Physics 111a -- Fluid Mechanics

Albion Lawrence

Jan 18, 2022

CONTENTS

These are the lecture notes for Physics 111a. They are meant to complement the class textbook [Acheson, 1990].

I will probably say other things during the lecture so you are encouraged to take our own notes as well.

Fluid mechanics is a rich and complex subject. I will not cover all of it, and certainly the book does not. The book and the course are a starting point for further explanation (as is true for any class you will take!). Furthermore, if you ar econfused by a point in the book or the lectures, in addition to talking to me and to your classmates, I encourage you to seek out additional discussions in the literature.

CHAPTER

ONE

BASIC EQUATIONS AND CONCEPTS

This week we will introduce some basic concepts and descibe some simple fluid behaviors. Much of this, particularly the basic equations, will be described more rigorously down the line.

The natural place to start is with the question "What is a fluid?". But before this, unde the assumption that things you think are fluids are a proper subset of things that are fluids, let us look at some examples with behavior we will try to explain.

- A swimming dog taken from).

(Not in fact my dog, but a dead ringer for him). Here you can see a kind of wavefront bounding a wake arcing out from and behind the dog, and choppier water behind him.

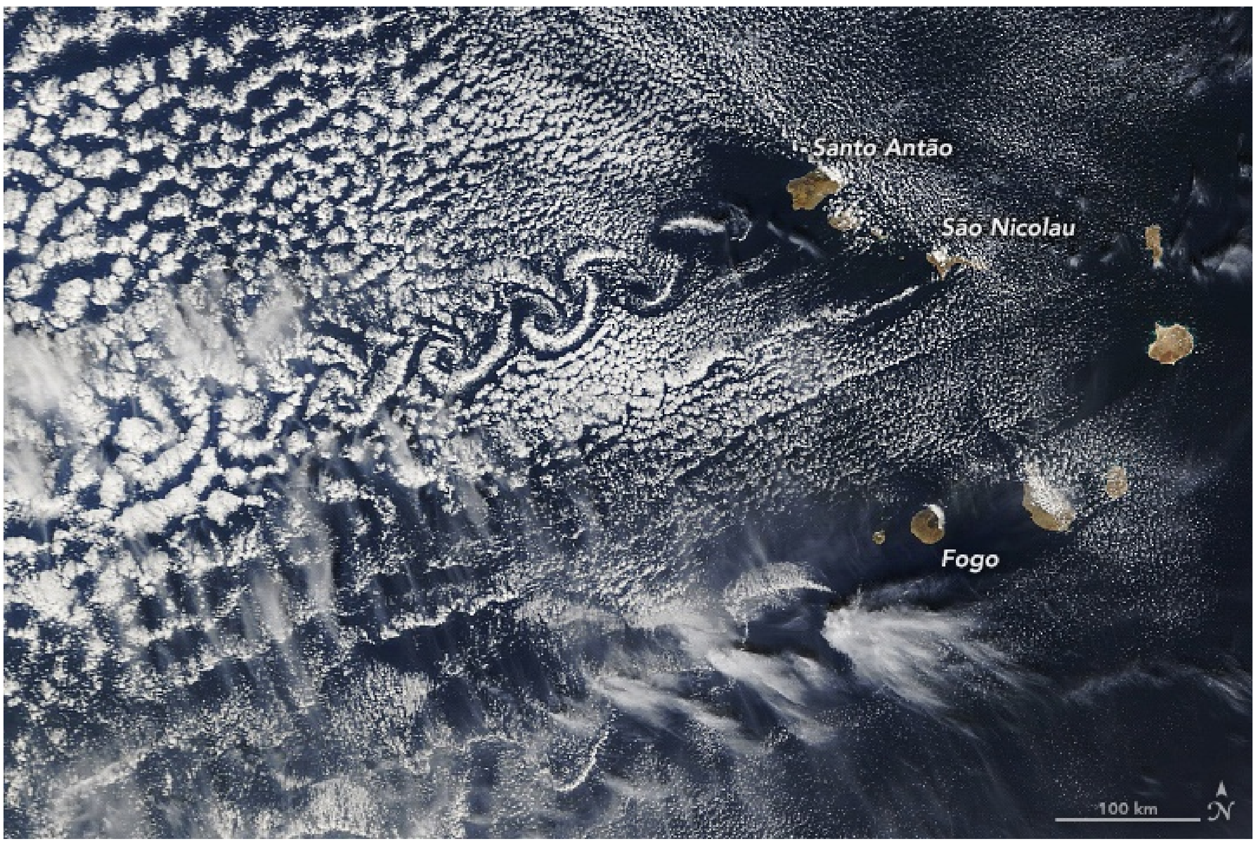
- Capillary waves emanating from a small disturbance:

In this case the surface tension is an important aspect of the dynamics. Note the wavelengths decrease with distance.

- “Gravity waves” emanating from a larger disturbance.

As Brad Marston pointed out:

- Generation and movement of vortices. Here is a picture (from of vortices in atmospheric flow (imprinting on the clouds) past islands in the Cape Verde archipelago:



These are all phenomena we will discuss in this course.

1.1 What is a Fluid?

1.1.1 1. Characterization

I highly recommend the introductory chapter of Batchelor's book [Batchelor, 2000] `for an in-depth discussion of this issue.

A *continuous medium* with material at every point.

This is of course a mathematical abstraction. A fluid is comprised of individual molecules and at sufficiently short distances the continuum approximation breaks down. In practice fluid mechanics is a *coarse-grained* description of materials.

The assumption is that if we coarse-grain over a large enough volume, the material can be effectively described by a set of quantities which vary smoothly in space and time, eg: 1. *velocity*: $\vec{v}(\vec{x}, t)$ 2. *density*: $\rho(\vec{x}, t)$ (we will often take this to be constant) 3. *temperature*: $T(\vec{x}, t)$ 4. For ocean water, *salinity* (salt content): $S(\vec{x}, t)$ 5. For the atmosphere, *specific humidity* $q = \rho_{vapor}/\rho$. and so on.

Density, temperature, and so on are themodynamic quantities that we typically define in equilibrium. A dynamic fluid will not be in equilibrium and in many interesting examples it is constantly being driven out of equilibrium. As an example, the solar heating of the atmosphere varies from the equator to th poles, varies with time over the seasons and over longer times as the Earth's roational axis and orbital characteristics change. In this class we will be assuming that the fluid is in local thermal equilibrium, such that thermodynamic quantities make sense but vary in space and time.

Deformation under shear

Fluids always deform under *shear stress*: essentially, a gradient of force across some planar surface. There may be friction, corresponding to a force between two layers sliding on top of each other, but there is no restoring force.

1.1.2 2. Examples

Note that for our purposes, a fluid could be a liquid, a gas, a plasma, and so on. Examples of fluids:

1. Planetary fluids: oceans, atmospheres, ice sheets, magma
2. Astrophysics and cosmology: protons and electrons in the early universe, solar interiors, accretion disks, proto-planetary disks, near-horizon geometry of black holes (spacetime as an effective fluid)
3. Condensed matter physics: superfluids, quantum hall fluids
4. Biological media (blood, mucous,…)
5. Plasma in heavy ion collisions

In other words this subject is everywhere: biological physics, soft and hard condensed matter physics, nuclear physics, astrophysics, cosmology, earth science, and so on.

1.2 Equations for Ideal Fluids

We will derive the equations of fluid dynamics somewhat more systematically later in the course. Our goal here is to write down some basic equations and make them plausible. Recommended texts: [Chorin *et al.*, 1990, Falkovich, 2018, Salmon, 1998].

1.2.1 1. Variables

A. Lagrangian description

In the continuum hypothesis we can consider a fluid as a collecion of infiniesimal parcels. Each parcel will have a trajectory $\vec{x}(t)$, and an associated velocity

$$\vec{v}(\vec{x}(t), t) = \frac{d}{dt}\vec{x}(t) \tag{1.1}$$

We then assign each fluid particle a density $\rho(\vec{x}(t), t)$, temperature, and so on. The description is natural in terms of understanding a fluid as a collection of parcels which are acted on by external forces and by each other. It is also natural for certain types of measurements such as autonomous floats in oceanography, and in understanding the transport of materials (pollutants in the ocean and atmosphere).

B. Eulerian description

In practice we often cannot or do not follow fluid parcels; rather we take measurements at points in space and time as determined by our apparatus: oceanographic moorings, measurements taken from ships or airplanes, and so on. The description of fluid quantities $\vec{v}(\vec{x}, t)$, $\rho(\vec{x}, t)$ at points in space and time is known as the *Eulerian description*. This description is the dominant one used in fluid dynamics. It is also the natural framework in which to do numerical experiments.

In this language, we have two interesting sets of integral curves which are discussed often. The first is the trajectory $\vec{x}(t)$. That is, given $\vec{v}(\vec{x}, t)$, and some initial condition $\vec{x}(t_0)$, the solutions to

$$fracddt\vec{x}(y) = \vec{v}(x, t) \tag{1.2}$$

has a unique solution which describes a line in $\mathbb{R}^d$.

1.2.2 2. Equations

$$\frac{d}{dt}\vec{p} = \vec{F} \tag{1.3}$$

where $\vec{p}$ is the momentum of a fluid parcel, and $\vec{F}$ is the force acting on this parcel. We also assume that the mass of the parcel is conserved: its density ρ and volume δV can change with time, but

$$\frac{d}{dt}m(\vec{x}(t).t) = \frac{d}{dt}\rho\delta V = 0 \tag{1.4}$$

A. Continuity equation

Consider a fluid inside a fixed volume W. The total mass inside this volume is

$$M(W,t) = \int_W d^d \rho(\vec{x},t) \tag{1.5}$$

Because the volume is fixed,

$$\frac{d}{dt} M = \int_W d^d x \frac{\partial \rho}{\partial t} \tag{1.6}$$

On the other hand, the total rate of mass flux into the volume is the density times the velocity flux through the boundary surface ∂W:

$$\frac{d}{dt} M = -\int_{\partial W} d\vec{A} \cdot \rho \vec{v} \tag{1.7}$$

$$= \int_W d^d x \vec{\nabla} \cdot (\rho \vec{v}) \tag{1.8}$$

where we have used the divergence theorem to rewrite the surface integral. These definitions of $d_t M$ must be the same for any volume W, so we finally have the continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{1.9}$$

The focus of this course (not exclusive!) will be on *incompressible fluids*. We can define these as fluids for which the density of a parcel does not change along a trajectory. In this case,

$$\frac{d}{dt} \rho(\vec{x}(t), t) = \frac{\partial}{\partial t} \rho + \vec{v} \cdot \vec{\nabla} \rho \equiv D_t \rho = 0 \tag{1.10}$$

Note that here we have defined he *convective derivative*

Combining this with the continuity equation gives us

$$\vec{\nabla} \cdot \vec{v} = 0 \tag{1.11}$$

Note that Acheson defines an ideal fluid as having constant density. This plus the continuity equation implies that the velocity is dovergence-free. In other references incompressibility, and not constant density, is part of the crierion for an ideal fluid. We should note that there are important examples of incompressible fluids which have variable density.

B. Euler's equation

The next step is to write dynamical equations for the motion of the fluid. We start with Newton's laws for a parcel. If the momentum is

$$\vec{p} = M \vec{v}(\vec{x}(t), t) = \rho \delta V \vec{v}(\vec{x}(t), t)$$

then combining mass conservation with the chain rule, we find:

$$\frac{d\vec{p}}{dt} = \delta V \rho \left(\frac{\partial}{\partial t} \vec{v} + \vec{v} \cdot \vec{\nabla} \vec{v} \right)$$

This is set equal to the force on the parcel.

To complete the equations, we need an expression for the force on a fluid parcel. The force will get contributions from sources external to the fluid, such as gravity, and from the force due to neighboring parcels.

If we take the external force *per unit mass* on the particle to be $\vec{f}$, the total force will $\vec{F} = \rho\delta V\vec{f}$. An important example is a constant gravitational field acting on the fluid, for which $\vec{f} = -g\vec{z}$, where g is the gravitational acceleration amd $\vec{z}$ the vertical direction.

An *ideal fluid*, in addition to being incompressible, is one in which the force exerted on a fluid parcel by neighboring parcels takes a specific form. In particular, there is no shear stress exerted by one parcel on another: the force is always perpendicular to the surface of the parcel, and takes the form

$$\vec{f}_A = -p(\vec{x},t)\hat{n}$$

where p is some scalar quantity and $\vec{n}$ is he unit normal pointing outward from the surface boundary. Noe that by Newton's thurd law, the parcel will exert the force $p(\vec{x},t)\hat{n}$ on the neighboring parcel, for which $-\hat{n}$ is the unit normal pointing outwards from that parcel. Integating $\vec{f}_A$ over the surface of the parcel, we find the total force on a parcel in volume W is

$$\vec{F} = -\int_{\partial W} d\vec{A}p = -\int_W \vec{\nabla}p \sim -\delta V\vec{\nabla}p$$

where δV is the volume of W. We identify p with the *pressure*. Adding all of the forces together and dividing the whole momentum equation by ρ we get *Euler's equation*:

$$D_t\vec{v} = -\frac{1}{\rho}\vec{\nabla}p - \frac{1}{\rho}\vec{F}_{ext}$$

1.3 Hydrostatics

The focus of his course is fluid *dynamics* in which fluid parcels have interesting nonrivial motion. Nonethelss I would like to pause here and make a couple of statements about static fluid configurations in the presence of nontrivial forces.

1.3.1 1. Conservative force field

First, if the force per unit mass is conservative, then

$$\frac{1}{\rho}\vec{F}_{ext} = \vec{\nabla}\Psi$$

Note this is not necessarily the same as $\vec{F}_{ext}$ being a gradient. Note also that it does include cases such as a constant vertical gravitational field for which $\Psi = -gz$. In this case, if the fluid parcels are static,

$$\vec{\nabla}p = \rho\vec{\nabla}\Psi$$

For constant ρ we would just have $p = \rho\Psi$. More generally, taking the curl of both sides and recalling that the curl of the gradient vanishes,

$$\vec{\nabla}\rho \times \vec{\nabla}\Psi = 0$$

which is only possible if the gradients of ρ and Ψ are parallel, and thus their level lines coincide.

1.3.2 2. Example: Isothermal atmospere

Let us consider the case that the atmosphere is uniform in the directions horizontal to the ground, so that p, ρ depend on the vertical coordinate z only, and

$$\partial_z p = -\rho g$$

If ρ is constant, then $p(z) = p(0) - g\rho z$. If ρ is not constant but the atmosphere is an ideal gas at fixed temperature, $p = \rho T/m$ is the ideal gas law, where m is the mass per molecule (for a monomolecular gas) so hat ρ/m is he number of molecules per unit volume. For constant T ("isothermal gas"), the hydrostatic equation becomes:

$$\partial_z p = \frac{T}{m}\partial_z \rho = -\rho g$$

This has the exponential solution $\$\rho(z) = \rho(0)e^{-mgz/T} \Rightarrow p(z) = \frac{T\rho(0)}{m}e^{-mgz/T} where\text{z} = 0 istheground and \text{z}\$$ increases aboveground.

The actual atmosphere is more complex: the temperature itself drops in the troposhere up to the *tropopause*, around $\sim$ 10km high, and then after remaining constant for another $\sim$ 30 km, begins to rise wih height in the stratosphere. The actual profile $p(z)$ is somewhere between exponentially decaying and linearly decaying. In general, the atmosphere and ocean are *stratified fluids* with vertically varying densities.

1.4 Bernoulli's Equation

1.4.1 1. Isentropic motion

Here we will make recourse to a bit of thermodynamics. We can define the *enthalpy* as

$$W = E + pV \tag{1.12}$$

where E is the inernal energy. Using the first law of thermodynamics, $dE = TdS - pdV$, where S is the entropy, we find

$$dW = TdS + Vdp = TdS + \frac{1}{\rho}dp \tag{1.13}$$

For "isentropic" fluids, with no input of hea and no heat exchange between fluid parcels, $dS = 0$ for each parcel, and $\frac{dp}{d\rho} = dW$. Thus

$$\frac{\partial}{\partial t}\vec{v} + \vec{v} \cdot \vec{\nabla}\vec{v} = -\vec{\nabla} W$$

1.4.2 2. Bernoulli's theorem.

Let us consider a velocity field $\vec{v}$ which is constant in time. In this case, the trajectories of fluid parcels follow the streamlines of the velocity field. In this case, *Bernoulli's theorem* states that $W + \frac{1}{2}\vec{v}^2$ is constant along streamlines.

The proof starts with the vector identity

$$\vec{A} \times (\vec{\nabla} \times \vec{B}) = A^i \vec{\nabla} B_i - \vec{A} \cdot \vec{\nabla}\vec{B}$$

(where we have used Einstein summation notation). Applying this to $\vec{A} = \vec{B} = \vec{v}$.

$$\vec{v} \times (\vec{\nabla} \times \vec{v}) = \frac{1}{2}\vec{\nabla}\vec{v}^2 - \vec{v} \cdot \vec{\nabla}\vec{v}$$

Applying this to the Euler equation, and taking the case $\partial_t \vec{v} = 0$, we find

$$\vec{\nabla}(W + \frac{1}{2}\vec{v}^2) = \vec{v} \times (\vec{\nabla} \times \vec{v})$$

Now the RHS is perpendicular to $\vec{v}$ since is a cross product with this vector, and thus

$$\vec{v} \cdot \vec{\nabla}(W + \frac{1}{2}\vec{v}^2) = 0$$

But $\vec{v} \cdot \vec{\nabla}$ corresponds precisely to the derivative along streamlines.

CHAPTER

TWO

BIBLIOGRAPHY

BIBLIOGRAPHY

[Ach90] DJ Acheson. *Elementary fluid dynamics: Oxford University Press*. Oxford, England, 1990.

[Bat00] G.K. Batchelor. *An Introduction to Fluid Dynamics*. Cambridge Mathematical Library. Cambridge University Press, 2000. ISBN 9780521663960.

[CMM90] Alexandre Joel Chorin, Jerrold E Marsden, and Jerrold E Marsden. *A mathematical introduction to fluid mechanics*. Springer, 3rd edition, 1990.

[Fal18] Gregory Falkovich. *Fluid Mechanics*. Cambridge University Press, 2 edition, 2018. doi:10.1017/9781316416600.

[Sal98] Rick Salmon. *Lectures on geophysical fluid dynamics*. Oxford University Press, 1998.