Hermitian Modular Forms for Fields of Low Discriminant

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in Mathematics

by
Albert Zeyer

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Supervisor: Prof. Dr. Aloys Krieg
Second examiner: Dr. Martin Raum

written at the
Lehrstuhl A für Mathematik
Prof. Dr. A. Krieg

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Chapter 1

Introduction

We develop an algorithm to compute Fourier expansions of Hermitian modular forms of degree 2 over $\mathrm{Sp}_2(\mathcal{O})$ for $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$, $\Delta \in \{3, 4, 8\}$.

In [PY07], spaces of Siegel modular cusp forms are calculated.

A similar algorithm is also [Rau12, Algorithm 4.3] for Jacobi forms.

We are doing the same for Hermitian modular forms.

Chapter 2

Preliminaries

$\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$ are all **integers**. $\mathbb{Q}$ are all the **rational numbers**, $\mathbb{R}$ are the **real numbers** and $\mathbb{C}$ are the **complex numbers**. $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$, $\mathbb{R}^\times$ and $\mathbb{C}^\times$ denotes all non-zero numbers.

Let $\mathrm{Mat}_n(R)$ be the set of all $n \times n$ **matrices** over some commutative ring R. Likewise, $\mathrm{Mat}_n^T(R)$ are the **symmetric** $n \times n$ matrices. X^T is the **transposed** matrix of $X \in \mathrm{Mat}_n(R)$. $\overline{Z}$ is the **conjugated** matrix of $Z \in \mathrm{Mat}_n(\mathbb{C})$. For $R \subseteq \mathbb{C}$, $\overline{R} \subseteq R$, the set of **Hermitian matrices** in R is defined as

$$\mathrm{Her}_n(R) = \left\{ Z \in \mathrm{Mat}_n(R) \,\middle|\, \overline{Z}^T = Z \right\}.$$

A matrix $Y \in \mathrm{Mat}_n(\mathbb{C})$ is greater 0 if and only if $\forall x \in \mathbb{C}^n - \{0\} : Y[x] := \overline{x}^T Y x \in \mathbb{R}^+$. Such symmetric matrices are called the **positive definitive matrices**, defined by

$$\mathcal{P}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid X > 0\}$$

for $R \subseteq \mathbb{C}$. Note that $\mathcal{P}_n(\mathbb{C}) \subseteq \mathrm{Her}_n(\mathbb{C})$.

For $A, X \in \mathrm{Mat}_n(\mathbb{C})$, we define $A[X] := \overline{X}^T AX$. The **denominator** of a matrix $Z \in \mathrm{Mat}_n(\mathbb{Q})$ is the smallest number $x \in \mathbb{N}$ such that $xZ \in \mathrm{Mat}_n(\mathbb{Z})$. We also write $\mathrm{denom}(Z) = x$. $1_n \in \mathrm{Mat}_n(\mathbb{Z})$ denotes the **identity matrix**. We use the **Gauß notation** $[a, b, c] := \left(\begin{smallmatrix} a & b \\ \overline{b} & c \end{smallmatrix}\right) \in \mathrm{Her}_n(\mathbb{C})$.

The **general linear group** is defined by

$$\mathrm{GL}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid \det(X) \text{ is a unit in } R\}$$

and the **special linear group** by

$$\mathrm{SL}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid \det(X) = 1\}.$$

The **orthogonal group** is defined by

$$\mathrm{O}_n(R) = \left\{X \in \mathrm{GL}_n(R) \,\middle|\, X^T 1_n X = 1_n\right\} \subseteq \mathrm{GL}_n(R).$$

The **symplectic group** is defined by

$$\mathrm{Sp}_n(R) = \left\{ X \in \mathrm{GL}_{2n}(R) \,\middle|\, \overline{X}^T J_n X = J_n \right\} \subseteq \mathrm{GL}_{2n}(R) \subseteq \mathrm{Mat}_{2n}(R)$$

where $J_n := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \in \mathrm{SL}_{2n}(R)$ (as in [Der01]). (Note that some authors (e.g. [PY07]) define J_n negatively.) $\mathrm{Sp}_n(R)$ is also called the **unitary group**. Note that [Der01] uses $\mathrm{U}_n(R) = \mathrm{Sp}_n(R)$. Also note that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_1(\mathbb{Z}) \Leftrightarrow ad - bc = 1 \Leftrightarrow M \in \mathrm{SL}_2(\mathbb{Z})$. Thus, $\mathrm{Sp}_1(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$.

In addition, for a ring $R \subseteq \mathbb{C}$, define

$$\begin{aligned} \mathrm{Rot}(U) &= \begin{pmatrix} \overline{U}^T & \\ & U^{-1} \end{pmatrix} \in \mathrm{Sp}_2(R), && U \in \mathrm{GL}_2(R) \\ \mathrm{Trans}(H) &= \begin{pmatrix} 1_2 & H \\ & 1_2 \end{pmatrix} \in \mathrm{Sp}_2(R), && H \in \mathrm{Her}_2(R) \end{aligned}$$

and note that we have $J_2 = \begin{pmatrix} & -1_2 \\ 1_2 & \end{pmatrix} \in \mathrm{Sp}_2(R)$. Those tree types of matrices form a generator set for the group $\mathrm{Sp}_2(R)$.

For $Z \in \mathrm{Mat}_n(\mathbb{C})$, we call

$$\Re(Z) = \frac{1}{2}\left(Z + \overline{Z}^T\right) \in \mathrm{Mat}_n(\mathbb{C})$$

the **real** part and

$$\Im(Z) = \frac{1}{2i}\left(Z - \overline{Z}^T\right) \in \mathrm{Mat}_n(\mathbb{C})$$

the **imaginary** part of Z and we have $Z = \Re(Z) + i\Im(Z)$. Note that we usually have $\Re(Z), \Im(Z) \notin \mathrm{Mat}_n(\mathbb{R})$ but we have $\Re(Z), \Im(Z) \in \mathrm{Her}_n(\mathbb{C})$.

We say that some function $f\colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{A} \subseteq \mathrm{Mat}_n(R)$, $\mathcal{B} \subseteq R$ is k**-invariant** under some $\mathcal{X} \subseteq \mathrm{Mat}_n(R)$ where $\mathcal{A}[\mathcal{X}] \subseteq \mathcal{A}$ if and only if $\det(U)^k f(T[U]) = f(T)$ for all $T \in \mathcal{A}$, $U \in \mathcal{X}$.

2.1 Siegel modular forms

Siegel modular forms aren't directly used in this work. However, the idea of this work is inspired by [PY07] and they are using them. Also, they are a generalization of Elliptic modular forms.

Let $\mathcal{H}_n := \left\{ Z \in \mathrm{Mat}_n^T(\mathbb{C}) \,\middle|\, \Im(Z) > 0 \right\}$ be the **Siegel upper half space**. We call $\mathrm{Sp}_n(\mathbb{Z})$ the **Siegel modular group**.

A **Siegel modular form** of degree $n \in \mathbb{N}$ with weight $k \in \mathbb{Z}$ for some $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{Z})$, Γ subgroup of $\mathrm{Sp}_n(\mathbb{Z})$, is a holomorphic function

$$f\colon \mathcal{H}_n \to \mathbb{C}$$

with

$$(1)\ \ f\left((AZ+B)\cdot(CZ+D)^{-1}\right) = \det(CZ+D)^k \cdot f(Z) \quad \forall \ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma, Z \in \mathcal{H}_n$$

$$(2)\ \ \text{for } n=1\colon f(Z) = O(1) \quad \text{for } Z \to i\infty$$

Note that many authors define the transformed function $f|M$ for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ with

$$(f|M)(Z) := f(M \cdot Z) \cdot \det(CZ+D)^{-k}$$

with $Z \in \mathcal{H}_n$, where $M \cdot Z := (AZ+B)\cdot(CZ+D)^{-1}$. Then the first property of Siegel modular forms can be written as

$$f|M = f \quad \forall\, M \in \Gamma.$$

$\mathcal{M}_k^{\mathcal{H}_n}(\Gamma)$ denotes the vector space of such Siegel modular forms.

2.2 Elliptic modular forms

These are functions over $\mathcal{H}_1 = \{z \in \mathbb{C} \mid \Im(z) > 0\} \subseteq \mathbb{C}$ which is called the **Poincaré upper half plane**.

We have Γ as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A **Elliptic modular form** with weight $k \in \mathbb{Z}$ over Γ is a holomorphic function

$$f\colon \mathcal{H}_1 \to \mathbb{C}$$

with

$$(1)\ \ f\left(\frac{a\tau+b}{b\tau+c}\right) = (c\tau+d)^k \cdot f(\tau) \quad \forall \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma, \tau \in \mathcal{H}_1$$

$$(2)\ \ f(\tau) = O(1) \quad \text{for } \tau \to i\infty$$

Note that most authors write $M\tau := \frac{a\tau+b}{b\tau+c}$.

$\mathcal{M}_k(\Gamma)$ denotes the vector space of such Elliptic modular forms.

Note that we have $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Sp}_1(\mathbb{Z})$. We can see that Elliptic modular forms are Siegel modular forms of degree $n = 1$. Thus we have $\mathcal{M}_k(\Gamma) = \mathcal{M}_k^{\mathcal{H}_1}(\Gamma)$.

In this work, we use a specific subgroup of $\mathrm{Sp}_1(\mathbb{Z})$. We define

$$\Gamma_0(l) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_1(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{l} \right\} \subseteq \mathrm{Sp}_1(\mathbb{Z}) \subseteq \mathrm{Mat}_2(\mathbb{Z})$$

as a subgroup of $\mathrm{Sp}_1(\mathbb{Z})$.

2.3 Hermitian modular forms

Let $\mathbb{H}_n := \{Z \in \mathrm{Mat}_n(\mathbb{C}) \mid \Im(Z) > 0\}$ be the **Hermitian upper half space**. Note that these matrices are not symmetric as the Siegel upper half space $\mathcal{H}_n$ but we have $\mathcal{H}_n \subseteq \mathbb{H}_n$ and $\mathcal{H}_1 = \mathbb{H}_1 \subseteq \mathbb{C}$.

Let $\Delta \in \mathbb{N}$ so that we have the imaginary quadratic number field $\mathbb{Q}(\sqrt{-\Delta})$ where $-\Delta$ is the fundamental discriminant. Then, let $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ be the maximum order. We call $\mathrm{Sp}_n(\mathcal{O})$ the **Hermitian modular group**. Let Γ be a subgroup of $\mathrm{Sp}_n(\mathcal{O})$. Let $\nu\colon \Gamma \to \mathbb{C}^\times$ be an Abel character of $\mathrm{Sp}_n(\mathcal{O})$.

A **Hermitian modular form** of degree $n \in \mathbb{N}$ with weight $k \in \mathbb{Z}$ over Γ and ν is a holomorphic function

$$f\colon \mathbb{H}_n \to \mathbb{C}$$

with

(1) $f((AZ+B)\cdot(CZ+D)^{-1}) = \nu(M)\det(CZ+D)^k f(Z), \quad M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma, Z \in \mathbb{H}_n,$

(2) for $n = 1$: f is holomorphic in all cusps.

Again as for Siegel modular forms, most authors write $M \cdot Z := (AZ+B)\cdot(CZ+D)^{-1}$.

$\mathcal{M}_k^{\mathbb{H}_n}(\Gamma, \nu)$ denotes the vector space of such Hermitian modular forms.

In this work, we will concentrate on Hermitian modular forms of degree $n = 2$. We will start with $\Delta \in \{3, 4, 8\}$.

2.3.1 Properties

Because $-\Delta$ is fundamental, we have two possible cases:

1. $\Delta \equiv 3 \pmod 4$ and Δ is square-free, or
2. $\Delta \equiv 0 \pmod 4$, $\Delta/4 \equiv 1, 2 \pmod 4$ and $\Delta/4$ is square-free.

And for the **maximum order** $\mathcal{O}$, we have (compare [Der01])

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}\frac{-\Delta + i\sqrt{\Delta}}{2},$$
$$\mathcal{O}^{\#} = \mathbb{Z}\frac{i}{\sqrt{\Delta}} + \mathbb{Z}\frac{1 + i\sqrt{\Delta}}{2}.$$

From now on, we will always work with Hermitian modular forms of degree $n = 2$. We also use $\Gamma = \mathrm{Sp}_2(\mathcal{O})$ for simplicity.

Chapter 3

Theory

Lemma 3.1. *Let* $f\colon \mathbb{H}_2 \to \mathbb{C}$ *be a Hermitian modular form of weight* k *with* $\nu \equiv 1$. *Let* $S \in \mathcal{P}_2(\mathcal{O})$. *Then,* $\tau \mapsto f(S\tau)\colon \mathbb{H}_1 \subseteq \mathbb{C} \to \mathbb{C}$ *is an Elliptic modular form of weight* $2k$ *to* $\Gamma_0(l)$, *where* l *is the denominator of* S^{-1}.

Proof. Define $\Gamma^H := \mathrm{Sp}_2(\mathcal{O})$ as the translation group for f. Let $\tau \in \mathbb{H}_1$. With $S = [s, t, u] \in \mathcal{P}_2(\mathbb{C})$ we have

$$\begin{aligned}
\Im(S\tau) &= \frac{1}{2i}\left(S\tau - \overline{S}^T\overline{\tau}\right)\\
&= \frac{1}{2i}S(\tau - \overline{\tau})\\
&= \frac{1}{2i}S \cdot 2i\Im(\tau)\\
&= S\Im(\tau) > 0,
\end{aligned}$$

thus $S\tau \in \mathbb{H}_2$. Thus, $\tau \mapsto f(S\tau)$ is a function $\mathbb{H}_1 \to \mathbb{C}$.

Let $l := \det(S)$. That is the denominator of S^{-1}. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(l) \subseteq \mathrm{SL}_2(\mathbb{Z})$. We have

$$\begin{aligned}
& S\frac{a\tau + b}{c\tau + d}\\
= \;& (a(S\tau) + bS) \cdot ((cS^{-1})(S\tau) + d)^{-1}\\
= \;& \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \cdot S\tau.
\end{aligned}$$

Define

$$M := \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \in \mathrm{Mat}_4(\mathbb{C}).$$

With $l|c$, we also have $cS^{-1} = \frac{c}{l}[u, -t, s] \in \mathrm{Mat}_2(\mathcal{O})$, thus we have $M \in \mathrm{Mat}_4(\mathcal{O})$. Recall that we have $S = \overline{S}^T$ and $ad - bc = 1$. Verify that we have $M \in \mathrm{Sp}_2(\mathcal{O}) = \Gamma^H$:

$$\begin{aligned}
& \overline{M}^T J_2 M \\
= \;& \overline{\begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix}}^T J_2 \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \\
= \;& \begin{pmatrix} (-acS^{-1} + ac\overline{S^{-1}}^T) & (-ad1_2 + cb\overline{S^{-1}}^T S) \\ (-bc\overline{S}^T S^{-1} + ad1_2) & (-bd\overline{S}^T + bdS) \end{pmatrix} \\
= \;& J_2.
\end{aligned}$$

Thus, because f is a Hermitian modular form, we have

$$\begin{aligned}
& f\left(S\frac{a\tau + b}{c\tau + d}\right) \\
= \;& f\left(M \cdot S\tau\right) \\
= \;& \nu(M) \cdot \det(cS^{-1}S\tau + d1_2)^k \cdot f(S\tau) \\
= \;& (c\tau + d)^{2k} \cdot f(S\tau).
\end{aligned}$$

Thus, f is an Elliptic modular form of weight $2k$ to $\Gamma_0(l)$. $\square$

Remark 3.2. Let us analyze the case $\nu \not\equiv 1$. According to [Der01], only for $\Delta \equiv 0 \pmod 4$, there is a single non-trivial Abel character ν. This ν has the following properties (see [Der01]):

$$\begin{aligned}
\nu(J_2) &= 1, & & \\
\nu(\mathrm{Trans}(H)) &= (-1)^{h_1 + h_4 + |h_2|^2}, & H &= [h_1, h_2, h_4] \in \mathrm{Her}_2(\mathcal{O}) \\
\nu(\mathrm{Rot}(U)) &= (-1)^{|1+u_1+u_4|^2 |1+u_2+u_3|^2 + |u_1 u_4|^2} & U &= \left(\begin{smallmatrix} u_1 & u_2 \\ u_3 & u_4 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O})
\end{aligned}$$

Consider the proof of the previos lemma. To calculate $\nu(M)$ with the given equations, we need to represent M in the creation system J_2, $\mathrm{Trans}(H)$ and $\mathrm{Rot}(U)$.

We must consider two different cases. Recall that we have $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, i.e. $ad - bc = 1$, $S = [s, t, u] \in \mathcal{P}_2(\mathcal{O})$ and

$$M = \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \in \mathrm{Sp}_2(\mathcal{O}).$$

Case 1: $c = 0$. Then we have $ad = 1$. Define $T := \frac{b}{d} S$. Then we have

$$\begin{aligned} & \operatorname{Trans}\left(\frac{b}{d}S\right) \operatorname{Rot}\left(\frac{1}{d}1_2\right) \\ = & \begin{pmatrix} 1_2 & \frac{b}{d}S \\ & 1_2 \end{pmatrix} \begin{pmatrix} \frac{1}{d}1_2 & \\ & d1_2 \end{pmatrix} \\ = & \begin{pmatrix} \frac{1}{d}1_2 & bS \\ & d1_2 \end{pmatrix} \\ = & M. \end{aligned}$$

And we have

$$\begin{aligned} \nu\left(\operatorname{Trans}\left(\frac{b}{d}S\right)\right) &= (-1)^{\frac{b}{d}s + \frac{b}{d}u + \left|\frac{b}{d}t\right|^2}, \\ \nu\left(\operatorname{Rot}\left(\frac{1}{d}1_2\right)\right) &= (-1)^{\left|1+\frac{2}{d}\right|^2 + \left|\frac{1}{d^2}\right|^2} = 1. \end{aligned}$$

Case 2: $c \neq 0$. Then we have

$$\begin{aligned} & \operatorname{Trans}\left(\frac{a}{c}S\right) \operatorname{Rot}\left(-\frac{1}{c}S\right) (-J_2) \operatorname{Trans}\left(-\frac{d}{c}S\right)^{-1} \\ = & \begin{pmatrix} 1_2 & \frac{a}{c}S \\ & 1_2 \end{pmatrix} \begin{pmatrix} -\frac{1}{c}\overline{S}^T & \\ & -cS^{-1} \end{pmatrix} (-J_2) \begin{pmatrix} 1_2 & -\frac{d}{c}S \\ & 1_2 \end{pmatrix}^{-1} \\ = & \begin{pmatrix} -\frac{1}{c}\overline{S}^T & -a1_2 \\ & -cS^{-1} \end{pmatrix} \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix} \begin{pmatrix} 1_2 & \frac{d}{c}S \\ & 1_2 \end{pmatrix} \\ = & \begin{pmatrix} -\frac{1}{c}\overline{S}^T & a1_2 \\ & -cS^{-1} \end{pmatrix} \begin{pmatrix} & 1_2 \\ -1_2 & -\frac{d}{c}S \end{pmatrix} \\ = & \begin{pmatrix} a1_2 & -\frac{1}{c}\overline{S}^T + \frac{ad}{c}S \\ cS^{-1} & d1_2 \end{pmatrix} \\ = & M. \end{aligned}$$

And we have

$$\nu\left(\mathrm{Trans}\left(\frac{a}{c}S\right)\right) = (-1)^{\frac{a}{c}s+\frac{a}{c}u+\left|\frac{a}{c}t\right|^2},$$

$$\nu\left(\mathrm{Rot}\left(-\frac{1}{c}S\right)\right) = (-1)^{\left|1-\frac{1}{c}s-\frac{1}{c}u\right|^2\left|1-\frac{2}{c}\Re(t)\right|^2+\left|\frac{su}{c^2}\right|^2},$$

$$\nu\left(-J_2\right) = -1,$$

$$\nu\left(\mathrm{Trans}\left(-\frac{d}{c}S\right)\right)^{-1} = (-1)^{-\frac{d}{c}s-\frac{d}{c}u+\left|\frac{d}{c}t\right|^2}.$$

As a conclusion for now, it looks complicated to restrict $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, i.e. the translation group Γ^E for the Elliptic modular forms, to satisfy $\nu(M) = 1$. For example, for the case $c = 0$, one fulfilling condition would be $2|b$. Thus, we will use $\nu \equiv 1$ for the rest of our work.

Lemma 3.3. *Prop 7.3. von Poor für herm Modulformen.* $\Gamma(\mathcal{L}) \supseteq \Gamma_0(l)$ *for* $l \in \mathbb{Z}^+, ls^{-1} \in \mathcal{P}_n(\mathcal{O})$.

We want to calculate a generating set for the Fourier expansions of Hermitian modular forms. Now we will formulate the main algorithm of our work.

Algorithm 3.4. We have the Hermitian modular form degree $n = 2$ fixed, as well as some Δ (for now, $\Delta \in \{3, 4, 8\}$). Then we select some form weight $k \in \mathbb{Z}$ ($k \in \{1, \ldots, 20\}$ or so), let $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ be the maximum order (see chapter 2.3.1) and some subgroup Γ of $\mathrm{Sp}_2(\mathcal{O})$. Then we select an abel character $\nu\colon \Gamma \to \mathbb{C}^\times$ of $\mathrm{Sp}_2(\mathcal{O})$.

We define the index set

$$\Lambda := \left\{ 0 \leq \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \mathrm{Mat}_2(\mathcal{O}^\#) \,\middle|\, a, c \in \mathbb{Z} \right\}.$$

Fix $B \in \mathbb{N}$ as a limit. Select a precision

$$\mathcal{F} := \left\{ \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \Lambda \,\middle|\, 0 \leq a, c < B \right\} \subseteq \Lambda.$$

1. We start with $l = 1$ and increase it but only use the square-free numbers.

2. Set $\mathcal{S} = \{\}$,

3. Enumerate matrices $S \in \mathrm{Mat}_2^T(\mathbb{Z})$, and set $\mathcal{S} \leftarrow \mathcal{S} \cup \{S\}$ and for each time you add a new matrix perform the following steps.

4. We set

$$\mathcal{M}^H_{k,\mathcal{S},\mathcal{F}} := \left\{(a[S])_{S\in\mathcal{S}} \,\middle|\, a \in \mathbb{Q}^{\mathcal{F}} \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\right\} \subseteq \bigoplus_{S\in\mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)},$$

where

$$a[S] := \mathbb{N}_0 \to \mathbb{Q}, n \mapsto \sum_{T\in\Lambda,\mathrm{tr}(ST)=n} a(T),$$

The elements $a \in \mathbb{Q}^{\mathcal{F}}$ are Fourier expansions of Elliptic modular forms ($\mathbb{H}_1 \to \mathbb{C}$) and $a(T) \in \mathbb{Q}$ for $T \in \mathcal{F} \subseteq \mathrm{Mat}_2(\mathcal{O}^{\#})$ are the Fourier coefficients. Recall that a being invariant under $\mathrm{GL}_2(\mathcal{O})$ means that we have

$$\det(U)^k a(T[U]) = a(T) \;\forall\, U \in \mathrm{GL}_2(\mathcal{O}).$$

With $[a, b, c] \in \mathcal{F}$, we have $0 \leq a, c < B$, thus there are only a finite number of possibilities. Because $0 \leq [a, b, c]$, we get $ac - |b|^2 \geq 0$ and thus b is also always limited. Thus, $\mathcal{F}$ is finite but it might be huge for even small B. Restricting the elements in $\mathcal{F}$ by the $\mathrm{GL}_2(\mathcal{O})$-invariation makes the set $\{x \in \mathcal{F} \mid x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\} \subseteq \mathcal{F}$ much smaller and better to handle in computer calculations. We use this set to identify a base of the finite dimension vector space $\left\{x \in \mathbb{Q}^{\mathcal{F}} \,\middle|\, x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\right\}$.

We identify

$$\bigoplus_{S\in\mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)} = \mathbb{Q}^N, \quad N = \sum_{S\in\mathcal{S}} \mathcal{F}(S).$$

For a given $S \in \mathcal{S}$ and limit $B \in \mathbb{N}$ which restricts $\mathcal{F} \subset \Lambda$, $\mathcal{F}(S) \in \mathbb{N}_0$ is the limit such that for any $T \in \Lambda - \mathcal{F}$, tr$(ST) \geq \mathcal{F}(S)$. Thus, for calculating the Fourier coefficients $T \in \Lambda$ with tr$(ST) \in \{0, \ldots, \mathcal{F}(S) - 1\}$, it is sufficient to enumerate the $T \in \mathcal{F}$.

Let $S = [s, t, u]$ and $T = [a, b, c]$. Recall that $S \in \mathrm{Mat}_2^T(\mathbb{Z})$. Then we have

$$\mathrm{tr}(ST) = as + \bar{t}b + t\bar{b} + cu = as + cu + 2t\Re(b).$$

Because $T \geq 0$, we have $ac \geq |b|^2$ and thus $\Re(b) \leq \sqrt{ac} \leq \max(a, c)$. Thus, $2t\Re(b) \geq -2\,|t| \max(a, c)$. We also have $as + cu \geq \max(a, c)(s + u)$. Assuming $T \in \Lambda - \mathcal{F}$, we have $\max(a, c) \geq B$. For such T, we get

$$\mathrm{tr}(ST) \geq B \cdot (s + u - 2\,|t|).$$

Given $S > 0$, we have $su > t^2$. Then we have

$$\begin{aligned} & s + u - 2\,|t| > 0 \\ \Leftrightarrow \quad & su + u^2 - 2\,|t|\,u > 0 \\ \Leftrightarrow \quad & \left(t^2 + u^2 - 2\,|t|\,u\right) + (su - t^2) > 0 \\ \Leftrightarrow \quad & (|t| - u)^2 + (su - t^2) > 0. \end{aligned}$$

All inequalities were sharp estimations[1], thus we get

$$\mathcal{F}(S) = B \cdot (s + u - 2\,|t|).$$

We want to calculate the matrix of the linear function

$$\left\{x \in \mathbb{Q}^{\mathcal{F}} \;\middle|\; x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\right\} \to \bigoplus_{S \in \mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)}, \;\; a \mapsto (a[S])_{S \in \mathcal{S}}.$$

The base of the destination room is canonical. The dimension is N. The base of the source room can be identified by $\{x \in \mathcal{F} \mid x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\}$.

And we set

$$\mathcal{M}_{k,\mathcal{S},\mathcal{F}} := \bigoplus_{S \in \mathcal{S}} \mathcal{F}\mathcal{E}_{\mathcal{F}(S)}(\mathcal{M}_k(\Gamma_0(l_S)))$$

where $\mathcal{M}_k(\Gamma_0(l_S))$ is the vectorspace of Elliptic modular forms over $\Gamma_0(l_S)$.

5. If

$$\dim \mathcal{M}^H_{k,\mathcal{S},\mathcal{F}} \cap \mathcal{M}_{k,\mathcal{S},\mathcal{F}} = \dim \mathcal{M}^{\mathbb{H}_2}_k(\Gamma, \nu),$$

then we are ready and we can reconstruct the Fourier expansion in the following way:

...

If not, then return to step 3, and enlarge $\mathcal{S}$.

[1] For example, let $S = [2, -1, 1]$. Then you have $s + u - 2\,|t| = 1$. With $c = B$ and $a = b = 1$, you hit the limit $\mathrm{tr}(ST) = 2 + B - 2 = B = \mathcal{F}(S)$.

Chapter 4

Implementation

In this chapter, we are describing the implementation.

Chapter 5

Conclusion

Blub

Chapter 6

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