

Problem set 2

Due date: Monday, October 10th at noon. Submit your Matlab code, plots and written report.

Exercise 1

The objective of the exercise is to work on a simple option model in which the underlying equity process is not lognormal and derive an *implied volatility surface*.

Inspired by Merton's model, we assume that

- A firm's assets V follow a lognormal stochastic process

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t \quad (1)$$

The value of the assets at the initial time t is $V_t = 100$. The assets yearly volatility is $\sigma = 0.3$.

- The firm has debt $D = 70$ due in 10 years.
- The (constant) riskless interest rate is $r = 4\%$.

Under this conditions, the equity of the firm is a residual claim: in 10 years, equity will be worth $\max(0, V_T - D)$. Hence equity is a call option on the firm's assets, with strike $D = 70$, expiration $T = 10$. **Compute the current value of the firm's equity.**

Now consider a call option on the firm's equity, with expiration $\tau < T$ and strike K . This can be regarded as a compound option (call on call) on the firm's assets V .

- Use the analytic formula for the compound option (call on call) to find the value of a call option on the firm's equity with strike K and maturity $\tau < T$.
- Write a small Matlab program to obtain the numerical value of the option for different strikes ($K = 60\%, 80\%, 100\%, 120\%$ of the current equity value) and maturities ($\tau = 2, 4, 6, 8$ years).
- Derive the Black-Scholes implied volatility of the firm's equity corresponding to all these strikes and expirations.
- For every maturity τ , plot the implied volatility as a function of strike.

Background: compound option

For this exercise you will have to use the formula to price a *compound option*, an option to buy at T_1 an option with expiration $T_2 > T_1$ for a predetermined price. Consider a (non-dividend paying) stock following a stochastic process

$$\frac{dS}{S} = \mu dt + \sigma dW \quad (2)$$

A call with strike K_1 and expiration T_1 on a call with strike K_2 and expiration $T_2 > T_1$ is worth

$$C(t, S) = SN_2 \left(a_1, b_1, \sqrt{\frac{(T_1 - t)}{(T_2 - t)}} \right) - K_2 e^{-r(T_2 - t)} N_2 \left(a_2, b_2, \sqrt{\frac{(T_1 - t)}{(T_2 - t)}} \right) - e^{-r(T_1 - t)} K_1 N(a_2) \quad (3)$$

where $N_2(x_1, x_2, \rho)$ is the bivariate cumulative distribution function of two standard normal variables with correlation ρ , $N(x)$ is the univariate standard normal cumulative distribution function,

$$a_1 = \frac{\ln(S/S^*) + (r + \sigma^2/2)(T_1 - t)}{\sigma\sqrt{T_1 - t}} \quad (4)$$

$$a_2 = a_1 - \sigma\sqrt{T_1 - t} \quad (5)$$

$$b_1 = \frac{\ln(S/K_2) + (r + \sigma^2/2)(T_2 - t)}{\sigma\sqrt{T_2 - t}} \quad (6)$$

$$b_2 = b_1 - \sigma\sqrt{T_2 - t} \quad (7)$$

S^* is a stock price for which the price of the underlying option will be equal to K_1 at T_1 , so the lowest stock price for which the compound option will be exercised.

In Matlab, for the bivariate normal distribution $N_2(a, b, \rho)$ use the command `mvncdf(x, μ , σ)` with $x = [a, b]$, $\mu = [0, 0]$, $\sigma = [1, \rho; \rho, 1]$

Exercise 2

Consider a non-dividend-paying stock with current value $S = 100$. In the risk-neutral measure the stock follows a jump-diffusion process

$$\frac{dS}{S} = (r - \lambda\gamma)dt + \sigma dW_t + \gamma dN_t \quad (8)$$

Take $\gamma = -0.07$ (i.e. the stock price is subject to downward 7% jumps), $\lambda = 0.25$ (i.e. on average the stock experiences a jump every 4 years), $\sigma = 0.2$, $r = 0.05$. Compute the value of a call option with expirations $T = 0.02, 0.08, 0.25, 0.5$ (i.e. approximately 1 week, 1 month, 3 months, 6 months). and strikes $K/S = 0.8, 0.9, 1, 1.1$. For every expiration, plot the Black-Scholes implied volatility of the option as a function of strike.