

# HERMITIAN MODULAR FORMS FOR FIELDS OF LOW DISCRIMINANT

DIPLOMA THESIS  
in Mathematics

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# Chapter 1

## Introduction

We develop an algorithm to compute Fourier expansions of Hermitian modular forms of degree 2 over  $\mathrm{Sp}_2(\mathcal{O})$  for  $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ ,  $\Delta \in \{3, 4, 8\}$ .

In [PY07], spaces of Siegel modular cusp forms are calculated.

A similar algorithm is also [Rau12, Algorithm 4.3] for Jacobi forms.

We are doing the same for Hermitian modular forms.

## Chapter 2

### Preliminaries

$\mathbb{N}$  denotes the set  $\{1, 2, 3, \dots\}$ ,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$  and  $\mathbb{Z}$  are all **integers**.  $\mathbb{Q}$  are all the **rational numbers**,  $\mathbb{R}$  are the **real numbers** and  $\mathbb{C}$  are the **complex numbers**.  $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$ ,  $\mathbb{R}^\times$  and  $\mathbb{C}^\times$  denotes all non-zero numbers.

Let  $\text{Mat}_n(R)$  be the set of all  $n \times n$  **matrices** over some commutative ring  $R$ . Likewise,  $\text{Mat}_n^T(R)$  are the **symmetric**  $n \times n$  matrices.  $X^T$  is the **transposed** matrix of  $X \in \text{Mat}_n(R)$ .  $\bar{Z}$  is the **conjugated** matrix of  $Z \in \text{Mat}_n(\mathbb{C})$ . For  $R \subseteq \mathbb{C}$ ,  $\bar{R} \subseteq R$ , the set of **Hermitian matrices** in  $R$  is defined as

$$\text{Her}_n(R) = \left\{ Z \in \text{Mat}_n(R) \mid \bar{Z}^T = Z \right\}.$$

A matrix  $Y \in \text{Mat}_n(\mathbb{C})$  is greater 0 if and only if  $\forall x \in \mathbb{C}^n - \{0\} : Y[x] := \bar{x}^T Y x \in \mathbb{R}^+$ . Such symmetric matrices are called the **positive definite matrices**, defined by

$$\mathcal{P}_n(R) = \{X \in \text{Mat}_n(R) \mid X > 0\}$$

for  $R \subseteq \mathbb{C}$ . Note that  $\mathcal{P}_n(\mathbb{C}) \subseteq \text{Her}_n(\mathbb{C})$ .

For  $A, X \in \text{Mat}_n(\mathbb{C})$ , we define  $A[X] := \bar{X}^T A X$ . The **denominator** of a matrix  $Z \in \text{Mat}_n(\mathbb{Q})$  is the smallest number  $x \in \mathbb{N}$  such that  $xZ \in \text{Mat}_n(\mathbb{Z})$ . We also write  $\text{denom}(Z) = x$ .  $1_n \in \text{Mat}_n(\mathbb{Z})$  denotes the **identity matrix**. We use the **Gauß notation**  $[a, b, c] := \begin{pmatrix} a & b \\ & c \end{pmatrix} \in \text{Her}_n(\mathbb{C})$ .

The **general linear group** is defined by

$$\text{GL}_n(R) = \{X \in \text{Mat}_n(R) \mid \det(X) \text{ is a unit in } R\}$$

and the **special linear group** by

$$\text{SL}_n(R) = \{X \in \text{Mat}_n(R) \mid \det(X) = 1\}.$$

The **orthogonal group** is defined by

$$\text{O}_n(R) = \{X \in \text{GL}_n(R) \mid X^T 1_n X = 1_n\} \subseteq \text{GL}_n(R).$$

The **symplectic group** is defined by

$$\text{Sp}_n(R) = \left\{ X \in \text{GL}_{2n}(R) \mid \bar{X}^T J_n X = J_n \right\} \subseteq \text{GL}_{2n}(R) \subseteq \text{Mat}_{2n}(R)$$

where  $J_n := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \in \mathrm{SL}_{2n}(R)$  (as in [Der01]). (Note that some authors (e.g. [PY07]) define  $J_n$  negatively.)  $\mathrm{Sp}_n(R)$  is also called the **unitary group**. Note that [Der01] uses  $\mathrm{U}_n(R) = \mathrm{Sp}_n(R)$ . Also note that  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_1(\mathbb{Z}) \Leftrightarrow ad - bc = 1 \Leftrightarrow M \in \mathrm{SL}_2(\mathbb{Z})$ . Thus,  $\mathrm{Sp}_1(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$ .

In addition, for a ring  $R \subseteq \mathbb{C}$ , define

$$\begin{aligned} \mathrm{Rot}(U) &= \begin{pmatrix} \overline{U}^T & \\ & U^{-1} \end{pmatrix} \in \mathrm{Sp}_2(R), & U &\in \mathrm{GL}_2(R) \\ \mathrm{Trans}(H) &= \begin{pmatrix} 1_2 & H \\ & 1_2 \end{pmatrix} \in \mathrm{Sp}_2(R), & H &\in \mathrm{Her}_2(R) \end{aligned}$$

and note that we have  $J_2 = \begin{pmatrix} & -1_2 \\ 1_2 & \end{pmatrix} \in \mathrm{Sp}_2(R)$ . Those tree types of matrices form a generator set for the group  $\mathrm{Sp}_2(R)$ .

For  $Z \in \mathrm{Mat}_n(\mathbb{C})$ , we call

$$\Re(Z) = \frac{1}{2} (Z + \overline{Z}^T) \in \mathrm{Mat}_n(\mathbb{C})$$

the **real part** and

$$\Im(Z) = \frac{1}{2i} (Z - \overline{Z}^T) \in \mathrm{Mat}_n(\mathbb{C})$$

the **imaginary part** of  $Z$  and we have  $Z = \Re(Z) + i\Im(Z)$ . Note that we usually have  $\Re(Z), \Im(Z) \notin \mathrm{Mat}_n(\mathbb{R})$  but we have  $\Re(Z), \Im(Z) \in \mathrm{Her}_n(\mathbb{C})$ .

We say that some function  $f: \mathcal{A} \rightarrow \mathcal{B}$  with  $\mathcal{A} \subseteq \mathrm{Mat}_n(R)$ ,  $\mathcal{B} \subseteq R$  is  **$k$ -invariant** under some  $\mathcal{X} \subseteq \mathrm{Mat}_n(R)$  where  $\mathcal{A}[\mathcal{X}] \subseteq \mathcal{A}$  if and only if  $\det(U)^k f(T[U]) = f(T)$  for all  $T \in \mathcal{A}$ ,  $U \in \mathcal{X}$ .

## 2.1 Siegel modular forms

Siegel modular forms aren't directly used in this work. However, the idea of this work is inspired by [PY07] and they are using them. Also, they are a generalization of Elliptic modular forms.

Let  $\mathcal{H}_n := \{Z \in \mathrm{Mat}_n^T(\mathbb{C}) \mid \Im(Z) > 0\}$  be the **Siegel upper half space**. We call  $\mathrm{Sp}_n(\mathbb{Z})$  the **Siegel modular group**.

A **Siegel modular form** of degree  $n \in \mathbb{N}$  with weight  $k \in \mathbb{Z}$  for some  $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{Z})$ ,  $\Gamma$  subgroup of  $\mathrm{Sp}_n(\mathbb{Z})$ , is a holomorphic function

$$f: \mathcal{H}_n \rightarrow \mathbb{C}$$

with

- (1)  $f((AZ + B) \cdot (CZ + D)^{-1}) = \det(CZ + D)^k \cdot f(Z) \quad \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma, Z \in \mathcal{H}_n$
- (2) for  $n = 1$ :  $f(Z) = O(1) \quad \text{for } Z \rightarrow i\infty$

Note that many authors define the transformed function  $f|M$  for  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$  with

$$(f|M)(Z) := f(M \cdot Z) \cdot \det(CZ + D)^{-k}$$

with  $Z \in \mathcal{H}_n$ , where  $M \cdot Z := (AZ + B) \cdot (CZ + D)^{-1}$ . Then the first property of Siegel modular forms can be written as

$$f|M = f \quad \forall M \in \Gamma.$$

$\mathcal{M}_k^{\mathcal{H}_n}(\Gamma)$  denotes the vector space of such Siegel modular forms.

## 2.2 Elliptic modular forms

These are functions over  $\mathcal{H}_1 = \{z \in \mathbb{C} \mid \Im(z) > 0\} \subseteq \mathbb{C}$  which is called the **Poincaré upper half plane**.

We have  $\Gamma$  as a subgroup of  $\text{SL}_2(\mathbb{Z})$ . A **Elliptic modular form** with weight  $k \in \mathbb{Z}$  over  $\Gamma$  is a holomorphic function

$$f: \mathcal{H}_1 \rightarrow \mathbb{C}$$

with

- (1)  $f\left(\frac{a\tau + b}{b\tau + c}\right) = (c\tau + d)^k \cdot f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tau \in \mathcal{H}_1$
- (2)  $f(\tau) = O(1) \quad \text{for } \tau \rightarrow i\infty$

Note that most authors write  $M\tau := \frac{a\tau + b}{b\tau + c}$ .

$\mathcal{M}_k(\Gamma)$  denotes the vector space of such Elliptic modular forms.

Note that we have  $\text{SL}_2(\mathbb{Z}) = \text{Sp}_1(\mathbb{Z})$ . We can see that Elliptic modular forms are Siegel modular forms of degree  $n = 1$ . Thus we have  $\mathcal{M}_k(\Gamma) = \mathcal{M}_k^{\mathcal{H}_1}(\Gamma)$ .

In this work, we use a specific subgroup of  $\text{Sp}_1(\mathbb{Z})$ . We define

$$\Gamma_0(l) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}_1(\mathbb{Z}) \mid c \equiv 0 \pmod{l} \right\} \subseteq \text{Sp}_1(\mathbb{Z}) \subseteq \text{Mat}_2(\mathbb{Z})$$

as a subgroup of  $\text{Sp}_1(\mathbb{Z})$ .

## 2.3 Hermitian modular forms

Let  $\mathbb{H}_n := \{Z \in \text{Mat}_n(\mathbb{C}) \mid \Im(Z) > 0\}$  be the **Hermitian upper half space**. Note that these matrices are not symmetric as the Siegel upper half space  $\mathcal{H}_n$  but we have  $\mathcal{H}_n \subseteq \mathbb{H}_n$  and  $\mathcal{H}_1 = \mathbb{H}_1 \subseteq \mathbb{C}$ .

Let  $\Delta \in \mathbb{N}$  so that we have the imaginary quadratic number field  $\mathbb{Q}(\sqrt{-\Delta})$  where  $-\Delta$  is the fundamental discriminant. Then, let  $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$  be the maximum order. We call  $\text{Sp}_n(\mathcal{O})$  the **Hermitian modular group**. Let  $\Gamma$  be a subgroup of  $\text{Sp}_n(\mathcal{O})$ . Let  $\nu: \Gamma \rightarrow \mathbb{C}^\times$  be an Abel character of  $\text{Sp}_n(\mathcal{O})$ .

A **Hermitian modular form** of degree  $n \in \mathbb{N}$  with weight  $k \in \mathbb{Z}$  over  $\Gamma$  and  $\nu$  is a holomorphic function

$$f: \mathbb{H}_n \rightarrow \mathbb{C}$$

with

- (1)  $f((AZ + B) \cdot (CZ + D)^{-1}) = \nu(M) \det(CZ + D)^k f(Z)$ ,  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma, Z \in \mathbb{H}_n$ ,
- (2) for  $n = 1$ :  $f$  is holomorphic in all cusps.

Again as for Siegel modular forms, most authors write  $M \cdot Z := (AZ + B) \cdot (CZ + D)^{-1}$ .

$\mathcal{M}_k^{\mathbb{H}_n}(\Gamma, \nu)$  denotes the vector space of such Hermitian modular forms.

In this work, we will concentrate on Hermitian modular forms of degree  $n = 2$ . We will start with  $\Delta \in \{3, 4, 8\}$ .

### 2.3.1 Properties

Because  $-\Delta$  is fundamental, we have two possible cases:

1.  $\Delta \equiv 3 \pmod{4}$  and  $\Delta$  is square-free, or
2.  $\Delta \equiv 0 \pmod{4}$ ,  $\Delta/4 \equiv 1, 2 \pmod{4}$  and  $\Delta/4$  is square-free.

And for the **maximum order**  $\mathcal{O}$ , we have (compare [Der01])

$$\begin{aligned} \mathcal{O} &= \mathbb{Z} + \mathbb{Z} \frac{-\Delta + i\sqrt{\Delta}}{2}, \\ \mathcal{O}^\# &= \mathbb{Z} \frac{i}{\sqrt{\Delta}} + \mathbb{Z} \frac{1 + i\sqrt{\Delta}}{2}. \end{aligned}$$

From now on, we will always work with Hermitian modular forms of degree  $n = 2$ . We also use  $\Gamma = \text{Sp}_2(\mathcal{O})$  for simplicity.

## Chapter 3

### Theory

**Lemma 3.1.** *Let  $f: \mathbb{H}_2 \rightarrow \mathbb{C}$  be a Hermitian modular form of weight  $k$  with  $\nu \equiv 1$ . Let  $S \in \mathcal{P}_2(\mathcal{O})$ . Then,  $\tau \mapsto f(S\tau): \mathbb{H}_1 \subseteq \mathbb{C} \rightarrow \mathbb{C}$  is an Elliptic modular form of weight  $2k$  to  $\Gamma_0(l)$ , where  $l$  is the denominator of  $S^{-1}$ .*

*Proof.* Define  $\Gamma^H := \mathrm{Sp}_2(\mathcal{O})$  as the translation group for  $f$ . Let  $\tau \in \mathbb{H}_1$ . With  $S = [s, t, u] \in \mathcal{P}_2(\mathbb{C})$  we have

$$\begin{aligned} \Im(S\tau) &= \frac{1}{2i} (S\tau - \overline{S}^T \overline{\tau}) \\ &= \frac{1}{2i} S(\tau - \overline{\tau}) \\ &= \frac{1}{2i} S \cdot 2i\Im(\tau) \\ &= S\Im(\tau) > 0, \end{aligned}$$

thus  $S\tau \in \mathbb{H}_2$ . Thus,  $\tau \mapsto f(S\tau)$  is a function  $\mathbb{H}_1 \rightarrow \mathbb{C}$ .

Let  $l := \det(S)$ . That is the denominator of  $S^{-1}$ . Let  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(l) \subseteq \mathrm{SL}_2(\mathbb{Z})$ . We have

$$\begin{aligned} &S \frac{a\tau + b}{c\tau + d} \\ &= (a(S\tau) + bS) \cdot ((cS^{-1})(S\tau) + d)^{-1} \\ &= \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \cdot S\tau. \end{aligned}$$

Define

$$M := \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \in \mathrm{Mat}_4(\mathbb{C}).$$

With  $l|c$ , we also have  $cS^{-1} = \frac{c}{l}[u, -t, s] \in \mathrm{Mat}_2(\mathcal{O})$ , thus we have  $M \in \mathrm{Mat}_4(\mathcal{O})$ . Recall



that we have  $S = \overline{S}^T$  and  $ad - bc = 1$ . Verify that we have  $M \in \text{Sp}_2(\mathcal{O}) = \Gamma^H$ :

$$\begin{aligned}
& \overline{M}^T J_2 M \\
&= \overline{\begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix}}^T J_2 \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \\
&= \begin{pmatrix} (-acS^{-1} + ac\overline{S^{-1}}^T) & (-ad1_2 + cb\overline{S^{-1}}^T S) \\ (-bc\overline{S}^T S^{-1} + ad1_2) & (-bd\overline{S}^T + bdS) \end{pmatrix} \\
&= J_2.
\end{aligned}$$

Thus, because  $f$  is a Hermitian modular form, we have

$$\begin{aligned}
& f\left(S \frac{a\tau + b}{c\tau + d}\right) \\
&= f(M \cdot S\tau) \\
&= \nu(M) \cdot \det(cS^{-1}S\tau + d1_2)^k \cdot f(S\tau) \\
&= (c\tau + d)^{2k} \cdot f(S\tau).
\end{aligned}$$

Thus,  $f$  is an Elliptic modular form of weight  $2k$  to  $\Gamma_0(l)$ . □

**Remark 3.2.** Let us analyze the case  $\nu \neq 1$ . According to [Der01], only for  $\Delta \equiv 0 \pmod{4}$ , there is a single non-trivial Abel character  $\nu$ . This  $\nu$  has the following properties (see [Der01]):

$$\begin{aligned}
\nu(J_2) &= 1, \\
\nu(\text{Trans}(H)) &= (-1)^{h_1+h_4+|h_2|^2}, & H &= [h_1, h_2, h_4] \in \text{Her}_2(\mathcal{O}) \\
\nu(\text{Rot}(U)) &= (-1)^{|1+u_1+u_4|^2|1+u_2+u_3|^2+|u_1u_4|^2}, & U &= \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \in \text{GL}_2(\mathcal{O})
\end{aligned}$$

Consider the proof of the previos lemma. To calculate  $\nu(M)$  with the given equations, we need to represent  $M$  in the creation system  $J_2$ ,  $\text{Trans}(H)$  and  $\text{Rot}(U)$ .

We must consider two different cases. Recall that we have  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ , i.e.  $ad - bc = 1$ ,  $S = [s, t, u] \in \mathcal{P}_2(\mathcal{O})$  and

$$M = \begin{pmatrix} a1_2 & bS \\ cS^{-1} & d1_2 \end{pmatrix} \in \text{Sp}_2(\mathcal{O}).$$

Case 1:  $c = 0$ . Then we have  $ad = 1$ . Define  $T := \frac{b}{d}S$ . Then we have

$$\begin{aligned}
 & \text{Trans} \left( \frac{b}{d}S \right) \text{Rot} \left( \frac{1}{d}1_2 \right) \\
 = & \begin{pmatrix} 1_2 & \frac{b}{d}S \\ & 1_2 \end{pmatrix} \begin{pmatrix} \frac{1}{d}1_2 & \\ & d1_2 \end{pmatrix} \\
 = & \begin{pmatrix} \frac{1}{d}1_2 & bS \\ & d1_2 \end{pmatrix} \\
 = & M.
 \end{aligned}$$

And we have

$$\begin{aligned}
 \nu \left( \text{Trans} \left( \frac{b}{d}S \right) \right) &= (-1)^{\frac{b}{d}s + \frac{b}{d}u + |\frac{b}{d}t|^2}, \\
 \nu \left( \text{Rot} \left( \frac{1}{d}1_2 \right) \right) &= (-1)^{|1 + \frac{2}{d}|^2 + |\frac{1}{d^2}|^2} = 1.
 \end{aligned}$$

Case 2:  $c \neq 0$ . Then we have

$$\begin{aligned}
 & \text{Trans} \left( \frac{a}{c}S \right) \text{Rot} \left( -\frac{1}{c}S \right) (-J_2) \text{Trans} \left( -\frac{d}{c}S \right)^{-1} \\
 = & \begin{pmatrix} 1_2 & \frac{a}{c}S \\ & 1_2 \end{pmatrix} \begin{pmatrix} -\frac{1}{c}\overline{S}^T & \\ & -cS^{-1} \end{pmatrix} (-J_2) \begin{pmatrix} 1_2 & -\frac{d}{c}S \\ & 1_2 \end{pmatrix}^{-1} \\
 = & \begin{pmatrix} -\frac{1}{c}\overline{S}^T & -a1_2 \\ & -cS^{-1} \end{pmatrix} \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix} \begin{pmatrix} 1_2 & \frac{d}{c}S \\ & 1_2 \end{pmatrix} \\
 = & \begin{pmatrix} -\frac{1}{c}\overline{S}^T & a1_2 \\ & -cS^{-1} \end{pmatrix} \begin{pmatrix} & 1_2 \\ -1_2 & -\frac{d}{c}S \end{pmatrix} \\
 = & \begin{pmatrix} a1_2 & -\frac{1}{c}\overline{S}^T + \frac{ad}{c}S \\ cS^{-1} & d1_2 \end{pmatrix} \\
 = & M.
 \end{aligned}$$

And we have

$$\begin{aligned}\nu\left(\text{Trans}\left(\frac{a}{c}S\right)\right) &= (-1)^{\frac{a}{c}s + \frac{a}{c}u + \left|\frac{a}{c}t\right|^2} \\ \nu\left(\text{Rot}\left(-\frac{1}{c}S\right)\right) &= (-1)^{\left|1 - \frac{1}{c}s - \frac{1}{c}u\right|^2 \left|1 - \frac{2}{c}\Re(t)\right|^2 + \left|\frac{su}{c^2}\right|^2} \\ \nu(-J_2) &= -1 \\ \nu\left(\text{Trans}\left(-\frac{d}{c}S\right)\right)^{-1} &= (-1)^{-\frac{d}{c}s - \frac{d}{c}u + \left|\frac{d}{c}t\right|^2}.\end{aligned}$$

As a conclusion for now, it looks complicated to restrict  $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ , i.e. the translation group  $\Gamma^E$  for the Elliptic modular forms, to satisfy  $\nu(M) = 1$ . For example, for the case  $c = 0$ , one fulfilling condition would be  $2|b|$ . Thus, we will use  $\nu \equiv 1$  for the rest of our work.

**Lemma 3.3.** *Prop 7.3. von Poor für herm Modulformen.*  $\Gamma(\mathcal{L}) \supseteq \Gamma_0(l)$  for  $l \in \mathbb{Z}^+$ ,  $ls^{-1} \in \mathcal{P}_n(\mathcal{O})$ .

We want to calculate a generating set for the Fourier expansions of Hermitian modular forms. Now we will formulate the main algorithm of our work.

**Algorithm 3.4.** We have the Hermitian modular form degree  $n = 2$  fixed, as well as some  $\Delta$  (for now,  $\Delta \in \{3, 4, 8\}$ ). Then we select some form weight  $k \in \mathbb{Z}$  ( $k \in \{1, \dots, 20\}$  or so), let  $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$  be the maximum order (see chapter 2.3.1) and some subgroup  $\Gamma$  of  $\text{Sp}_2(\mathcal{O})$ . Then we select an abel character  $\nu: \Gamma \rightarrow \mathbb{C}^\times$  of  $\text{Sp}_2(\mathcal{O})$ .

We define the index set

$$\Lambda := \left\{ 0 \leq \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \text{Mat}_2(\mathcal{O}^\#) \mid a, c \in \mathbb{Z} \right\}.$$

Fix  $B \in \mathbb{N}$  as a limit. Select a precision

$$\mathcal{F} := \left\{ \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \Lambda \mid 0 \leq a, c < B \right\} \subseteq \Lambda.$$

1. We start with  $l = 1$  and increase it but only use the square-free numbers.
2. Set  $\mathcal{S} = \{\}$ ,

3. Enumerate matrices  $S \in \text{Mat}_2^T(\mathbb{Z})$ , and set  $\mathcal{S} \leftarrow \mathcal{S} \cup \{S\}$  and for each time you add a new matrix perform the following steps.
4. We set

$$\mathcal{M}_{k,S,\mathcal{F}}^H := \{(a[S])_{S \in \mathcal{S}} \mid a \in \mathbb{Q}^{\mathcal{F}} \text{ is } \text{GL}_2(\mathcal{O}) \text{ invariant}\} \subseteq \bigoplus_{S \in \mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)},$$

where

$$a[S] := \mathbb{N}_0 \rightarrow \mathbb{Q}, n \mapsto \sum_{T \in \Lambda, \text{tr}(ST)=n} a(T),$$

The elements  $a \in \mathbb{Q}^{\mathcal{F}}$  are Fourier expansions of Elliptic modular forms ( $\mathbb{H}_1 \rightarrow \mathbb{C}$ ) and  $a(T) \in \mathbb{Q}$  for  $T \in \mathcal{F} \subseteq \text{Mat}_2(\mathcal{O}^\#)$  are the Fourier coefficients. Recall that  $a$  being invariant under  $\text{GL}_2(\mathcal{O})$  means that we have

$$\det(U)^k a(T[U]) = a(T) \quad \forall U \in \text{GL}_2(\mathcal{O}).$$

With  $[a, b, c] \in \mathcal{F}$ , we have  $0 \leq a, c < B$ , thus there are only a finite number of possibilities. Because  $0 \leq [a, b, c]$ , we get  $ac - |b|^2 \geq 0$  and thus  $b$  is also always limited. Thus,  $\mathcal{F}$  is finite but it might be huge for even small  $B$ . Restricting the elements in  $\mathcal{F}$  by the  $\text{GL}_2(\mathcal{O})$ -invariance makes the set  $\{x \in \mathcal{F} \mid x \text{ is } \text{GL}_2(\mathcal{O}) \text{ invariant}\} \subseteq \mathcal{F}$  much smaller and better to handle in computer calculations. We use this set to identify a base of the finite dimension vector space  $\{x \in \mathbb{Q}^{\mathcal{F}} \mid x \text{ is } \text{GL}_2(\mathcal{O}) \text{ invariant}\}$ .

We identify

$$\bigoplus_{S \in \mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)} = \mathbb{Q}^N, \quad N = \sum_{S \in \mathcal{S}} \mathcal{F}(S).$$

For a given  $S \in \mathcal{S}$  and limit  $B \in \mathbb{N}$  which restricts  $\mathcal{F} \subset \Lambda$ ,  $\mathcal{F}(S) \in \mathbb{N}_0$  is the limit such that for any  $T \in \Lambda - \mathcal{F}$ ,  $\text{tr}(ST) \geq \mathcal{F}(S)$ . Thus, for calculating the Fourier coefficients  $T \in \Lambda$  with  $\text{tr}(ST) \in \{0, \dots, \mathcal{F}(S) - 1\}$ , it is sufficient to enumerate the  $T \in \mathcal{F}$ .

Let  $S = [s, t, u]$  and  $T = [a, b, c]$ . Recall that  $S \in \text{Mat}_2^T(\mathbb{Z})$ . Then we have

$$\text{tr}(ST) = as + \bar{t}b + t\bar{b} + cu = as + cu + 2t\Re(b).$$

Because  $T \geq 0$ , we have  $ac \geq |b|^2$  and thus  $\Re(b) \leq \sqrt{ac} \leq \max(a, c)$ . Thus,  $2t\Re(b) \geq -2|t|\max(a, c)$ . We also have  $as + cu \geq \max(a, c)(s + u)$ . Assuming  $T \in \Lambda - \mathcal{F}$ , we have  $\max(a, c) \geq B$ . For such  $T$ , we get

$$\text{tr}(ST) \geq B \cdot (s + u - 2|t|).$$

Given  $S > 0$ , we have  $su > t^2$ . Then we have

$$\begin{aligned}
 & s + u - 2|t| > 0 \\
 \Leftrightarrow & \quad su + u^2 - 2|t|u > 0 \\
 \Leftrightarrow & \quad (t^2 + u^2 - 2|t|u) + (su - t^2) > 0 \\
 \Leftrightarrow & \quad (|t| - u)^2 + (su - t^2) > 0.
 \end{aligned}$$

All inequalities were sharp estimations<sup>1</sup>, thus we get

$$\mathcal{F}(S) = B \cdot (s + u - 2|t|).$$

We want to calculate the matrix of the linear function

$$\{x \in \mathbb{Q}^{\mathcal{F}} \mid x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\} \rightarrow \bigoplus_{S \in \mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)}, \quad a \mapsto (a[S])_{S \in \mathcal{S}}.$$

The base of the destination room is canonical. The dimension is  $N$ . The base of the source room can be identified by  $\{x \in \mathcal{F} \mid x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\}$ .

And we set

$$\mathcal{M}_{k,S,\mathcal{F}} := \bigoplus_{S \in \mathcal{S}} \mathcal{FE}_{\mathcal{F}(S)}(\mathcal{M}_k(\Gamma_0(l_S)))$$

where  $\mathcal{M}_k(\Gamma_0(l_S))$  is the vectorspace of Elliptic modular forms over  $\Gamma_0(l_S)$ .

5. If

$$\dim \mathcal{M}_{k,S,\mathcal{F}}^H \cap \mathcal{M}_{k,S,\mathcal{F}} = \dim \mathcal{M}_k^{\mathbb{H}_2}(\Gamma, \nu),$$

then we are ready and we can reconstruct the Fourier expansion in the following way:

...

If not, then return to step 3, and enlarge  $\mathcal{S}$ .

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<sup>1</sup>For example, let  $S = [2, -1, 1]$ . Then you have  $s + u - 2|t| = 1$ . With  $c = B$  and  $a = b = 1$ , you hit the limit  $\mathrm{tr}(ST) = 2 + B - 2 = B = \mathcal{F}(S)$ .

## **Chapter 4**

### **Implementation**

In this chapter, we are describing the implementation.

## **Chapter 5**

## **Conclusion**

Blub

## Chapter 6

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