Hermitian Modular Forms for Fields of Low Discriminant

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in Mathematics

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Chapter 1

Introduction

We develop an algorithm to compute Fourier expansions of Hermitian modular forms of degree 2 over $\mathrm{Sp}_2(\mathcal{O})$ for $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$, $\Delta \in \{3, 4, 8\}$.

In [PY07], spaces of Siegel modular cusp forms are calculated.

A similar algorithm is also [Rau12, Algorithm 4.3] for Jacobi forms.

We are doing the same for Hermitian modular forms.

Chapter 2

Preliminaries

$\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$ are all **integers**. $\mathbb{Q}$ are all the **rational numbers**, $\mathbb{R}$ are the **real numbers** and $\mathbb{C}$ are the **complex numbers**. $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$, $\mathbb{R}^\times$ and $\mathbb{C}^\times$ denotes all non-zero numbers.

Let $\mathrm{Mat}_n(R)$ be the set of all $n \times n$ **matrices** over some commutative ring R. Likewise, $\mathrm{Mat}_n^T(R)$ are the **symmetric** $n \times n$ matrices. X^T is the **transposed** matrix of $X \in \mathrm{Mat}_n(R)$. $\overline{Z}$ is the **conjugated** matrix of $Z \in \mathrm{Mat}_n(\mathbb{C})$. For $R \subseteq \mathbb{C}$, $\overline{R} \subseteq R$, the set of **Hermitian matrices** in R is defined as

$$\mathrm{Her}_n(R) = \left\{ Z \in \mathrm{Mat}_n(R) \,\middle|\, \overline{Z}^T = Z \right\}.$$

A matrix $Y \in \mathrm{Mat}_n(\mathbb{C})$ is greater 0 if and only if $\forall x \in \mathbb{C}^n - \{0\} : Y[x] := \overline{x}^T Y x \in \mathbb{R}^+$. Such matrices are called the **positive definitive matrices**, defined by

$$\mathcal{P}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid X > 0\}$$

for $R \subseteq \mathbb{C}$. Note that $\mathcal{P}_n(R) \subseteq \mathrm{Her}_n(R)$, i.e. all positive definite matrices are Hermitian. For a matrix over $\mathbb{R}$, it means that it is also symmetric.

For $A, X \in \mathrm{Mat}_n(\mathbb{C})$, we define $A[X] := \overline{X}^T A X$. The **denominator** of a matrix $Z \in \mathrm{Mat}_n(\mathbb{Q})$ is the smallest number $x \in \mathbb{N}$ such that $xZ \in \mathrm{Mat}_n(\mathbb{Z})$. We also write $\mathrm{denom}(Z) = x$. $1_n \in \mathrm{Mat}_n(\mathbb{Z})$ denotes the **identity matrix**. We use the **Gauß notation** $[a, b, c] := \left(\begin{smallmatrix} a & b \\ \overline{b} & c \end{smallmatrix}\right) \in \mathrm{Her}_n(\mathbb{C})$.

The **general linear group** is defined by

$$\mathrm{GL}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid \det(X) \text{ is a unit in } R\}$$

and the **special linear group** by

$$\mathrm{SL}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid \det(X) = 1\}.$$

The **orthogonal group** is defined by

$$\mathrm{O}_n(R) = \left\{X \in \mathrm{GL}_n(R) \mid X^T 1_n X = 1_n\right\} \subseteq \mathrm{GL}_n(R).$$

The **symplectic group** is defined by

$$\mathrm{Sp}_n(R) = \left\{ X \in \mathrm{GL}_{2n}(R) \,\middle|\, \overline{X}^T J_n X = J_n \right\} \subseteq \mathrm{GL}_{2n}(R) \subseteq \mathrm{Mat}_{2n}(R)$$

where $J_n := \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right) \in \mathrm{SL}_{2n}(R)$ (as in [Der01]). (Note that some authors (e.g. [PY07]) define J_n negatively.) $\mathrm{Sp}_n(R)$ is also called the **unitary group**. Note that [Der01] uses $\mathrm{U}_n(R) = \mathrm{Sp}_n(R)$. Also note that $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_1(\mathbb{Z}) \Leftrightarrow ad - bc = 1 \Leftrightarrow M \in \mathrm{SL}_2(\mathbb{Z})$. Thus, $\mathrm{Sp}_1(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$.

In addition, for a ring $R \subseteq \mathbb{C}$, define

$$\mathrm{Rot}(U) := \begin{pmatrix} \overline{U}^T & \\ & U^{-1} \end{pmatrix} \in \mathrm{Sp}_2(R), \qquad U \in \mathrm{GL}_2(R)$$

$$\mathrm{Trans}(H) := \begin{pmatrix} 1_2 & H \\ & 1_2 \end{pmatrix} \in \mathrm{Sp}_2(R), \qquad H \in \mathrm{Her}_2(R)$$

and note that we have $J_2 = \left(\begin{smallmatrix} & -1_2 \\ 1_2 & \end{smallmatrix}\right) \in \mathrm{Sp}_2(R)$. Those tree types of matrices form a generator set for the group $\mathrm{Sp}_2(R)$.

For $Z \in \mathrm{Mat}_n(\mathbb{C})$, we call

$$\Re(Z) := \frac{1}{2}\left(Z + \overline{Z}^T\right) \in \mathrm{Mat}_n(\mathbb{C})$$

the **real** part and

$$\Im(Z) := \frac{1}{2i}\left(Z - \overline{Z}^T\right) \in \mathrm{Mat}_n(\mathbb{C})$$

the **imaginary** part of Z and we have $Z = \Re(Z) + i\Im(Z)$. Note that we usually have $\Re(Z), \Im(Z) \notin \mathrm{Mat}_n(\mathbb{R})$ but we have $\Re(Z), \Im(Z) \in \mathrm{Her}_n(\mathbb{C})$.

We say that some function $f\colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{A} \subseteq \mathrm{Mat}_n(R)$, $\mathcal{B} \subseteq R$ is k**-invariant** under some $\mathcal{X} \subseteq \mathrm{Mat}_n(R)$ where $\mathcal{A}[\mathcal{X}] \subseteq \mathcal{A}$ if and only if $\det(U)^k f(T[U]) = f(T)$ for all $T \in \mathcal{A}$, $U \in \mathcal{X}$.

2.1 Elliptic modular forms

Elliptic modular forms are holomorphic functions over the set

$$\mathcal{H}_1 := \{z \in \mathbb{C} \mid \Im(z) > 0\} \subseteq \mathbb{C}$$

which is called the **Poincaré upper half plane**.

Let f be a holomorphic function $\mathcal{H}_1 \to \mathbb{C}$. **Modular forms** are functions which are invariant with regard to a specific **translation**. In this case, the translation is given by some $M \in \mathrm{Sp}_1(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$ and a **weight** $k \in \mathbb{Z}$.

Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_1(\mathbb{Z})$ and $\tau \in \mathcal{H}_1$. We write

$$M\tau := \frac{a\tau + b}{c\tau + d}.$$

Note that we have $\Im(M\tau) = \frac{\Im(\tau)}{(c\Re(\tau)+d)^2+(c\Im(\tau))^2} > 0$ and thus $M\tau \in \mathcal{H}_1$. We define the **translated function** $f|M \colon \mathcal{H}_1 \to \mathbb{C}$ as

$$(f|M)(\tau) := (c\tau + d)^{-k} \cdot f(M\tau).$$

Let Γ be a subgroup of $\mathrm{Sp}_1(\mathbb{Z})$. We also call Γ the **translation group**.

An **Elliptic modular form** with weight $k \in \mathbb{Z}$ over Γ is a holomorphic function

$$f \colon \mathcal{H}_1 \to \mathbb{C}$$

with

$$\begin{aligned} &(1)\ \ f|M = f \quad \forall\, M \in \Gamma, \\ &(2)\ \ f(\tau) = O(1) \quad \text{for } \tau \to i\infty. \end{aligned}$$

Thus, (1) yields the equation

$$f\left(\frac{a\tau + b}{b\tau + c}\right) = (c\tau + d)^k \cdot f(\tau) \quad \forall\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma, \tau \in \mathcal{H}_1.$$

$\mathcal{M}_k(\Gamma)$ denotes the vector space of such Elliptic modular forms.

In this work, we use a specific subgroup of $\mathrm{Sp}_1(\mathbb{Z})$. We define

$$\Gamma_0(l) := \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{Sp}_1(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{l} \right\} \subseteq \mathrm{Sp}_1(\mathbb{Z}) \subseteq \mathrm{Mat}_2(\mathbb{Z})$$

as a subgroup of $\mathrm{Sp}_1(\mathbb{Z})$.

An **Elliptic modular cusp form** is an Elliptic modular form $f \colon \mathcal{H}_1 \to \mathbb{C}$ with

$$\lim_{t \to \infty} f(it) = 0.$$

We can represent the cusps with $\Gamma \backslash \mathbb{Q}$.

More general cusps: $\Gamma \backslash \mathrm{SL}_2(\mathbb{Q}) \div \Gamma_{\infty,\mathbb{Q}}$, where $\Gamma_{\infty,\mathbb{Q}}$ are the upper triangular matrices in $\mathrm{GL}_2(\mathbb{Z})$.

2.2 Siegel modular forms

Siegel modular forms are a generalization of Elliptic modular forms for higher dimensions. Let

$$\mathcal{H}_n := \left\{ Z \in \mathrm{Mat}_n^T(\mathbb{C}) \,\middle|\, \Im(Z) > 0 \right\}$$

be the **Siegel upper half space**. We call $\mathrm{Sp}_n(\mathbb{Z})$ the **Siegel modular group**. Siegel modular forms are holomorphic functions $\mathcal{H}_n \to \mathbb{C}$ for a given **degree** $n \in \mathbb{N}$.

The **translation group** Γ is a subgroup of $\mathrm{Sp}_n(\mathbb{Z})$. For $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$ and $Z \in \mathcal{H}_n$, we write

$$M \cdot Z := (AZ + B) \cdot (CZ + D)^{-1}.$$

Again, we can confirm that $M \cdot Z \in \mathcal{H}_n$. Generalizing the Elliptic translation, the Siegel **translated function** $f|M \colon \mathcal{H}_n \to \mathbb{C}$ is defined as

$$(f|M)(Z) := \det(CZ + D)^{-k} \cdot f(M \cdot Z)$$

A **Siegel modular form** of degree $n \in \mathbb{N}$ with weight $k \in \mathbb{Z}$ over Γ is a holomorphic function

$$f \colon \mathcal{H}_n \to \mathbb{C}$$

with

$$\begin{aligned} &(1)\ \ f|M = f \quad \forall\, M \in \Gamma, \\ &(2)\ \ \text{for } n = 1\colon f(Z) = O(1) \quad \text{for } Z \to i\infty \end{aligned}$$

$\mathcal{M}_k^{\mathcal{H}_n}(\Gamma)$ denotes the vector space of such Siegel modular forms.

Note that Elliptic modular forms are Siegel modular forms of degree $n = 1$. Thus we have $\mathcal{M}_k(\Gamma) = \mathcal{M}_k^{\mathcal{H}_1}(\Gamma)$.

Siegel modular forms aren't directly used in this work. However, the idea of this work is inspired by [PY07] and they are using them.

2.3 Hermitian modular forms

Let

$$\mathbb{H}_n := \{Z \in \mathrm{Mat}_n(\mathbb{C}) \mid \Im(Z) > 0\}$$

be the **Hermitian upper half space**. Note that these matrices are not symmetric as the Siegel upper half space $\mathcal{H}_n$ but we have $\mathcal{H}_n \subseteq \mathbb{H}_n$ and $\mathcal{H}_1 = \mathbb{H}_1 \subseteq \mathbb{C}$.

Hermitian modular forms are holomorphic functions $\mathbb{H}_n \to \mathbb{C}$. They are a generalization of Siegel modular forms where the **translation group** Γ is not a subgroup of $\mathrm{Sp}_n(\mathbb{Z})$ but a subgroup of $\mathrm{Sp}_n(\mathcal{O})$ for some $\mathcal{O} \subseteq \mathbb{C}$.

More specificially, let $\Delta \in \mathbb{N}$ so that we have the imaginary quadratic number field $\mathbb{K} := \mathbb{Q}(\sqrt{-\Delta})$ where $-\Delta$ is the fundamental discriminant. Then, let $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ be the maximum order. We call $\mathrm{Sp}_n(\mathcal{O})$ the **Hermitian modular group**. Let Γ be a subgroup of $\mathrm{Sp}_n(\mathcal{O})$.

Again, with $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathcal{O})$, $Z \in \mathbb{H}_n$, $M \cdot Z := (AZ + B) \cdot (CZ + D)^{-1} \in \mathbb{H}_n$ as for Siegel modular forms and the **weight** $k \in \mathbb{Z}$, we define the **translated function** $f|M \colon \mathbb{H}_n \to \mathbb{C}$ as

$$(f|M)(Z) := \det(CZ + D)^{-k} \cdot f(M \cdot Z).$$

A **Hermitian modular form** of **degree** $n \in \mathbb{N}$ with **weight** $k \in \mathbb{Z}$ over Γ is a holomorphic function

$$f \colon \mathbb{H}_n \to \mathbb{C}$$

with

(1) $f|M = f \quad \forall\, M \in \Gamma, Z \in \mathbb{H}_n$,

(2) for $n = 1$: f is holomorphic in all cusps.

$\mathcal{M}_k^{\mathbb{H}_n}(\Gamma)$ denotes the vector space of such Hermitian modular forms.

As it can be done for Siegel modular forms, we generalize this further by introducing a **Multiplicative character** $\nu \colon \Gamma \to \mathbb{C}^\times$. Thus, for $M_1, M_2 \in \Gamma$, we have $\nu(M_1) \cdot \nu(M_2) = \nu(M_1 \cdot M_2)$.

A **Hermitian modular form** over Γ and ν is a holomorphic function

$$f \colon \mathbb{H}_n \to \mathbb{C}$$

with

(1) $f|M = \nu(M) \cdot f \quad \forall\, M \in \Gamma, Z \in \mathbb{H}_n$,

(2) for $n = 1$: f is holomorphic in all cusps.

$\mathcal{M}_k^{\mathbb{H}_n}(\Gamma, \nu)$ denotes the vector space of such Hermitian modular forms.

For $f \in \mathcal{M}_k^{\mathbb{H}_n}(\Gamma, \nu)$, define the **Siegel Φ-operator** as

$$(f|\Phi)(Z') := \lim_{t \to \infty} f \begin{pmatrix} Z' & 0 \\ 0 & it \end{pmatrix}, \quad Z' \in \mathbb{H}_{n-1}.$$

Then (see [Der01]), $f|\Phi \colon \mathbb{H}_{n-1} \to \mathbb{C}$ is a well-defined Hermitian modular form of degree $n - 1$.

A Hermitian modular form $f \in \mathcal{M}_k^{\mathbb{H}_n}(\Gamma, \nu)$ is a **Hermitian modular cusp form**, if and only if for all $R \in \mathrm{Sp}_n(\mathbb{K})$, it holds

$$(f|R)|\Phi \equiv 0.$$

In this work, we will always use Hermitian modular forms of degree $n = 2$.

2.3.1 Properties

Because $-\Delta$ is fundamental, we have two possible cases:

1. $\Delta \equiv 3 \pmod 4$ and Δ is square-free, or
2. $\Delta \equiv 0 \pmod 4$, $\Delta/4 \equiv 1, 2 \pmod 4$ and $\Delta/4$ is square-free.

And for the **maximal order** $\mathcal{O}$, we have (compare [Der01])

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}\frac{-\Delta + i\sqrt{\Delta}}{2},$$
$$\mathcal{O}^{\#} = \mathbb{Z}\frac{i}{\sqrt{\Delta}} + \mathbb{Z}\frac{1 + i\sqrt{\Delta}}{2}.$$

From now on, we will always work with Hermitian modular forms of degree $n = 2$. We also use $\Gamma = \mathrm{Sp}_2(\mathcal{O})$ for simplicity.

Chapter 3

Theory

Lemma 3.1. *Let $f\colon \mathbb{H}_2 \to \mathbb{C}$ be a Hermitian modular form of weight k with $\nu \equiv 1$. Let $S \in \mathcal{P}_2(\mathcal{O})$. Then, $\tau \mapsto f(S\tau)\colon \mathbb{H}_1 \subseteq \mathbb{C} \to \mathbb{C}$ is an Elliptic modular form of weight $2k$ to $\Gamma_0(l)$, where l is the denominator of S^{-1}.*

We write

$$f[S]\colon \mathbb{H}_1 \to \mathbb{C}, \quad \tau \mapsto f(S\tau).$$

Proof. Define $\Gamma^H := \mathrm{Sp}_2(\mathcal{O})$ as the translation group for f. Let $\tau \in \mathbb{H}_1$. With $S = [s,t,u] \in \mathcal{P}_2(\mathbb{C})$ we have

$$\begin{aligned}
\Im(S\tau) &= \frac{1}{2i}\left(S\tau - \overline{S}^T\overline{\tau}\right)\\
&= \frac{1}{2i}S(\tau - \overline{\tau})\\
&= \frac{1}{2i}S \cdot 2i\Im(\tau)\\
&= S\Im(\tau) > 0,
\end{aligned}$$

thus $S\tau \in \mathbb{H}_2$. Thus, $\tau \mapsto f(S\tau)$ is a function $\mathbb{H}_1 \to \mathbb{C}$.

Let $l := \det(S)$. That is the denominator of S^{-1}. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(l) \subseteq \mathrm{SL}_2(\mathbb{Z})$. We have

$$\begin{aligned}
& S\frac{a\tau + b}{c\tau + d}\\
= \ & (a(S\tau) + bS) \cdot ((cS^{-1})(S\tau) + d)^{-1}\\
= \ & \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \cdot S\tau.
\end{aligned}$$

Define

$$M := \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \in \mathrm{Mat}_4(\mathbb{C}).$$

With $l|c$, we also have $cS^{-1} = \frac{c}{l}[u,-t,s] \in \mathrm{Mat}_2(\mathcal{O})$, thus we have $M \in \mathrm{Mat}_4(\mathcal{O})$. Recall that we have $S = \overline{S}^T$ and $ad - bc = 1$. Verify that we have $M \in \mathrm{Sp}_2(\mathcal{O}) = \Gamma^H$:

$$\begin{aligned}
& \overline{M}^T J_2 M \\
= \;& \overline{\begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix}}^T J_2 \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \\
= \;& \begin{pmatrix} (-acS^{-1} + ac\overline{S^{-1}}^T) & (-ad1_2 + cb\overline{S^{-1}}^T S) \\ (-bc\overline{S}^T S^{-1} + ad1_2) & (-bd\overline{S}^T + bdS) \end{pmatrix} \\
= \;& J_2.
\end{aligned}$$

Thus, because f is a Hermitian modular form, we have

$$\begin{aligned}
& f[S]\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau\right) \\
= \;& f\left(S\frac{a\tau + b}{c\tau + d}\right) \\
= \;& f\left(M \cdot S\tau\right) \\
= \;& \nu(M) \cdot \det(cS^{-1}S\tau + d1_2)^k \cdot f(S\tau) \\
= \;& (c\tau + d)^{2k} \cdot f[S](\tau).
\end{aligned}$$

This is the same as

$$(f[S])|_{2k}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = f[S].$$

It follows that $f[S]$ is an Elliptic modular form of weight $2k$ to $\Gamma_0(l)$. $\square$

Remark 3.2. Let us analyze the case $\nu \not\equiv 1$. According to [Der01], only for $\Delta \equiv 0 \pmod 4$, there is a single non-trivial Abel character ν. This ν has the following properties (see [Der01]):

$$\begin{aligned}
\nu(J_2) &= 1, & & \\
\nu(\mathrm{Trans}(H)) &= (-1)^{h_1 + h_4 + |h_2|^2}, & & H = [h_1, h_2, h_4] \in \mathrm{Her}_2(\mathcal{O}) \\
\nu(\mathrm{Rot}(U)) &= (-1)^{|1+u_1+u_4|^2 |1+u_2+u_3|^2 + |u_1 u_4|^2} & & U = \left(\begin{smallmatrix} u_1 & u_2 \\ u_3 & u_4 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O})
\end{aligned}$$

Consider the proof of the previous lemma. To calculate $\nu(M)$ with the given equations, we need to represent M in the generating system J_2, $\mathrm{Trans}(H)$ and $\mathrm{Rot}(U)$.

We must consider two different cases. Recall that we have $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, i.e. $ad - bc = 1$, $S = [s,t,u] \in \mathcal{P}_2(\mathcal{O})$ and

$$M = \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \in \mathrm{Sp}_2(\mathcal{O}).$$

Case 1: $c = 0$. Then we have $ad = 1$. Define $T := \frac{b}{d}S$. Then we have

$$\begin{aligned}
& \operatorname{Trans}\left(\frac{b}{d}S\right)\operatorname{Rot}\left(\frac{1}{d}1_2\right) \\
= \; & \begin{pmatrix} 1_2 & \frac{b}{d}S \\ & 1_2 \end{pmatrix}\begin{pmatrix} \frac{1}{d}1_2 & \\ & d1_2 \end{pmatrix} \\
= \; & \begin{pmatrix} \frac{1}{d}1_2 & bS \\ & d1_2 \end{pmatrix} \\
= \; & M.
\end{aligned}$$

And we have

$$\begin{aligned}
\nu\left(\operatorname{Trans}\left(\frac{b}{d}S\right)\right) &= (-1)^{\frac{b}{d}s + \frac{b}{d}u + \left|\frac{b}{d}t\right|^2}, \\
\nu\left(\operatorname{Rot}\left(\frac{1}{d}1_2\right)\right) &= (-1)^{\left|1+\frac{2}{d}\right|^2 + \left|\frac{1}{d^2}\right|^2} = 1.
\end{aligned}$$

Case 2: $c \neq 0$. Then we have

$$\begin{aligned}
& \operatorname{Trans}\left(\frac{a}{c}S\right)\operatorname{Rot}\left(-\frac{1}{c}S\right)(-J_2)\operatorname{Trans}\left(-\frac{d}{c}S\right)^{-1} \\
= \; & \begin{pmatrix} 1_2 & \frac{a}{c}S \\ & 1_2 \end{pmatrix}\begin{pmatrix} -\frac{1}{c}\overline{S}^T & \\ & -cS^{-1} \end{pmatrix}(-J_2)\begin{pmatrix} 1_2 & -\frac{d}{c}S \\ & 1_2 \end{pmatrix}^{-1} \\
= \; & \begin{pmatrix} -\frac{1}{c}\overline{S}^T & -a1_2 \\ & -cS^{-1} \end{pmatrix}\begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix}\begin{pmatrix} 1_2 & \frac{d}{c}S \\ & 1_2 \end{pmatrix} \\
= \; & \begin{pmatrix} -\frac{1}{c}\overline{S}^T & a1_2 \\ & -cS^{-1} \end{pmatrix}\begin{pmatrix} & 1_2 \\ -1_2 & -\frac{d}{c}S \end{pmatrix} \\
= \; & \begin{pmatrix} a1_2 & -\frac{1}{c}\overline{S}^T + \frac{ad}{c}S \\ cS^{-1} & d1_2 \end{pmatrix} \\
= \; & M.
\end{aligned}$$

And we have

$$\begin{aligned}
\nu\left(\mathrm{Trans}\left(\frac{a}{c}S\right)\right) &= (-1)^{\frac{a}{c}s+\frac{a}{c}u+\left|\frac{a}{c}t\right|^2},\\
\nu\left(\mathrm{Rot}\left(-\frac{1}{c}S\right)\right) &= (-1)^{\left|1-\frac{1}{c}s-\frac{1}{c}u\right|^2\left|1-\frac{2}{c}\Re(t)\right|^2+\left|\frac{su}{c^2}\right|^2},\\
\nu\left(-J_2\right) &= -1,\\
\nu\left(\mathrm{Trans}\left(-\frac{d}{c}S\right)\right)^{-1} &= (-1)^{-\frac{d}{c}s-\frac{d}{c}u+\left|\frac{d}{c}t\right|^2}.
\end{aligned}$$

As a conclusion for now, it looks complicated to restrict $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, i.e. the translation group Γ^E for the Elliptic modular forms, to satisfy $\nu(M) = 1$. For example, for the case $c = 0$, one fulfilling condition would be $2|b$.

To avoid such complications, we will use $\nu \equiv 1$ for the rest of our work.

Preliminaries 3.3. We want to calculate a generating set for the Fourier expansions of Hermitian modular forms.

We define the index set

$$\Lambda := \left\{ 0 \leq \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \mathrm{Mat}_2(\mathcal{O}^{\#}) \,\middle|\, a, c \in \mathbb{Z} \right\}$$

as the index for the Fourier coefficients of the Fourier expansions of our Hermitian modular forms.

Remark 3.4. For a holomorphic function $f\colon \mathbb{H}_2 \to \mathbb{C}$, we write its Fourier expansion as

$$f(Z) = \sum_{T\in\Lambda} a(T) \cdot e^{2\pi i\cdot \mathrm{tr}(TZ)}$$

with its Fourier coefficients $a\colon \Lambda \to \mathbb{Q}$.

For any $S \in \mathcal{P}_2(\mathcal{O})$, for the restricted function $f[S]\colon \mathbb{H}_1 \to \mathbb{C}$, this gives us

$$f[S](\tau) = \sum_{T\in\Lambda} a(T) \cdot e^{2\pi i\cdot \mathrm{tr}(TS\tau)}.$$

We use $a[S]\colon \mathbb{N}_0 \to \mathbb{Q}$ for the Fourier coefficients of $f[S]$, i.e. we have

$$f[S](\tau) = \sum_{n\in\mathbb{N}_0} a[S](n) \cdot e^{2\pi i n\tau}.$$

This gives us

$$a[S](n) = \sum_{T \in \Lambda, \mathrm{tr}(ST)=n} a(T).$$

Fix $B_{\mathcal{F}} \in \mathbb{N}$ as a limit. Select a precision of the Fourier coefficient index

$$\mathcal{F} := \mathcal{F}_B := \left\{ \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \Lambda \,\middle|\, 0 \leq a, c < B_{\mathcal{F}} \right\} \subseteq \Lambda.$$

The main algorithm is going to be described in 3.8. It will start with the vectorspace of all possible Fourier expansions for the precision index set $\mathcal{F}$ and reduce that vectorspace.

Lemma 3.5. *Given a Hermitian modular form f and its Fourier expansion coefficients $a\colon \mathcal{F}_B \to \mathbb{Q}$ of the precision index set $\mathcal{F}_B$ and a matrix $S = [s, t, u] \in \mathcal{P}_2(\mathcal{O})$, the precision of the Fourier expansion of $f[S]$ is given by*

$$\mathcal{F}(S) = B \cdot (s + u - 2\,|t|).$$

Thus, we can calculate the Fourier expansion coefficients (as described in remark 3.4)

$$a[S]\colon\ \{k \in N_0 \mid k < \mathcal{F}(S)\} \to \mathbb{Q}.$$

Proof. For a given $S \in \mathcal{S}$ and limit $B \in \mathbb{N}$ which restricts $\mathcal{F} \subset \Lambda$, $\mathcal{F}(S) \in \mathbb{N}_0$ is the limit such that for any $T \in \Lambda - \mathcal{F}$, $\mathrm{tr}(ST) \geq \mathcal{F}(S)$. Thus, for calculating the Fourier coefficients $T \in \Lambda$ with $\mathrm{tr}(ST) \in \{0, \ldots, \mathcal{F}(S) - 1\}$, it is sufficient to enumerate the $T \in \mathcal{F}$.

Let $S = [s, t, u]$ and $T = [a, b, c]$. Recall that $S \in \mathcal{P}_2(\mathcal{O})$. Then we have

$$\mathrm{tr}(ST) = as + \bar{t}b + t\bar{b} + cu = as + cu + 2\Re(\bar{t}b).$$

Because $T \geq 0$, we have $ac \geq |b|^2$ and thus

$$|b| \leq \sqrt{ac} \leq \max(a, c).$$

Thus,

$$2\Re(\bar{t}b) \geq -2\,|t|\,|b| \geq -2\,|t| \max(a, c).$$

We also have $as + cu \geq \max(a,c)(s+u)$. Assuming $T \in \Lambda - \mathcal{F}$, we have $\max(a,c) \geq B$. For such T, we get

$$\operatorname{tr}(ST) \geq B \cdot (s + u - 2\,|t|).$$

Given $S > 0$, we have $su > |t|^2$. Then we have

$$\begin{aligned} & s + u - 2\,|t| > 0 \\ \Leftrightarrow\quad & su + u^2 - 2\,|t|\,u > 0 \\ \Leftrightarrow\quad & \left(|t|^2 + u^2 - 2\,|t|\,u\right) + (su - |t|^2) > 0 \\ \Leftrightarrow\quad & (|t| - u)^2 + (su - |t|^2) > 0. \end{aligned}$$

Thus, for $B > 0$, we have

$$B \cdot (s + u - 2\,|t|) > 0.$$

All inequalities were sharp estimations[1], thus we get

$$\mathcal{F}(S) = B \cdot (s + u - 2\,|t|).$$

□

Remark 3.6. Let $\mathcal{M}_i$ be a sub vector space of Fourier expansions $a\colon \mathcal{F} \to \mathbb{Q}$. Remark 3.4 and lemma 3.5 gives us the tools to reduce $\mathcal{M}_i$ to a sub vector space $\mathcal{M}_{i+1} \subset \mathcal{M}_i$.

For a given $S \in \mathcal{P}_2(\mathcal{O})$, calculating the restrictions $a \mapsto a[S]$ for all $a \in \mathcal{M}_i$, we must only get Fourier expansions of Elliptic modular forms.

Remark 3.7. With $[a,b,c] \in \mathcal{F}$, we have $0 \leq a, c < B$, thus there are only a finite number of possible $(a,c) \in \mathbb{N}_0^2$. Because $0 \leq [a,b,c]$, we get $ac - |b|^2 \geq 0$ and thus b is also always limited. Thus, $\mathcal{F}$ is finite but it might be huge for even small B. For example[2],

$$\begin{aligned} &\text{for } D = -3, B = 10, \quad \text{we have } \#\mathcal{F} = 13398. \\ &\text{for } D = -3, B = 20, \quad \text{we have } \#\mathcal{F} = 252598. \end{aligned}$$

[1] For example, let $S = [2, -1, 1]$. Then you have $s + u - 2\,|t| = 1$. With $c = B$ and $a = b = 1$, you hit the limit $\operatorname{tr}(ST) = 2 + B - 2 = B = \mathcal{F}(S)$.

[2] This example was calculated with the code at [Zey].

Because we want $a \in \mathbb{Q}^{\mathcal{F}}$ to be Fourier expansions of Hermitian modular forms, we can assume that a is invariant under $\mathrm{GL}_2(\mathcal{O})$. This means that we have

$$\det(U)^k a(T[U]) = a(T) \quad \forall\, U \in \mathrm{GL}_2(\mathcal{O}).$$

Restricting the elements in $\mathcal{F}$ by the $\mathrm{GL}_2(\mathcal{O})$-invariation makes the set $\mathcal{F}^{\mathrm{GL}_2(\mathcal{O})} \subseteq \mathcal{F}$ much smaller and better to handle in computer calculations. For example,

$$\text{for } D = -3, B = 10, \quad \text{we have } \#\left(\mathcal{F}^{\mathrm{GL}_2(\mathcal{O})}\right) = 420,$$
$$\text{for } D = -3, B = 20, \quad \text{we have } \#\left(\mathcal{F}^{\mathrm{GL}_2(\mathcal{O})}\right) = 4840.$$

We use this set to identify a base of the finite dimension vector space $\left(\mathbb{Q}^{\mathcal{F}}\right)^{\mathrm{GL}_2(\mathcal{O})}$.

Algorithm 3.8. We have the Hermitian modular form degree $n = 2$ fixed, as well as some Δ (for now, $\Delta \in \{3, 4, 8\}$). Then we select some form weight $k \in \mathbb{Z}$ ($k \in \{1, \dots, 20\}$ or so), let $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ be the maximal order (see chapter 2.3.1) and some subgroup Γ of $\mathrm{Sp}_2(\mathcal{O})$. Then we select an abel character $\nu\colon \Gamma \to \mathbb{C}^\times$ of $\mathrm{Sp}_2(\mathcal{O})$ (we just use $\nu \equiv 1$, see remark 3.2).

1. Enumerate matrices $S \in \mathcal{P}_2(\mathcal{O})$ and for each matrix perform the following steps.

2. We set

   $$\mathcal{M}^H_{k,\mathcal{S},\mathcal{F}} := \left\{(a[S])_{S\in\mathcal{S}} \,\middle|\, a \in \mathbb{Q}^{\mathcal{F}} \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\right\} \subseteq \bigoplus_{S\in\mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)}.$$

   The elements $a \in \mathbb{Q}^{\mathcal{F}}$ are Fourier expansions of Elliptic modular forms ($\mathbb{H}_1 \to \mathbb{C}$) and $a(T) \in \mathbb{Q}$ for $T \in \mathcal{F} \subseteq \mathrm{Mat}_2(\mathcal{O}^\#)$ are the Fourier coefficients.

   We identify

   $$\bigoplus_{S\in\mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)} = \mathbb{Q}^N, \;\; N = \sum_{S\in\mathcal{S}} \mathcal{F}(S).$$

   See lemma 3.5.

   We want to calculate the matrix of the linear function

   $$\mathbb{Q}^{\mathcal{F}^{\mathrm{GL}_2(\mathcal{O})}} \to \bigoplus_{S\in\mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)}, \;\; a \mapsto (a[S])_{S\in\mathcal{S}}.$$

   The base of the destination room is canonical. The dimension is N. The base of the source room can be identified by $\mathcal{F}^{\mathrm{GL}_2(\mathcal{O})}$.

And we set

$$\mathcal{M}_{k,\mathcal{S},\mathcal{F}} := \bigoplus_{S \in \mathcal{S}} \mathcal{F}\mathcal{E}_{\mathcal{F}(S)}(\mathcal{M}_k(\Gamma_0(l_S)))$$

where $\mathcal{M}_k(\Gamma_0(l_S))$ is the vectorspace of Elliptic modular forms over $\Gamma_0(l_S)$.

3. If

$$\dim \mathcal{M}^H_{k,\mathcal{S},\mathcal{F}} \cap \mathcal{M}_{k,\mathcal{S},\mathcal{F}} = \dim \mathcal{M}^{\mathbb{H}_2}_k(\Gamma, \nu),$$

then we are ready and we can reconstruct the Fourier expansion in the following way:

...

If not, then return to step 1, and enlarge $\mathcal{S}$.

Chapter 4

Implementation

In this chapter, we are describing the implementation.

Chapter 5

Conclusion

Blub

Chapter 6

References

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[Zey] Code repository for this work: `https://github.com/albertz/diplom-thesis-math/`.