Hermitian Modular Forms for Fields of Low Discriminant

Diploma Thesis
in Mathematics

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Chapter 1

Introduction

We develop an algorithm to compute Fourier expansions of Hermitian Modular Forms of degree 2 over $\mathrm{Sp}_2(\mathcal{O})$ for $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{D})$, $D \in \{-3, -4, -8\}$.

In [PY07], spaces of Siegel Modular Cusp forms are calculated.

A similar algorithm is also [Rau12, Algorithm 4.3] for Jacobi forms.

We are doing the same for Hermitian Modular forms.

Chapter 2

Background results

2.1 Preliminaries

Let $\mathrm{M}_n(\mathcal{K})$ be the set of all $n \times n$ matrices over some field $\mathcal{K}$. Likewise, $\mathrm{M}_n^T(\mathcal{K})$ are the symetric $n \times n$ matrices. A matrix $Y \in \mathrm{M}_n(\mathbb{R})$ is greater 0 iff $\forall x \in \mathbb{R}^n - \{0\} : Y[x] := x^T Y x > 0$. Let $\mathbb{H}_n := \left\{ Z = X + iY \in \mathrm{M}_n^T(\mathbb{C}) \,\middle|\, Y > 0 \right\}$. Thus, $\mathbb{H}_1$ is the Poincaré upper half plane.

The general linear group is defined by $\mathrm{GL}_n(\mathcal{K}) = \{X \in \mathrm{M}_n(\mathcal{K}) \mid \det(X) \neq 0\}$ and the special linear group by $\mathrm{SL}_n(\mathcal{K}) = \{X \in \mathrm{M}_n(\mathcal{K}) \mid \det(X) = 1\}$. The orthogonal group is defined by $\mathrm{O}_n(\mathcal{K}) = \left\{ X \in \mathrm{GL}_n(\mathcal{K}) \,\middle|\, X^T 1_n X = 1_n \right\}$. The symplectic group is defined by $\mathrm{Sp}_n(\mathcal{K}) = \left\{ X \in \mathrm{GL}_n(\mathcal{K}) \,\middle|\, X^T J_n X = J_n \right\}$ where $J_n := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \in \mathrm{SL}_{2n}(\mathcal{K})$.

A **Siegel Modular Cusp form** of degree $n \in \mathbb{N}$ is a holomorphic function

$$f \colon \mathbb{H}_n \to \mathbb{C}$$

with

(1) $f|_k y = f \;\; \forall\, y \in \Gamma \subseteq \mathrm{Sp}_n(\mathbb{Z})$

(2) for $n = 1$: $f(Z) = O(1)$ for $Z \to i\infty$

where

$$\left(f|_k \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right)(Z) = f((AZ+B)(CZ+D)^{-1}) \cdot \det(CZ+D)^{-k}$$

with $Z \in \mathbb{H}_n$, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$.

A **Hermitian Modular form** of degree $n \in \mathbb{N}$ and weight $k \in \mathbb{Z}$ is a holomorphic function

$$f \colon \mathbb{H}_n \to \mathbb{C}$$

with

$$f(M \cdot Z) = \nu(M) \det(CZ+D)^k f(Z), \quad M = \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathcal{O}),$$

where $Z \in \mathbb{H}_n$ and ν is an abel character of $\mathrm{Sp}_n(\mathcal{O})$.

In this work, we will concentrate on Hermitian Modular forms of degree 2.

Chapter 3

Theory

Lemma 3.1. *Let $f\colon \mathrm{M}_2(\mathbb{C}) \to \mathbb{C}$ be a Hermitian Modular form of weight k. Then, $f(S\tau)\colon \mathbb{H}_1 \subseteq \mathbb{C} \to \mathbb{C}$ is an eliptic modular form of weight $2k$ for some matrix $S \in \mathrm{M}_2(\mathbb{Z})$ with $\Gamma(S) \subseteq \mathrm{SL}_2(\mathbb{Z})$.*

Algorithm 3.2.

1. Select a set of matrices $\mathcal{S} \subseteq \mathrm{M}_2^T(\mathbb{Z})$ with $0 < S \in \mathcal{S}$. Make $\mathcal{S}$ big enough. Now, for some $S \in \mathcal{S}$:
2. Fix $B \in \mathbb{N}$ as a limit. Or select a precision
$$\mathcal{F} = \left\{ \left(\begin{array}{cc} a & b \\ \bar{b} & c \end{array} \right) \middle| \, 0 \le ac < B \right\} \subseteq \Lambda,$$
where
$$\Lambda := \left\{ 0 \le \left(\begin{array}{cc} a & b \\ \bar{b} & c \end{array} \right) \in \mathrm{M}_2(\mathcal{O}^{\#}) \, \middle| \, a, c \in \mathbb{Z} \right\}.$$
3. $$\mathcal{M}^H_{k,\mathcal{S},\mathcal{F}} = \left\{ fS \,\middle|\, f \in \mathbb{Q}^{\mathcal{F}} \right\},$$
$$\mathcal{M}_{k,S} = \mathcal{F}\mathcal{E}_{\mathcal{F}(S)}(\mathrm{M}_k(\Gamma(S)))$$
4. If
$$\dim \mathcal{M}^H_{k,\mathcal{S},\mathcal{F}} \cap \bigoplus_{S \in \mathcal{S}} \mathcal{M}_{k,S} = \dim M^H_k,$$
then we are ready and we can reconstruct the Fourier expansion in the following way: ...

Chapter 4

Implementation

In this chapter, we are describing the implementation.

Chapter 5

Conclusion

Blub

Chapter 6

References

[PY07] C. Poor and D.S. Yuen. Computations of spaces of siegel modular cusp forms. *Journal of the Mathematical Society of Japan*, 59(1):185–222, 2007.

[Rau12] M. Raum. Computing Jacobi Forms and Linear Equivalences of Special Divisors. *ArXiv e-prints*, December 2012.