APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

STUDY MATERIALS

KTU ASSIST

DYNAMIC PROGRAMMING:

Matrix Chain Multiplication

Dynamic Programming Recap

- Bottom-up design:
  - Start at the bottom
  - Solve small sub-problems
  - Store solutions
  - Reuse previous results for solving larger sub-problems

Matrix-Chain multiplication

- We are given a sequence

$$\langle A_1,\ A_2, \ldots,\ A_n \rangle$$

- And we wish to compute

$$A_1 A_2 \cdots A_n$$

Matrix-Chain multiplication (cont.)

- Matrix multiplication is assosiative, and so all parenthesizations yield the same product.
- For example, if the chain of matrices is A_1, A_2, A_3, and A_4 then the product $A_1.A_2.A_3.A_4$ can be fully paranthesized in five distinct way:

$$(A_1(A_2(A_3A_4)))$$

$$(A_1((A_2A_3)A_4))$$

$$((A_1A_2)(A_3A_4))$$

$$((A_1(A_2A_3))A_4)$$

$$(((A_1A_2)A_3)A_4)$$

Matrix-Chain multiplication

MATRIX-MULTIPLY (A,B**)**

if *columns* [A] $\neq$ *rows* [B]

then error "incompatible dimensions"

else for $i \leftarrow 1$ **to** *rows* [A]

do for $j \leftarrow 1$ **to** *columns* [B]

do $C[i, j] \leftarrow 0$

for $k \leftarrow 1$ **to** *columns* [A]

do $C[i, j] \leftarrow C[i, j] + A[i, k] * B[k, j]$

return C

Example

$$\underset{4\times 6}{\begin{bmatrix}1&5&9&7&3&4\\2&1&9&7&2&6\\9&5&2&2&3&5\\6&6&1&3&1&7\end{bmatrix}} \times \underset{6\times 3}{\begin{bmatrix}5&1&3\\9&5&1\\8&7&6\\9&6&8\\8&1&3\\2&2&9\end{bmatrix}} = \underset{4\times 3}{\begin{bmatrix}217&142&163\\182&126&177\\158&73&114\\141&76&120\end{bmatrix}}$$

72 Multiplications in Total!

Matrix-Chain multiplication (cont.)

Cost of the matrix multiplication:

An example:

$$\langle A_1 A_2 A_3 \rangle$$

$$A_1: \quad 10 \times 100$$

$$A_2: \quad 100 \times 5$$

$$A_3: \quad 5 \times 50$$

Matrix-Chain multiplication (cont.)

If we multiply $((A_1 A_2) A_3)$ we perform $10 \cdot 100 \cdot 5 = 5000$ scalar multiplications to compute the 10×5 matrix product $A_1 A_2$, plus another $10 \cdot 5 \cdot 50 = 2500$ scalar multiplications to multiply this matrix by A_3, for a total of 7500 scalar multiplications.

If we multiply $(A_1 (A_2 A_3))$ we perform $100 \cdot 5 \cdot 50 = 25\,000$ scalar multiplications to compute the 100×50 matrix product $A_2 A_3$, plus another $10 \cdot 100 \cdot 50 = 50\,000$ scalar multiplications to multiply A_1 by this matrix, for a total of $75\,000$ scalar multiplications.

Matrix-Chain multiplication (cont.)

- The problem:

Given a chain $\langle A_1, A_2, \ldots, A_n \rangle$ of n matrices, where matrix A_i has dimension p_{i-1}x p_i, fully paranthesize the product $A_1A_2 \ldots A_n$ in a way that minimizes the number of scalar multiplications.

Matrix-Chain multiplication (cont.)

Given

dimensions $p_0, p_1, \ldots, p_n$

corresponding to matrix sequence $A_1, A_2, \ldots, A_n$

where A_i has dimension $p_{i-1} \times p_i$,

determine the "multiplication sequence" that minimizes the number of scalar multiplications in computing $A_1 A_2 \cdots A_n$. That is, determine how to parenthisize the multiplications.

$$
\begin{aligned}
A_1 A_2 A_3 A_4 &= (A_1 A_2)(A_3 A_4) \\
&= A_1(A_2(A_3 A_4)) = A_1((A_2 A_3) A_4) \\
&= ((A_1 A_2) A_3)(A_4) = (A_1(A_2 A_3))(A_4)
\end{aligned}
$$

Developing a Dynamic Programming Algorithm

Step 1: Determine the structure of an optimal solution (in this case, a parenthesization).

Decompose the problem into subproblems: For each pair $1 \le i \le j \le n$, determine the multiplication sequence for $A_{i..j} = A_i A_{i+1} \cdots A_j$ that minimizes the number of multiplications.

Clearly, $A_{i..j}$ is a $p_{i-1} \times p_j$ matrix.

Original Problem: determine sequence of multiplication for $A_{1..n}$.

Contd…

- Optimal Substructure Property: If final "optimal" solution of $A_{i..j}$ involves splitting into $A_{i..k}$ and $A_{k+1..j}$ at final step then parenthesization of $A_{i..k}$ and $A_{k+1..j}$ in final optimal solution must also be optimal for the subproblems "standing alone"
- If parenthisization of $A_{i..k}$ was not optimal we could replace it by a better parenthesization and get a cheaper final solution
- If parenthisization of $A_{k+1..j}$ was not optimal we could replace it by a better parenthesization and get a cheaper final solution

Matrix-Chain multiplication (cont.)

Step 2: A recursive solution:

- Let m[i,j] be the minimum number of scalar multiplications needed to compute the matrix $A_{i\ldots j}$ where $1 \le i \le j \le n$.
- Thus, the cost of a cheapest way to compute $A_{1\ldots n}$ would be m[1,n].
- Assume that **op** splits the product $A_{i\ldots j}$ between A_k and A_{k+1}.where $i \le k < j$.
- Then m[i,j] =The minimum cost for computing $A_{i\ldots k}$ and $A_{k+1\ldots j}$ + the cost of multiplying these two matrices.

Matrix-Chain multiplication (cont.)

Recursive defination for the minimum cost of paranthesization:

$$m[i,j]=\begin{cases}0 & \text{if} \quad i=j \\ \min\limits_{i\le k<j}\{m[i,k]+m[k+1,j]+p_{i-1}p_kp_j\} & \text{if} \quad i<j.\end{cases}$$

Matrix-Chain multiplication (cont.)

Step 3: Computing the optimal costs

We compute the optimal cost by using a tabular, bottom-up approach.

Matrix-Chain multiplication (cont.)

Step 4: Constructing an optimal solution

An optimal solution can be constructed from the computed information stored in the table $s[1...n, 1...n]$.

We know that the final matrix multiplication is

$$A_{1...s[1,n]}A_{s[1,n]+1...n}$$

The earlier matrix multiplication can be computed recursively.

Example

Matrix Chain Multiplication - Dynamic Programming

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 \quad \text{if} \quad i=j \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

i \ j	1	2	3	4	5
1					
2	x				
3	x	x			
4	x	x	x		
5	x	x	x	x	

Step 1: Fill the table for i = j

Contd…

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

i \ j	1	2	3	4	5
1	0				
2	x	0			
3	x	x	0		
4	x	x	x	0	
5	x	x	x	x	0

Step 2: Fill the table for:

i=1, j=2

i=2, j=3

i=3, j=4

i=4, j=5

Contd…

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$\underset{\substack{4x10\\p_0\ p_1}}{A_1} \times \underset{\substack{10x3\\p_1\ p_2}}{A_2} \times \underset{\substack{3x12\\p_2\ p_3}}{A_3} \times \underset{\substack{12x20\\p_3\ p_4}}{A_4} \times \underset{\substack{20x7\\p_4\ p_5}}{A_5}$$

i \ j	1	2	3	4	5
1	0				
2	x	0			
3	x	x	0		
4	x	x	x	0	
5	x	x	x	x	0

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$\underset{4\times10}{A_1} \times \underset{10\times3}{A_2} \times \underset{3\times12}{A_3} \times \underset{12\times20}{A_4} \times \underset{20\times7}{A_5}$$

$p_0\ p_1 \quad p_1\ p_2 \quad p_2\ p_3 \quad p_3\ p_4 \quad p_4\ p_5$

i \ j	1	2	3	4	5
1	0	120			
2	x	0			
3	x	x	0		
4	x	x	x	0	
5	x	x	x	x	0

$M[1,2] = \min_{1 \le k < 2} \{M[1,1] + M[1+1,2] + p_0p_1p_2\}$

$M[1,2] = \min_{1 \le k < 2} \{0 + 0 + 4\text{x}10\text{x}3\}$

$M[1,2] = 120$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$$\underset{\substack{4x10 \\ p_0\ p_1}}{A_1} \times \underset{\substack{10x3 \\ p_1\ p_2}}{A_2} \times \underset{\substack{3x12 \\ p_2\ p_3}}{A_3} \times \underset{\substack{12x20 \\ p_3\ p_4}}{A_4} \times \underset{\substack{20x7 \\ p_4\ p_5}}{A_5}$$

i \ j	1	2	3	4	5
1	0	120			
2	x	0	360		
3	x	x	0		
4	x	x	x	0	
5	x	x	x	x	0

$M[1,2] = \min_{1 \le k<2} \{M[1,1] + M[1+1,2] + p_0p_1p_2\}$

$M[1,2] = \min_{1 \le k<2} \{0 + 0 + 4x10x3\}$

$M[1,2] = 120$

$M[2,3] = \min_{2 \le k<3} \{M[2,2] + M[2+1,3] + p_1p_2p_3\}$

$M[2,3] = \min_{2 \le k<3} \{0 + 0 + 10x3x12\}$

$M[2,3] = 360$

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j \end{cases}$$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$\underset{\substack{4\text{x}10 \\ p_0\ p_1}}{A_1} \times \underset{\substack{10\text{x}3 \\ p_1\ p_2}}{A_2} \times \underset{\substack{3\text{x}12 \\ p_2\ p_3}}{A_3} \times \underset{\substack{12\text{x}20 \\ p_3\ p_4}}{A_4} \times \underset{\substack{20\text{x}7 \\ p_4\ p_5}}{A_5}$$

i \ j	1	2	3	4	5
1	0	120			
2	x	0	360		
3	x	x	0		
4	x	x	x	0	
5	x	x	x	x	0

$M[1,2] = \min_{1 \le k < 2} \{M[1,1] + M[1+1,2] + p_0p_1p_2\}$

$M[1,2] = \min_{1 \le k < 2} \{0 + 0 + 4\text{x}10\text{x}3\}$

$M[1,2] = 120$

$M[2,3] = \min_{2 \le k < 3} \{M[2,2] + M[2+1,3] + p_1p_2p_3\}$

$M[2,3] = \min_{2 \le k < 3} \{0 + 0 + 10\text{x}3\text{x}12\}$

$M[2,3] = 360$

$M[3,4] = \min_{3 \le k < 4} \{M[3,3] + M[3+1,4] + p_2p_3p_4\}$

$M[3,4] = \min_{3 \le k < 4} \{0 + 0 + 3\text{x}12\text{x}20\}$

$M[2,3] = 720$

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
p_0 p_1	p_1 p_2	p_2 p_3	p_3 p_4	p_4 p_5

i \ j	1	2	3	4	5
1	0	120			
2	x	0	360		
3	x	x	0	720	
4	x	x	x	0	
5	x	x	x	x	0

$M[4,5] = \min_{4 \leq k < 5} \{M[4,4] + M[4+1,5] + p_3p_4p_5\}$

$M[4,5] = \min_{4 \leq k < 5} \{0 + 0 + 12x20x7\}$

$M[1,2] = 1680$

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$\begin{array}{ccccc} A_1 \times & A_2 \times & A_3 \times & A_4 \times & A_5 \\ 4\text{x}10 & 10\text{x}3 & 3\text{x}12 & 12\text{x}20 & 20\text{x}7 \\ p_0\ p_1 & p_1\ p_2 & p_2\ p_3 & p_3\ p_4 & p_4\ p_5 \end{array}$$

i \ j	1	2	3	4	5
1	0	120			
2	x	0	360		
3	x	x	0	720	
4	x	x	x	0	1680
5	x	x	x	x	0

$M[1,3] = \min_{1 \le k < 3}$

k=1

$= M[1,1] + M[1+1,3] + p_0p_1p_3$

$= 0 + 360 + 4\text{x}10\text{x}12$

$= 840$

k=2

$= M[1,2] + M[2+1,3] + p_0p_2p_3$

$= 120 + 0 + 4\text{x}3\text{x}12$

$= 264$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

i \ j	1	2	3	4	5
1	0	120	264		
2	x	0	360	1320	
3	x	x	0	720	
4	x	x	x	0	1680
5	x	x	x	x	0

$$A_1 \times A_2 \times A_3 \times A_4 \times A_5$$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
p_0 p_1	p_1 p_2	p_2 p_3	p_3 p_4	p_4 p_5

$M[3,5] = \min_{3 \le k < 5}$

k=3

$= M[3,3] + M[3+1,5] + p_2p_3p_5$

$= 0 + 1680 + 3x12x7$

$= 1932$

k=4

$= M[3,4] + M[4+1,5] + p_2p_4p_5$

$= 720 + 0 + 3x20x7$

$= 1140$

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
p_0 p_1	p_1 p_2	p_2 p_3	p_3 p_4	p_4 p_5

i \ j	1	2	3	4	5
1	0	120	264		
2	x	0	360	1320	
3	x	x	0	720	1140
4	x	x	x	0	1680
5	x	x	x	x	0

$M[1,4] = \min_{1 \le k < 4}$

k=1

$= M[1,1] + M[1+1,4] + p_0p_1p_4$

$= 0 + 1320 + 4x10x20$

$= 2120$

k=2

$= M[1,2] + M[2+1,4] + p_0p_2p_4$

$= 120 + 720 + 4x3x20$

$= 1080$

k=3

$= M[1,3] + M[3+1,4] + p_0p_3p_4$

$= 264 + 0 + 4x12x20$

$= 1224$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$\underset{\substack{4x10\\p_0\ p_1}}{A_1} \times \underset{\substack{10x3\\p_1\ p_2}}{A_2} \times \underset{\substack{3x12\\p_2\ p_3}}{A_3} \times \underset{\substack{12x20\\p_3\ p_4}}{A_4} \times \underset{\substack{20x7\\p_4\ p_5}}{A_5}$$

i \ j	1	2	3	4	5
1	0	120	264	1080	
2	x	0	360	1320	
3	x	x	0	720	1140
4	x	x	x	0	1680
5	x	x	x	x	0

$M[2,5] = \min_{2 \leq k < 5}$

k=2

$= M[2,2] + M[2+1,5] + p_1p_2p_5$

= 0 + 1140+ 10x3x7

= 1350

k=3

$= M[2,3] + M[3+1,5] + p_1p_3p_5$

= 360 + 1680 + 10x12x7

= 2880

k=4

$= M[2,4] + M[4+1,5] + p_1p_4p_5$

= 1320 + 0 + 10x20x7

= 2720

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
p_0 p_1	p_1 p_2	p_2 p_3	p_3 p_4	p_4 p_5

i \ j	1	2	3	4	5
1	0	120	264	1080	
2	x	0	360	1320	1350
3	x	x	0	720	1140
4	x	x	x	0	1680
5	x	x	x	x	0

$M[1,5] = \min_{1 \le k < 5}$

k=1

$= M[1,1] + M[1+1,5] + p_0p_1p_5$

= 0 + 1350+ 4x10x7

= 1630

k=2

$= M[1,2] + M[2+1,5] + p_0p_2p_5$

= 120 + 1140 + 4x3x7

= 1344

k=3

$= M[1,3] + M[3+1,5] + p_0p_3p_5$

= 264 + 1680 + 4x12x7

= 2280

k=4

$= M[1,4] + M[4+1,5] + p_0p_4p_5$

= 1080 + 0 + 4x20x7

= 1640

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$$\begin{array}{ccccccccc} A_1 & \times & A_2 & \times & A_3 & \times & A_4 & \times & A_5 \\ 4\text{x}10 & & 10\text{x}3 & & 3\text{x}12 & & 12\text{x}20 & & 20\text{x}7 \\ p_0\ p_1 & & p_1\ p_2 & & p_2\ p_3 & & p_3\ p_4 & & p_4\ p_5 \end{array}$$

We now know that we can multiply A_1 to A_5 in as few as 1344 multiplication operations!

But where do we put our brackets?

We must focus on the selected k values

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$$(A_1 \times A_2)(A_3 \times A_4 \times A_5)$$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
p_0 p_1	p_1 p_2	p_2 p_3	p_3 p_4	p_4 p_5

k=2

$$M[1,5] = \underline{M[1,2]} + M[3,5] + p_0p_2p_5$$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$$(A_1 \times A_2)((A_3 \times A_4)\ A_5)$$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
p_0 p_1	p_1 p_2	p_2 p_3	p_3 p_4	p_4 p_5

k=2

$M[1,5] = \underline{M[1,2]} + \underline{M[3,5]} + p_0p_2p_5$

k=4

$M[3,5] = M[3,4] + M[5,5] + p_2p_4p_5$

$$(A_1 \times A_2)((A_3 \times A_4)\ A_5)$$

4x10 10x3 3x12 12x20 20x7

(120 4x3)(720 3x20 20x7)

420

4x3 3x7

84

4x7

$$120 + 720 + 420 + 84 = \underline{1344}$$

Matrix-Chain multiplication (Contd.)

MATRIX-CHAIN-ORDER(*p*)

$n \leftarrow length[p]$-1

for $i \leftarrow 1$ **to** n

    do $m[i,i] \leftarrow 0$

for $l \leftarrow 2$ **to** n

        do for $i \leftarrow 1$ **to** $n-l+1$

                do $j \leftarrow i+l-1$

                    $m[i,j] \leftarrow \infty$

                    for $k \leftarrow i$ **to** $j-1$

                        do $q \leftarrow m[i,k] + m[k+1,j]+p_{i-1}\, p_k\, p_j$

                            if $q < \mathrm{m}[i,j]$

                                then $m[i,j] \leftarrow q$

                                    $s[i,j] \leftarrow k$

return *m* and *s*

Matrix-Chain multiplication (Contd.)

```
PRINT-OPTIMAL-PARENS (s, i, j)
if i=j
    then print "A_i"
    else print " ( "
        PRINT-OPTIMAL-PARENS (s, i, s[i,j])
        PRINT-OPTIMAL-PARENS (s, s[i,j]+1, j)
        Print " ) "
```

Matrix-Chain multiplication (Contd.)

RUNNING TIME:

Recursive solution takes exponential time.

Matrix-chain order yields a running time of $O(n^3)$

END