Computational finance, take-home exam 2: theoretical part

Karl Albin Henriksson

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Exercise 2

a)

We can write the PDE that v satisfies in terms of φ and ψ. We obtain the following

$$-(\varphi'(T-t,\nu)+x\psi'(T-t,\nu))v(t,x)+\kappa(\theta-x)\psi(T-t,\nu)v(t,x)+\frac{\lambda^2}{2}\psi(T-t,\nu)^2v(t,x)=0 \implies$$

$$\implies -(\varphi'(T-t,\nu)+x\psi'(T-t,\nu))+\kappa(\theta-x)\psi(T-t,\nu)+\frac{\lambda^2}{2}\psi(T-t,\nu)^2=0.$$

From here, we collect the terms with and without x and set both to 0.

$$\begin{cases} -\varphi'(T-t,\nu)+\kappa\theta\psi(T-t,\nu)+\frac{\lambda^2}{2}\psi(T-t,\nu)^2=0 \\ -\psi'(T-t,\nu)-\kappa\psi(T-t,\nu)=0. \end{cases} \tag{1}$$

From the boundary condition

$$v(T,x)=e^{i\nu x},$$

we can deduce that

$$\begin{cases} \varphi(0,\nu)=0 \\ \psi(0,\nu)=i\nu. \end{cases}$$

b)

We start with the second ODE in system 1. Its solution is $\psi(T-t)=C\cdot e^{-\kappa(T-t)}$ with $C=i\nu$ from the initial condition. By inserting this into the first equation, we obtain

$$-\varphi'(T-t,\nu)+\kappa\theta i\nu e^{-\kappa(T-t)}+\frac{\lambda^2}{2}(i\nu e^{-\kappa(T-t)})^2=$$

$$=-\varphi'(T-t,\nu)+\kappa\theta i\nu e^{-\kappa(T-t)}-\frac{\lambda^2}{2}\nu^2e^{-2\kappa(T-t)}=0.$$

To solve for φ, we make the ansatz

$$\varphi(T-t,\nu)=ae^{-\kappa(T-t)}+be^{-2\kappa(T-t)}+c$$

for some constants a, b and c and insert it to equation 2 to obtain

$$\kappa a e^{-\kappa(T-t)} + 2\kappa b e^{-2\kappa(T-t)} + \kappa\theta i\nu e^{-\kappa(T-t)} - \frac{\lambda^2}{2}\nu^2 e^{-2\kappa(T-t)} = 0.$$

We see that we require $a = -\theta i\nu$ and $b = \frac{\lambda^2\nu^2}{4\kappa}$ for the equality to hold. From the initial condition for φ, we also obtain

$$c = -(a+b) = -\frac{\lambda^2\nu^2}{4\kappa} + \theta i\nu.$$

Thus, we have the solutions

$$\begin{cases} \psi(T-t,\nu) = i\nu e^{-\kappa(T-t)} \\ \varphi(T-t,\nu) = \theta i\nu(1-e^{-\kappa(T-t)}) + \frac{\lambda^2\nu^2}{4\kappa}(e^{-2\kappa(T-t)}-1). \end{cases}$$

Using the result from above, we obtain that

$$v(t,x) = \mathbb{E}[e^{i\nu X_T} \mid X_t = x] = v(t,x) = e^{\theta i\nu(1-e^{-\kappa(T-t)}) + \frac{\lambda^2\nu^2}{4\kappa}(e^{-2\kappa(T-t)}-1) + i\nu e^{-\kappa(T-t)}x}$$

The charateristic function of X_t, given that $X_0 = x$ is thus

$$\mathbb{E}[e^{i\nu X_t} \mid X_0 = x] = e^{\theta i\nu(1-e^{-\kappa t}) + \frac{\lambda^2\nu^2}{4\kappa}(e^{-2\kappa t}-1) + i\nu e^{-\kappa t}x}.$$

c)

To prove that X_t is normal, recall that the charateristic function of $Y \sim N(\mu, \sigma^2)$ is

$$\mathbb{E}[e^{i\nu Y}] = e^{i\nu\mu - \frac{\nu^2\sigma^2}{2}}.$$

By comparing the characteristic function of the normal variable Y with that of X_t above and identifying the mean and variance, we see that X_t has a mean of

$$\mathbb{E}[X_t] = \theta(1-e^{-\kappa t}) + e^{-\kappa t}x$$

and variance

$$Var(X_t) = \frac{\lambda^2}{2\kappa}(1-e^{-2\kappa t}).$$

Now since the characteristic function of a random variable uniquely determines its distribution, we know that X_t must be normal with the mean and variance determined above.

d)

The SDE can equivalently be written as

$$dX_t + \kappa X_t dt = \theta\kappa dt + \lambda dW_t \implies e^{-\kappa t}d(X_t e^{\kappa t}) = \theta\kappa dt + \lambda dW_t.$$

From here, we multiply by $e^{\kappa t}$, integrate from 0 to t, and use $X_0 = x$.

$$X_t e^{\kappa t} - x = \theta\kappa\int_0^t e^{\kappa s}ds + \lambda\int_0^t e^{\kappa s}dW_s \implies X_t = e^{-\kappa t}x + \theta(1-e^{-\kappa t}) + \lambda\int_0^t e^{-\kappa(t-s)}dW_s.$$

Using the result above, we obtain

$$\mathbb{E}[X_t] = \mathbb{E}[e^{-\kappa t}x + \theta(1 - e^{-\kappa t}) + \lambda \int_0^t e^{-\kappa(t-s)} dW_s] = e^{-\kappa t}x + \theta(1 - e^{-\kappa t})$$

since the integral of any deterministic function with respect to dW_s will be 0. As for the variance, we have

$$Var(X_t) = Var(e^{-\kappa t}x + \theta(1 - e^{-\kappa t})) + \lambda \int_0^t e^{-\kappa(t-s)} dW_s) = Var(\lambda \int_0^t e^{-\kappa(t-s)} dW_s)$$

$$= \mathbb{E}[(\lambda \int_0^t e^{-\kappa(t-s)} dW_s))^2].$$

where in the last step we used the definition of variance and that the integral has 0 mean. Now using the Itô isometry, we obtain

$$\mathbb{E}[(\lambda \int_0^t e^{-\kappa(t-s)} dW_s))^2] = \lambda^2 \int_0^t (e^{-\kappa(t-s)})^2 dt = \frac{\lambda^2}{2}(1 - e^{-2\kappa t}).$$

We see that the mean and variance are exactly the same as before.