t|).$$

Given $S > 0$, we have $su > t^2$. Then we have

$$\begin{aligned} & s + u - 2\,|t| > 0 \\ \Leftrightarrow \quad & su + u^2 - 2\,|t|\,u > 0 \\ \Leftrightarrow \quad & \left(t^2 + u^2 - 2\,|t|\,u\right) + (su - t^2) > 0 \\ \Leftrightarrow \quad & (|t| - u)^2 + (su - t^2) > 0. \end{aligned}$$

All inequalities were sharp estimations[1], thus we get

$$\mathcal{F}(S) = B \cdot (s + u - 2\,|t|).$$

We want to calculate the matrix of the linear function

$$\left\{x \in \mathbb{Q}^{\mathcal{F}} \;\middle|\; x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\right\} \to \bigoplus_{S \in \mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)}, \;\; a \mapsto (a[S])_{S \in \mathcal{S}}.$$

The base of the destination room is canonical. The dimension is N. The base of the source room can be identified by $\{x \in \mathcal{F} \mid x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\}$.

And we set

$$\mathcal{M}_{k,\mathcal{S},\mathcal{F}} := \bigoplus_{S \in \mathcal{S}} \mathcal{F}\mathcal{E}_{\mathcal{F}(S)}(\mathcal{M}_k(\Gamma_0(l_S)))$$

where $\mathcal{M}_k(\Gamma_0(l_S))$ is the vectorspace of Elliptic modular forms over $\Gamma_0(l_S)$.

5. If

$$\dim \mathcal{M}^{H}_{k,\mathcal{S},\mathcal{F}} \cap \mathcal{M}_{k,\mathcal{S},\mathcal{F}} = \dim \mathcal{M}^{\mathbb{H}_2}_{k}(\Gamma, \nu),$$

then we are ready and we can reconstruct the Fourier expansion in the following way:

...

If not, then return to step 3, and enlarge $\mathcal{S}$.

[1] For example, let $S = [2, -1, 1]$. Then you have $s + u - 2\,|t| = 1$. With $c = B$ and $a = b = 1$, you hit the limit $\mathrm{tr}(ST) = 2 + B - 2 = B = \mathcal{F}(S)$.

Chapter 4

Implementation

In this chapter, we are describing the implementation.

Chapter 5

Conclusion

Blub

Chapter 6

References

[Der01] T. Dern. *Hermitesche Modulformen zweiten Grades*. Mainz, 2001.

[PY07] C. Poor and D.S. Yuen. Computations of spaces of siegel modular cusp forms. *Journal of the Mathematical Society of Japan*, 59(1):185–222, 2007.

[Rau12] M. Raum. Computing Jacobi Forms and Linear Equivalences of Special Divisors. *ArXiv e-prints*, December 2012.