Hermitian Modular Forms for Fields of Low Discriminant

Diploma Thesis
in Mathematics

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Chapter 1

Introduction

We develop an algorithm to compute Fourier expansions of Hermitian modular forms of degree 2 over $\mathrm{Sp}_2(\mathcal{O})$ for $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$, $\Delta \in \{3, 4, 8\}$.

In [PY07], spaces of Siegel modular cusp forms are calculated.

A similar algorithm is also [Rau12, Algorithm 4.3] for Jacobi forms.

We are doing the same for Hermitian modular forms.

Chapter 2

Preliminaries

$\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$ are all **integers**. $\mathbb{Q}$ are all the **rational numbers**, $\mathbb{R}$ are the **real numbers** and $\mathbb{C}$ are the **complex numbers**. $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$, $\mathbb{R}^\times$ and $\mathbb{C}^\times$ denotes all non-zero numbers.

Let $\mathrm{Mat}_n(R)$ be the set of all $n \times n$ **matrices** over some commutative ring R. Likewise, $\mathrm{Mat}_n^T(R)$ are the **symmetric** $n \times n$ matrices. X^T is the **transposed** matrix of $X \in \mathrm{Mat}_n(R)$. $\overline{Z}$ is the **conjugated** matrix of $Z \in \mathrm{Mat}_n(\mathbb{C})$. A matrix $Y \in \mathrm{Mat}_n(\mathbb{C})$ is greater 0 if and only if $\forall x \in \mathbb{C}^n - \{0\} : Y[x] := \overline{x}^T Y x \in \mathbb{R}^+$. Such symmetric matrices are called the **positive definitive matrices**, defined by

$$\mathcal{P}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid X > 0\}.$$

For $A, X \in \mathrm{Mat}_n(\mathbb{C})$, we define $A[X] := \overline{X}^T A X$. The **denominator** of a matrix $Z \in \mathrm{Mat}_n(\mathbb{Q})$ is the smallest number $x \in \mathbb{N}$ such that $xZ \in \mathrm{Mat}_n(\mathbb{Z})$.

The **general linear group** is defined by

$$\mathrm{GL}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid \det(X) \text{ is a unit in } R\}$$

and the **special linear group** by

$$\mathrm{SL}_n(R) = \{X \in \mathrm{Mat}_n(R) \mid \det(X) = 1\}.$$

The **orthogonal group** is defined by

$$\mathrm{O}_n(R) = \left\{X \in \mathrm{GL}_n(R) \,\middle|\, X^T 1_n X = 1_n\right\}.$$

For $R \subseteq \mathbb{C}$, $\overline{R} \subseteq R$, the set of **Hermitian matrices** in R is defined as

$$\mathrm{Her}_n(R) = \left\{Z \in \mathrm{Mat}_n(R) \,\middle|\, \overline{Z}^T = Z\right\}.$$

The **symplectic group** is defined by

$$\mathrm{Sp}_n(R) = \left\{X \in \mathrm{GL}_{2n}(R) \,\middle|\, \overline{X}^T J_n X = J_n\right\} \subseteq \mathrm{Mat}_{2n}(R)$$

where $J_n := \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right) \in \mathrm{SL}_{2n}(R)$. $\mathrm{Sp}_n(R)$ is also called the **unitary group**. For $Z \in \mathrm{Mat}_n(\mathbb{C})$, we call

$$\Re(Z) = \frac{1}{2}\left(Z + \overline{Z}^T\right) \in \mathrm{Mat}_n(\mathbb{C})$$

the **real** part and

$$\Im(Z) = \frac{1}{2i}\left(Z - \overline{Z}^T\right) \in \mathrm{Mat}_n(\mathbb{C})$$

the **imaginary** part of Z and we have $Z = \Re(Z) + i\Im(Z)$. Note that we usually have $\Re(Z), \Im(Z) \notin \mathrm{Mat}_n(\mathbb{R})$ but we have $\Re(Z), \Im(Z) \in \mathrm{Her}_n(\mathbb{C})$.

We say that some function $f\colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{A} \subseteq \mathrm{Mat}_n(R)$, $\mathcal{B} \subseteq R$ is k**-invariant** under some $\mathcal{X} \subseteq \mathrm{Mat}_n(R)$ where $\mathcal{A}[\mathcal{X}] \subseteq \mathcal{A}$ if and only if $\det(U)^k f(T[U]) = f(T)$ for all $T \in \mathcal{A}$, $U \in \mathcal{X}$.

2.1 Siegel modular forms

Let $\mathcal{H}_n := \left\{Z \in \mathrm{Mat}_n^T(\mathbb{C}) \,\middle|\, \Im(Z) > 0\right\}$ be the **Siegel upper half space**. Thus, $\mathcal{H}_1$ is the **Poincaré upper half plane**. We call $\mathrm{Sp}_n(\mathbb{Z})$ the **Siegel modular group**.

A **Siegel modular cusp form** of degree $n \in \mathbb{N}$ for some $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{Z})$, Γ subgroup of $\mathrm{Sp}_n(\mathbb{Z})$, is a holomorphic function

$$f\colon \mathcal{H}_n \to \mathbb{C}$$

with

$$(1)\;\; f|_k y = f \;\; \forall\, y \in \Gamma$$

$$(2)\;\; \text{for } n = 1\colon\; f(Z) = O(1) \;\; \text{for } Z \to i\infty$$

where

$$\left(f|_k \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)(Z) = f((AZ+B)(CZ+D)^{-1}) \cdot \det(CZ+D)^{-k}$$

with $Z \in \mathcal{H}_n$, $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$.

$\mathcal{M}_k^{\mathcal{H}_n}(\Gamma)$ denotes the vector space of such Siegel modular forms.

2.2 Elliptic modular forms

We define

$$\Gamma_0(l) := \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_1(\mathbb{Z}) \,\middle|\, C \equiv 0 \pmod{l}\right\} \subseteq \mathrm{Sp}_1(\mathbb{Z}) \subseteq \mathrm{Mat}_2(\mathbb{Z})$$

as a subgroup of $\mathrm{Sp}_1(\mathbb{Z})$.

Elliptic modular forms are Siegel modular cusp forms of degree 1 with weight $k \in \mathbb{N}$ over $\Gamma_0(l)$ for some $l \in \mathbb{N}$.

$\mathcal{M}_k(\Gamma)$ denotes the vector space of such Elliptic modular forms with weight $k \in \mathbb{N}$.

2.3 Hermitian modular forms

Let $\mathbb{H}_n := \{Z \in \mathrm{Mat}_n(\mathbb{C}) \mid \Im(Z) > 0\}$ be the **Hermitian upper half space**. Note that these matrices are not symmetric as $\mathcal{H}_n$ but we have $\mathcal{H}_n \subseteq \mathbb{H}_n$ and $\mathcal{H}_1 = \mathbb{H}_1 \subseteq \mathbb{C}$.

Let $\Delta \in \mathbb{N}$ so that we have the imaginary quadratic number field $\mathbb{Q}(\sqrt{-\Delta})$ where $-\Delta$ is the fundamental discriminant. Then, let $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ be the maximum order. We call $\mathrm{Sp}_n(\mathcal{O})$ the **Hermitian modular group**. Let Γ be a subgroup of $\mathrm{Sp}_n(\mathcal{O})$. Let $\nu\colon \Gamma \to \mathbb{C}^\times$ be an abel character of $\mathrm{Sp}_n(\mathcal{O})$.

A **Hermitian modular form** of degree $n \in \mathbb{N}$ with weight $k \in \mathbb{Z}$ over Γ and ν is a holomorphic function

$$f\colon \mathbb{H}_n \to \mathbb{C}$$

with

(1) $f(M \cdot Z) = \nu(M) \det(CZ + D)^k f(Z), \quad M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma, Z \in \mathbb{H}_n,$

(2) for $n = 1\colon$ f is holomorphic in all cusps.

$\mathcal{M}_k^{\mathbb{H}_n}(\Gamma, \nu)$ denotes the vector space of such Hermitian modular forms.

In this work, we will concentrate on Hermitian modular forms of degree 2. We will start with $\Delta \in \{3, 4, 8\}$.

Because $-\Delta$ is fundamental, we have two possible cases:

1. $\Delta \equiv 3 \pmod 4$ and Δ is square-free, or
2. $\Delta \equiv 0 \pmod 4$, $\Delta/4 \equiv 1, 2 \pmod 4$ and $\Delta/4$ is square-free.

And for the maximum order $\mathcal{O}$, we have (compare [Der01])

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}\frac{-\Delta + i\sqrt{\Delta}}{2},$$
$$\mathcal{O}^\# = \mathbb{Z}\frac{i}{\sqrt{\Delta}} + \mathbb{Z}\frac{1 + i\sqrt{\Delta}}{2}.$$

From now on, we will always work with Hermitian modular forms of degree 2, i.e. we will always have $n = 2$, except if otherwise stated.

Chapter 3

Theory

Lemma 3.1. *Let* $f\colon \mathbb{H}_2 \to \mathbb{C}$ *be a Hermitian modular form of weight* k*. Let* $S \in \mathcal{P}_2(\mathbb{C})$*. Then,* $f(S\tau)\colon \mathbb{H}_1 \subseteq \mathbb{C} \to \mathbb{C}$ *is an elliptic modular form of weight* $2k$ *to* $\Gamma_0(l)$*, where* l *is the denominator of* S^{-1}*.*

Lemma 3.2. *Prop 7.3. von Poor für herm Modulformen.* $\Gamma(\mathcal{L}) \supseteq \Gamma_0(l)$ *for* $l \in \mathbb{Z}^+, ls^{-1} \in \mathcal{P}_n(\mathcal{O})$*.*

We want to calculate a generating set for the Fourier expansions of Hermitian modular forms. Now we will formulate the main algorithm of our work.

Algorithm 3.3. We have the Hermitian modular form degree $n = 2$ fixed, as well as some Δ (for now, $\Delta \in \{3, 4, 8\}$). Then we select some form weight $k \in \mathbb{Z}$ ($k \in \{1, \ldots, 20\}$ or so), let $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ be the maximum order and some subgroup Γ of $\mathrm{Sp}_2(\mathcal{O})$. Then we select an abel character $\nu\colon \Gamma \to \mathbb{C}^\times$ of $\mathrm{Sp}_2(\mathcal{O})$.

We define the index set

$$\Lambda := \left\{ 0 \le \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \mathrm{Mat}_2(\mathcal{O}^{\#}) \;\middle|\; a, c \in \mathbb{Z} \right\}.$$

Fix $B \in \mathbb{N}$ as a limit. Select a precision

$$\mathcal{F} := \left\{ \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \Lambda \;\middle|\; 0 \le a, c < B \right\} \subseteq \Lambda.$$

1. We start with $l = 1$ and increase it but only use the square-free numbers.
2. Set $\mathcal{S} = \{\}$,
3. Enumerate matrices $S \in \mathrm{Mat}_2^T(\mathbb{Z})$, and set $\mathcal{S} \leftarrow \mathcal{S} \cup \{S\}$ and for each time you add a new matrix perform the following steps.

4. We set

$$\mathcal{M}^H_{k,\mathcal{S},\mathcal{F}} := \left\{(a[S])_{S\in\mathcal{S}} \,\middle|\, a \in \mathbb{Q}^{\mathcal{F}} \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\right\} \subseteq \bigoplus_{S\in\mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)},$$

where

$$a[S] := \mathbb{N} \to \mathbb{Q}, \tau \mapsto a(S\tau),$$

The elements a are Fourier expansions of Elliptic modular forms ($\mathbb{H}_1 \to \mathbb{C}$) and $a(T) \in \mathbb{Q}$ for $T \in \mathcal{F} \subseteq \mathrm{Mat}_2(\mathcal{O}^{\#})$ are the Fourier coefficients. Recall that a being invariant under $\mathrm{GL}_2(\mathcal{O})$ means that we have

$$\det(U)^k a(T[U]) = a(T) \;\forall\, U \in \mathrm{GL}_2(\mathcal{O}).$$

As $\mathcal{F}$ is finite, so is $\{x \in \mathcal{F} \mid x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\} \subseteq \mathcal{F}$ and $\left\{x \in \mathbb{Q}^{\mathcal{F}} \,\middle|\, x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\right\}$ is of finite dimension. Define

$$I_{\mathcal{F}} := \left\{T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{F} \,\middle|\, \forall\, U \in \mathrm{GL}_2(\mathcal{O})\colon\; \det T[U] \geq \det T\right\}.$$

The set $I_{\mathcal{F}}$ is finite. And we have the canonical maps $r_I\colon \mathcal{F} \to I_{\mathcal{F}}$, $r_U\colon \mathcal{F} \to \mathrm{GL}_2(\mathcal{O})$ such that $r_I(T)[r_U(T)] = T$. Then,

$$a(T) = \det(r_U(T))^k a(r_I(T))$$

and we have Thus, to represent $\left\{x \in \mathbb{Q}^{\mathcal{F}} \,\middle|\, x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\right\}$, we can use $\mathbb{Q}^{I_{\mathcal{F}}}$. We identify

$$\bigoplus_{S\in\mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)} = \mathbb{Q}^N, \;\; N = \sum_S \mathcal{F}(S).$$

We want to calculate the matrix of the linear function $\left\{x \in \mathbb{Q}^{\mathcal{F}} \,\middle|\, x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\right\} \to \bigoplus_{S\in\mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)}$, $a \mapsto (a[S])_{S\in\mathcal{S}}$. The base of the destination room is canonical. The base of the source room can be identified by $\{x \in \mathcal{F} \mid x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\}$.

And we set

$$\mathcal{M}_{k,\mathcal{S},\mathcal{F}} := \bigoplus_{S\in\mathcal{S}} \mathcal{F}\mathcal{E}_{\mathcal{F}(S)}(\mathcal{M}_k(\Gamma_0(l_S)))$$

where $\mathcal{M}_k(\Gamma_0(l_S))$ is the vectorspace of Elliptic modular forms over $\Gamma_0(l_S)$.

5. If

$$\dim \mathcal{M}^{H}_{k,\mathcal{S},\mathcal{F}} \cap \mathcal{M}_{k,\mathcal{S},\mathcal{F}} = \dim \mathcal{M}^{\mathbb{H}_2}_{k}(\Gamma, \nu),$$

   then we are ready and we can reconstruct the Fourier expansion in the following way:

   ...

   If not, then return to step 3, and enlarge $\mathcal{S}$.

Chapter 4

Implementation

In this chapter, we are describing the implementation.

Chapter 5

Conclusion

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Chapter 6

References

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