Computational finance, take-home exam 1: theoretical part

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October 2022

Exercise 1

a)

Applying Itô's lemma on S_t gives

$$dS_t = S_t(\mu - \frac{\sigma^2}{2})dt + \sigma S_t dW_t + \frac{\sigma^2}{2} S_t dt = S_t(\mu dt + \sigma dW_t).$$

b)

a)

The payoff at maturity of a portfolio containing a call and put option is clearly

$$max(S_t - K, 0) + max(K - S_t, 0) = \begin{cases} S_t - K & \text{if } S_t > K \\ K - S_t & \text{otherwise} \end{cases} = |S_t - K|.$$

Thus the present value is the same since the price of a given payoff is unique in accordance with no arbitrage.

b)

With a similar argument as before, we just need to prove that the payoff at maturity is the same.

$$S_t - max(K - S_t) = \begin{cases} S_t & \text{if } S_t < K \\ K & \text{otherwise} \end{cases} = min(S_t, K).$$

Discounting the LHS to present value, We get S_0 minus the price of the put option.

Exercise 2

a)

The SDE can equivalently be written as

$$\kappa X_t dt + dX_t = e^{-\kappa t} d(e^{\kappa t} X_t) = \kappa\theta dt + \lambda W_t.$$

Now multiplying by $e^{\kappa t}$ and integrating yields

$$e^{\kappa t} X_t = x_0 + \int_0^t \kappa\theta e^{\kappa s} ds + \int_0^t \lambda e^{\kappa t} dW_s \implies X_t = x_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \int_0^t \lambda e^{\kappa(s-t)} dW_s.$$

b)

$$\mathbb{E}[x_0e^{-\kappa t}+\int_0^t \kappa\theta e^{\kappa(s-t)}ds+\int_0^t \lambda e^{\kappa(s-t)}dW_s]=x_0e^{-\kappa t}+\int_0^t \kappa\theta e^{\kappa(s-t)}ds$$

since any deterministic function integrated w.r.t dW_t has expected value 0. Clearly the first term of the RHS disappears as $t\to\infty$. We have

$$\int_0^t \kappa\theta e^{\kappa(s-t)}ds=[\frac{\kappa}{\kappa}\theta e^{\kappa(s-t)}]_0^t=\theta e^{\kappa(t-t)}-\theta e^{\kappa(0-t)}=\theta(1-e^{-\kappa t}).$$

Now if we let $t\to\infty$, all that remains is θ.

c)

First part

$$\mathbb{E}[X_t^2]=\mathbb{E}[(x_0e^{-\kappa t}+\int_0^t \kappa\theta e^{\kappa(s-t)}ds+\int_0^t \lambda e^{\kappa(s-t)}dW_s)^2]=$$

$$(x_0e^{-\kappa t}+\int_0^t \kappa\theta e^{\kappa(s-t)}ds)^2+2(x_0e^{-\kappa t}+\int_0^t \kappa\theta e^{\kappa(s-t)}ds)\,\mathbb{E}[\int_0^t \lambda e^{\kappa(s-t)}dW_s]+\mathbb{E}[(\int_0^t \lambda e^{\kappa(s-t)}dW_s)^2]=$$

$$(x_0e^{-\kappa t}+\int_0^t \kappa\theta e^{\kappa(s-t)}ds)^2+\int_0^t(\lambda^2e^{2\kappa(s-t)}ds=(x_0e^{-\kappa t}+\theta(1-e^{-\kappa t}))^2+\lambda^2\frac{1-e^{-2\kappa t}}{2\kappa}=$$

$$=x_0^2e^{-2\kappa t}+2x_0^{-\kappa t}-2x_0^{-2\kappa t}+\theta^2-2\theta^2e^{-\kappa t}+\theta^2e^{-2\kappa t}+\frac{\lambda^2}{2\kappa}-\frac{\lambda^2}{2\kappa}e^{-\kappa t}$$

Second part

We have

$$\begin{cases}\frac{\partial v}{\partial t}=-a'(T-t)-b'(T-t)x-c'(T-t)x^2\\ \frac{\partial v}{\partial x}=b(T-t)+2c(T-t)x\\ \frac{\partial^2 v}{\partial x^2}=2c(T-t)\end{cases}$$

Substitute in the PDE:

$$-a'(T-t)-b'(T-t)x-c'(T-t)x^2+\kappa(\theta-x)(b(T-t)+2c(T-t)x)+\frac{\lambda^2}{2}2c(T-t)=0.$$

Now this needs to be 0 in all powers of x, thus we have the system

$$\begin{cases}-a'(T-t)+\kappa\theta b(T-t)+\frac{\lambda^2}{2}2c(T-t)=0\\ -b'(T-t)-\kappa b(T-t)+2\kappa\theta c(T-t)=0\\ -c'(T-t)-2\kappa c(T-t)=0\end{cases}.$$

Thus from the last equation we find that $c(T-t)=C\cdot e^{-2\kappa(T-t)}$ where $C=1$ due to the initial condition. To find b in the second equation, we make the ansatz

$$b=xe^{-\kappa(T-t)}+ye^{-2\kappa(T-t)}$$

for some real numbers x,y. Substituting in, we obtain

$$-(-\kappa x e^{-\kappa(T-t)} - 2\kappa y e^{-2\kappa(T-t)}) - \kappa(x e^{-\kappa(T-t)} + y e^{-2\kappa(T-t)}) + 2\kappa\theta e^{-2\kappa(T-t)} = 0$$

From this equation, we see that $y = -2\theta$, and from the initial condition, we must therefore have $x = 2\theta$ and thus

$$b(T-t) = 2\theta(e^{-\kappa(T-t)} - e^{-2\kappa(T-t)})$$

To find a, we make a similar ansatz:

$$a = z e^{-\kappa(T-t)} + w e^{-2\kappa(T-t)} + v.$$

Substituting in the first equation:

$$-(-\kappa z e^{-\kappa(T-t)} - 2\kappa w e^{-2\kappa(T-t)}) + \kappa\theta(2\theta(e^{-\kappa(T-t)} - e^{-2\kappa(T-t)})) + \lambda^2 e^{-2\kappa(T-t)} = 0.$$

We immediately see that $z = -2\theta^2$. A bit of algebra also gives us $w = -\frac{\lambda^2}{2\kappa} + \theta^2$. In order for the initial condition to hold, we also require $v = -(z+w) = \theta^2 + \frac{\lambda^2}{2\kappa}$. Thus,

$$a(T-t) = -2\theta^2 e^{-\kappa(T-t)} + (-\frac{\lambda^2}{2\kappa} + \theta^2)e^{-2\kappa(T-t)} + \theta^2 + \frac{\lambda^2}{2\kappa}.$$

Now that we have a,b and c. We can easily calculate $\mathbb{E}[X_T^2]$ as

$$v(t, x_0) = \mathbb{E}[(X_T^{t,x_0})^2] = a(T-t) + b(T-t)x + c(T-t)x^2 =$$

$$= -2\theta^2 e^{-\kappa t} + (-\frac{\lambda^2}{2\kappa} + \theta^2)e^{-2\kappa t} + \theta^2 + \frac{\lambda^2}{2\kappa} + 2\theta(e^{-\kappa t} - e^{-2\kappa t})x_0 + e^{-2\kappa t}x_0^2.$$

Comparing the last row with what we obtained using the other method, we see that the results match.

Exercise 3

We use the martingale property of a discounted derivative and Itô's lemma to solve this exercise. Applying Itô's lemma gives us

$$d(e^{-r(T-t)}U(t,S,V)) = e^{-r(T-t)}(-rU dt + \frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial V}dV +$$

$$+\frac{1}{2}(\frac{\partial^2 U}{\partial S^2}d < S,S > +2\frac{\partial^2 U}{\partial S \partial V}d < S,V > +\frac{\partial^2 U}{\partial V^2}d < V,V >)).$$

We have

$$\begin{cases} d < S,S >= (S^2 V\rho^2 + S^2 V(1-\rho^2))dt = S^2 V dt \\ d < S,V >= SV\rho\nu dt \\ d < V,V >= \nu^2 V dt. \end{cases}$$

Now we will use the martingale property, which means that we will let the deterministic part of dU be 0, this will give us a PDE. Thus, combining the results from above with the deterministic parts of dV and dS gives us

$$dU_{deterministic}(t,S,V) = e^{-r(T-t)}(-rU + \frac{\partial U}{\partial t} + \frac{\partial U}{\partial S}rS + \frac{\partial U}{\partial V}\kappa(\theta - V) + \frac{1}{2}(\frac{\partial^2 U}{\partial S^2}S^2 V + 2\frac{\partial^2 U}{\partial S \partial V}SV\rho\nu + \frac{\partial^2 U}{\partial V^2}\nu^2 V))dt.$$

Since we want this to be 0, the PDE that $U(t,s,v)$ satisfies is

$$-rU + \frac{\partial U}{\partial t} + \frac{\partial U}{\partial s}rs + \frac{\partial U}{\partial v}\kappa(\theta - v) + \frac{1}{2}(\frac{\partial^2 U}{\partial s^2}s^2 v + 2\frac{\partial^2 U}{\partial s \partial v}sv\rho\nu + \frac{\partial^2 U}{\partial v^2}\nu^2 v) = 0.$$