# Interior point method using slacks

Albin Fredriksson

February 2025

## 1 Introduction

We consider the optimization problem

$$\begin{aligned}
\underset{x}{\text{minimize}} \quad & 0.5x'Hx + c'x \\
\text{subject to} \quad & l_A \leq Ax \leq u_A \\
& l_x \leq x \leq u_x
\end{aligned}$$

where it is allowed that $(l_A)_i = (u_A)_i$ (equality constraints), $(l_A)_i = -\infty$ or $(u_A)_i = \infty$ (constraint unbounded from below/above), and $(l_x)_i = -\infty$ and/or $(u_x)_i = \infty$ (variable unbounded from below/above).

Add slacks to the constraints:

$$\begin{aligned}
\underset{x}{\text{minimize}} \quad & 0.5x'Hx + c'x \\
\text{subject to} \quad & Ax - s = 0 \\
& l_A \leq s \leq u_A \\
& l_x \leq x \leq u_x
\end{aligned}$$

Add slacks to the bounds:

$$\begin{aligned}
\underset{x}{\text{minimize}} \quad & 0.5x'Hx + c'x \\
\text{subject to} \quad & Ax - s = 0 \\
& s - g = l_A \\
& s + t = u_A \\
& x - y = l_x \\
& x + z = u_x \\
& g, t, y, z \geq 0
\end{aligned}$$

Now, the only inequalities are the nonnegativity constraints for the slack variables. We handle these inequalities by log-barriers:

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & 0.5x'Hx + c'x - \mu e' \log(g) - \mu e' \log(t) - \mu e' \log(y) - \mu e' \log(w) \\
\text{subject to} \quad & Ax - s = 0 \\
& s - g = l_A \\
& s + t = u_A \\
& x - y = l_x \\
& x + z = u_x
\end{aligned}
$$

The Lagrangian of this problem is:

$$
\begin{aligned}
L(x, s, g, t, y, z, \lambda_A, \lambda_g, \lambda_t, \lambda_y, \lambda_z) = {} & 0.5x'Hx + c'x \\
& - \mu e' \log(g) - \mu e' \log(t) - \mu e' \log(y) - \mu e' \log(z) \\
& + \lambda_A'(Ax - s) \\
& - \lambda_g'(s - g - l_A) + \lambda_t'(s + t - u_A) \\
& - \lambda_y'(x - y - l_x) + \lambda_z'(x + z - u_x)
\end{aligned}
$$

And the first-order optimality conditions are the following:

$$
\begin{aligned}
\nabla_x L &= Hx + c + A'\lambda_A - \lambda_y + \lambda_z = 0 \\
\nabla_{\lambda_A} L &= Ax - s = 0 \\
-\nabla_{\lambda_g} L &= s - g - l_A = 0 \\
\nabla_{\lambda_t} L &= s + t - u_A = 0 \\
-\nabla_{\lambda_y} L &= x - y - l_x = 0 \\
\nabla_{\lambda_z} L &= x + z - u_x = 0 \\
\nabla_s L &= -\lambda_A - \lambda_g + \lambda_t = 0 \\
\nabla_g L &= -\mu G^{-1} e + \lambda_g = 0 \\
\nabla_t L &= -\mu T^{-1} e + \lambda_t = 0 \\
\nabla_y L &= -\mu Y^{-1} e + \lambda_y = 0 \\
\nabla_z L &= -\mu Z^{-1} e + \lambda_z = 0
\end{aligned}
$$

Rearrangement of the first-order optimality conditions and taking a step gives us:

$$\begin{aligned}
H(x+\Delta x)+A'(\lambda_A+\Delta\lambda_A)-\lambda_y-\Delta\lambda_y+\lambda_z+\Delta\lambda_z &= -c\\
A(x+\Delta x)-s-\Delta s &= 0\\
s+\Delta s-g-\Delta g-l_A &= 0\\
s+\Delta s+t+\Delta t-u_A &= 0\\
x+\Delta x-y-\Delta y-l_x &= 0\\
x+\Delta x+z+\Delta z-u_x &= 0\\
-\lambda_A-\Delta\lambda_A-\lambda_g-\Delta\lambda_g+\lambda_t+\Delta\lambda_t &= 0\\
(G+\Delta G)(\Lambda_g+\Delta\Lambda_g)e &= \mu e\\
(T+\Delta T)(\Lambda_t+\Delta\Lambda_t)e &= \mu e\\
(Y+\Delta Y)(\Lambda_y+\Delta\Lambda_y)e &= \mu e\\
(Z+\Delta Z)(\Lambda_z+\Delta\Lambda_z)e &= \mu e
\end{aligned}$$

We rearrange this to get the deltas on left and drop the nonlinear terms in delta:

$$\begin{aligned}
H\Delta x+A'\Delta\lambda_A-\Delta\lambda_y+\Delta\lambda_z &= -c-Hx-A'\lambda_A+\lambda_y-\lambda_z\\
A\Delta x-\Delta s &= -Ax+s\\
\Delta s-\Delta g &= -s+g+l_A\\
\Delta s+\Delta t &= -s-t+u_A\\
\Delta x-\Delta y &= -x+y+l_x\\
\Delta x+\Delta z &= -x-z+u_x\\
-\Delta\lambda_A-\Delta\lambda_g+\Delta\lambda_t &= \lambda_A+\lambda_g-\lambda_t\\
\Lambda_g\Delta Ge+G\Delta\Lambda_g e &= \mu e-G\Lambda_g e\\
\Lambda_t\Delta Te+T\Delta\Lambda_t e &= \mu e-T\Lambda_t e\\
\Lambda_y\Delta Ye+Y\Delta\Lambda_y e &= \mu e-Y\Lambda_y e\\
\Lambda_z\Delta Ze+Z\Delta\Lambda_z e &= \mu e-Z\Lambda_z e
\end{aligned}$$

We can formulate these equations (for computing the Newton step) on matrix form:

$$\begin{pmatrix}
H & A' & 0 & 0 & -I & I & 0 & 0 & 0 & 0 & 0\\
A & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & I & 0 & 0\\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0\\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I\\
0 & -I & -I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & G & 0 & 0 & 0 & 0 & \Lambda_g & 0 & 0 & 0\\
0 & 0 & 0 & T & 0 & 0 & 0 & 0 & \Lambda_t & 0 & 0\\
0 & 0 & 0 & 0 & Y & 0 & 0 & 0 & 0 & \Lambda_y & 0\\
0 & 0 & 0 & 0 & 0 & Z & 0 & 0 & 0 & 0 & \Lambda_z
\end{pmatrix}
\begin{pmatrix}
\Delta x\\ \Delta\lambda_A\\ \Delta\lambda_g\\ \Delta\lambda_t\\ \Delta\lambda_y\\ \Delta\lambda_z\\ \Delta s\\ \Delta g\\ \Delta t\\ \Delta y\\ \Delta z
\end{pmatrix}
= -\begin{pmatrix}
Hx+A'\lambda_A-\lambda_y+\lambda_z+c\\
Ax-s\\
s-g-l_A\\
s+t-u_A\\
x-y-l_x\\
x+z-u_x\\
-\lambda_A-\lambda_g+\lambda_t\\
G\Lambda_g e-\mu e\\
T\Lambda_t e-\mu e\\
Y\Lambda_y e-\mu e\\
Z\Lambda_z e-\mu e
\end{pmatrix}
= -\begin{pmatrix}
r_x\\ r_{\lambda_A}\\ r_{\lambda_g}\\ r_{\lambda_t}\\ r_{\lambda_y}\\ r_{\lambda_z}\\ r_s\\ r_g\\ r_t\\ r_y\\ r_z
\end{pmatrix}$$

Utilizing that $Z\Delta\lambda_z + \Lambda_z\Delta z = -r_z$, we get

$$\Delta z = -\Lambda_z^{-1} r_z - \Lambda_z^{-1} Z \Delta\lambda_z$$

and, similarly, for $y$, $t$, and $g$. We add row eleven multiplied by $-\Lambda_z^{-1}$ to row six etc.

$$\begin{pmatrix} H & A' & 0 & 0 & -I & I & 0 \\ A & 0 & 0 & 0 & 0 & 0 & -I \\ 0 & 0 & \Lambda_g^{-1}G & 0 & 0 & 0 & I \\ 0 & 0 & 0 & -\Lambda_t^{-1}T & 0 & 0 & I \\ I & 0 & 0 & 0 & \Lambda_y^{-1}Y & 0 & 0 \\ I & 0 & 0 & 0 & 0 & -\Lambda_z^{-1}Z & 0 \\ 0 & -I & -I & I & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta\lambda_A \\ \Delta\lambda_g \\ \Delta\lambda_t \\ \Delta\lambda_y \\ \Delta\lambda_z \\ \Delta s \end{pmatrix} = - \begin{pmatrix} r_x \\ r_{\lambda_A} \\ r_{\lambda_g} + \Lambda_g^{-1} r_g \\ r_{\lambda_t} - \Lambda_t^{-1} r_t \\ r_{\lambda_y} + \Lambda_y^{-1} r_y \\ r_{\lambda_z} - \Lambda_z^{-1} r_z \\ r_s \end{pmatrix}$$

We now use that

$$\Delta\lambda_y = -Y^{-1}\Lambda_y \Delta x - Y^{-1}\Lambda_y r_{\lambda_y} - Y^{-1} r_y$$

and

$$\Delta\lambda_z = Z^{-1}\Lambda_z \Delta x + Z^{-1}\Lambda_z r_{\lambda_z} - Z^{-1} r_z$$

to remove the fifth and sixth rows and the $\Delta\lambda_y$ and $\Delta\lambda_z$ columns, i.e., add row five multiplied by $Y^{-1}\Lambda_y$ and row six multiplied by $Z^{-1}\Lambda_z$ to row one to get:

$$\begin{pmatrix} H + Y^{-1}\Lambda_y + Z^{-1}\Lambda_z & A' & 0 & 0 & 0 \\ A & 0 & 0 & 0 & -I \\ 0 & 0 & \Lambda_g^{-1}G & 0 & I \\ 0 & 0 & 0 & -\Lambda_t^{-1}T & I \\ 0 & -I & -I & I & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta\lambda_A \\ \Delta\lambda_g \\ \Delta\lambda_t \\ \Delta s \end{pmatrix} = - \begin{pmatrix} r_x + Y_y^{-1} r_{\lambda_y} + Y^{-1} r_y + Z^{-1}\Lambda_z r_{\lambda_z} - Z^{-1} r_z \\ r_{\lambda_A} \\ r_{\lambda_g} + \Lambda_g^{-1} r_g \\ r_{\lambda_t} - \Lambda_t^{-1} r_t \\ r_s \end{pmatrix}$$

Now, we use that

$$\Delta\lambda_g = -G^{-1}\Lambda_g \Delta s - G^{-1}\Lambda_g r_{\lambda_g} - G^{-1} r_g$$

and

$$\Delta\lambda_t = T^{-1}\Lambda_t \Delta s + T^{-1}\Lambda_t r_{\lambda_t} - T^{-1} r_t$$

i.e., add row three multiplied by $G^{-1}\Lambda_g$ and four multiplied by $T^{-1}\Lambda_t$ to row five to get

$$\begin{pmatrix} H + Y^{-1}\Lambda_y + Z^{-1}\Lambda_z & A' & 0 \\ A & 0 & -I \\ 0 & -I & G^{-1}\Lambda_g + T^{-1}\Lambda_t \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta\lambda_A \\ \Delta s \end{pmatrix} = - \begin{pmatrix} r_x + Y_y^{-1} r_{\lambda_y} + Y^{-1} r_y + Z^{-1}\Lambda_z r_{\lambda_z} - Z^{-1} r_z \\ r_{\lambda_A} \\ r_s + G^{-1}\Lambda_g r_{\lambda_g} + G^{-1} r_g + T^{-1}\Lambda_t r_{\lambda_t} - T^{-1} r_t \end{pmatrix}$$

Finally, we add the third row multiplied by $(G^{-1}\Lambda_g + T^{-1}\Lambda_t)^{-1}$ to the second row, i.e., use the fact that

$$-\Delta\lambda_A + (G^{-1}\Lambda_g + T^{-1}\Lambda_t)\Delta s = -(r_s + G^{-1}\Lambda_g r_{\lambda_g} + G^{-1}r_g + T^{-1}\Lambda_t r_{\lambda_t} - T^{-1}r_t)$$

to arrive at

$$\begin{pmatrix} H + Y^{-1}\Lambda_y + Z^{-1}\Lambda_z & A' \\ A & -(G^{-1}\Lambda_g + T^{-1}\Lambda_t)^{-1} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta\lambda_A \end{pmatrix} = -\begin{pmatrix} r_x + Y^{-1}\Lambda_y r_{\lambda_y} + Y^{-1}r_y + Z^{-1}\Lambda_z r_{\lambda_z} - Z^{-1}r_z \\ r_{\lambda_A} + (G^{-1}\Lambda_g + T^{-1}\Lambda_t)^{-1}(r_s + G^{-1}\Lambda_g r_{\lambda_g} + G^{-1}r_g + T^{-1}\Lambda_t r_{\lambda_t} - T^{-1}r_t) \end{pmatrix}$$

which is used to find $\Delta x$ and $\Delta\lambda_A$. Then these can be used to get:

$$\begin{aligned}
\Delta s &= (G^{-1}\Lambda_g + T^{-1}\Lambda_t)^{-1}(\Delta\lambda_A - (r_s + G^{-1}\Lambda_g r_{\lambda_g} + G^{-1}r_g + T^{-1}\Lambda_t r_{\lambda_t} - T^{-1}r_t)) \\
\Delta\lambda_g &= -G^{-1}\Lambda_g\Delta s - G^{-1}\Lambda_g r_{\lambda_g} - G^{-1}r_g \\
\Delta\lambda_t &= T^{-1}\Lambda_t\Delta s + T^{-1}\Lambda_t r_{\lambda_t} - T^{-1}r_t \\
\Delta\lambda_y &= -Y^{-1}\Lambda_y\Delta x - Y^{-1}\Lambda_y r_{\lambda_y} - Y^{-1}r_y \\
\Delta\lambda_z &= Z^{-1}\Lambda_z\Delta x + Z^{-1}\Lambda_z r_{\lambda_z} - Z^{-1}r_z \\
\Delta g &= -\Lambda_g^{-1}r_g - \Lambda_g^{-1}G\Delta\lambda_g \\
\Delta t &= -\Lambda_t^{-1}r_t - \Lambda_t^{-1}T\Delta\lambda_t \\
\Delta y &= -\Lambda_y^{-1}r_y - \Lambda_y^{-1}Y\Delta\lambda_y \\
\Delta z &= -\Lambda_z^{-1}r_z - \Lambda_z^{-1}Z\Delta\lambda_z
\end{aligned}$$