

# HERMITIAN MODULAR FORMS FOR FIELDS OF LOW DISCRIMINANT

DIPLOMA THESIS  
in Mathematics

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# Chapter 1

## Introduction

We develop an algorithm to compute Fourier expansions of Hermitian modular forms of degree 2 over  $\mathrm{Sp}_2(\mathcal{O})$  for  $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$ ,  $\Delta \in \{3, 4, 8\}$ .

In [PY07], spaces of Siegel modular cusp forms are calculated.

A similar algorithm is also [Rau12, Algorithm 4.3] for Jacobi forms.

We are doing the same for Hermitian modular forms.

## Chapter 2

### Preliminaries

$\mathbb{N}$  denotes the set  $\{1, 2, 3, \dots\}$ ,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$  and  $\mathbb{Z}$  are all **integers**.  $\mathbb{Q}$  are all the **rational numbers**,  $\mathbb{R}$  are the **real numbers** and  $\mathbb{C}$  are the **complex numbers**.  $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$ ,  $\mathbb{R}^\times$  and  $\mathbb{C}^\times$  denotes all non-zero numbers.

Let  $\text{Mat}_n(R)$  be the set of all  $n \times n$  **matrices** over some commutative ring  $R$ . Likewise,  $\text{Mat}_n^T(R)$  are the **symmetric**  $n \times n$  matrices.  $X^T$  is the **transposed** matrix of  $X \in \text{Mat}_n(R)$ .  $\bar{Z}$  is the **conjugated** matrix of  $Z \in \text{Mat}_n(\mathbb{C})$ . A matrix  $Y \in \text{Mat}_n(\mathbb{C})$  is greater 0 if and only if  $\forall x \in \mathbb{C}^n - \{0\} : Y[x] := \bar{x}^T Y x \in \mathbb{R}^+$ . Such symmetric matrices are called the **positive definitive matrices**, defined by

$$\mathcal{P}_n(R) = \{X \in \text{Mat}_n(R) \mid X > 0\}.$$

For  $A, X \in \text{Mat}_n(\mathbb{C})$ , we define  $A[X] := \bar{X}^T A X$ . The **denominator** of a matrix  $Z \in \text{Mat}_n(\mathbb{Q})$  is the smallest number  $x \in \mathbb{N}$  such that  $xZ \in \text{Mat}_n(\mathbb{Z})$ .

The **general linear group** is defined by

$$\text{GL}_n(R) = \{X \in \text{Mat}_n(R) \mid \det(X) \text{ is a unit in } R\}$$

and the **special linear group** by

$$\text{SL}_n(R) = \{X \in \text{Mat}_n(R) \mid \det(X) = 1\}.$$

The **orthogonal group** is defined by

$$\text{O}_n(R) = \{X \in \text{GL}_n(R) \mid X^T 1_n X = 1_n\}.$$

For  $R \subseteq \mathbb{C}$ ,  $\bar{R} \subseteq R$ , the set of **Hermitian matrices** in  $R$  is defined as

$$\text{Her}_n(R) = \left\{ Z \in \text{Mat}_n(R) \mid \bar{Z}^T = Z \right\}.$$

The **symplectic group** is defined by

$$\text{Sp}_n(R) = \left\{ X \in \text{GL}_{2n}(R) \mid \bar{X}^T J_n X = J_n \right\} \subseteq \text{Mat}_{2n}(R)$$

where  $J_n := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \in \text{SL}_{2n}(R)$ .  $\text{Sp}_n(R)$  is also called the **unitary group**. For  $Z \in \text{Mat}_n(\mathbb{C})$ , we call

$$\Re(Z) = \frac{1}{2} \left( Z + \bar{Z}^T \right) \in \text{Mat}_n(\mathbb{C})$$

the **real** part and

$$\Im(Z) = \frac{1}{2i} (Z - \bar{Z}^T) \in \text{Mat}_n(\mathbb{C})$$

the **imaginary** part of  $Z$  and we have  $Z = \Re(Z) + i\Im(Z)$ . Note that we usually have  $\Re(Z), \Im(Z) \notin \text{Mat}_n(\mathbb{R})$  but we have  $\Re(Z), \Im(Z) \in \text{Her}_n(\mathbb{C})$ .

We say that some function  $f: \mathcal{A} \rightarrow \mathcal{B}$  with  $\mathcal{A} \subseteq \text{Mat}_n(R)$ ,  $\mathcal{B} \subseteq R$  is  **$k$ -invariant** under some  $\mathcal{X} \subseteq \text{Mat}_n(R)$  where  $\mathcal{A}[\mathcal{X}] \subseteq \mathcal{A}$  if and only if  $\det(U)^k f(T[U]) = f(T)$  for all  $T \in \mathcal{A}$ ,  $U \in \mathcal{X}$ .

## 2.1 Siegel modular forms

Let  $\mathcal{H}_n := \{Z \in \text{Mat}_n^T(\mathbb{C}) \mid \Im(Z) > 0\}$  be the **Siegel upper half space**. Thus,  $\mathcal{H}_1$  is the **Poincaré upper half plane**. We call  $\text{Sp}_n(\mathbb{Z})$  the **Siegel modular group**.

A **Siegel modular cusp form** of degree  $n \in \mathbb{N}$  for some  $\Gamma \subseteq \text{Sp}_n(\mathbb{Z})$ ,  $\Gamma$  subgroup of  $\text{Sp}_n(\mathbb{Z})$ , is a holomorphic function

$$f: \mathcal{H}_n \rightarrow \mathbb{C}$$

with

- (1)  $f|_k y = f \quad \forall y \in \Gamma$
- (2) for  $n = 1$ :  $f(Z) = O(1)$  for  $Z \rightarrow i\infty$

where

$$\left(f|_k \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)(Z) = f((AZ + B)(CZ + D)^{-1}) \cdot \det(CZ + D)^{-k}$$

with  $Z \in \mathcal{H}_n$ ,  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ .

$\mathcal{M}_k^{\mathcal{H}_n}(\Gamma)$  denotes the vector space of such Siegel modular forms.

## 2.2 Elliptic modular forms

We define

$$\Gamma_0(l) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_1(\mathbb{Z}) \mid C \equiv 0 \pmod{l} \right\} \subseteq \text{Sp}_1(\mathbb{Z}) \subseteq \text{Mat}_2(\mathbb{Z})$$

as a subgroup of  $\text{Sp}_1(\mathbb{Z})$ .

**Elliptic modular forms** are Siegel modular cusp forms of degree 1 with weight  $k \in \mathbb{N}$  over  $\Gamma_0(l)$  for some  $l \in \mathbb{N}$ .

$\mathcal{M}_k(\Gamma)$  denotes the vector space of such Elliptic modular forms with weight  $k \in \mathbb{N}$ .

### 2.3 Hermitian modular forms

Let  $\mathbb{H}_n := \{Z \in \text{Mat}_n(\mathbb{C}) \mid \Im(Z) > 0\}$  be the **Hermitian upper half space**. Note that these matrices are not symmetric as  $\mathcal{H}_n$  but we have  $\mathcal{H}_n \subseteq \mathbb{H}_n$  and  $\mathcal{H}_1 = \mathbb{H}_1 \subseteq \mathbb{C}$ .

Let  $\Delta \in \mathbb{N}$  so that we have the imaginary quadratic number field  $\mathbb{Q}(\sqrt{-\Delta})$  where  $-\Delta$  is the fundamental discriminant. Then, let  $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$  be the maximum order. We call  $\text{Sp}_n(\mathcal{O})$  the **Hermitian modular group**. Let  $\Gamma$  be a subgroup of  $\text{Sp}_n(\mathcal{O})$ . Let  $\nu: \Gamma \rightarrow \mathbb{C}^\times$  be an abel character of  $\text{Sp}_n(\mathcal{O})$ .

A **Hermitian modular form** of degree  $n \in \mathbb{N}$  with weight  $k \in \mathbb{Z}$  over  $\Gamma$  and  $\nu$  is a holomorphic function

$$f: \mathbb{H}_n \rightarrow \mathbb{C}$$

with

- (1)  $f(M \cdot Z) = \nu(M) \det(CZ + D)^k f(Z)$ ,  $M = \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma, Z \in \mathbb{H}_n$ ,
- (2) for  $n = 1$ :  $f$  is holomorphic in all cusps.

$\mathcal{M}_k^{\mathbb{H}_n}(\Gamma, \nu)$  denotes the vector space of such Hermitian modular forms.

In this work, we will concentrate on Hermitian modular forms of degree 2. We will start with  $\Delta \in \{3, 4, 8\}$ .

Because  $-\Delta$  is fundamental, we have two possible cases:

1.  $\Delta \equiv 3 \pmod{4}$  and  $\Delta$  is square-free, or
2.  $\Delta \equiv 0 \pmod{4}$ ,  $\Delta/4 \equiv 1, 2 \pmod{4}$  and  $\Delta/4$  is square-free.

And for the maximum order  $\mathcal{O}$ , we have (compare [Der01])

$$\begin{aligned} \mathcal{O} &= \mathbb{Z} + \mathbb{Z} \frac{-\Delta + i\sqrt{\Delta}}{2}, \\ \mathcal{O}^\# &= \mathbb{Z} \frac{i}{\sqrt{\Delta}} + \mathbb{Z} \frac{1 + i\sqrt{\Delta}}{2}. \end{aligned}$$

From now on, we will always work with Hermitian modular forms of degree 2, i.e. we will always have  $n = 2$ , except if otherwise stated.

## Chapter 3

### Theory

**Lemma 3.1.** *Let  $f: \mathbb{H}_2 \rightarrow \mathbb{C}$  be a Hermitian modular form of weight  $k$ . Let  $S \in \mathcal{P}_2(\mathbb{C})$ . Then,  $f(S\tau): \mathbb{H}_1 \subseteq \mathbb{C} \rightarrow \mathbb{C}$  is an elliptic modular form of weight  $2k$  to  $\Gamma_0(l)$ , where  $l$  is the denominator of  $S^{-1}$ .*

**Lemma 3.2.** *Prop 7.3. von Poor für herm Modulformen.  $\Gamma(\mathcal{L}) \supseteq \Gamma_0(l)$  for  $l \in \mathbb{Z}^+$ ,  $ls^{-1} \in \mathcal{P}_n(\mathcal{O})$ .*

We want to calculate a generating set for the Fourier expansions of Hermitian modular forms. Now we will formulate the main algorithm of our work.

**Algorithm 3.3.** We have the Hermitian modular form degree  $n = 2$  fixed, as well as some  $\Delta$  (for now,  $\Delta \in \{3, 4, 8\}$ ). Then we select some form weight  $k \in \mathbb{Z}$  ( $k \in \{1, \dots, 20\}$  or so), let  $\mathcal{O} \subseteq \mathbb{Q}(\sqrt{-\Delta})$  be the maximum order and some subgroup  $\Gamma$  of  $\mathrm{Sp}_2(\mathcal{O})$ . Then we select an abel character  $\nu: \Gamma \rightarrow \mathbb{C}^\times$  of  $\mathrm{Sp}_2(\mathcal{O})$ .

We define the index set

$$\Lambda := \left\{ 0 \leq \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \mathrm{Mat}_2(\mathcal{O}^\#) \mid a, c \in \mathbb{Z} \right\}.$$

Fix  $B \in \mathbb{N}$  as a limit. Select a precision

$$\mathcal{F} := \left\{ \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \Lambda \mid 0 \leq a, c < B \right\} \subseteq \Lambda.$$

1. We start with  $l = 1$  and increase it but only use the square-free numbers.
2. Set  $\mathcal{S} = \{\}$ ,
3. Enumerate matrices  $S \in \mathrm{Mat}_2^T(\mathbb{Z})$ , and set  $\mathcal{S} \leftarrow \mathcal{S} \cup \{S\}$  and for each time you add a new matrix perform the following steps.

4. We set

$$\mathcal{M}_{k,S,\mathcal{F}}^H := \{(a[S])_{S \in \mathcal{S}} \mid a \in \mathbb{Q}^{\mathcal{F}} \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\} \subseteq \bigoplus_{S \in \mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)},$$

where

$$a[S] := \mathbb{N} \rightarrow \mathbb{Q}, \tau \mapsto a(S\tau),$$

The elements  $a$  are Fourier expansions of Elliptic modular forms ( $\mathbb{H}_1 \rightarrow \mathbb{C}$ ) and  $a(T) \in \mathbb{Q}$  for  $T \in \mathcal{F} \subseteq \mathrm{Mat}_2(\mathcal{O}^\#)$  are the Fourier coefficients. Recall that  $a$  being invariant under  $\mathrm{GL}_2(\mathcal{O})$  means that we have

$$\det(U)^k a(T[U]) = a(T) \quad \forall U \in \mathrm{GL}_2(\mathcal{O}).$$

$\mathcal{F}$  is not finite but  $\{x \in \mathcal{F} \mid x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\} \subseteq \mathcal{F}$  is. Thus,  $\{x \in \mathbb{Q}^{\mathcal{F}} \mid x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\}$  is of finite dimension.

We identify

$$\bigoplus_{S \in \mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)} = \mathbb{Q}^N, \quad N = \sum_S \mathcal{F}(S).$$

We want to calculate the matrix of the linear function

$$\{x \in \mathbb{Q}^{\mathcal{F}} \mid x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\} \rightarrow \bigoplus_{S \in \mathcal{S}} \mathbb{Q}^{\mathcal{F}(S)}, \quad a \mapsto (a[S])_{S \in \mathcal{S}}.$$

The base of the destination room is canonical. The dimension is  $N$ . The base of the source room can be identified by  $\{x \in \mathcal{F} \mid x \text{ is } \mathrm{GL}_2(\mathcal{O}) \text{ invariant}\}$ .

And we set

$$\mathcal{M}_{k,S,\mathcal{F}} := \bigoplus_{S \in \mathcal{S}} \mathcal{FE}_{\mathcal{F}(S)}(\mathcal{M}_k(\Gamma_0(l_S)))$$

where  $\mathcal{M}_k(\Gamma_0(l_S))$  is the vectorspace of Elliptic modular forms over  $\Gamma_0(l_S)$ .

5. If

$$\dim \mathcal{M}_{k,S,\mathcal{F}}^H \cap \mathcal{M}_{k,S,\mathcal{F}} = \dim \mathcal{M}_k^{\mathbb{H}_2}(\Gamma, \nu),$$

then we are ready and we can reconstruct the Fourier expansion in the following way:

...

If not, then return to step 3, and enlarge  $\mathcal{S}$ .



## **Chapter 4**

### **Implementation**

In this chapter, we are describing the implementation.

## Chapter 5

### Conclusion

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## Chapter 6

### References

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