BUAN 5260: Mathematical Models for Decision Making

(Constrained Optimization)

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This week

- Minimization of constrained optimization
- General linear programming review
- Duality theory
- Practice problems
- Linear programming under uncertainty

Graphical solution of a Linear Programming model

Recall the Boot/Shoe problem from last week:

- As a profit max we had the following set up and graph

max $_{B,S}$: $\pi = 40B + 25S$

subject to:

$$30B + 20S \leq 1200$$
$$10B + 10S \leq 500$$
$$B \leq 30$$
$$\text{and } B \geq 0,\ S \geq 0$$

We noted 5 possible corner solutions:

i (0, 0) $\rightarrow \pi = 40*0 + 25*0 = \$0$

ii (0, 50) $\rightarrow \pi = 40*0 + 25*50 = \$1250$

iii (20, 30) $\rightarrow \pi = 40*20 + 25*30 = \$1550$

iv (30, 15) $\rightarrow \pi = 40*30 + 25*15 = \$1575$

v (30, 0) $\rightarrow \pi = 40*30 + 25*0 = \$1200$

How does the concept of iso-profit help us solve this?

How solve using minimization instead?

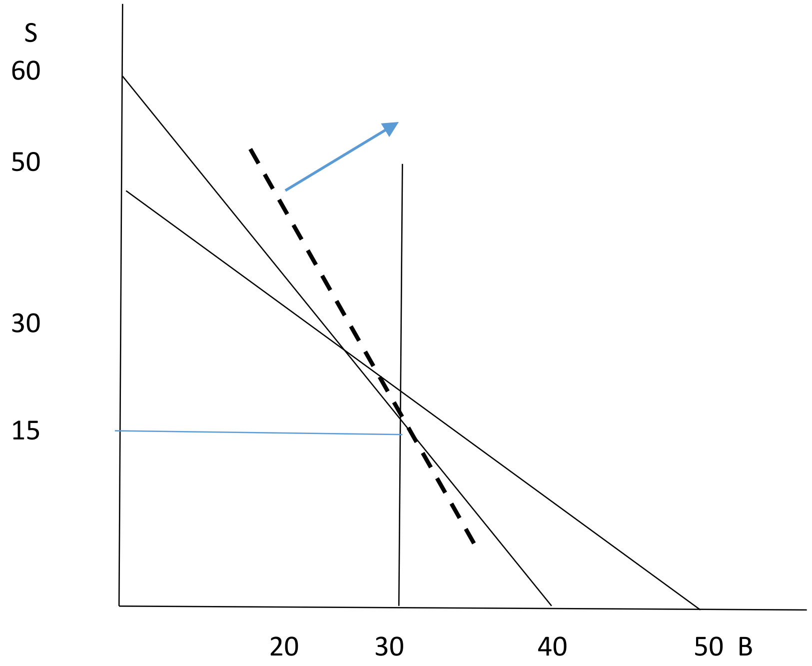
min$_{B,S}$: $\pi = -40B + -25S$

subject to:

$$30B + 20S \leq 1200$$
$$10B + 10S \leq 500$$
$$B \leq 30$$
$$\text{and } B \geq 0,\ S \geq 0$$

Profit values are same **but negative**

- But this works since a large negative number is smaller
- Which combination of B and S give the minimum??



Graphical solution of a Linear Programming model

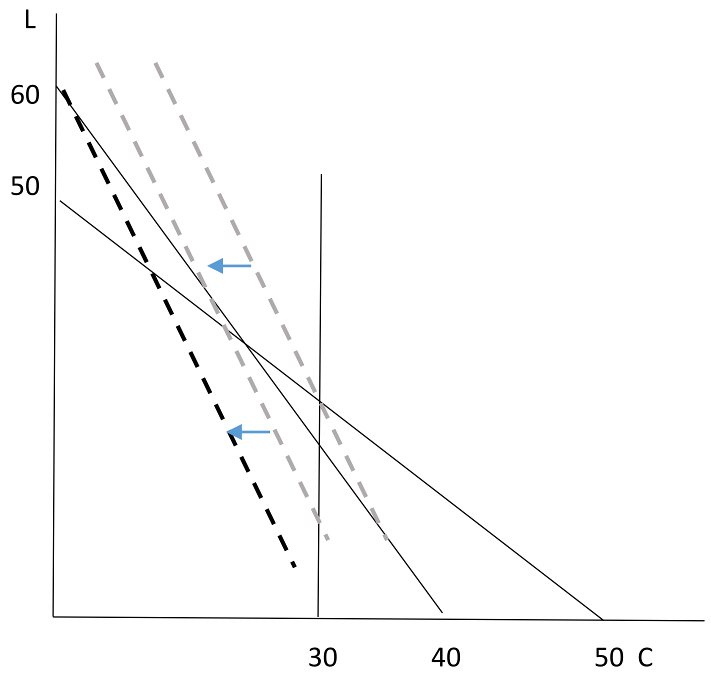
Recall the Capital/Labor Ag problem from last week:

- These are combinations of C and L to produce a given level of output (production function not specified)
- As a cost min we had the following set up and graph

$\min_{B,S}$: Cost = 40*C* + 25*L*

subject to:

$$\begin{aligned} 30C + 20L &\geq 1200 \\ 10C + 10L &\geq 500 \\ C \qquad &\leq 30 \\ \text{and } C \geq 0,\ L &\geq 0 \end{aligned}$$



How does the concept of iso-cost help us solve this?

- We noted 3 possible corner solutions:

i (30, 15) → p = 40*30 + 25*15 = $1575

ii (20, 30) → p = 40*20 + 25*30 = $1550

iii (0, 60) → p = 40*0 + 25*60 = $1500

More on Linear Programming

Linear programming model

- Allocate *m resources* (constraints) to *n activities* (variables)
  - Z: value of objective function
  - x_j: level of activity j (for j = 1, 2, 3, … , n)
    - x_j's are the decision variables
  - c_j: Increase in Z from each unit increase in activity j (objective function values)
    - c_j increases Z for either a max or a min
  - b_i: Amount of resource i to be allocated to activities (for i = 1, 2, 3, … , m)
    - Can indicate the maximum available or minimum required of a resource
  - a_{ij}: Amount of resource i used by each unit of activity j
    - Links the activity (decision variable) to the resource (constraint)

General modeling structure

In symbolic form, the linear programming model is:

Choose values of the *decision variables* $x_1, x_2, \ldots, x_n$ to

Maximize $Z = c_1x_1 + \ldots + c_nx_n$ ← Objective Function

subject to

$$\left.\begin{array}{l} a_{11}x_1 + \ldots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n \le b_2 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n \le b_m \end{array}\right\} \leftarrow \text{Functional Constraints}$$

and

$$x_1 \ge 0, \ldots, x_n \ge 0 \qquad \leftarrow \text{Nonnegativity Constraints}$$

for known *parameters* $c_1, \ldots, c_n$; $a_{11}, \ldots, a_{mn}$; $b_1, \ldots, b_m$.

Forms of linear programming models

The key difference is the mathematical structure—**not the application**

- Resource-allocation: Allocate limited resources among activities
  - Most common, like demonstration and participation activity problems
  - Homework problems 1 and 4
- Cost-benefit-trade-off: Best trade-off between costs and benefits
  - Like radiation therapy, air pollution, and scheduling examples in 3.4
  - Homework problem 2
- Blending: blend ingredients into final product according to specifications
  - Like reclaiming solid waste example in 3.4
- Fixed-requirements: meet specific requirements with **equality**
  - **Similar set up as resource-allocation except constraints meet with equality**
  - Like network example in 3.4
  - Homework problem #3

Duality Theory

Duality theory (Chapter 6)

Every constrained optimization model can be viewed from two perspectives:

- Primal: the original problem of interest
- Dual: an “inverse” form of the original problem
- **For example: If we interpret the primal LP as a classical "Resource Allocation" problem, its dual can be interpreted as a "Resource Valuation" problem**
  - Why do we care about a “Resource Valuation” problem?
  - Find the “value” of the resource in terms of the objective function
    - Indicates how one more unit of the resource will change the objective function
    - Would increasing a non-binding constraint impact profits (like labor in Boots/Shoes)?
    - What is the “value” (i.e. profit) of more labor?
      - The additional profit from more boots and shoes

Duality theory

Why do we care about the dual problem?

1. If programming, sometimes save computing time solving the dual rather than primal
2. From a business or policy perspective, the dual variables give us critical information
    - **Shadow price** (shadow value or in-situ value)
    - Shadow price: the *value* of one more unit of a resource
        - e.g. How much profit increases with one more unit of a resource
    - **We can estimate the *value* of a resource as it impacts the objective function from the dual**
        - e.g. what is *value* of an hour of labor to producing boots and shoes
            - We know the profit of each: π_B = \$40 and π_S = \$25
            - But we do not know the π from 1 unit of labor
            - Must also consider machine and demand constraint
            - The dual can tell us the *value* of 1 unit of labor **accounting for all dimensions of production**
        - **Knowing the cost of a resource is not enough, need to know net value of the resource to identify the most *valuable* resource**

Duality theory

We will not be programming the Dual problem

- But you need to understand it to appreciate how we get **shadow prices**

Primal Problem

Maximize $\quad Z = \sum_{j=1}^{n} c_j x_j,$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \qquad \text{for } i = 1, 2, \ldots, m$$

and

$$x_j \ge 0, \qquad \text{for } j = 1, 2, \ldots, n.$$

Dual Problem

Minimize $\quad W = \sum_{i=1}^{m} b_i y_i,$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \ge c_j, \qquad \text{for } j = 1, 2, \ldots, n$$

and

$$y_i \ge 0, \qquad \text{for } i = 1, 2, \ldots, m.$$

Duality theory

Four things to realize:

1. The coeff of the objective fn in primal are constraint values in dual
   - i.e. the c’s in the primal are the b’s in the dual
2. The constraint values of the primal are the coeff of the objective fn in the dual
   - i.e. the b’s in the primal are the c’s in the dual
3. The coefficient matrix of the constraints in the primal are transposed in the dual
   - If here are m constraints and n variables in the primal
   - There are n constraints and m variables in the dual
4. The inequality constraints are flipped

Duality theory

The boots/shoes problem

Primal

$$\max_{B,S} \quad \pi = 40B + 25S$$

subject to:

$$30B + 20S \le 1200$$
$$10B + 10S \le 500$$
$$B \le 30$$
$$\text{and } B \ge 0,\ S \ge 0$$

Dual

$$\min_{L,M,D} C = 1200L_D + 500M_D + 30D_D$$

subject to:

$$30L_D + 10M_D + D_D \ge 40$$
$$20L_D + 10M_D \ge 25$$
$$\text{and } L_D \ge 0,\ M_D \ge 0,\ D_D \ge 0$$

Now you write out the Dual

Duality, part 2

Duality theory

In the Dual problem we are estimating the ***value*** of one more unit *Labor*, *Machine* and *Demand* in **units of the objective function**

- ***L_D, M_D* and *D_D* are $\$\pi$/unit for one more unit of the resource**
- Note if "π" were *time* we'd find time/unit
  - **That is, the shadow price is in units of the primal objective function**

Duality theory

Recall solution from primal problem:

- π = 1575, *Labor* = 1200, *Machine* = 450, *Demand* = 30, *B* = 30, and *S* = 15
  - What are the units of each above?
    - π is $, labor is hours, machine is hours, *Demand* is pairs, *B* is pairs, *S* is pairs
    - These are all physical units except the objective function
- The solution to the dual is: L_D = 1.25, M_D = 0, D_D = 2.50, B_D = 0 & S_D = 0
  - Find with R-lpSolveAPI: get.dual.solution(*model*)
    - Will go through that in a bit
  - What are the units of the dual values above?
    - L_D is $/person hour, M_D is $/machine hour, D_D is $/pair, *B* is $/pair, *S* is $/pair
    - These are all dollar values per unit of good
    - They are in the ***<u>units of the objective function of the primal</u>***

Duality theory

Economic interpretation of duality

- Consider: L_D = \$1.25, \$$M_D$ = 0, \$$D_D$ = 2.50
  - L_D = \$1.25 indicates that adding more Labor hour will increase profit by \$1.25
  - M_D = 0 indicates that adding more Machine hours will NOT increase profit
  - D_D = \$2.50 indicates that increasing Demand will increase profit by \$2.50
    - Wait, how can that be – increasing demand for boots should bring in an additional \$40!!
    - But realize will decrease *S* and change the use of *L* and *M*
    - There is a lot more going on that you can easily calculate in your head

Duality theory

Think about that for a minute

- The profit of boots is $40 per pair
- The profit of shoes is $25 per pair
- We have no idea what the wage is because it is rolled up in the profit number
  - <u>**But the dual tells us the profit from hiring one more unit of labor is $1.25**</u>
  - Similar for *M*achine *hours* and *Demand*
- Profit will increase linearly at $1.25/labor hour till <u>**hit another constraint**</u>
  - That is, if there are 1201 labor hours, profit will be $1,576.25 instead of $1,575
  - This relates to sensitivity analysis we will discuss in Chapter 7

Duality theory

Further, if we look at the Dual objective function

$$\boldsymbol{C = 1200L + 500M + 30D}$$

- 1200, 500 and 30 are all physical units
- L_D, M_D and D_D are all $/unit

$$C = 1200*1.25 + 450*0 + 30*2.50 = \$1575$$

- So (L_D*$Labor$) is the contribution to profit from labor
  - L_D is indirectly related to the wage, but incorporates all other factors

Duality theory

A few things to consider about the dual

1. The Dual will always be in opposite units of the Primal
   - If the primal variables are in physical units, dual objective function will be in $
   - If the primal variables are in $, dual will be in physical units
2. This highlights a concept **complementary slackness**
   - If a resource is exhausted, it’s marginal value is its marginal contribution to profit
     - In this example, adding labor or demand for boots will increase profit
   - If a resource is not exhausted, it’s marginal value is zero
     - In this example, adding machines will not increase profit
   - Complementary Slackness says that if the value of a dual variable is greater than zero (slack) then the corresponding primal constraint binding
     - It also says that if the primal constraint is slack then the corresponding dual variable is zero

Go to 5260_S20_W2.Rmd

Duality, part 3

go to R slides

Programming Under Uncertainty

Linear programming under uncertainty (Ch 7)

Sensitivity analysis—conceptual

- From the graphical depiction we can see how the optimal solution depends on *a*'s, *b*'s & *c*'s
- If the objective function or constraints change, then the optimal solution may change
  - *c*'s change the slope of the objective function
  - *b*'s change will shift the constraints in or out
  - *a*'s change the intercept and slope of constraints

Linear programming under uncertainty

We've solved the model as though the parameters are known with certainty

- What if parameters are "best guess", "expert opinion" or statistically estimated from data

Need to determine if the model gives **<u>robust</u>** (stable) and **<u>realistic</u>** solutions

- Is the solution sensitive to small changes in the parameters?
- Does the model behave in a way that makes sense?
    - Do the machines sit idle while people are always working?
        - i.e. 450 < 500 and $M_D = 0$
    - Does it take more labor and similar machine hours for boots and shoes?
        - $a_{11} = 30 > a_{12} = 20$ and $a_{11} = a_{12} = 10$
    - Is demand of $B = 30$ hold true with experience?
    - **<u>Must work with operations, production and marketing to verify!</u>**

Linear programming under uncertainty

What are sensitive constraints, parameters and variables?

- We may want to think of the model parameters more generally
- Recall the *Boots* and *Shoes* problem and which roles the *a*'s, *b*'s and *c*'s play

$\max_{B,S}$ $\pi = c_1B + c_2S$ $c_1 = 40$ *and* $c_2 = 25$

subject to:

$a_{11}B + a_{12}S \leq b_1$ $a_{11} = 30$, $a_{12} = 20$ and $b_1 = 1200$

$a_{21}B + a_{22}S \leq b_2$ $a_{21} = 10$, $a_{22} = 10$ and $b_2 = 500$

$a_{31}B + a_{32}S \leq b_3$ $a_{31} = 1$, $a_{32} = 0$ and $b_3 = 30$

Linear programming under uncertainty

What are sensitive constraints, parameters and variables?

- *c*'s are parameters for profitability (**price, cost**)
  - Simplifying assumption that know profitability by unit of output
    - Is linear relationship for revenue and cost realistic?
  - *c*'s relate the decision variable to the objective
- *a*'s are constraint parameters (**technology**)
  - Relate variable to resource use and constraint
- *b*'s are resource constraint or requirement parameters (**labor, machines, inputs**)

Programming Under Uncertainty, part 2

Linear programming under uncertainty

Which of the *a*’s, *b*’s, and *c*’s impact the results the most and least?

- Earlier we discussed impact of *b*’s on profitability—duals
  - Determine **what constraint should be increased next to increase profits**
  - How determine this??
    - Determine the **shadow value** of each constraint—dual values Ch 6
- What about *a*’s and *c*’s impact on the solution
  - How sensitive are the model results to *a*’s and *c*’s
    - **Perform sensitivity analysis on critical *a*’s and *c*’s**
    - **If near a “switching value” the solution will be sensitive to it**

Linear programming under uncertainty

Perform sensitivity analysis to determine the **stability of the model and solution**

- Do "**what-if**" analysis to determine what happens if change:

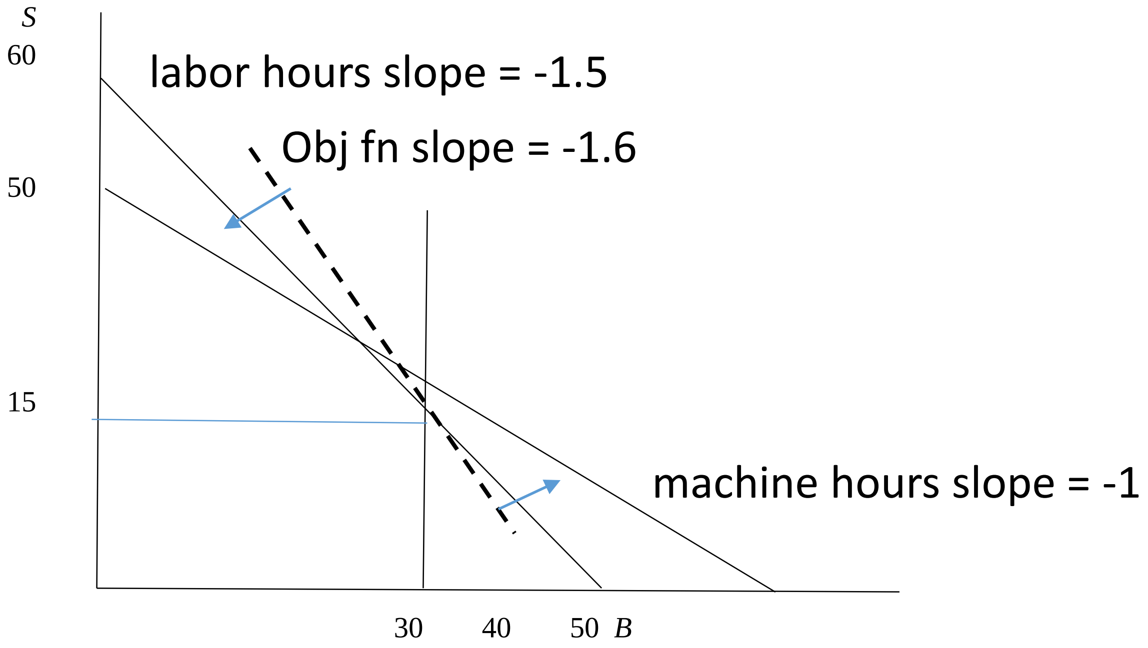
The text demonstrates sensitivity analysis by changing values of the parameters and seeing the impact on the solution

- We'll take a different approach to build intuition
- R can be used to find exact **"switching values"** of *b*'s and *c*'s

Linear programming under uncertainty

Use visualization to illustrate sensitivity analysis

- Easy to show with two decision variables model



Linear programming under uncertainty

- **Objective function sensitivity analysis**
- What range of c's will leave the solution set of B =30 and S =15 unchanged?
  - Changes in a c will change the slope of the objective function
  - Change in the slope may change solution set
    - Slope represents the **relative trade-off** between variables
  - For example: ***show how solution change as slope changes due to c_1***
    - If c_1 decreased from 40 to c_1 = 35
    - Slope of obj fn would change to **–**1.4 (**–35/25)**
    - New solution would be B = 20, S = 20 – **how relate to graph and solution in R?**
      - However, if c_1 increase to ∞ solution not change
- That indicates the **solution is sensitive to decreases in c_1 but not increases**
  - Must determine the likelihood of c_1 decreasing

Linear programming under uncertainty

- However, slope of objective function depends on all *c*'s:
  $-c_1/c_2 = -40/25 = -1.6$
  - For example: ***<u>show how solution change as slope changes due to c_2</u>***
    - If c_2 increased from 25 to $c_2 = 30$
      - Slope of obj fn ($-c_1/c_2$) would change to –1.333 (–40/30)
      - New solution would be $B = 20$, $S = 20$
    - If c_2 increase to 40 solution change to –1 (–40/40)
      - New solution would be $B = 0$, $S = 50$
      - If c_2 increase beyond 40 solution stay at $B = 0$, $S = 50$
- The solution is sensitive to critical changes in $-c_1/c_2$
  - Which relates to c_1 decreasing $c_1 < 37.5$ if $c_2 = 25$ remains unchanged
    - Can find critical value of coefficients using R or looking at ratios
  - Or c_2 increasing $c_2 > 26.67$ or $c_2 > 40$
  - These values change slope from –1.6 to move toward –1.5 or –1
    - Which are the slopes of the constraints

Linear programming under uncertainty

The sensitivity depends on the **relative magnitude of the slopes of the obj fn and constraints**

- That is, **how does c_1/c_2 compare to the different a_{m1}/a_{m2}**?
- **Slope measure the importance of the decision variable to the obj fn and constraint**
  - Which decision variable increases the obj fn most?
  - Which decision variable impacts the constraint most?

Functional constraint sensitivity analysis

- There are two general elements of the constraints to analyze
- The quantity of the resource—*b*
- The coefficient that relates the decision variable to the resource—*a*
  - i.e. boots use 50% more labor than shoes

Linear programming under uncertainty

We've already discussed the impact of changing *b*—the quantity constraint

- The dual values indicated which would be the most impactful to the obj fn
- But, *may* not know which is the most cost effective
  - Labor cost probably **<u>is</u>** incorporated in *c*—profit coefficient
    - So likely know increasing labor is cost effective
    - *Aside – usually know price of output and cost of input rather than profit, but still not know how additional unit of input impacts profit, is still measured by shadow price*
  - But cost of increasing demand **<u>not</u>** likely included in *c*'s
    - So not know if increasing demand is cost effective

Linear programming under uncertainty

- **Sensitivity analysis involves testing the impact of *a*'s, *b*'s and *c*'s on the solution**
  - **<u>Make sure the solution is stable and gives realistic results</u>**
- Sensitivity analysis is a critical component of validating a model
  - Is the solution realistic?
  - Is the solution stable?
  - Do the sensitivities make sense?