HW3 SF2525

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Problem 1

a)

We want to estimate the price of a European call option

$$f(0, S_0) = e^{-rT}\,\mathbb{E}[\max(S(T) - K, 0)],$$

where r is the risk free interest rate, T is the maturity of the option, and K is the strike price. The stock price $S(T)$ is given by

$$S(T) = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z}, \;\; Z \sim N(0,1), \;\; S(0) = S_0.$$

To estimate $\mathbb{E}[\max(S(T) - K, 0)]$, a Monte Carlo method will be implemented in Matlab, simulating $S_i(T)$ for $i = 1, 2...N$,

$$S_i(T) = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z_i}, \;\; Z_i \sim N(0,1),$$

and using the approximation

$$\mathbb{E}[\max(S(T) - K, 0)] \approx \frac{1}{N}\sum_{i=1}^{N} \max(S_i(T) - K, 0)].$$

The law of large numbers tells us that this estimate converges to the true value as N increases. Since N is sufficiently large, the central limit theorem ensures that the error is $N(0, \frac{\sigma^2}{N})$ distributed. We will compare the simulated values to the theoretical price of a European call option, obtained using the Black & Scholes option pricing formula

$$C = S(T)N(d_1) - e^{-rT}KN(d_2) \tag{1}$$

where

$$d_1 = \frac{log(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \;\; d_2 = d_1 - \sigma\sqrt{T},$$

and $N(x)$ denotes the cumulative normal distribution at x. The results can be seen in table 1.

N	$f(0, S_0)$	$\hat{\sigma}$	s.e.= $\frac{\hat{\sigma}}{\sqrt{N}}$	$\lvert f(0, S_0) - C \rvert$
10^2	2.1076	3.0186	0.3019	0.2119
10^3	2.2509	3.3055	0.1045	0.0686
10^4	2.3460	3.3960	0.0340	0.0265
10^5	3.167	3.3541	0.0106	0.0027
10^6	2.3172	3.3696	0.0034	0.0023
10^7	3.179	3.3681	0.0011	0.0015
10^8	3.191	3.3697	0.0004	0.0004

Table 1: A table showing the MC simulated values of the call option obtained for different values of N, along with the sample standard deviation $\hat{\sigma}$ and the standard error (s.e.). In this particular example, we used $(S_0, K, r, \sigma, T) = (35, 35, 0.04, 0.2, 0.5)$. For reference, the theoretical price of a European call option with these values is $C = 2.3195$.

b)

In this assignment, we want to compute Δ, defined as the change in the price of the call option, given an infinitesimal change in the underlying asset.

$$\Delta = \frac{df(0,S_0)}{dS}.$$

This will be done by using the approximation

$$\Delta_1 \approx \frac{f(0,S_0+\Delta S) - f(0,S_0)}{\Delta S}. \tag{2}$$

As in the previous assignment, our numerical solutions can be compared to a theoretical value. From equation 1, we see that

$$\frac{\partial C}{\partial S} = \Delta = N(d_1).$$

This will be our reference value. We will also use the more accurate second order method

$$\Delta_2 \approx \frac{f(0,S_0+\Delta S) - f(0,S_0-\Delta S)}{2\Delta S} \tag{3}$$

as a comparison. As before, we will perform calculations for different values of N. The results can be seen in table 2. As we can see in the last two columns, the second order method is the more accurate of the two.

N	Δ_1	Δ_2	$\hat{\sigma_1}$	$\hat{\sigma_2}$	s.e.$_1 = \frac{\hat{\sigma_1}}{\sqrt{N}}$	s.e.$_2 = \frac{\hat{\sigma_2}}{\sqrt{N}}$	$\lvert\Delta_1 - N(d_1)\rvert$	$\lvert\Delta_2 - N(d_1)\rvert$
10^2	0.6036	0.5974	0.5672	0.5662	0.0567	0.0566	0.0196	0.0134
10^3	0.5849	0.5826	0.5582	0.5582	0.0177	0.0177	0.001	0.001
10^4	0.5893	0.5855	0.5550	0.5546	0.0056	0.0055	0.0053	0.0015
10^5	0.5883	0.5844	0.5548	0.5543	0.0018	0.0018	0.0043	$3.7 \cdot 10^{-4}$
10^6	0.5875	0.5836	0.5551	0.5546	$5.551 \cdot 10^{-4}$	$5.5461 \cdot 10^{-4}$	0.0035	$3.5728 \cdot 10^{-4}$
10^7	0.5880	0.5841	0.5549	0.5544	$1.7548 \cdot 10^{-4}$	$1.7533 \cdot 10^{-4}$	0.0040	$5.8903 \cdot 10^{-4}$
10^8	0.5879	0.5840	0.5549	0.5545	$5.5494 \cdot 10^{-5}$	$5.5446 \cdot 10^{-5}$	0.0039	$3.3781 \cdot 10^{-5}$

Table 2: A table showing the MC simulated values of Δ obtained for different values of N, along with the sample standard deviation $\hat{\sigma}$ and the standard error (s.e.). Parameters with index 1 use the estimate of Δ used in equation 2, and parameters with index 2 use the estimate used in equation 3. In this particular example, we used $(S_0, K, r, \sigma, T) = (35, 35, 0.04, 0.2, 0.5)$. For reference, the theoretical Δ with these values is $\Delta = N(d_1) = 0.5840$.

Problem 2

a)

We want to solve the SDE

$$dY(t) = (-\alpha(2+Y(t)) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2})dt + 0.4\sqrt{\alpha}d\hat{Z}(t), \tag{4}$$

where

$$d\hat{Z}(t) = \rho dW(t) + \sqrt{1-\rho^2} dZ(t).$$

Here, $W(t)$ and $Z(t)$ are two independent Brownian motions. Equivalently, equation 4 can be written as

$$e^{-\alpha t} d(e^{\alpha t} Y(t)) = (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1-\rho^2})dt + 0.4\sqrt{\alpha} d\hat{Z}(t)$$

Multiplying by $e^{\alpha t}$ and integrating from 0 to t yields

$$e^{\alpha t} Y(t) - Y(0) = (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}) \int_0^t e^{\alpha s} ds + 0.4\sqrt{\alpha} \int_0^t e^{\alpha s} d\hat{Z}(s).$$

Finally, solving for $Y(t)$:

$$Y(t) = e^{-\alpha t}(Y(0) + (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}) \int_0^t e^{\alpha s} ds + 0.4\sqrt{\alpha} \int_0^t e^{\alpha s} d\hat{Z}(s)) =$$

$$= e^{-\alpha t}(Y(0) + (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}) \int_0^t e^{\alpha s} ds + 0.4\sqrt{\alpha}(\rho \int_0^t e^{\alpha s} dW(s) + \sqrt{1-\rho^2} \int_0^t e^{\alpha s} dZ(t))).$$

Now, since

$$\mathbb{E}[\int_0^t e^{\alpha s} dW(s)] = \mathbb{E}[\int_0^t e^{\alpha s} dZ(t)] = 0,$$

we have

$$\mathbb{E}[Y(t)] = e^{-\alpha t}(Y(0) + (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}) \int_0^t e^{\alpha s} ds) =$$

$$= e^{-\alpha t}(Y(0) + (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1-\rho^2})(\frac{e^{\alpha t}-1}{\alpha})) = Y(0)e^{-\alpha t} + (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1-\rho^2})(\frac{1-e^{-\alpha t}}{\alpha}).$$

Consequently,

$$\lim_{t\to\infty} \mathbb{E}[Y(t)] = (-2 + 0.4\frac{1}{\sqrt{\alpha}}\sqrt{1-\rho^2}).$$

Using the identity $Var(Y(t)) = \mathbb{E}[(Y(t) - \mathbb{E}[Y(t)])^2]$ gives us

$$Var(Y(t)) = \mathbb{E}[(e^{-\alpha t} 0.4\sqrt{\alpha}(\rho \int_0^t e^{\alpha s} dW(s) + \sqrt{1-\rho^2} \int_0^t e^{\alpha s} dZ(t)))^2] =$$

$$= e^{-2\alpha t} 0.16\alpha(\mathbb{E}[\rho^2(\int_0^t e^{\alpha s} dW(s))^2] + 2\rho\sqrt{1-\rho^2}\,\mathbb{E}[\int_0^t e^{\alpha s} dW(s) \int_0^t e^{\alpha s} dZ(t)] + \mathbb{E}[(1-\rho^2)(\int_0^t e^{\alpha s} dZ(s))^2].$$

Now

$$\mathbb{E}[\int_0^t e^{\alpha s} dW(s) \int_0^t e^{\alpha s} dZ(s)] = \mathbb{E}[\int_0^t e^{\alpha s} dW(s)]\,\mathbb{E}[\int_0^t e^{\alpha s} dZ(s)] = 0$$

due to the independence of $W(t)$ and $Z(t)$, and

$$\mathbb{E}[(\int_0^t e^{\alpha s} dW(s))^2] = \mathbb{E}[(\int_0^t e^{\alpha s} dZ(s))^2] = \mathbb{E}[\int_0^t (e^{\alpha s})^2 ds] = \frac{e^{2\alpha t}-1}{2\alpha}$$

from the Itô Isometry. Thus, we obtain

$$Var(Y(t)) = e^{-2\alpha t}0.16\alpha(\rho^2\frac{e^{2\alpha t}-1}{2\alpha}+(1-\rho^2)\frac{e^{2\alpha t}-1}{2\alpha}) = e^{-2\alpha t}0.16\alpha\frac{e^{2\alpha t}-1}{2\alpha} = 0.08(1-e^{-2\alpha t}).$$

Clearly,

$$\lim_{t\to\infty}Var(Y(t)) = 0.08.$$

We notice that $\mathbb{E}[Y(t)]$ is exactly the solution to the ODE

$$y'(t) = -\alpha(2+y(t)) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}$$

and the corresponding limit is the same limit as for $y(t)$ in the ODE as $t \to \infty$.

b)

i)

Since the solution here is very similar to the solution of a), this one will be kept shorter.

With

$$dX(t) = -\alpha X(t)dt + \sqrt{\alpha}dW(t),$$

we have

$$e^{-\alpha t}d(e^{\alpha t}X(t)) = \sqrt{\alpha}dW(t) \implies X(t) = e^{-\alpha t}(X(0) + \sqrt{\alpha}\int_0^t e^{\alpha s}dW(s)).$$

Consequently,

$$\mathbb{E}[X(t)] = X(0)e^{-\alpha t},$$

and

$$Var(X(t)) = \mathbb{E}[(X(t)-\mathbb{E}[X(t)])^2] = E[(e^{-\alpha t}\sqrt{\alpha}\int_0^t e^{\alpha s}dW(s))^2] = e^{-2\alpha t}\alpha\,\mathbb{E}[\int_0^t (e^{\alpha s})^2 ds] = \frac{1-e^{-2\alpha t}}{2}.$$

We have

$$\lim_{t\to\infty}\mathbb{E}[X(t)] = 0$$

and

$$\lim_{t\to\infty}Var(X(t)) = \frac{1}{2}.$$

ii)

With a forward Euler discretization, we have

$$X_{n+1} = X_n - \alpha\Delta t X_n + \sqrt{\alpha}\Delta W_n.$$

where $t_{n+1} - t_n = \Delta t_i = \Delta t$ since we have uniform timesteps, and $\Delta W_n \sim N(0, \Delta t)$. We have

$$\mathbb{E}[X_{n+1}] = (1 - \alpha\Delta t)\,\mathbb{E}[X_n] = (1 - \alpha\Delta t)^2\,\mathbb{E}[X_{n-1}] = ... = (1 - \alpha\Delta t)^{n+1} X_0.$$

Now, due to the independence of X_k and ΔW_k

$$\mathbb{E}[X_{n+1}^2] = \mathbb{E}[(X_n - \alpha\Delta t X_n + \sqrt{\alpha}\Delta W_n)^2] = (1 - \alpha\Delta t)^2\,\mathbb{E}[X_n^2] + \alpha\Delta t =$$

$$= (1 - \alpha\Delta t)^2\,\mathbb{E}[(X_{n-1} - \alpha\Delta t X_{n-1} + \sqrt{\alpha}\Delta W_{n-1})^2] + \alpha\Delta t =$$

$$= (1-\alpha\Delta t)^2((1-\alpha\Delta t)^2\,\mathbb{E}[X_{n-1}^2]+\alpha\Delta t)+\alpha\Delta t = ... = (1-\alpha\Delta t)^{2(n+1)} X_0^2 + \alpha\Delta t \sum_{k=0}^{n} (1-\alpha\Delta t)^{2k}.$$

Thus,

$$Var(X_{n+1}) = \mathbb{E}[X_{n+1}^2] - \mathbb{E}[X_{n+1}]^2 = (1-\alpha\Delta t)^{2(n+1)} X_0^2 + \alpha\Delta t \sum_{k=0}^{n} (1-\alpha\Delta t)^{2k} - ((1-\alpha\Delta t)^{n+1} X_0)^2 =$$

$$= \alpha\Delta t \sum_{k=0}^{n} (1 - \alpha\Delta t)^{2k}.$$

Now since

$$\lim_{t\to\infty} \mathbb{E}[X_n] = \lim_{n\to\infty} \mathbb{E}[X_n],$$

and

$$|1 - \alpha\Delta t| < 1,$$

we have

$$\lim_{t\to\infty} \mathbb{E}[X_n] = \lim_{n\to\infty} (1 - \alpha\Delta t)^n X_0 = 0.$$

Similarly, for the variance,

$$\lim_{t\to\infty} Var(X_n) = \lim_{n\to\infty} \alpha\Delta t \sum_{k=0}^{n-1} (1-\alpha\Delta t)^{2k} = \lim_{n\to\infty} \alpha\Delta t \sum_{k=0}^{n-1} ((1-\alpha\Delta t)^2)^k$$

$$= \alpha\Delta t \frac{1}{1 - (1 - \alpha\Delta t)^2} = \alpha\Delta t \frac{1}{2\alpha\Delta t - (\alpha\Delta t)^2} = \frac{1}{2 - \alpha\Delta t}$$

where we have used the formula for a converging infinite geometric sum.

iii)

Using the backward Euler discretization, we have

$$X_{n+1} = X_n - \alpha X_{n+1}\Delta t + \sqrt{\alpha}\Delta W_n \implies X_{n+1} = \frac{1}{1 + \alpha\Delta t}(X_n + \sqrt{\alpha}\Delta W_n).$$

This implies

$$\mathbb{E}[X_{n+1}] = \frac{1}{1 + \alpha\Delta t}\,\mathbb{E}[X_n] = (\frac{1}{1 + \alpha\Delta t})^2\,\mathbb{E}[X_{n-1}] = ... = (\frac{1}{1 + \alpha\Delta t})^{n+1} X_0.$$

For the variance, we need to calculate $\mathbb{E}[X_{n+1}^2]$, we have

$$\mathbb{E}[X_{n+1}^2] = \mathbb{E}[(\frac{1}{1+\alpha\Delta t}(X_n + \sqrt{\alpha}\Delta W_n))^2] = (\frac{1}{1+\alpha\Delta t})^2\,\mathbb{E}[X_n^2] + \frac{\alpha\Delta t}{(1+\alpha\Delta t)^2} =$$

$$(\frac{1}{1+\alpha\Delta t})^2\,\mathbb{E}[(\frac{1}{1+\alpha\Delta t}(X_{n-1} + \sqrt{\alpha}\Delta W_{n-1}))^2] + \frac{\alpha\Delta t}{(1+\alpha\Delta t)^2} =$$

$$= (\frac{1}{1+\alpha\Delta t})^4(\mathbb{E}[X_{n-1}^2]+\alpha\Delta t)+\frac{\alpha\Delta t}{(1+\alpha\Delta t)^2} = ... = (\frac{1}{1+\alpha\Delta t})^{2(n+1)}X_0^2+\alpha\Delta t\sum_{k=1}^{n+1}(\frac{1}{1+\alpha\Delta t})^{2k}.$$

Thus,

$$Var(X_n) = \mathbb{E}[X_n^2]-\mathbb{E}[X_n]^2 = (\frac{1}{1+\alpha\Delta t})^{2n}X_0^2+\alpha\Delta t\sum_{k=1}^{n}(\frac{1}{1+\alpha\Delta t})^{2k}-((\frac{1}{1+\alpha\Delta t})^n X_0)^2 =$$

$$= \alpha\Delta t\sum_{k=1}^{n}(\frac{1}{1+\alpha\Delta t})^{2k}.$$

Clearly

$$\lim_{t\to\infty}\mathbb{E}[X_n] = \lim_{n\to\infty}(\frac{1}{1+\alpha\Delta t})^n X_0 = 0,$$

since $|\frac{1}{1+\alpha\Delta t}| < 1$. As for the variance,

$$\lim_{t\to\infty}Var(X_n) = \lim_{n\to\infty}\alpha\Delta t\sum_{k=1}^{n-1}(\frac{1}{1+\alpha\Delta t})^{2k} = \alpha\Delta t(-1+\lim_{n\to\infty}\sum_{k=0}^{n-1}((\frac{1}{1+\alpha\Delta t})^2)^k) =$$

$$= -\alpha\Delta+\frac{\alpha\Delta t}{1-(\frac{1}{1+\alpha\Delta t})^2} = -\alpha\Delta+\frac{\alpha\Delta t}{\frac{2\alpha\Delta t+(\alpha\Delta t)^2}{1+2\alpha\Delta t+(\alpha\Delta t)^2}} = -\alpha\Delta+\frac{1+2\alpha\Delta t+(\alpha\Delta t)^2}{2+\alpha\Delta t} = \frac{1}{2+\alpha\Delta t},$$

again, making use of the formula for infinite geometric sums.

iv)

As seen in i)-iii), the expectation for the exact solution is the same as those for forward and backward euler as $t \to \infty$. The variances are sligthly different. The exact solution has a variance of $\frac{1}{2}$ while the forward Euler has a variance of $\frac{1}{2-\alpha\Delta t}$ and the backward Euler has a variance of $\frac{1}{2+\alpha\Delta t}$ as $t \to \infty$. The variance of both the backward and forward Euler method converges to $\frac{1}{2}$ if we let the timestep approach 0.

c)

Consider a special case of the Log Ornstein-Uhlenbeck model, where the underlying asset S_t satisfies

$$\begin{cases} dS_t = rS_t dt + e^{Y_t} S_t dW_t \\ Y_t = (-\alpha(2+Y_t) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2})dt + 0.4\sqrt{\alpha} dZ_t \\ Z_t = \rho dW_t + \sqrt{1-\rho^2} dB_t. \end{cases} \tag{5}$$

Here, W_t and B_t are two independent Brownian motions and e^{Y_t} is the stochastic volatility.

We want to estimate the option price

$$e^{-rT}\mathbb{E}[\max(S_T - K, 0)$$

where e^{-rT} is the discount factor, K is the strike price, and T is the time at maturity of the option. This will be achieved by performing Monte Carlo simulations on a discretized system.

Now, using the time discretization $t_n = n\Delta t$, $n = 1, 2..N$ where $\Delta t = \frac{T}{N}$, the problem discretizes to

$$S_{n+1} - S_n = rS_n\Delta t + e^{Y_n} S_n (W_{n+1} - W_n),$$
$$Y_{n+1} - Y_n = (-\alpha(2+Y_n) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2})\Delta t + 0.4\sqrt{\alpha}(Z_{n+1} - Z_n),$$
$$Z_{n+1} - Z_n = \rho(W_{n+1} - W_n) + \sqrt{1-\rho^2}(B_{n+1} - B_n),$$

where $W_{n+1} - W_n$, and $B_{n+1} - B_n$ are $N(0, \Delta t)$ distributed. We will consider an at the money option with parameters $\alpha = 100, \rho = -0.3, r = 0.04, T = 0.75, Y_0 = -1$ and $S_0 = K = 100$. We use the Monte Carlo estimate of the expectation. This estimate approaches the true expectation as we increase the number of simulations M in accordance with the law of large numbers. We have

$$\mathbb{E}[e^{-rT}\max(S(T) - K, 0) \approx e^{-rT}\frac{1}{M}\sum_{j=1}^{M}\max(S_j(T) - K, 0),$$

where $S_j(T)$ are samples from the forward euler discretization at time T. Using $M = 5 \cdot 10^4$ simulations, and a timestep of $\Delta t = 0.75 \cdot 10^-3$, the Monte Carlo estimate

$$e^{-rT}\frac{1}{M}\sum_{j=1}^{M}\max(S_j(T) - K, 0) = 6.91$$

is obtained. Figure 1 shows 10 simulations of S_t.

d)

Now, let $\hat{C}$ be the Monte Carlo estimate of the option price. We want an estimate of $\hat{C}$ such that the true value C has a high confidence of being within the range

$$(1 - \text{TOL})\hat{C} \leq C \leq (1 + \text{TOL})\hat{C} \implies 6.56 \leq C \leq 7.25$$

where $\text{TOL} = 5 \cdot 10^{-2}$ and $\hat{C} = 6.91$. Here, we will make use of the antithetic variables method in order to keep the computational cost down. For each simulated $X \sim N(0, \Delta t)$, we will also use $-X$ in the computation, which will result in a two samples of $\max(S_j(T) - K, 0)$ that are negatively correlated. As a result, variance will be reduced, and we do not need to draw as many samples for a fixed standard error. In this estimate, we have used $M = 450$ and $N = 10^3 \implies \Delta t = 0.75 \cdot 10^{-3}$. This resulted in a sample variance of $\hat{\sigma}^2 = 22.9$ which implies a standard error of s.e. $= \frac{\hat{\sigma}}{\sqrt{M}} = 0.22$ Now since $Z_{0.05} = 1.64$, there is a 90% probability that the true C is within

$$\hat{C} - Z_{0.05}\frac{\hat{\sigma}}{\sqrt{M}} \leq C \leq \hat{C} + Z_{0.05}\frac{\hat{\sigma}}{\sqrt{M}} \implies 6.55 \leq C \leq 7.27,$$

which is close to the tolerated range. With these values of M and N, the time taken to obtain an estimate was 0.21 seconds. If we instead use $TOL = 5 \cdot 10^{-3}$, we require

$$(1 - \text{TOL})\hat{C} \leq C \leq (1 + \text{TOL})\hat{C} \implies 6.875 \leq C \leq 6.945.$$

Using $M = 450 \cdot 100$, we obtain s.e. $= \frac{\hat{\sigma}}{\sqrt{M}} = 0.022$ and thus

$$\hat{C} - Z_{0.05}\frac{\hat{\sigma}}{\sqrt{M}} \leq C \leq \hat{C} + Z_{0.05}\frac{\hat{\sigma}}{\sqrt{M}} \implies 6.873 \leq C \leq 6.946,$$

which, again is close to the tolerated range. This time, the computational time was 21.68 seconds. In each estimate, we need to draw $M(N-1)$ samples from the normal distribution, resulting in $450 \cdot 999$ samples for $\text{TOL} = 5 \cdot 10^{-2}$ and $450000 \cdot 999$ samples for $\text{TOL} = 5 \cdot 10^{-3}$.

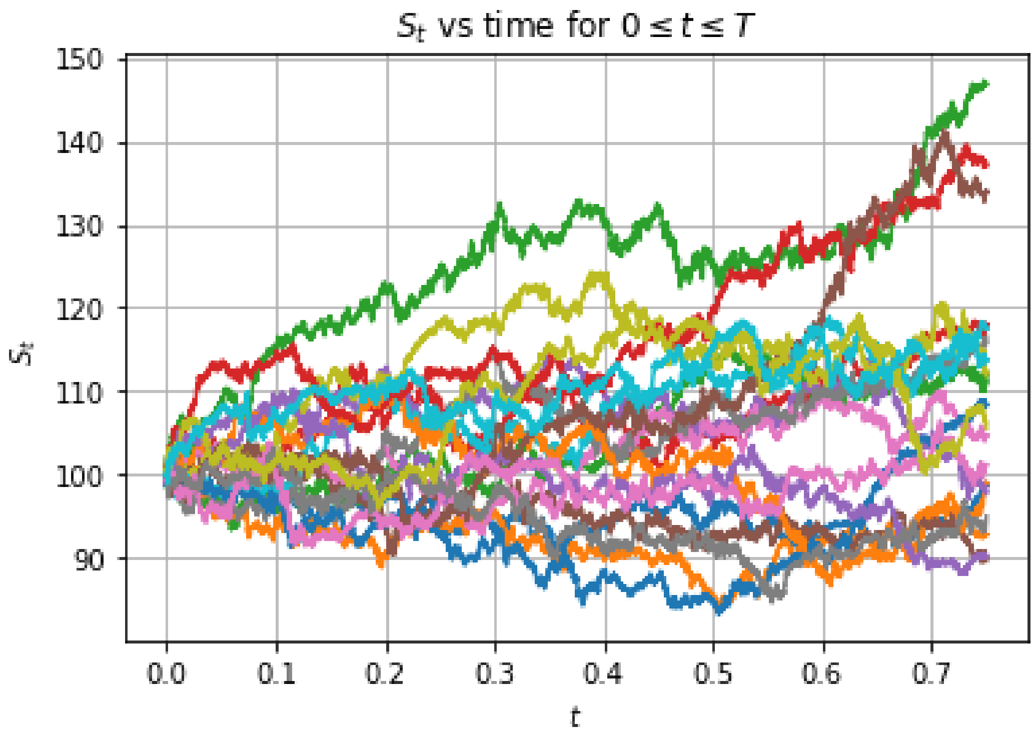

Figure 1: 10 simulations of S_t using the forward Euler discretization of the SDE system 5.