QP

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1 Introduction

Primal:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & 0.5x'Hx + c'x \\ \text{subject to} \quad & Ax = b \\ & Cx \geq d \end{aligned}$$

Replace $Cx \geq d$ with $Cx - s = d$ and $s \geq 0$.
Log-barrier:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & 0.5x'Hx + c'x - \mu e' \log(s) \\ \text{subject to} \quad & Ax = b \\ & Cx - s = d \end{aligned}$$

Lagrangian:

$$L(x, s, y, z) = 0.5x'Hx + c'x - \mu e' \log(s) + y'(Ax - b) - z'(Cx - s - d)$$

First-order optimality conditions:

$$\begin{aligned} \nabla_x L(x, s, y, z) &= Hx + c + A'y - C'z = 0 \\ \nabla_s L(x, s, y, z) &= z - \mu S^{-1} e = 0 \\ \nabla_y L(x, s, y, z) &= Ax - b = 0 \\ -\nabla_z L(x, s, y, z) &= Cx - s - d = 0 \end{aligned}$$

Rearrangement of the first-order optimality conditions:

$$\begin{aligned} Hx + A'y - C'z &= -c \\ Ax - b &= 0 \\ Cx - s - d &= 0 \\ SZe &= \mu e \end{aligned}$$

Step:

$$\begin{aligned} H(x+\Delta x) + A'(y+\Delta y) - C'(z+\Delta z) &= -c \\ A(x+\Delta x) - b &= 0 \\ C(x+\Delta x) - (s+\Delta s) - d &= 0 \\ (S+\Delta S)(Z+\Delta Z)e &= \mu e \end{aligned}$$

Rearrange with delta on left and drop nonlinear terms in delta:

$$\begin{aligned} H\Delta x + A'\Delta y - C'\Delta z &= -c - Hx - A'y + C'z \\ A\Delta x &= b - Ax \\ C\Delta x - \Delta s &= d - Cx + s \\ S\Delta z + Z\Delta s &= \mu e - SZe \end{aligned}$$

Newton step:

$$\begin{pmatrix} H & A' & C' & 0 \\ A & 0 & 0 & 0 \\ C & 0 & 0 & I \\ 0 & 0 & -S & -Z \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ -\Delta z \\ -\Delta s \end{pmatrix} = - \begin{pmatrix} Hx + A'y - C'z + c \\ Ax - b \\ Cx - s - d \\ SZe - \mu e \end{pmatrix} = - \begin{pmatrix} r_x \\ r_y \\ r_z \\ r_s \end{pmatrix}$$

Utilizing that $S\Delta z + Z\Delta s = -r_s$, we get

$$\Delta s = -Z^{-1}(r_s + S\Delta z)$$

or, written in another way

$$-Z^{-1}S\Delta z = \Delta s + Z^{-1}r_s$$

we can remove Δs from the third equation and reduce the system to

$$\begin{pmatrix} H & A' & C' \\ A & 0 & 0 \\ C & 0 & -Z^{-1}S \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ -\Delta z \end{pmatrix} = - \begin{pmatrix} r_x \\ r_y \\ r_z + Z^{-1}r_s \end{pmatrix}$$

This can be solved using a symmetric indefinite solver.

Using $C\Delta x + Z^{-1}S\Delta z = -(r_z + Z^{-1}r_s)$, we get

$$\Delta z = -ZS^{-1}(C\Delta x + r_z + Z^{-1}r_s)$$

which means that

$$-C\Delta z = CZS^{-1}C\Delta x + CZS^{-1}(r_z + Z^{-1}r_s)$$

so we can reduce the system even more:

$$\begin{pmatrix} H + C'ZS^{-1}C & A' \\ A & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = - \begin{pmatrix} r_x + C'ZS^{-1}(r_z + Z^{-1}r_s) \\ r_y \end{pmatrix}$$

If we had used a penalty $(1/2\mu)(Ax - b)'(Ax - b)$ with $y = (Ax - b)/\mu$, we would have had $-\mu I$ instead of the zero. In that case we could have used $A\Delta x - \mu\Delta y = -r_y$ to get

$$\Delta y = (A\Delta x + r_y)/\mu$$

which could be used to reduce the system to

$$(H + C'ZS^{-1}C + A'A/\mu)\Delta x = -r_x + C'S^{-1}(Zr_z + r_s) + A'r_y/\mu$$