Homework 3 Advanced Derivatives

Arya Salahshour, Dayan Gualberti, Karl Albin Henriksson

October 2022

Exercise 1

We have

$$\Pi(S_T^n) = e^{-rT}\,\mathbb{E}[S_T^n] = e^{-rT}\,\mathbb{E}[(S_0 e^{(r-q+\lambda\gamma-\frac{\sigma^2}{2})T+\sigma W_T^Q}(1-\gamma)^{N_T})^n] =$$

$$= e^{-rT}\sum_{j=0}^{\infty} P(N_T=j)\,\mathbb{E}[(S_0 e^{(r-q+\lambda\gamma-\frac{\sigma^2}{2})T+\sigma W_T^Q}(1-\gamma)^j)^n] =$$

$$= e^{-rT} S_0^n e^{n(r-q+\lambda\gamma-\frac{\sigma^2}{2})T}\,\mathbb{E}[e^{n\sigma W_T^Q}]\sum_{j=0}^{\infty} P(N_T=j)((1-\gamma)^j)^n.$$

Since $W_T^Q \sim N(0,T)$, we have

$$\mathbb{E}[e^{n\sigma W_T^Q}] = e^{\frac{n^2\sigma^2}{2}T}.$$

Further,

$$\sum_{j=0}^{\infty} P(N_T=j)((1-\gamma)^j)^n = \sum_{j=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^j}{j!}((1-\gamma)^n)^j =$$

$$= \sum_{j=0}^{\infty} e^{-\lambda T}\frac{(\lambda T(1-\gamma)^n)^j}{j!} = e^{-\lambda T} e^{\lambda T(1-\gamma)^n}\sum_{j=0}^{\infty} e^{-\lambda T(1-\gamma)^n}\frac{(\lambda T(1-\gamma)^n)^j}{j!} =$$

$$= e^{\lambda T((1-\gamma)^n-1)}.$$

In the last step, we used that the sum of probabilities of a poisson distribution with parameter $\lambda(1-\gamma)^n$ is equal to 1. Putting everything together, we obtain

$$\Pi(S_T^n) = e^{-rT} S_0^n e^{n(r-q+\lambda\gamma-\frac{\sigma^2}{2})T} e^{\frac{n^2\sigma^2}{2}T} e^{\lambda T((1-\gamma)^n-1)} = S_0^n e^{-rT+n(r-q+\lambda\gamma-\frac{\sigma^2}{2})T+\frac{n^2\sigma^2}{2}T+\lambda T((1-\gamma)^n-1)}.$$

Exercise 2

From the slides 22 to 25 of Lecture2, we have that a stock behaving as such:

$$S_T = S_t e^{(r-q+\lambda\gamma-r\frac{\sigma^2}{2})(T-t)+\sigma(W_T^Q-W_t^Q)}(1-\gamma)^{(N_T-N_t)} \tag{1}$$

will have the following survival probability:

$$P^S(t,\lambda) = e^{-\lambda t} \tag{2}$$

So, now we can compute the probability that the time the next jump will be:

- Longer than two years: $\boxed{P^S(2,\ 0.2) \approxeq 0.670}$
- Shorter than 3 years $\iff$ At least 1 jump occurred during the 3 years: $\boxed{1 - P^S(3,\ 0.2) \approxeq 0.451}$
- Between 2 and 3 years $\iff$ Equivalent to the intersection between the last 2 points. Using $P(X \cup Y) = P(X) + P(B) - P(X \cap Y) \leftrightarrow P(X \cap Y) = P(X) + P(B) - P(X \cup Y)$, with $P(X \cup Y) = 1$. Putting everything together, we finally get:

$$\boxed{P^S(2,\ 0.2) + 1 - P^S(3,\ 0.2) - 1 \approxeq 0.121}$$

Exercise 3

To approximate the second derivative of the call option with respect to strike, we will use the following approximation of the PDF:

$$\phi(S, T; S_0) = \frac{\partial^2 C(S, T; S_0)}{\partial K^2} \approx \frac{C(\cdot, K - \Delta K) - 2C(\cdot, K) + C(\cdot, K + \Delta K)}{(\Delta K)^2}.$$

Using this, we obtain the implied pdf's seen in figure 1.

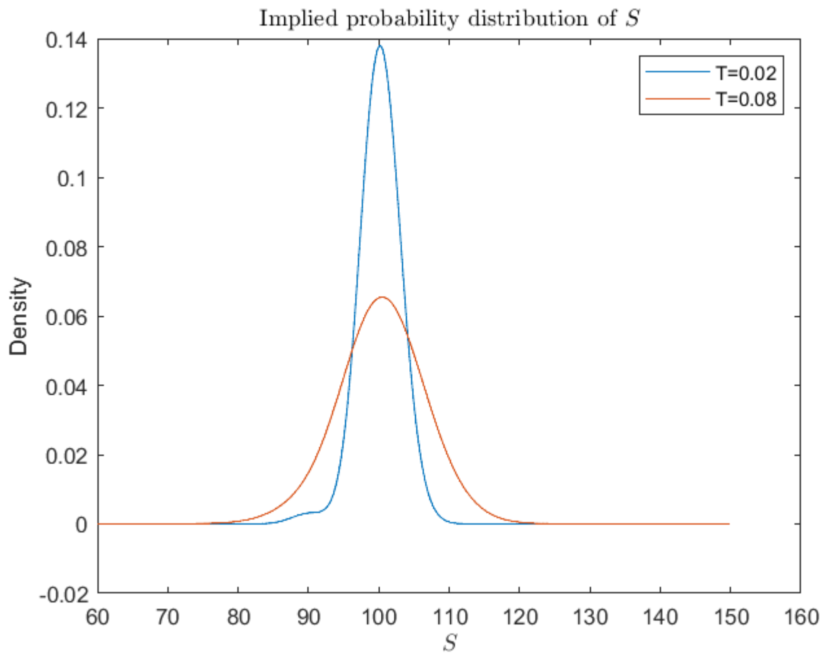

Figure 1: Implied probability distribution of S for the given maturities.

Bonus

We have

$$S_T = S_0 e^{(r - q + \lambda\gamma - \frac{\sigma^2}{2})T + \sigma W_T^Q} (1 - \gamma)^{N_T}.$$

Using this, we can generate, say, $M = 10^7$ values of $W_T \sim N(0, T)$ & $N_T \sim \text{Poisson}(\lambda T)$ and use them to get an empirical distribution of S_T. Doing this, we obtain the graph in figure 2. As we can see, it is virtually identical to figure 1.

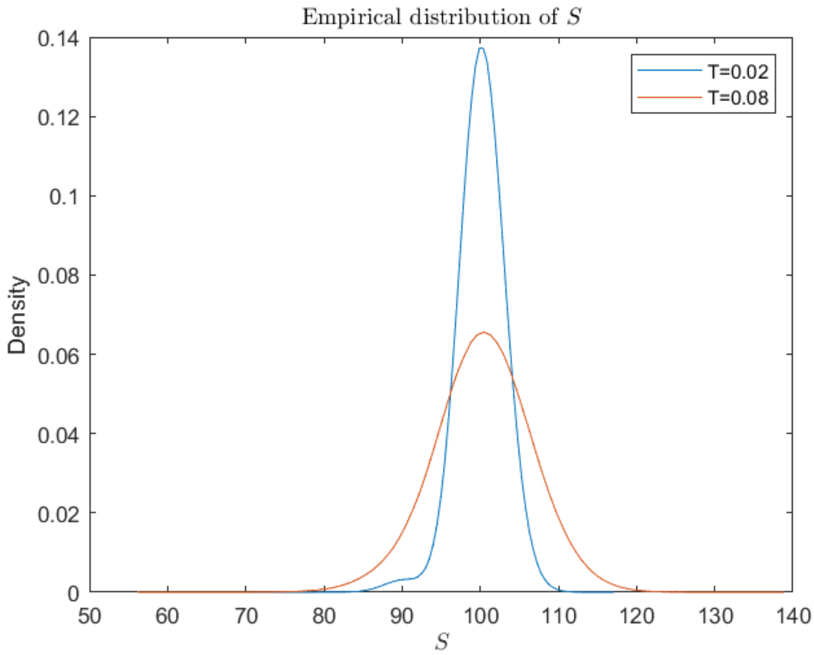

Figure 2: Empirical distribution of S for the given maturities.