

# HW3 SF2525

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March 2022

## Problem 1

a)

We want to estimate the price of a European call option

$$f(0, S_0) = e^{-rT} \mathbb{E}[\max(S(T) - K, 0)],$$

where  $r$  is the risk free interest rate,  $T$  is the maturity of the option, and  $K$  is the strike price. The stock price  $S(T)$  is given by

$$S(T) = S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z}, \quad Z \sim N(0, 1), \quad S(0) = S_0.$$

To estimate  $\mathbb{E}[\max(S(T) - K, 0)]$ , a Monte Carlo method will be implemented in Matlab, simulating  $S_i(T)$  for  $i = 1, 2, \dots, N$ ,

$$S_i(T) = S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z_i}, \quad Z_i \sim N(0, 1),$$

and using the approximation

$$\mathbb{E}[\max(S(T) - K, 0)] \approx \frac{1}{N} \sum_{i=1}^N \max(S_i(T) - K, 0).$$

The law of large numbers tells us that this estimate converges to the true value as  $N$  increases. Since  $N$  is sufficiently large, the central limit theorem ensures that the error is  $N(0, \frac{\sigma^2}{N})$  distributed. We will compare the simulated values to the theoretical price of a European call option, obtained using the Black & Scholes option pricing formula

$$C = S(T)N(d_1) - e^{-rT}KN(d_2) \tag{1}$$

where

$$d_1 = \frac{\log(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

and  $N(x)$  denotes the cumulative normal distribution at  $x$ . The results can be seen in table 1.

$N$	$f(0, S_0)$	$\hat{\sigma}$	s.e. = $\frac{\hat{\sigma}}{\sqrt{N}}$	$ f(0, S_0) - C $
$10^2$	2.1076	3.0186	0.3019	0.2119
$10^3$	2.2509	3.3055	0.1045	0.0686
$10^4$	2.3460	3.3960	0.0340	0.0265
$10^5$	3.167	3.3541	0.0106	0.0027
$10^6$	2.3172	3.3696	0.0034	0.0023
$10^7$	3.179	3.3681	0.0011	0.0015
$10^8$	3.191	3.3697	0.0004	0.0004

Table 1: A table showing the MC simulated values of the call option obtained for different values of  $N$ , along with the sample standard deviation  $\hat{\sigma}$  and the standard error (s.e.). In this particular example, we used  $(S_0, K, r, \sigma, T) = (35, 35, 0.04, 0.2, 0.5)$ . For reference, the theoretical price of a European call option with these values is  $C = 2.3195$ .

b)

In this assignment, we want to compute  $\Delta$ , defined as the change in the price of the call option, given an infinitesimal change in the underlying asset.

$$\Delta = \frac{df(0, S_0)}{dS}.$$

This will be done by using the approximation

$$\Delta_1 \approx \frac{f(0, S_0 + \Delta S) - f(0, S_0)}{\Delta S}. \quad (2)$$

As in the previous assignment, our numerical solutions can be compared to a theoretical value. From equation 1, we see that

$$\frac{\partial C}{\partial S} = \Delta = N(d_1).$$

This will be our reference value. We will also use the more accurate second order method

$$\Delta_2 \approx \frac{f(0, S_0 + \Delta S) - f(0, S_0 - \Delta S)}{2\Delta S} \quad (3)$$

as a comparison. As before, we will perform calculations for different values of  $N$ . The results can be seen in table 2. As we can see in the last two columns, the second order method is the more accurate of the two.

$N$	$\Delta_1$	$\Delta_2$	$\hat{\sigma}_1$	$\hat{\sigma}_2$	s.e. <sub>1</sub> = $\frac{\hat{\sigma}_1}{\sqrt{N}}$	s.e. <sub>2</sub> = $\frac{\hat{\sigma}_2}{\sqrt{N}}$	$ \Delta_1 - N(d_1) $	$ \Delta_2 - N(d_1) $
$10^2$	0.6036	0.5974	0.5672	0.5662	0.0567	0.0566	0.0196	0.0134
$10^3$	0.5849	0.5826	0.5582	0.5582	0.0177	0.0177	0.001	0.001
$10^4$	0.5893	0.5855	0.5550	0.5546	0.0056	0.0055	0.0053	0.0015
$10^5$	0.5883	0.5844	0.5548	0.5543	0.0018	0.0018	0.0043	$3.7 \cdot 10^{-4}$
$10^6$	0.5875	0.5836	0.5551	0.5546	$5.551 \cdot 10^{-4}$	$5.5461 \cdot 10^{-4}$	0.0035	$3.5728 \cdot 10^{-4}$
$10^7$	0.5880	0.5841	0.5549	0.5544	$1.7548 \cdot 10^{-4}$	$1.7533 \cdot 10^{-4}$	0.0040	$5.8903 \cdot 10^{-4}$
$10^8$	0.5879	0.5840	0.5549	0.5545	$5.5494 \cdot 10^{-5}$	$5.5446 \cdot 10^{-5}$	0.0039	$3.3781 \cdot 10^{-5}$

Table 2: A table showing the MC simulated values of  $\Delta$  obtained for different values of  $N$ , along with the sample standard deviation  $\hat{\sigma}$  and the standard error (s.e.). Parameters with index 1 use the estimate of  $\Delta$  used in equation 2, and parameters with index 2 use the estimate used in equation 3. In this particular example, we used  $(S_0, K, r, \sigma, T) = (35, 35, 0.04, 0.2, 0.5)$ . For reference, the theoretical  $\Delta$  with these values is  $\Delta = N(d_1) = 0.5840$ .

## Problem 2

a)

We want to solve the SDE

$$dY(t) = (-\alpha(2 + Y(t)) + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2})dt + 0.4\sqrt{\alpha}d\hat{Z}(t), \quad (4)$$

where

$$d\hat{Z}(t) = \rho dW(t) + \sqrt{1 - \rho^2} dZ(t).$$

Here,  $W(t)$  and  $Z(t)$  are two independent Brownian motions. Equivalently, equation 4 can be written as

$$e^{-\alpha t} d(e^{\alpha t} Y(t)) = (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2})dt + 0.4\sqrt{\alpha} d\hat{Z}(t)$$

Multiplying by  $e^{\alpha t}$  and integrating from 0 to  $t$  yields

$$e^{\alpha t} Y(t) - Y(0) = (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2}) \int_0^t e^{\alpha s} ds + 0.4\sqrt{\alpha} \int_0^t e^{\alpha s} d\hat{Z}(s).$$

Finally, solving for  $Y(t)$ :

$$\begin{aligned} Y(t) &= e^{-\alpha t} (Y(0) + (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2}) \int_0^t e^{\alpha s} ds + 0.4\sqrt{\alpha} \int_0^t e^{\alpha s} d\hat{Z}(s)) = \\ &= e^{-\alpha t} (Y(0) + (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2}) \int_0^t e^{\alpha s} ds + 0.4\sqrt{\alpha} (\rho \int_0^t e^{\alpha s} dW(s) + \sqrt{1 - \rho^2} \int_0^t e^{\alpha s} dZ(s))). \end{aligned}$$

Now, since

$$\mathbb{E}[\int_0^t e^{\alpha s} dW(s)] = \mathbb{E}[\int_0^t e^{\alpha s} dZ(s)] = 0,$$

we have

$$\begin{aligned} \mathbb{E}[Y(t)] &= e^{-\alpha t} (Y(0) + (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2}) \int_0^t e^{\alpha s} ds) = \\ &= e^{-\alpha t} (Y(0) + (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2}) (\frac{e^{\alpha t} - 1}{\alpha})) = Y(0)e^{-\alpha t} + (-2\alpha + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2}) (\frac{1 - e^{-2\alpha t}}{\alpha}). \end{aligned}$$

Consequently,

$$\lim_{t \rightarrow \infty} \mathbb{E}[Y(t)] = (-2 + 0.4 \frac{1}{\sqrt{\alpha}} \sqrt{1 - \rho^2}).$$

Using the identity  $Var(Y(t)) = \mathbb{E}[(Y(t) - \mathbb{E}[Y(t)])^2]$  gives us

$$\begin{aligned} Var(Y(t)) &= \mathbb{E}[(e^{-\alpha t} 0.4\sqrt{\alpha} (\rho \int_0^t e^{\alpha s} dW(s) + \sqrt{1 - \rho^2} \int_0^t e^{\alpha s} dZ(s)))^2] = \\ &= e^{-2\alpha t} 0.16\alpha (\mathbb{E}[\rho^2 (\int_0^t e^{\alpha s} dW(s))^2] + 2\rho\sqrt{1 - \rho^2} \mathbb{E}[\int_0^t e^{\alpha s} dW(s) \int_0^t e^{\alpha s} dZ(s)] + \mathbb{E}[(1 - \rho^2) (\int_0^t e^{\alpha s} dZ(s))^2]). \end{aligned}$$

Now

$$\mathbb{E}[\int_0^t e^{\alpha s} dW(s) \int_0^t e^{\alpha s} dZ(s)] = \mathbb{E}[\int_0^t e^{\alpha s} dW(s)] \mathbb{E}[\int_0^t e^{\alpha s} dZ(s)] = 0$$

due to the independence of  $W(t)$  and  $Z(t)$ , and

$$\mathbb{E}[(\int_0^t e^{\alpha s} dW(s))^2] = \mathbb{E}[(\int_0^t e^{\alpha s} dZ(s))^2] = \mathbb{E}[\int_0^t (e^{\alpha s})^2 ds] = \frac{e^{2\alpha t} - 1}{2\alpha}$$

from the Itô Isometry. Thus, we obtain

$$Var(Y(t)) = e^{-2\alpha t} 0.16\alpha (\rho^2 \frac{e^{2\alpha t} - 1}{2\alpha} + (1-\rho^2) \frac{e^{2\alpha t} - 1}{2\alpha}) = e^{-2\alpha t} 0.16\alpha \frac{e^{2\alpha t} - 1}{2\alpha} = 0.08(1-e^{-2\alpha t}).$$

Clearly,

$$\lim_{t \rightarrow \infty} Var(Y(t)) = 0.08.$$

**b)**

**i)**

Since the solution here is very similar to the solution of a), this one will be kept shorter.

With

$$dX(t) = -\alpha X(t)dt + \sqrt{\alpha}dW(t),$$

we have

$$e^{-\alpha t} d(e^{\alpha t} X(t)) = \sqrt{\alpha} dW(t) \implies X(t) = e^{-\alpha t} (X(0) + \sqrt{\alpha} \int_0^t e^{\alpha s} dW(s)).$$

Consequently,

$$\mathbb{E}[X(t)] = X(0)e^{-\alpha t},$$

and

$$Var(X(t)) = \mathbb{E}[(X(t) - \mathbb{E}[X(t)])^2] = E[(e^{-\alpha t} \sqrt{\alpha} \int_0^t e^{\alpha s} dW(s))^2] = e^{-2\alpha t} \alpha \mathbb{E}[\int_0^t (e^{\alpha s})^2 ds] = \frac{1 - e^{-2\alpha t}}{2}.$$

We have

$$\lim_{t \rightarrow \infty} \mathbb{E}[X(t)] = 0$$

and

$$\lim_{t \rightarrow \infty} Var(X(t)) = \frac{1}{2}.$$

**ii)**

With a forward Euler discretization, we have

$$X_{n+1} = X_n - \alpha \Delta t X_n + \sqrt{\alpha} \Delta W_n.$$

where  $t_{n+1} - t_n = \Delta t_i = \Delta t$  since we have uniform timesteps, and  $\Delta W_n \sim N(0, \Delta t)$ . We have

$$\mathbb{E}[X_{n+1}] = (1 - \alpha \Delta t) \mathbb{E}[X_n] = (1 - \alpha \Delta t)^2 \mathbb{E}[X_{n-1}] = \dots = (1 - \alpha \Delta t)^{n+1} X_0.$$

Now, due to the independence of  $X_k$  and  $\Delta W_k$

$$\mathbb{E}[X_{n+1}^2] = \mathbb{E}[(X_n - \alpha \Delta t X_n + \sqrt{\alpha} \Delta W_n)^2] = (1 - \alpha \Delta t)^2 \mathbb{E}[X_n^2] + \alpha \Delta t =$$

$$\begin{aligned}
&= (1 - \alpha\Delta t)^2 \mathbb{E}[(X_{n-1} - \alpha\Delta t X_{n-1} + \sqrt{\alpha}\Delta W_{n-1})^2] + \alpha\Delta t = \\
&= (1 - \alpha\Delta t)^2((1 - \alpha\Delta t)^2 \mathbb{E}[X_{n-1}^2] + \alpha\Delta t) + \alpha\Delta t = \dots = (1 - \alpha\Delta t)^{2(n+1)} X_0^2 + \alpha\Delta t \sum_{k=0}^n (1 - \alpha\Delta t)^{2k}.
\end{aligned}$$

Thus,

$$\begin{aligned}
Var(X_{n+1}) &= \mathbb{E}[X_{n+1}^2] - \mathbb{E}[X_{n+1}]^2 = (1 - \alpha\Delta t)^{2(n+1)} X_0^2 + \alpha\Delta t \sum_{k=0}^n (1 - \alpha\Delta t)^{2k} - ((1 - \alpha\Delta t)^{n+1} X_0)^2 = \\
&= \alpha\Delta t \sum_{k=0}^n (1 - \alpha\Delta t)^{2k}.
\end{aligned}$$

Now since

$$\lim_{t \rightarrow \infty} \mathbb{E}[X_n] = \lim_{n \rightarrow \infty} \mathbb{E}[X_n],$$

and

$$|1 - \alpha\Delta t| < 1,$$

we have

$$\lim_{t \rightarrow \infty} \mathbb{E}[X_n] = \lim_{n \rightarrow \infty} (1 - \alpha\Delta t)^n X_0 = 0.$$

Similarly, for the variance,

$$\begin{aligned}
\lim_{t \rightarrow \infty} Var(X_n) &= \lim_{n \rightarrow \infty} \alpha\Delta t \sum_{k=0}^{n-1} (1 - \alpha\Delta t)^{2k} = \lim_{n \rightarrow \infty} \alpha\Delta t \sum_{k=0}^{n-1} ((1 - \alpha\Delta t)^2)^k \\
&= \alpha\Delta t \frac{1}{1 - (1 - \alpha\Delta t)^2} = \alpha\Delta t \frac{1}{2\alpha\Delta t - (\alpha\Delta t)^2} = \frac{1}{2 - \alpha\Delta t}
\end{aligned}$$

where we have used the formula for a converging infinite geometric sum.

**iii)**

Using the backward Euler discretization, we have

$$X_{n+1} = X_n - \alpha X_{n+1} \Delta t + \sqrt{\alpha} \Delta W_n \implies X_{n+1} = \frac{1}{1 + \alpha\Delta t} (X_n + \sqrt{\alpha} \Delta W_n).$$

This implies

$$\mathbb{E}[X_{n+1}] = \frac{1}{1 + \alpha\Delta t} \mathbb{E}[X_n] = \left(\frac{1}{1 + \alpha\Delta t}\right)^2 \mathbb{E}[X_{n-1}] = \dots = \left(\frac{1}{1 + \alpha\Delta t}\right)^{n+1} X_0.$$

For the variance, we need to calculate  $\mathbb{E}[X_{n+1}^2]$ , we have

$$\begin{aligned}
\mathbb{E}[X_{n+1}^2] &= \mathbb{E}\left[\left(\frac{1}{1 + \alpha\Delta t} (X_n + \sqrt{\alpha} \Delta W_n)\right)^2\right] = \left(\frac{1}{1 + \alpha\Delta t}\right)^2 \mathbb{E}[X_n^2] + \frac{\alpha\Delta t}{(1 + \alpha\Delta t)^2} = \\
&\left(\frac{1}{1 + \alpha\Delta t}\right)^2 \mathbb{E}\left[\left(\frac{1}{1 + \alpha\Delta t} (X_{n-1} + \sqrt{\alpha} \Delta W_{n-1})\right)^2\right] + \frac{\alpha\Delta t}{(1 + \alpha\Delta t)^2} =
\end{aligned}$$

$$= \left(\frac{1}{1+\alpha\Delta t}\right)^4 (\mathbb{E}[X_{n-1}^2] + \alpha\Delta t) + \frac{\alpha\Delta t}{(1+\alpha\Delta t)^2} = \dots = \left(\frac{1}{1+\alpha\Delta t}\right)^{2(n+1)} X_0^2 + \alpha\Delta t \sum_{k=1}^{n+1} \left(\frac{1}{1+\alpha\Delta t}\right)^{2k}.$$

Thus,

$$\begin{aligned} \text{Var}(X_n) &= \mathbb{E}[X_n^2] - \mathbb{E}[X_n]^2 = \left(\frac{1}{1+\alpha\Delta t}\right)^{2n} X_0^2 + \alpha\Delta t \sum_{k=1}^n \left(\frac{1}{1+\alpha\Delta t}\right)^{2k} - \left(\left(\frac{1}{1+\alpha\Delta t}\right)^n X_0\right)^2 = \\ &= \alpha\Delta t \sum_{k=1}^n \left(\frac{1}{1+\alpha\Delta t}\right)^{2k}. \end{aligned}$$

Clearly

$$\lim_{t \rightarrow \infty} \mathbb{E}[X_n] = \lim_{n \rightarrow \infty} \left(\frac{1}{1+\alpha\Delta t}\right)^n X_0 = 0,$$

since  $|\frac{1}{1+\alpha\Delta t}| < 1$ . As for the variance,

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Var}(X_n) &= \lim_{n \rightarrow \infty} \alpha\Delta t \sum_{k=1}^{n-1} \left(\frac{1}{1+\alpha\Delta t}\right)^{2k} = \alpha\Delta t (-1 + \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \left(\frac{1}{1+\alpha\Delta t}\right)^{2k}) = \\ &= -\alpha\Delta t + \frac{\alpha\Delta t}{1 - \left(\frac{1}{1+\alpha\Delta t}\right)^2} = -\alpha\Delta t + \frac{\alpha\Delta t}{\frac{2\alpha\Delta t + (\alpha\Delta t)^2}{1+2\alpha\Delta t + (\alpha\Delta t)^2}} = -\alpha\Delta t + \frac{1 + 2\alpha\Delta t + (\alpha\Delta t)^2}{2 + \alpha\Delta t} = \frac{1}{2 + \alpha\Delta t}, \end{aligned}$$

again, making use of the formula for infinite geometric sums.

**iv)**

As seen in i)-iii), the expectation for the exact solution is the same as those for forward and backward euler as  $t \rightarrow \infty$ . The variances are slightly different. The exact solution has a variance of  $\frac{1}{2}$  while the forward Euler has a variance of  $\frac{1}{2-\alpha\Delta t}$  and the backward Euler has a variance of  $\frac{1}{2+\alpha\Delta t}$  as  $t \rightarrow \infty$ . The variance of both the backward and forward Euler method converges to  $\frac{1}{2}$  if we let the timestep approach 0.

**c)**

Consider a special case of the Log Ornstein-Uhlenbeck model, where the underlying asset  $S_t$  satisfies

$$\begin{cases} dS_t = rS_t dt + e^{Y_t} S_t dW_t \\ Y_t = (-\alpha(2 + Y_t) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2})dt + 0.4\sqrt{\alpha}dZ_t \\ Z_t = \rho dW_t + \sqrt{1-\rho^2}dB_t. \end{cases} \quad (5)$$

Here,  $W_t$  and  $B_t$  are two independent Brownian motions and  $e^{Y_t}$  is the stochastic volatility.

We want to estimate the option price

$$e^{-rT} \mathbb{E}[\max(S_T - K, 0)]$$

where  $e^{-rT}$  is the discount factor,  $K$  is the strike price, and  $T$  is the time at maturity of the option. This will be achieved by performing Monte Carlo simulations on a discretized system.

Now, using the time discretization  $t_n = n\Delta t$ ,  $n = 1, 2, \dots, N$  where  $\Delta t = \frac{T}{N}$ , the problem discretizes to

$$\begin{aligned} S_{n+1} - S_n &= rS_n\Delta t + e^{Y_n}S_n(W_{n+1} - W_n), \\ Y_{n+1} - Y_n &= (-\alpha(2 + Y_n) + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2})\Delta t + 0.4\sqrt{\alpha}(Z_{n+1} - Z_n), \\ Z_{n+1} - Z_n &= \rho(W_{n+1} - W_n) + \sqrt{1 - \rho^2}(B_{n+1} - B_n), \end{aligned}$$

where  $W_{n+1} - W_n$ , and  $B_{n+1} - B_n$  are  $N(0, \Delta t)$  distributed. We will consider an at the money option with parameters  $\alpha = 100, \rho = -0.3, r = 0.04, T = 0.75, Y_0 = -1$  and  $S_0 = K = 100$ . We use the Monte Carlo estimate of the expectation. This estimate approaches the true expectation as we increase the number of simulations  $M$  in accordance with the law of large numbers. We have

$$\mathbb{E}[e^{-rT} \max(S(T) - K, 0)] \approx e^{-rT} \frac{1}{M} \sum_{j=1}^M \max(S_j(T) - K, 0),$$

where  $S_j(T)$  are samples from the forward euler discretization at time  $T$ . Using  $M = 5 \cdot 10^4$  simulations, and a timestep of  $\Delta t = 0.75 \cdot 10^{-3}$ , the Monte Carlo estimate

$$e^{-rT} \frac{1}{M} \sum_{j=1}^M \max(S_j(T) - K, 0) = 6.91$$

is obtained. Figure 1 shows 10 simulations of  $S_t$ .

d)

Now, let  $\hat{C}$  be the Monte Carlo estimate of the option price. We want an estimate of  $\hat{C}$  such that the true value  $C$  has a high confidence of being within the range

$$(1 - \text{TOL})\hat{C} \leq C \leq (1 + \text{TOL})\hat{C} \implies 6.56 \leq C \leq 7.25$$

where  $\text{TOL} = 5 \cdot 10^{-2}$  and  $\hat{C} = 6.91$ . Here, we will make use of the antithetic variables method, in order to keep computational cost down. For each simulated  $X \sim N(0, \Delta t)$ , we will also use  $-X$  in the computation. As a result, we do not need to draw as many samples for a fixed standard error. In this estimate, we have used  $M = 850$  and  $N = 10^3 \implies \Delta t = 0.75 \cdot 10^{-3}$ . This resulted in a sample standard deviation of  $\hat{\sigma} = 9.35$  which implies a standard error of  $\text{s.e.} = \frac{\hat{\sigma}}{\sqrt{2M}} = 0.22$ . Note that we divide by  $2M$  due to the antithetic variables



method. Now since  $Z_{0.05} = 1.64$ , there is a 90% probability that the true  $C$  is within

$$\hat{C} - Z_{0.05} \frac{\hat{\sigma}}{\sqrt{M}} \leq C \leq \hat{C} + Z_{0.05} \frac{\hat{\sigma}}{\sqrt{M}} \implies 6.55 \leq C \leq 7.27,$$

which is close to the tolerated range. With these values of  $M$  and  $N$ , the time taken to obtain an estimate was 0.33 seconds. If we instead use  $TOL = 5 \cdot 10^{-3}$ , we require

$$(1 - TOL)\hat{C} \leq C \leq (1 + TOL)\hat{C} \implies 6.875 \leq C \leq 6.945.$$

Now we use  $M = 850 \cdot 100$ , for this  $M$ , we obtain s.e. =  $\frac{\hat{\sigma}}{\sqrt{2M}} = 0.022$  and thus

$$\hat{C} - Z_{0.05} \frac{\hat{\sigma}}{\sqrt{M}} \leq C \leq \hat{C} + Z_{0.05} \frac{\hat{\sigma}}{\sqrt{M}} \implies 6.873 \leq C \leq 6.946,$$

which again is close to the tolerated range. This time, the computational time was 197.54 seconds.

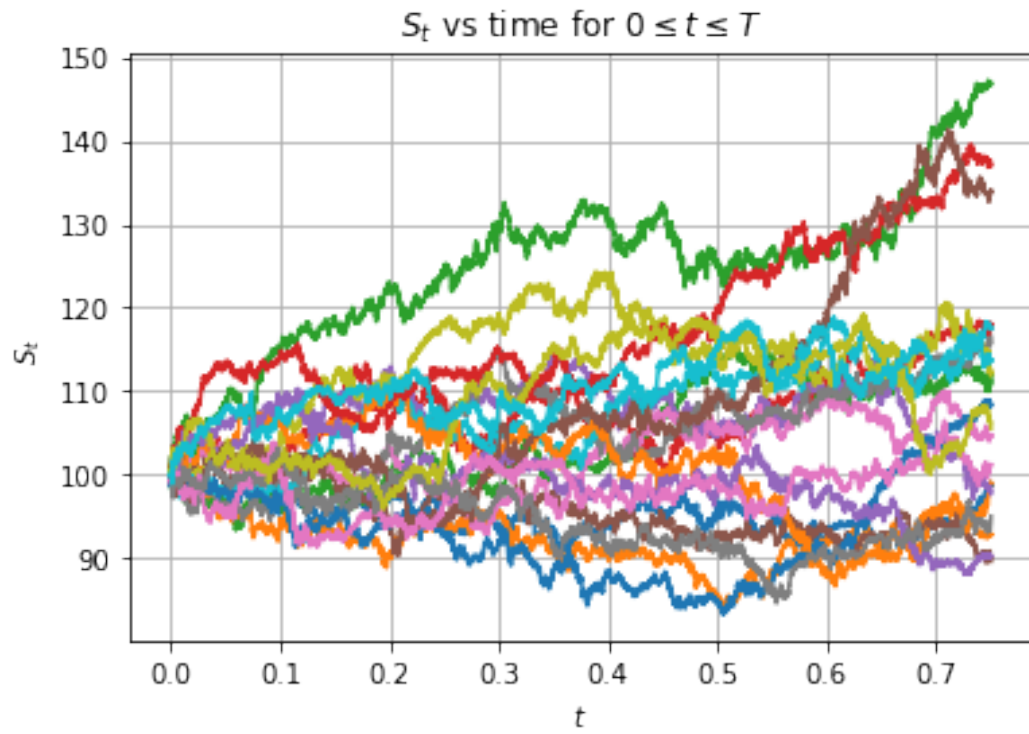


Figure 1: 10 simulations of  $S_t$  using the forward Euler discretization of the SDE system 5.