Homework 1 Advanced Derivatives

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September 2022

Question

Derive a formula for the price $O(t, S_1(t), S_2(t))$ of an Outperformance option, whose payout is

$$\max(0, \frac{S_1(T)}{S_1(0)} - \frac{S_2(T)}{S_2(0)}).$$

The stocks S_1 and S_2 pay no dividends and follow the stochastic processes

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dW_1$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dW_2$$

with $\mathbb{E}[dW_1 dW_2] = \rho dt$. $S_1(0)$ and $S_2(0)$ are the stock prices at time 0 and should be treated as constants. Follow the same logic as for the pricing of the exchange option, done in class.

Solution

Let there be a replicating portfolio

$$P = \Delta_1 S_1 + \Delta_2 S_2 + B$$

for some constants Δ_1 and Δ_2 and the risk-free asset B. Now since the payout X can be expressed as

$$X(\lambda S_1, \lambda S_2, t) = \lambda X(S_1, S_2, t),$$

we can use Euler's theorem and obtain

$$X = \frac{\partial X}{\partial S_1} S_1 + \frac{\partial X}{\partial S_2} S_2.$$

We see that the bond B is not present here, so $B = 0$. These arguments were obtained from the lecture notes. We apply Itô's lemma on X and obtain

$$dX = \frac{\partial X}{\partial t} dt + \frac{\partial X}{\partial S_1} dS_1 + \frac{\partial X}{\partial S_2} dS_2 + \frac{1}{2}(\frac{\partial^2 X}{\partial S_1^2}\sigma_1^2 S_1^2 + \frac{\partial^2 X}{\partial S_2^2}\sigma_2^2 S_2^2 + 2\frac{\partial^2 X}{\partial S_1 \partial S_2}\rho\sigma_1\sigma_2 S_1 S_2)dt.$$

Now since we want

$$X = P, dP = dX,$$

and

$$dP = \Delta_1 dS_1 + \Delta_2 S_2,$$

we require $\Delta_1 = \frac{\partial X}{\partial S_1}$ and $\Delta_2 = \frac{\partial X}{\partial S_2}$ in order to match the dS_1 and dS_2 terms in the expression for dX. We also need the dt term to vanish, so X must satisfy the PDE

$$\frac{\partial X}{\partial t} + \frac{1}{2}(\frac{\partial^2 X}{\partial S_1^2}\sigma_1^2 S_1^2 + \frac{\partial^2 X}{\partial S_2^2}\sigma_2^2 S_2^2 + 2\frac{\partial^2 X}{\partial S_1 \partial S_2}\rho\sigma_1\sigma_2 S_1 S_2) = 0.$$

Now we define $y = \frac{S_1/S_1(0)}{S_2/S_2(0)}$ and f s.t.

$$X(S_1, S_2, t) = S_2/S_2(0)f(y, t).$$

Using this notation, we have

$$\frac{\partial X}{\partial t} = S_2/S_2(0)\frac{\partial f}{\partial t}, \quad \frac{\partial X}{\partial S_1} = S_2/S_2(0)\frac{\partial f}{\partial y}\frac{\partial y}{\partial S_1} = \frac{1}{S_1(0)}\frac{\partial f}{\partial y},$$

$$\frac{\partial X}{\partial S_2} = \frac{1}{S_2(0)}f + S_2/S_2(0)\frac{\partial f}{\partial y}\frac{\partial y}{\partial S_2} = \frac{1}{S_2(0)}f - \frac{S_1/S_1(0)}{S_2}\frac{\partial f}{\partial y},$$

$$\frac{\partial^2 X}{\partial S_1^2} = \frac{1}{S_1^2(0) \cdot S_2/S_2(0)}\frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 X}{\partial S_1 \partial S_2} = -\frac{S_1}{S_1^2(0) \cdot S_2^2/S_2(0)}\frac{\partial^2 f}{\partial y^2},$$

$$\frac{\partial^2 X}{\partial S_2^2} = -\frac{S_1}{S_1(0) \cdot S_2^2}\frac{\partial f}{\partial y} + \frac{S_1}{S_1(0) \cdot S_2^2}\frac{\partial f}{\partial y} + \frac{(S_1/S_1(0))^2}{S_2^3/S_2(0)}\frac{\partial^2 f}{\partial y^2} = \frac{(S_1/S_1(0))^2}{S_2^3/S_2(0)}\frac{\partial^2 f}{\partial y^2}.$$

Plugging this into the PDE gives us

$$S_2/S_2(0)\frac{\partial f}{\partial t} + \frac{1}{2}(\frac{\sigma_1^2 S_1^2}{S_1^2(0) \cdot S_2/S_2(0)}\frac{\partial^2 f}{\partial y^2} + \frac{\sigma_2^2 S_2^2 (S_1/S_1(0))^2}{S_2^3/S_2(0)}\frac{\partial^2 f}{\partial y^2} - \frac{2S_1\rho\sigma_1\sigma_2 S_1 S_2}{S_1^2(0) \cdot S_2^2/S_2(0)}\frac{\partial^2 f}{\partial y^2}) = 0.$$

This can be simplified a bit by dividing by $\frac{S_2}{S_2(0)}$ and using our definition of y:

$$\frac{\partial f}{\partial t} + \frac{1}{2}y^2(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)\frac{\partial^2 f}{\partial y^2} = 0.$$

This is the BS PDE with $r = 0$. We also have the terminal condition

$$f(y, T) = \max(y - 1, 0)$$

since

$$f(y, T) = \frac{1}{S_2(T)/S_2(0)}X(S_1, S_2, T) = \frac{1}{S_2(T)/S_2(0)}\max(0, \frac{S_1(T)}{S_1(0)} - \frac{S_2(T)}{S_2(0)}) = \max(y-1, 0).$$

Thus, we simply apply the BS call option formula with strike 1 and volatility $\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$. We obtain

$$f(t, y) = yN(d_1) - N(d_2)$$

and

$$O(S_1(t), S_2(t), t) = \frac{S_2(t)}{S_2(0)} f(t, y) = \frac{S_2(t)}{S_2(0)} (yN(d_1) - N(d_2)) = \frac{S_1(t)}{S_1(0)} N(d_1) - \frac{S_2(t)}{S_2(0)} N(d_2)$$

where

$$d_1 = \frac{log(\frac{S_1(t)/S_1(0)}{S_2(t)/S_2(0)}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t)}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$