Streaming Data Management and Time Series Analysis

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Indice

1 Statistical Prediction

A *Statistical Prediction* is a guess about the value of a random variable Y based on the outcome of other random variables $X_1, \dots, X_m$. A *Predictor* is a function of the random variables: $\hat{Y} = p(X_1, \dots, X_m)$. An optimal prediction is the *Loss Function* which maps the prediction error to it cost; if the prediction error is zero, also the loss is 0 (exact guess = no losses). A loss function can be symmetric about 0 or asymmetric: $l(-x) = l(x)$, but the most used is generally asymmetric, like the *Quadratic loss function*: $l_2(x) = x^2$.

The predictor $\hat{Y} = \hat{p}(X_1, \dots, X_m)$ is optimal for Y with respect to the loss l if minimize il the expected loss among the class of measurable functions:

$$\mathbb{E}\ l(Y - \hat{Y}) = \min_{p \in M} \mathbb{E}\ l[Y - p(X_1, \dots, X_m)]$$

The optimal predictor under quadratic loss is the conditional expectation $\hat{Y} = \mathbb{E}[Y|X_1, \dots, X_m]$.

The properties of the conditional expectation are:

1. *Linearity*: $\mathbb{E}[aY + bZ + c|X] = a\mathbb{E}[Y|X] + b\mathbb{E}[Z|X] + c$, with a, b, c, constants;
2. *Orthogonality of the prediction error*: $\mathbb{E}\{(Y - \hat{Y})g(x)\} = \mathbb{E}\{(Y - \mathbb{E}[Y|X])g(x)\} = 0$;
3. *Functions of conditioning variables*: $\mathbb{E}[Yg(x)[X] = \mathbb{E}[Y|X]g(x)$;
4. *Indipendence with the conditioning variables*: $\mathbb{E}[Y|X] = \mathbb{E}[Y]$ if $X \perp\!\!\!\perp Y$
5. *Law of iterated expecatons*: $\mathbb{E}[Y] = \mathbb{E}\{\mathbb{E}[Y|X]\}$;
6. *Law of total variance*: $\mathbb{V}\text{ar}[Y] = \mathbb{E}[\mathbb{V}\text{ar}(Y|X)] + \mathbb{V}\text{ar}[\mathbb{E}[Y|X]]$.

1.1 Optimal Linear Predictor

Sometimes it can be easer to limit the search to a smaller class functions, like linear combination. The main advantage is that the covariance structure of the random variables is all needed to compute the prediction:

$$\mathbb{P}[Y|X_1, \dots, X_m] = \mu_Y + \Sigma_{YX}\Sigma_{XX}^{-1}(X - \mu_X)$$

The Properties of the optimal linear predictor are:

1. *Unbiasedness*: $\mathbb{E}[Y - \mathbb{P}[Y|X]] = 0$;
2. ***M**ean **S**quare **E**rror* (**MSE**) of the prediction: $MSE_{lin} = \Sigma_{YY} - \Sigma_{YX}\Sigma_{YY}^{-1}\Sigma_{XY}$
3. *Orthogonality of the prediction error*: $\mathbb{E}[(Y - \mathbb{P}[Y|X])X^t] = 0$;
4. *Linearity*: $\mathbb{P}[aY + bZ + c|Z] = a\mathbb{P}[Y|X] + b\mathbb{P}[Z|X] + c$
5. *Law of iterated projections*: if $\mathbb{E}(X - \mu_x)(Z - \mu_Z)^t = 0$, then $\mathbb{P}[Y|X, Z] = \mu_Y + \mathbb{P}[Y - \mu_Y|X] + \mathbb{P}[Y - \mu_Y|Z]$

2 Time Series Concepts

A *Time Series* is a sequence of observation ordered to a time index t taking values in an index set S. If S contains finite numbers we speak about of *Discrete* time series y_t, otherwise a *Continuos* time series $y(t)$.

The most important form of time homogeneity is *Stationarity*, defined as time-invariance of the whole probability of the data generating process (*strict* stationarity) or just of its two moments (*weak* stationarity).

The process $\{Y_t\}$ is *Strictly* stationary if $\forall k \in \mathbb{N}, h \in \mathbb{Z}$ and $(t_1, \dots, t_k) \in \mathbb{Z}^k$,

$$(Y_{t1}, \dots, Y_{tk},) \stackrel{d}{=} (Y_{t1+h}, \dots, Y_{tk+h},)$$

The process $\{Y_t\}$ is *Weakly* stationary if, $\forall h, t \in \mathbb{Z}$, with $\gamma(0) < \infty$

$$\mathbb{E}(Y_t) = \mu$$
$$\mathbb{C}\text{ov}(Y_t, Y_{t-h}) = \gamma(h)$$

If a time series is strictly stationary, then it is also weakly stationary if and only if $\mathbb{V}\text{ar}(Y_t) < \infty$. If a time series a Gaussian process, then strict and weak stationarity are equivalent.

The most elementary stationarity process is the white noise. A stochastic process is *White Noise* if has $\mu = 0, \sigma^2 > 0$ and covariance function

$$\gamma(h) = \begin{cases} \sigma^2 & \text{for } h = 0 \\ 0 & \text{for } h \neq 0 \end{cases}$$

The *Autocovariance Function* is a function characterized by a weakly stationary process, while the *Autocorellation Function* (**ACF**) is the scale independent version of the autocovariance function:

If $/Y_t/$ is a stationary process with autocovariance $\gamma()$, the its ACF is $p(h) = \mathbb{C}\text{or}(Y_t, Y_{t-h}) = \gamma(h)/\gamma(0)$.

An alternative summary of the linear dependence of a stationary process can be obtained from the *Partial Autocorellation Function*, that measures the correlation between Y_t and Y_{t-h} after their linear dependence on the interventing random variables has been removed

2.1 Moving Average Process

A ***M****oving* ***A****verage Process* (**MA**) is a process that estimates the trend-cycle at time t obtained by averaging values of the time series within k periods of t. The average eliminates some of the randomness in the data, leaving a smooth trend-cycle component.

$$Y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$$
$$Y_{t-1} = \varepsilon_{t-1} + \theta_1 \varepsilon_{t-2} + \dots + \theta_q \varepsilon_{t-q-1}$$

A $MA(q)$ process has $PACF = 0$ for $h > p$ and its characteristic equation $1 + \theta_1 X + \dots + \theta_q X^q = 0$ that has only external solution of the unitary circle is a process such that $Y_t = k + \psi_1 Y_{t-1} + \dots + \psi_q Y_{t-q} + \varepsilon_t$

2.2 Autoregressive Process

An Autoregressive Process is a process that forecasts the Y variable using a linear combination of past values of the variable. The term autoregression indicates that it is a regression of the variable against itself.

$$Y_t = k + \phi_1 Y_{t-1} + \dots + \phi_p Y_{t-p} + \varepsilon_t$$

An $AR(p)$ process is stationary for the characteristic equation $1 - \phi_1 X - \dots - \phi_p X^p = 0$ and all solutions are external of the unitary circle

If Y_t is stationary, $\mathbb{E}(Y_t) = \frac{k}{1-\phi_1-\dots-\phi_p}$.

The $AR(1)$ process is $Y_t = k + \phi Y_{t-1} + \varepsilon_1$, so $1 - \phi x = 0 \implies x = 1/\phi$. AR(1) is stationary if $|\phi| < 1$. If in $AR(1)$ $\phi = 1$ we obtain a non stationary process (integrate) called Random Walk:

$$Y_t = k + Y_{t-1} + \varepsilon$$

2.3 Integrated Process

An integrated process $\{Y_t\} \sim I(d)$ is a non stationary process, but its first difference is stationary:

$$\Delta Z_t = Z_t - Z_{t-1} \sim I(0)$$

A Z_t process is integrated of order d if it is not stationary, $\Delta^{d-1} Z_{t-1}$ non stationary, while $\Delta^d Z_t$ is stationary.

The process $\{Y_t\}$ is ARMA(p, q) if it is stationary and satisfies:

$$Y_t = \phi_1 Y_{t-1} + \dots + \phi_p Y_{t-p} + \theta_1 Z_{t-1} + \dots + \theta_1 Y_{t-1}$$

2.4 ARIMA Models

If we combine differencing with autoregression and a moving average model, we obtain a non-seasonal ***A****uto****R****egressive* ***I****ntegrated* ***M****oving* ***A****verage* (**ARIMA**).

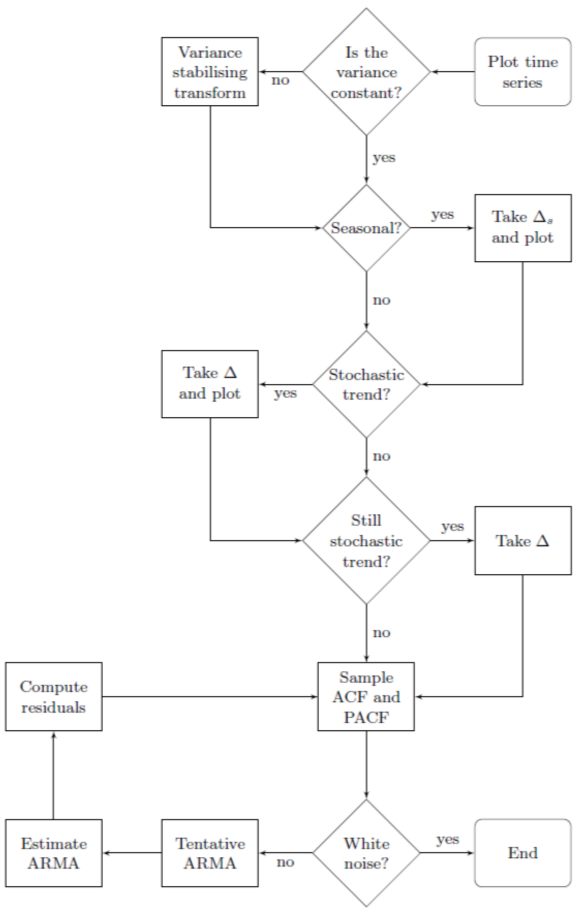
$$Y'_t = c + \phi_1 Y'_{t-1} + \dots + \phi_p Y'_{t-p} + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q} + \varepsilon_t$$

where Y'_t is the differenced series. This is an $ARIMA(p,d,q)$, where p is the order of the ***A***uto***R***egressive part, d is the degree of first differencing involved (***I***ntegrated) and q is the order of the ***M***oving ***A***verage part.

The backward shift operator $B(y_t) = y_{t-1}$ is a useful notational device when working with time series lags, so it has the effect of shifting the data back one period. The backward shift operator is convenient for describing the process of differencing. A first difference can be written as:

$$y'_t = y_t - y_{t-1} = y_t - B(y_t) = (1-B)y_t$$

Backshift notation is particularly useful when combining differences, as the operator can be treated using ordinary algebraic rules. In particular, terms involving B can be multiplied together.



2.5 ARIMA Regression

The time series models in the previous two chapters allow for the inclusion of information from past observations of a series, but not for the inclusion of other information that may also be relevant:

$$\Delta^d y_t = \beta_t \Delta^d X_t + \frac{\theta_q(B)}{\phi_q(B)}\varepsilon_t$$

$$\Delta^d (y_t - \beta_t X_t) = \frac{\theta_q(B)}{\phi_q(B)}\varepsilon_t$$

$$\phi(B)\Delta^d (y_t - \beta_t X_t) = \theta(B)\varepsilon_t$$

3 Unobserved Comoponents Model

A natural way humans tend to think about time series is as sum of non directly observable components such as trends, seasonality and cycle. ***U***nobserved ***C***omoponents ***M***odels (**UCM**) select the best features of stochastic framework as ARIMA models, but also they tend to perform very well in forecasting .

The observable time series of UCM is the sum of trend (μ_t), cycle (ψ_t), seasonality (γ_t) and white noise (ε_t):

$$Y_t = \mu_t + \psi_t + \gamma_t + \varepsilon_t$$

Some of these components could be skipped and some other could be added. Also, they can be seen as stochastic version of the deterministic functions of time. In the next sections we see how to build stochastically evolving components.

3.1 Trend

The *Trend* usually adopted in UCM is the local linear trend. Let us take a linear function defined as follows:

$$\mu_t = \mu_0 + \beta_0 t$$

where μ_0 is the intercept and β_0 the slope and write it in incremental form:

$$\mu_t = \mu_{t-1} + \beta_0$$

By adding the white noise η_n, we obtain a random walk with drift. In this case we can interpreted μ_t as a linear trend with a random walk intercept, but the slope remains unchanged.

It is possible obtain a time-varying slope in trends, easily achieved letting the slope evolves as a random walk:

$$\mu_t = \mu_{t-1} + \beta_0 + \eta_n$$
$$\beta_t = \beta_{t-1} + \xi_t$$

These equations define the local linear trend interpreted as a linear trend where both intercept and slope evolve in time as random walks

The local linear trend has different special case of interest obtained by fixing the value of the variances:

- *Deterministic Linear Trend* if $\sigma_\eta^2 = \sigma_\xi^2 = 0$;
- *Random Walk with Drift* β_0 if $\sigma_\xi^2 = 0$ (slope constant);
- *Random Walk* if $\sigma_\xi^2 = \beta_0 = 0$ (slope = 0);
- *Integrated Random Walk* if $\sigma_\eta^2 = 0$, with a very smooth trend.

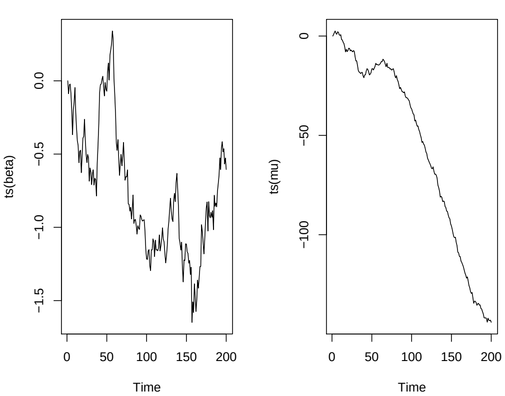
Obs.: the local linear trend can be also seen as an ARIMA process: if $\sigma_\xi^2 > 0$, $\mu_t \sim I(2)$ process as trend is non-stationary, while its second difference is stationary.

```
set.seed(20201006)
n <- 200              #-- Observations
beta <- numeric(n) #-- Slope
mu <- numeric(n)   #-- Trend

sd_eta <- 1        #-- Standard Deviation White Noise
sd_xi  <- 0.1      #-- Standard Deviation White Noise

for (t in 2:n){
  beta[t] <- beta[t-1] + rnorm(1,sd = sd_xi)                  #-- Slope
  mu[t] <- mu[t-1] + beta[t-1] + rnorm(1, sd = sd_eta)  #-- Intercept
}

par(mfrow = c(1,2))
plot(ts(beta))
plot(ts(mu))
```




```
par(mfrow = c(1,1))
```

3.2 State Space Form

The state space form is a system of equation in which one or more observable time series are linearly related to a set of unobservable state variables. It is defined by the following system of equation:

$$\begin{cases} Y_t &= d_t + Z_t\alpha_t + \varepsilon_t \quad \text{Observation equation} \\ \alpha_{t+1} &= c_t + T_t\alpha_t + R_t\eta_t \quad \text{State equation} \end{cases}$$

where $\varepsilon_t \sim WN(0, H_t)$, $\eta_t \sim WN(0, Q_t)$, uncorrelated random variables with zero mean and covariance matrix respectably H_t and Q_t.

The initialization is the process where the state vector $a_{1|0} = \mathbb{E}(\alpha_1)$ and $P_{1|0} = \mathbb{E}(\alpha_1 - a_{1|0})(\alpha_1 - a_{1|0})^t$.

Example. Let's transform a Local Linear Trend with White Noise in the State Space form. The LLT has this system of equation:

$$\mu_t = \mu_{t-1} + \beta_0 + \eta_n$$
$$\beta_t = \beta_{t-1} + \xi_t$$

with noise $y_t = \mu_t + \varepsilon_t$. The state space form of the LLT is the following:

$$\begin{cases} \alpha_{t+1} = T\alpha_t + R\eta_t = \begin{bmatrix}\mu_{t+1} \\ \beta_{t+1}\end{bmatrix} = \begin{bmatrix}1 & 1 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\mu_t \\ \beta_t\end{bmatrix} + \begin{bmatrix}\eta_t \\ \zeta_t\end{bmatrix} \quad \text{State} \\ Y_t = Z\alpha_t + \varepsilon_t = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\mu_t \\ \beta_t\end{bmatrix} + \varepsilon_t \quad \text{Observation equation} \end{cases}$$

where $\eta_t = \begin{bmatrix}\eta_t \\ \zeta^2\end{bmatrix}$ and $Q = \begin{bmatrix}\sigma_\eta^2 & 0 \\ 0 & \sigma_\zeta^2\end{bmatrix}$.

The initialization process is the following:

$$a_{1|0} = \begin{bmatrix}0 \\ 0\end{bmatrix} \qquad P_{1|0} = \begin{bmatrix}\infty & 0 \\ 0 & \infty\end{bmatrix}$$

3.3 Kalman Filter

Suppose that all parameters in the state space form are known; the only unknown quantities are unobservable components, specified as random variables, then the inference to carry out is the statistical prediction:

- Forecasting, the SP of α_t based on Y_s $(s < t)$;
- Filtering, SP of α_t based on Y_t;
- Smoothing, SP of α_t based on Y_s $(s > t)$.

It is possible to build the optimal linear predictor that is better than any other predictor assuming if ε_t, $v_t \sim WN$ and the initial state α_1 are jointly Gaussian.

Consider the following notation:

$$a_{t|s} = \mathbb{P}[\alpha_t|Y_t]$$
$$P_{t|s} = \mathbb{E}[\alpha_t - a_{t|s}][\alpha_t - a_{t|s}]^t$$

The *Kalman Filter* is an algorithm for computing the pair $\{a_{t|t-1}, P_{t|t-1}\}$ starting from $\{a_{t-1|t-1}, P_{t-1|t-1}\}$ and $\{a_{t|t}, P_{t|t}\}$ starting from $\{a_{t|t-1}, P_{t|t-1}\}$. Perciò proietta le stime di y_t partendo basate sulle osservazioni precedenti.

It also provides the sequence of innovations with the relative covariance matrix, used to compute the Gaussian likelihood of the model in state space form.

$$\begin{cases} a_{t|t-1} & = \mathbb{P}(\alpha_t|y_1, \dots, y_{t-1}) \\ P_{t|t-1} & = \mathbb{E}[\alpha_t - a_{t|t-1}][\alpha_t - a_{t|t-1}]^t \end{cases} \quad \begin{cases} \hat{y}_{t|t-1} & = \mathbb{P}(\alpha_t|y_1, \dots, y_{t-1}) \\ i_t & = y_t - \hat{y}_{t|t-1} \\ F_t & = \mathbb{E}[i_t i_t^t] = \mathbb{E}[y_t - \hat{y}_{t|t-1}][y_t - \hat{y}_{t|t-1}]^t \end{cases}$$

$$\begin{cases} a_{t|t} & = \mathbb{P}(\alpha_t|y_1, \dots, y_t) \\ P_{t|t} & = \mathbb{E}[\alpha_t - a_{t|t}][\alpha_t - a_{t|t}]^t \end{cases}$$

$$(a_{1|0}, P_{1|0}) \to (a_{1|1}, P_{1|1}) \to (a_{2|1}, P_{2|1}) \to (a_{2|2}, P_{2|2}) \to \dots \to (a_{n|n}, P_{n|n})$$