3.1 The Equation of Continuity

3.2 The Equation of Motion

3.3 The Equation of Energy

3.4 Initial and boundary conditions

3.2 The Equation of Motion

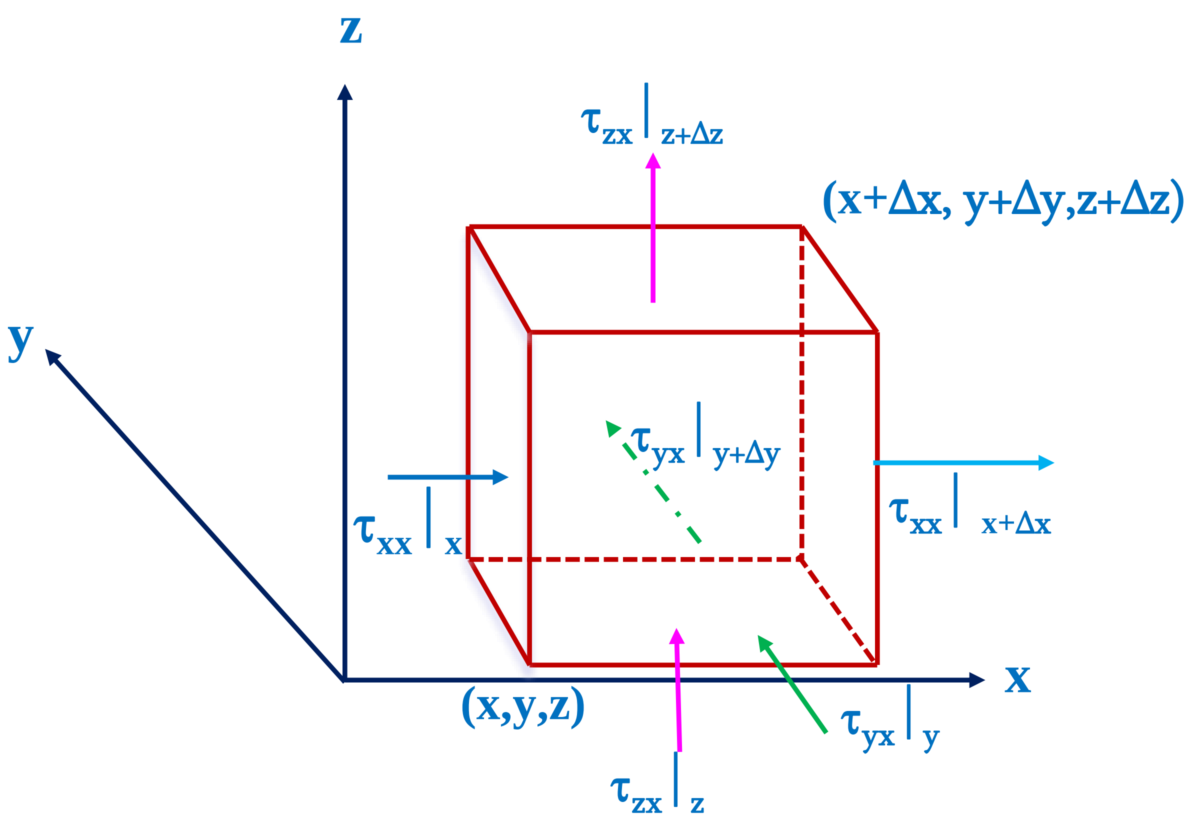
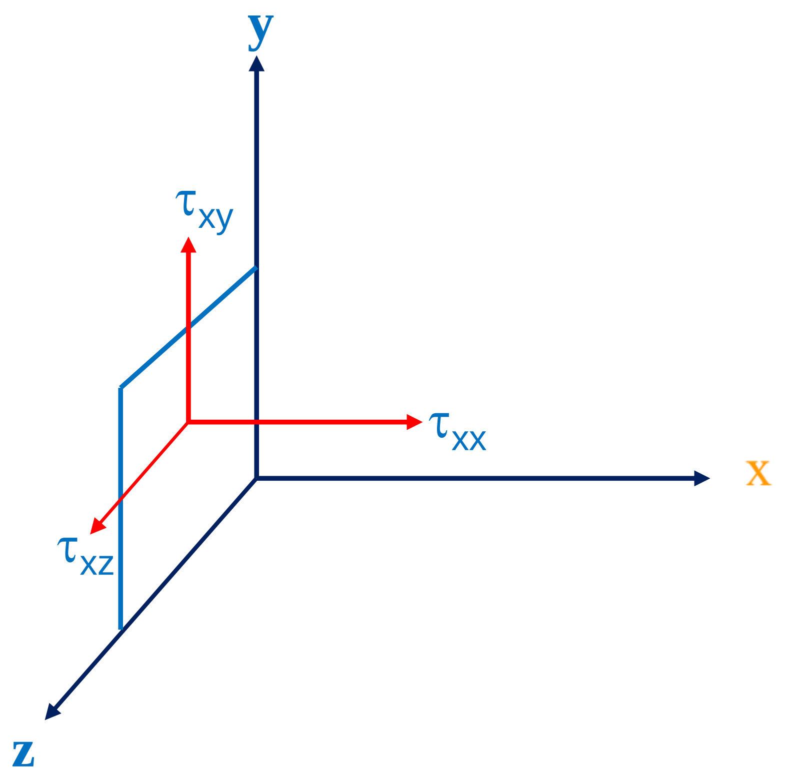


Fig. 3.2.1 Fixed volume element $\Delta x\ \Delta y\ \Delta z$ with arrows indicating the direction in which the x-component of momentum is transported through the surface.

3.2 The Equation of Motion



τ_{xx}**-Normal stress**
τ_{xy},τ_{xz}**-shear stress**

Fig. 3.2.1a Molecular transfer by velocity gradients

3.2 The Equation of Motion

- The fluid flows through all six faces of volume element. Eq. 3.20 is a vector equation with components in each of the three coordinate directions x, y, and z.
- We begin by considering the x-component of each term in Eq.3.20. The y- and z-components may be handled analogously.
- Let us first consider the rates of flow of the x-component of momentum into and out of the volume element shown in Fig, 3.2.1.
- Momentum flows into and out of the volume element by two mechanisms: by convection(i.e. by virtue of the bulk fluid flow) and by molecular transfer(i.e. by virtue of the velocity gradients).

3.2 The Equation of Motion

- The rate at which the x-component of momentum enters the face at x by convection is $\rho u_x u_x|_x \Delta y \Delta z$, and the rate at which it leaves at x+Δx is $\rho u_x u_x|_{x+\Delta x} \Delta y \Delta z$. The rate at which it enters at y is $\rho u_y u_x|_y \Delta x \Delta z$. Similar expressions may be written for other three faces.
- The net convective flow of x-momentum flow into the volume element is

$$\Delta y \Delta z \left(\rho u_x u_x|_{\mathrm{x}} - \rho u_x u_x|_{x+\Delta x}\right)$$
$$+\Delta x \Delta z \left(\rho u_y u_x|_{\mathrm{y}} - \rho u_y u_x|_{y+\Delta y}\right)$$
$$+\Delta x \Delta y \left(\rho u_z u_x|_{\mathrm{z}} - \rho u_z u_x|_{z+\Delta z}\right) \quad (3.21)$$

3.2 The Equation of Motion

- **Similarly, the rate at which the x-component of momentum enters the face at x by molecular transport is $\tau_{xx}|_x \Delta y \Delta z$, and the rate at which it leaves at x+Δx is $\tau_{xx}|_{x+\Delta x} \Delta y \Delta z$. The rate at which it enters at y is $\tau_{yx}|_y \Delta x \Delta z$. τ_{yx} is the flux of x-momentum through a face perpendicular to y-axis. When these six contributions are summed up, we get**

$$\Delta y \Delta z\left(\tau_{xx}|\mathrm{x} - \tau_{xx}|x + \Delta x\right)$$
$$+\Delta x \Delta z\left(\tau_{yx}\Big|\mathrm{y} - \tau_{yx}|y + \Delta y\right)$$
$$+\Delta x \Delta y\left(\tau_{zx}|\mathrm{z} - \tau_{zx}|z + \Delta z\right) \quad (3.22)$$

3.2 The Equation of Motion

- **Fluid pressure and the gravitational force per unit mass** $\vec{g}$
- **X-direction will be**

$$\Delta y \Delta z \left(p\big|x - p|x + \Delta x\right) + \rho g_x \Delta x \Delta y \Delta z \quad (3.23)$$

- **Pressure in a moving fluid is defined by the equation of state p=p(ρ, T) and is a scalar quantity.**
- **The rate of accumulation of x-component within the element is**

$$\frac{\partial \rho u_x}{\partial t} \Delta x \Delta y \Delta z$$

3.2 The Equation of Motion

- **The x-component of the equation of motion:**

$$\frac{\partial}{\partial \mathrm{t}}\rho \mathrm{u_x} = -\left(\frac{\partial}{\partial \mathrm{x}}\rho u_x u_x + \frac{\partial}{\partial \mathrm{y}}\rho u_y u_x + \frac{\partial}{\partial \mathrm{z}}\rho u_z u_x\right)$$

$$-\left(\frac{\partial}{\partial \mathrm{x}}\tau_{xx} + \frac{\partial}{\partial \mathrm{y}}\tau_{yx} + \frac{\partial}{\partial \mathrm{z}}\tau_{zx}\right) - \frac{\partial \mathrm{p}}{\partial \mathrm{x}} + \rho \mathrm{g}_{\mathrm{x}} \quad (3.24)$$

- **The y- and z-components, which may be obtained similarly, are**

$$\frac{\partial}{\partial \mathrm{t}}\rho \mathrm{u_y} = -\left(\frac{\partial}{\partial \mathrm{x}}\rho u_x u_y + \frac{\partial}{\partial \mathrm{y}}\rho u_y u_y + \frac{\partial}{\partial \mathrm{z}}\rho u_z u_y\right)$$

- $$-\left(\frac{\partial}{\partial \mathrm{x}}\tau_{xy} + \frac{\partial}{\partial \mathrm{y}}\tau_{yy} + \frac{\partial}{\partial \mathrm{z}}\tau_{zy}\right) - \frac{\partial \mathrm{p}}{\partial \mathrm{y}} + \rho \mathrm{g}_{\mathrm{y}} \quad (3.25)$$

3.2 The Equation of Motion

- $$\frac{\partial}{\partial t}\rho u_z = -\left(\frac{\partial}{\partial x}\rho u_x u_z + \frac{\partial}{\partial y}\rho u_y u_z + \frac{\partial}{\partial z}\rho u_z u_z\right) - \left(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right) - \frac{\partial p}{\partial z} + \rho g_z \quad (3.26)$$

- **ρu_x, ρu_y, ρu_z are the components of the mass velocity vectors,**
- **$\rho\vec{u}$; g_x, g_y, g_z are the components of the gravitational acceleration $\vec{g}$. The single vector equation for 3.24~3.26:**

- $$\frac{\partial}{\partial t}\rho\vec{u} = -[\nabla\cdot\rho\vec{u}\vec{u}] - \nabla p - [\nabla\cdot\tau] + \rho\vec{g} \quad (3.27)$$

3.2 The Equation of Motion

$\frac{\partial}{\partial t}\rho\vec{u}$ ——— rate of increase of momentum per unit volume

$[\nabla \cdot \rho\vec{u}\vec{u}]$ ———rate of momentum gain by convection puv

∇p———pressure force on element puv

$[\nabla \cdot \tau]$ – *rate* of momentum gain by viscous transfer puv

$\rho\vec{g}$ —— —gravitational force on elment puv

- **Combined with the continuity equation, Eq.(3.27) becomes:**

$$\rho\frac{D\vec{u}}{Dt} = -\nabla p - \nabla \cdot \vec{\tau} + \rho\vec{g} \quad (3.28)$$

mass puv times acceleration	pressure force on element puv	viscous force on element puv	gravitational force on element puv

3.2 The Equation of Motion

- In order to use these equations to determine velocity distributions, we must find out various stresses in terms of velocity gradients and fluid properties.

a) τ_{xy}、 τ_{yz}、 τ_{zx} ~velocity gradients. According to Newton's law of viscosity(for Newtonian fluids):

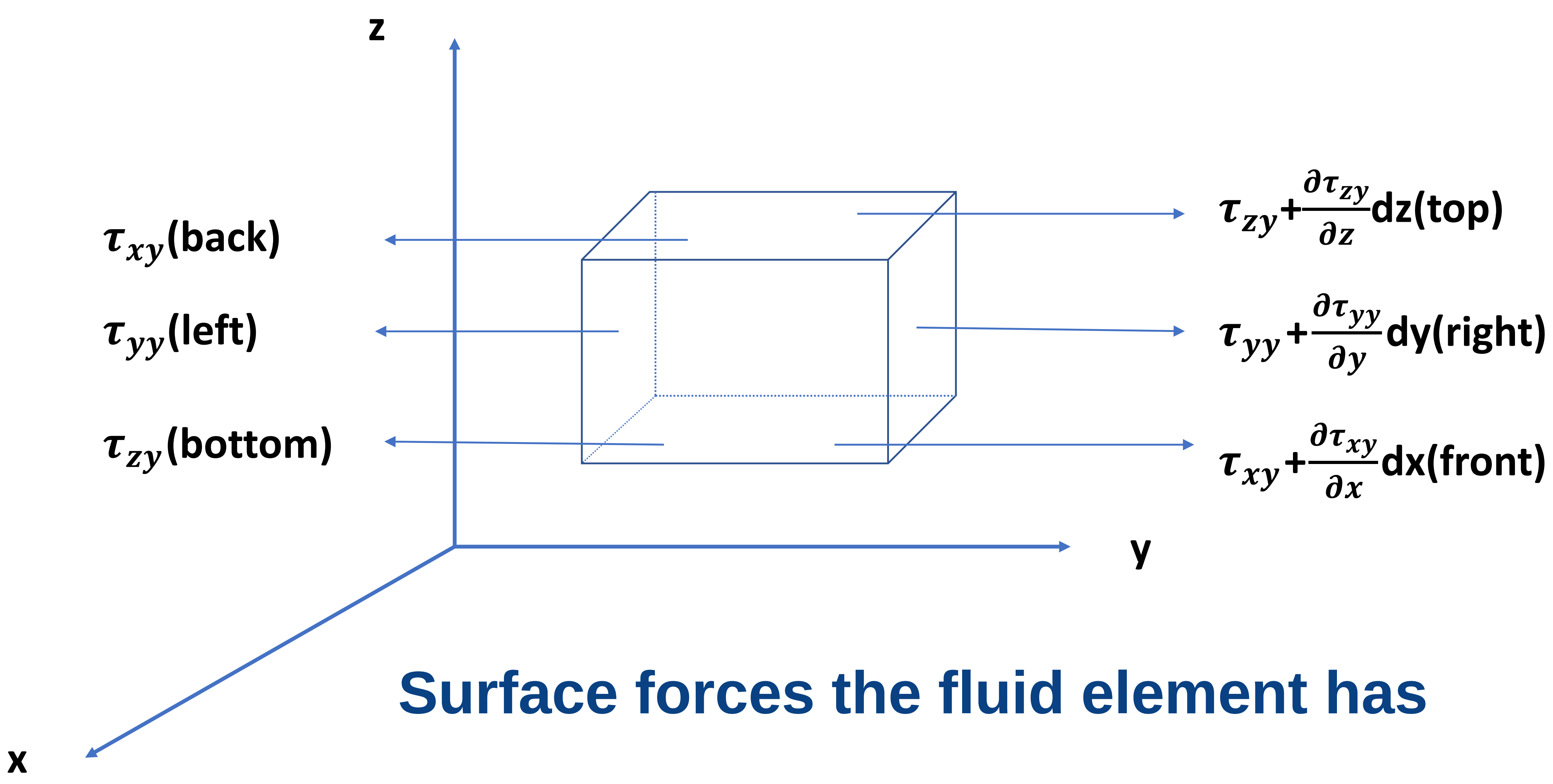
$$\tau_{yx} = \mu \frac{\partial u_x}{\partial y}$$

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)$$

τ_{xy}、 τ_{yz}、 τ_{zx} ~velocity gradients



Surface forces the fluid element has

τ_{xy}、 τ_{yz}、 τ_{zx} ~velocity gradients

According to Newton's second law of rotation

Σtorque=rotational inertia $\times$angular acceleration

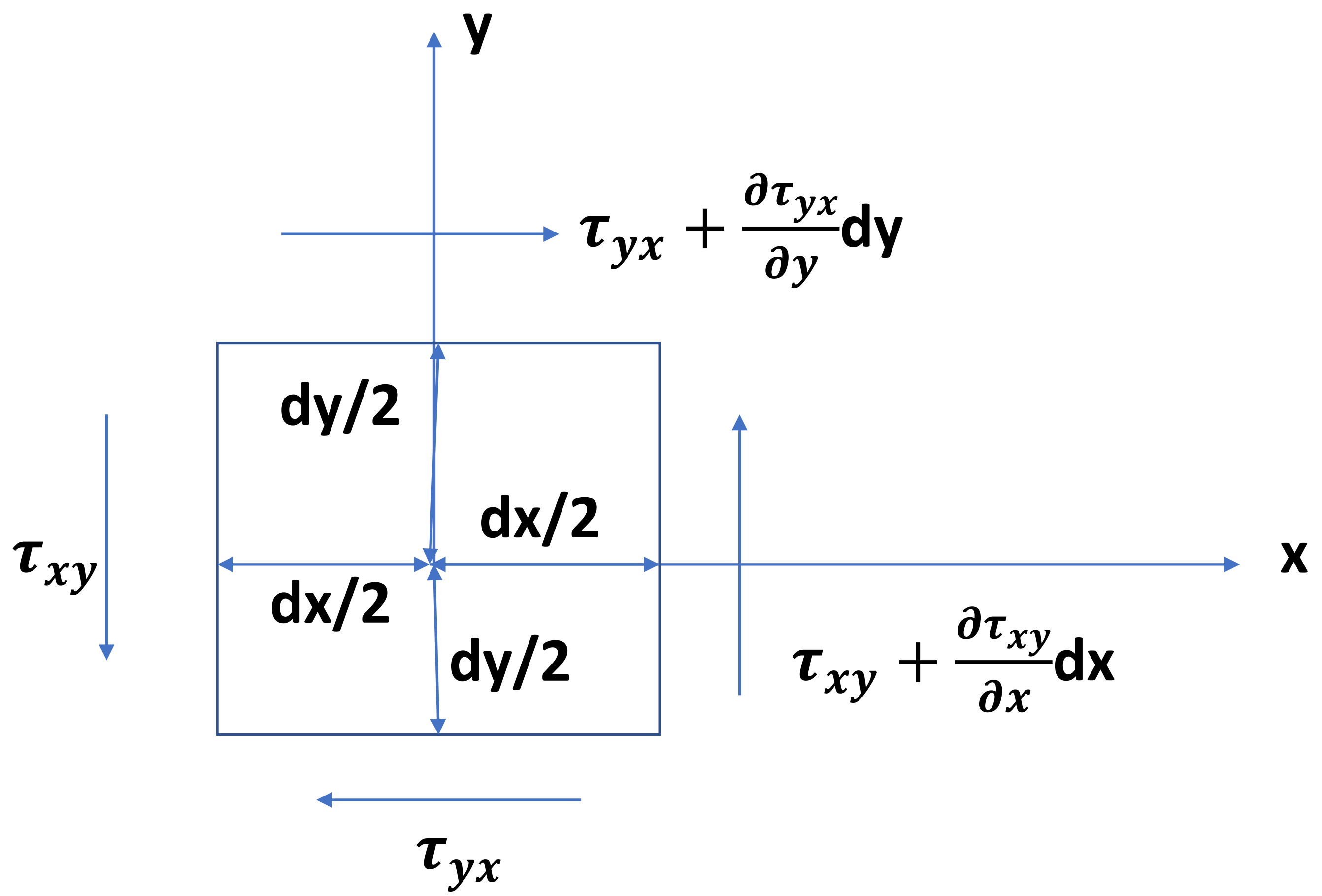
Rotational inertia=mass$\times$(radius of gyration)^2

$=\rho dxdydz \cdot r^2 \cdot \alpha$

Where α is angular acceleration

Σtorque= Σ (force $\times$ rotational distance)

τ_{xy}、 τ_{yz}、 τ_{zx} ~velocity gradients



Tangential stress for rotational axis

τ_{xy}、 τ_{yz}、 τ_{zx} ~velocity gradients

$$\Sigma torgue = \tau_{xy} dydz\left(\frac{dx}{2}\right) + (\tau_{xy} + \frac{\partial\tau_{xy}}{\partial x}\mathrm{dx})\mathrm{dydz}\left(\frac{dx}{2}\right) -$$

$$\tau_{yx} dxdz\left(\frac{dy}{2}\right) - (\tau_{yx} + \frac{\partial\tau_{yx}}{\partial y}\mathrm{dy})\mathrm{dxdz}\left(\frac{dy}{2}\right)$$

$$=\left(\tau_{xy} - \tau_{yx}\right)\mathrm{dxdydz} + \left(\frac{\partial\tau_{xy}}{\partial x}dx - \frac{\partial\tau_{yx}}{\partial y}dy\right)\frac{dxdydz}{2}$$

$$= \rho\,\mathrm{dxdydz}\cdot r^2\cdot\alpha$$

τ_{xy}、 τ_{yz}、 τ_{zx} ~velocity gradients

$$\left(\tau_{xy}-\tau_{yx}\right)+\left(\frac{\partial \tau_{xy}}{\partial x}dx-\frac{\partial \tau_{yx}}{\partial y}dy\right)\frac{1}{2}=\rho r^2\cdot\alpha$$

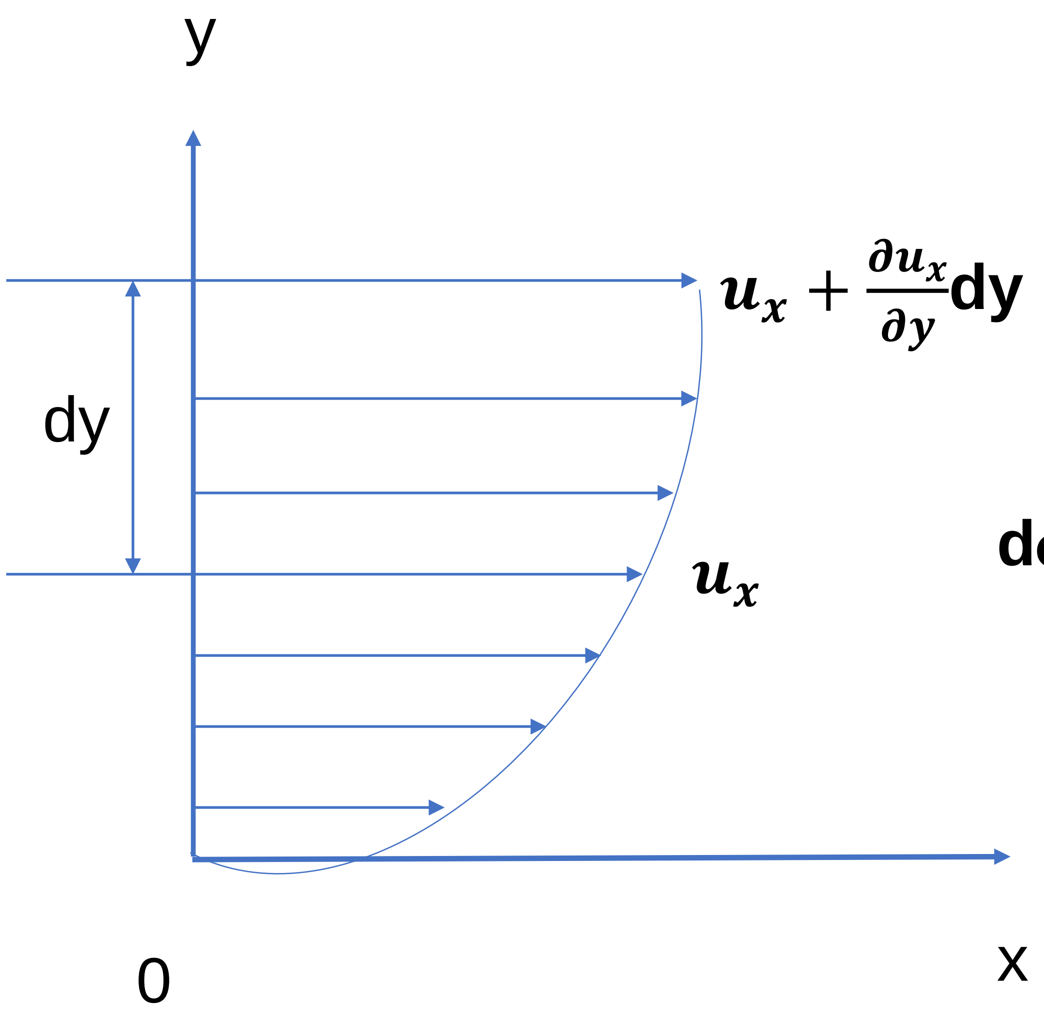
When $\Delta v(dxdydz)\to 0$, r,dx,dy,dz$\to$0, thus

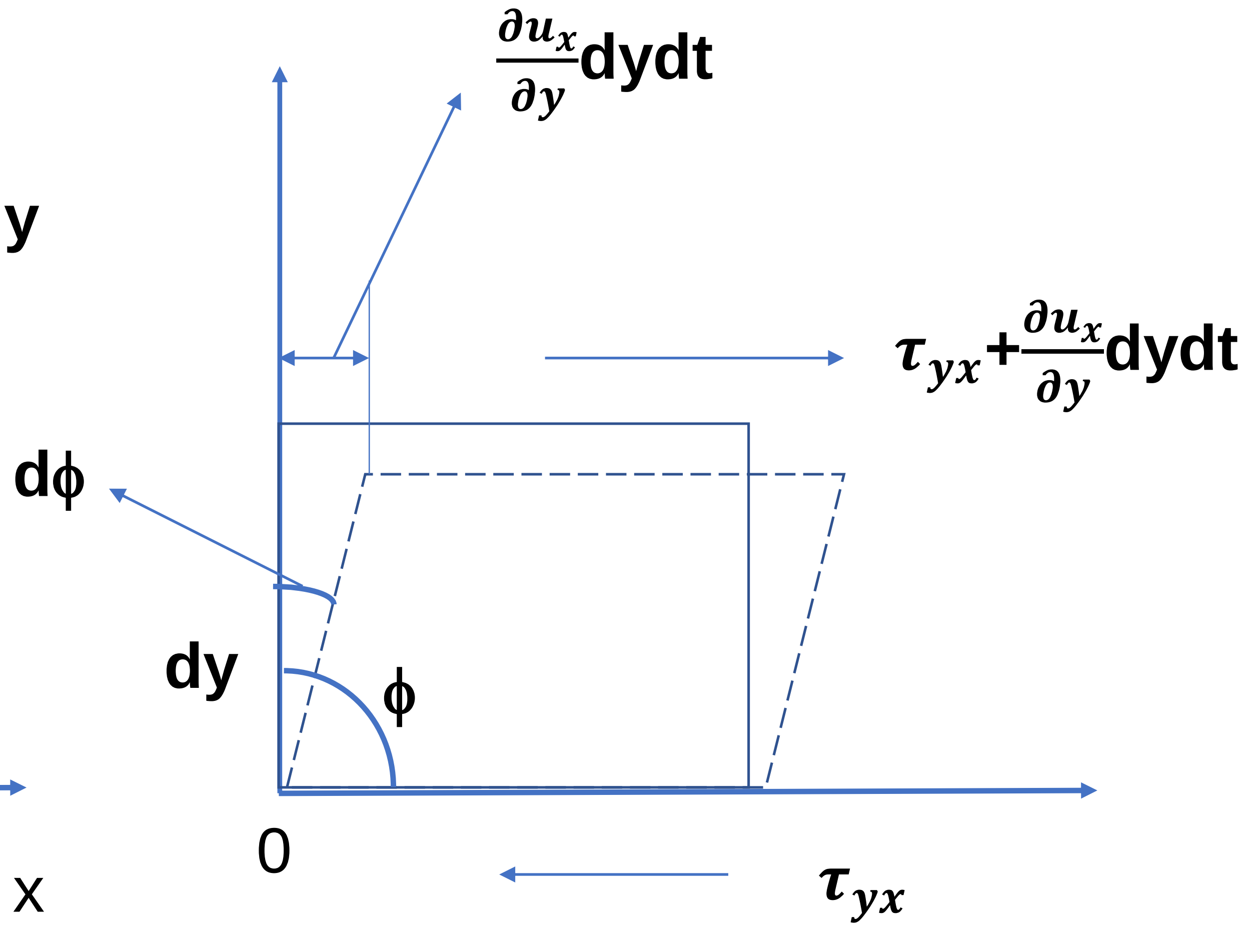
$$\tau_{xy}=\tau_{yx}$$

Similarly,

$$\tau_{xz}=\tau_{zx}$$
$$\tau_{yz}=\tau_{zy}$$

τ_{xy}、 τ_{yz}、 τ_{zx} ~velocity gradients





Tangential stress makes rectangular surface deforms at1D flow

τ_{xy}、 τ_{yz}、 τ_{zx} ~velocity gradients

$$\tan d\boldsymbol{\phi} = -\frac{\frac{\partial u_x}{\partial y}dydt}{dy}$$

where $\partial u_x/\partial y$ is shear rate or deformation rate

dt is time

dϕ is rotation angle, rad.

"-" stands for, when the upper fluid moves $\partial u_x/\partial y dydt$, ϕ reduces dϕ, i.e., dϕ is negative. Because dϕ is very small,

$$\tan d\phi \approx d\phi$$

τ_{xy}、 τ_{yz}、 τ_{zx} ~velocity gradients

$$d\phi = -\left(\frac{du_x}{dy}dydt\right)/dy$$

$$\frac{d\phi}{dt} = -\frac{du_x}{dy}$$

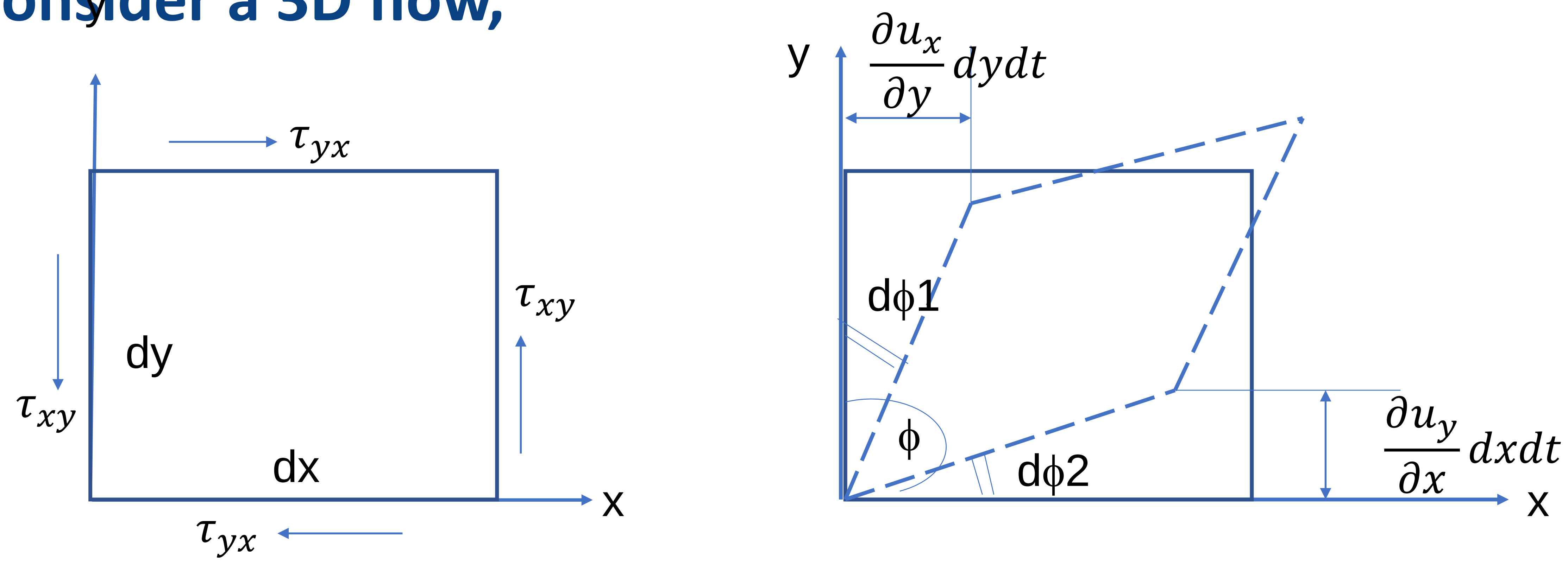
Multiplying μ, we obtain,

(angular deformation rate) $\mu\frac{d\phi}{dt} = -\mu\frac{du_x}{dy}$(shear rate)

$= \tau_{yx}$

τ_{xy}、 τ_{yz}、 τ_{zx} ~velocity gradients

Consider a 3D flow,



Before deformation **After deformation**

Tangential stress makes the plane deformation

τ_{xy}、 τ_{yz}、 τ_{zx} ~velocity gradients

The volume of the fluid element is dxdydz. During the flow, the volumetric deformation happens from rectangular hexahedron to rhombohedron(菱形六面体). For x-y plane, there are four tangential stresses take effect for the deformation, where , $\tau_{xy}\ and\ \tau_{yx}$ act on 4 planes normal to x=y plane. At the relative sides, $\tau_{xy} = \tau_{yx}$ in value but opposite direction.

After time dt, rectangular→rhombus(diamond)

ϕ From $\pi/2$ to $< \pi/2$.

τ_{xy}、 τ_{yz}、 τ_{zx} ~velocity gradients

The upper fluid moves $\frac{\partial u_x}{\partial y} dydt$ more than the lower fluid.

The right fluid moves $\frac{\partial u_y}{\partial x} dxdt$ more than the left fluid.

Both dϕ1 and dϕ2 are negative.

$$\tan \boldsymbol{d\phi 1} = -\frac{\partial u_x}{\partial y} dydt/dy \approx d\phi 1$$

$$\tan \boldsymbol{d\phi 2} = -\frac{\partial u_y}{\partial x} dxdt/dx \approx d\phi 2$$

τ_{xy}、 τ_{yz}、 τ_{zx} ~velocity gradients

For dϕ=dϕ1+dϕ2

$$\frac{d\phi}{dt} = \frac{d\phi_1}{dt} + \frac{d\phi_2}{dt}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

3.2 The Equation of Motion

b) Normal stress vs velocity gradients:

- **Normal stress is composed of pressure(P) and viscous stress(σ):**

$$\tau_{xx} = -P + \sigma_{xx}\,; \quad \tau_{yy} = -P + \sigma_{yy}\,; \quad \tau_{zz} = -P + \sigma_{zz}$$

- **For normal stress, stretching is positive, so pressure is negative. For static fluid or ideal fluid,**

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = -P$$

3.2 The Equation of Motion

For viscous flowing fluid, Stokes assumes：

$$P = -\frac{1}{3}(\tau_{xx}+\tau_{yy}+\tau_{zz})$$

$$\sigma_{xx} = 2\mu\frac{\partial u_x}{\partial x} + \lambda(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z})$$

thus,

$$\tau_{xx} = -P + 2\mu\frac{\partial u_x}{\partial x} + \lambda(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z})$$

$$\tau_{yy} = -P + 2\mu\frac{\partial u_y}{\partial y} + \lambda(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z})$$

$$\tau_{zz} = -P + 2\mu\frac{\partial u_z}{\partial z} + \lambda(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z})$$

3.2 The Equation of Motion

Summation the above three equations:

$$\lambda = -\frac{2}{3}\mu$$

Newtonian fluid, μ=const

$$\tau_{\mathbf{xx}} = -\mathrm{P} + 2\mu\frac{\partial \mathrm{u_x}}{\partial \mathrm{x}} - \frac{2}{3}\mu(\nabla \cdot \vec{\mathrm{u}})$$

The above equation is normal stress vs velocity gradients for Newtonian fluid.

3.2 The Equation of Motion

$$\tau_{\mathbf{ji}} = -(\mathrm{P} + \frac{2}{3}\mu\nabla\cdot\vec{\mathrm{u}})\delta_{\mathbf{ij}} + \mu(\frac{\partial \mathrm{u_i}}{\partial \mathrm{j}} + \frac{\partial \mathrm{u_j}}{\partial \mathrm{i}})$$

where δ_{ij} is called Kronecker δ

$$\delta_{\mathbf{ij}} = \begin{cases} \mathbf{0} & (\mathrm{i} \neq \mathrm{j}) \\ \mathbf{1} & (\mathrm{i} = \mathrm{j}) \end{cases} \begin{pmatrix} \mathrm{i} = \mathrm{x, y, z} \\ \mathrm{j} = \mathrm{x, y, z} \end{pmatrix}$$

3.2 The Equation of Motion

Substituting shear **stress, normal stress vs velocity gradients into Eq.(3.28) :**

$$\rho\frac{\mathbf{Du_x}}{\mathbf{Dt}} = \rho X - \frac{\partial P}{\partial x} + \mu\nabla^2 u_x + \frac{1}{3}\mu\frac{\partial}{\partial x}(\nabla\cdot\vec{u}) \qquad (\mathbf{3.29})$$

$$\rho\frac{\mathbf{Du_y}}{\mathbf{Dt}} = \rho Y - \frac{\partial P}{\partial y} + \mu\nabla^2 u_y + \frac{1}{3}\mu\frac{\partial}{\partial y}(\nabla\cdot\vec{u}) \qquad (\mathbf{3.30})$$

$$\rho\frac{\mathbf{Du_z}}{\mathbf{Dt}} = \rho Z - \frac{\partial P}{\partial z} + \mu\nabla^2 u_z + \frac{1}{3}\mu\frac{\partial}{\partial z}(\nabla\cdot\vec{u}) \qquad (\mathbf{3.31})$$

$$\rho\frac{D\vec{u}}{\mathbf{Dt}} = \rho\vec{g} - \nabla P + \mu\nabla^2\vec{u} + \frac{1}{3}\mu\nabla(\nabla\cdot\vec{u}) \qquad (\mathbf{3.32})$$

Eq.(3.32) was derived by C. L. M. H. Navier in 1827, and by G. G. Stokes in 1881 separately with different approaches. It is then called *Navier-Stokes equation* or *N-S equation*. The constrains for N-S equation are: Newtonian fluid, laminar flow, μ=constant, isotropy.

3.2 The Equation of Motion

For incompressible fluid, ρ and μ are constant,

$$\nabla \cdot \vec{u} = 0$$

$$\rho \frac{D\vec{u}}{Dt} = \rho\vec{g} - \nabla P + \mu\nabla^2\vec{u} \qquad (3.33)$$

For ideal fluid:

$$\mu = 0 \text{ or } \nabla \cdot \tau = 0$$

$$\rho \frac{D\vec{u}}{Dt} = \rho\vec{g} - \nabla P \qquad (3.34)$$

Equation 3.34 is the famous *Euler equation*, first derived in 1755.

3.2 The Equation of Motion

- **When the acceleration terms in the N-S equation are neglected:**

$$\rho \frac{D\vec{u}}{\mathbf{Dt}} = 0$$

$$\rho\,\vec{g} - \nabla P + \mu \nabla^2 \vec{u} = \mathbf{0} \qquad (\mathbf{3.35})$$

- **Equation 3.35 is called the *Stokes flow equation*. It is sometimes called the creeping flow equation, because the term $\rho[\vec{u} \cdot \nabla \vec{u}]$, which is quadratic in the velocity, can be discarded when the flow is extremely slow. Eq. 3.35 is important in lubrication theory, particle motions in suspension, flow through porous media.**

3.3 The Equation of Energy

- The conservation of energy, i.e., the first law of thermodynamics

$$
\begin{aligned}
&\text{rate of increase of total energy(internal and kinetic)}\\
&=\text{ rate of addition of total energy (internal and kinetic)}\\
&+\text{rate of addition of heat}\\
&+\text{rate of work by external force on the fluid} \quad (3.36)
\end{aligned}
$$

$$U = Q + W$$

$$\frac{DU}{Dt} = \frac{DQ}{Dt} + \frac{DW}{Dt}$$

3.3 The Equation of Energy

variables：

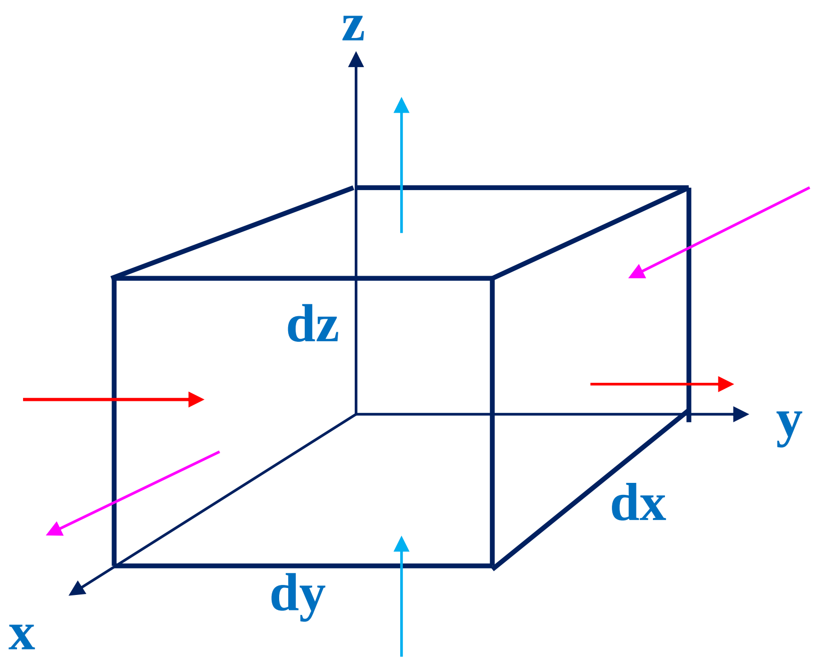
- q—heat flow rate（heat capacity）, J/h
- q′—heat formation rate per unit volume of fluid, $J/m^3 \cdot h$
- q/A—heat flux, $J/m^2 \cdot h$
- K — heat conductivity， $J/m \cdot K \cdot s$ or $W/m \cdot K$
- α—thermal diffusivity, m^2/s
- $\rho c_p T$—energy per unit volume of fluid, J/m^3
- h —convective heat transfer coefficient, $W/m^2 \cdot K$
- Φ — rate of heat by friction (dissipation rate), $J/m^3 \cdot h$
- U—internal energy per unit mass of fluid
- Q—addition of heat per unit mass of fluid
- W—work by surface stress per unit mass of fluid
- Addition of heat contains two parts: 1) heat conduction from environment; 2) chemical reaction or nuclear reaction.

3.3 The Equation of Energy

$$W = \int_{\upsilon_1}^{\upsilon_2} \mathrm{P}d\upsilon - l_w$$

$$\frac{\mathbf{DU}}{\mathbf{Dt}} = \frac{\mathbf{DQ}}{\mathbf{Dt}} - \mathrm{P}\frac{\mathrm{D}\upsilon}{\mathbf{Dt}} + \frac{\mathbf{Dl_w}}{\mathbf{Dt}}$$

3.3 The Equation of Energy



3.3 The Equation of Energy

x-component in(back):

$$\left(\frac{q}{A}\right)_x \mathbf{dydz}$$

x-component out(front):

$$\left[\left(\frac{q}{A}\right)_x + \frac{\partial\left(\frac{q}{A}\right)_x}{\partial x}\mathbf{dx}\right]\mathbf{dydz}$$

3.3 The Equation of Energy

net heat flow rate:

$$-\frac{\partial\left(\frac{q}{A}\right)_x}{\partial x}\,\mathbf{dxdydz}$$

For the whole element volume:

$$-\left(\frac{\partial\left(\frac{q}{A}\right)_x}{\partial x}+\frac{\partial\left(\frac{q}{A}\right)_y}{\partial y}+\frac{\partial\left(\frac{q}{A}\right)_z}{\partial z}\right)\mathbf{dxdydz}$$

3.3 The Equation of Energy

According to Fourier's first law of heat conduction:

$$\left(\frac{q}{A}\right)_x = -K\frac{\partial T}{\partial x}$$

When K is constant

$$K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)\mathbf{dxdydz} = K\nabla^2\mathbf{Tdxdydz}$$

3.3 The Equation of Energy

Therefore, the total heat addition:

$$\rho \frac{DQ}{Dt} dxdydz = K\nabla^2 T dxdydz + \dot{q} dxdydz$$

or

$$\rho \frac{DQ}{Dt} = K\nabla^2 T + \dot{q}$$

Work:

$$\rho \frac{DW}{Dt} = -P(\nabla \cdot \vec{u}) + \varphi$$

3.3 The Equation of Energy

$$\rho \frac{\mathrm{DU}}{\mathrm{Dt}} \mathrm{dxdydz} = \rho \frac{\mathrm{DQ}}{\mathrm{Dt}} \mathrm{dxdydz} - \mathrm{P}\rho \frac{\mathrm{D}\upsilon}{\mathrm{Dt}} \mathrm{dxdydz} + \rho \frac{\mathrm{Dl_w}}{\mathrm{Dt}} \mathrm{dxdydz}$$

Let

$$\Phi = \rho \frac{\mathrm{Dl_w}}{\mathrm{Dt}}$$

$$\rho \frac{\mathrm{DU}}{\mathrm{Dt}} + \mathrm{P}\rho \frac{\mathrm{D}\upsilon}{\mathrm{Dt}} = \mathrm{K}\nabla^2\mathrm{T} + \Phi + \mathbf{q}'$$

3.3 The Equation of Energy

Let H be the enthalpy for the liquid, H=U+Pυ. The substantial derivative of H is as follows,

$$\rho\frac{\mathbf{DH}}{\mathbf{Dt}} = \rho\frac{\mathbf{DU}}{\mathbf{Dt}} + \mathrm{P}\rho\frac{\mathrm{D}\upsilon}{\mathbf{Dt}} + \upsilon\rho\frac{\mathbf{DP}}{\mathbf{Dt}}$$

$$\rho\frac{\mathbf{DH}}{\mathbf{Dt}} - \frac{\mathbf{DP}}{\mathbf{Dt}} = \rho\frac{\mathbf{DU}}{\mathbf{Dt}} + \mathrm{P}\rho\frac{\mathrm{D}\upsilon}{\mathbf{Dt}}$$

The general form of the Energy equation:

$$\rho\frac{\mathbf{DH}}{\mathbf{Dt}} = \rho\frac{\mathbf{DP}}{\mathbf{Dt}} + \mathrm{K}\nabla^2\mathrm{T} + \Phi + \mathbf{q}'$$

3.3 The Equation of Energy

For incompressible fluid, no internal heat source :

$$\nabla \cdot \vec{u} = 0 \quad \Phi = 0 \quad q' = 0$$

$$c_p \approx c_v = \text{constant} \quad (\text{neglet U change with P})$$

$$\rho \frac{DH}{Dt} = \rho c_p \frac{DT}{Dt} + \frac{DP}{Dt}$$

$$\rho c_p \frac{DT}{Dt} = K\nabla^2 T$$

$$\frac{DT}{Dt} = \alpha \nabla^2 T$$

3.3 The Equation of Energy

In solid case, ρ=constant, no flow $\vec{u} = 0$ then

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

The above Equation is Fourier's second law of heat conduction. For steady-state heat conduction:

$$\nabla^2 T = 0$$

-Laplace equation in terms of temperature.

3.4 Initial and Boundary Conditions

1) Initial Conditions

t=0, transport phenomena should satisfy the initial states. Variables are : u_x、u_y、u_z、P and ρ。

$$
\begin{aligned}
u_x(x,y,z,0) &= f_1(x,y,z)\\
u_y(x,y,z,0) &= f_2(x,y,z)\\
u_z(x,y,z,0) &= f_3(x,y,z)\\
P(x,y,z,0) &= f_4(x,y,z)\\
\rho(x,y,z,0) &= f_5(x,y,z)
\end{aligned}
$$

f_1~f_5 is known functions.

3.4 Initial and Boundary Conditions

2) Boundary Conditions

- **At solid wall (viscous fluid)**

- **$y=0, u_x=0, u_y=0$** **$y=0, u_x=U, u_y=0$** **$y=0, u_x=0, u_y=V$**

- **For viscous flow, at the solid wall, fluid adheres to the wall. At any point in the wall, fluid velocity is equal to the velocity at that point.**

3.4 Initial and Boundary Conditions

2) Boundary Conditions

- **At solid wall (ideal fluid, μ=0)**

$$\overrightarrow{u_w} = \overrightarrow{u_s}$$

- y=0, u_y=0
- **For ideal fluid, μ=0 , velocity at solid wall equals to the velocity at that point.**

3.4 Initial and Boundary Conditions

2) Boundary Conditions

- **Infinitive distance**

$$\vec{u}\Big|_{\vec{r}\to\infty} = \vec{u}_\infty$$

$$P\Big|_{\vec{r}\to\infty} = P_\infty$$

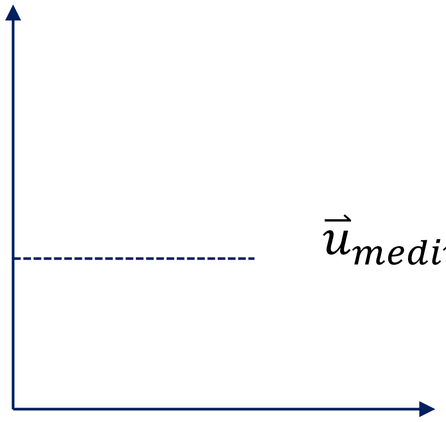
$$\rho\Big|_{\vec{r}\to\infty} = \rho_\infty$$

$$T\Big|_{\vec{r}\to\infty} = T_\infty$$

3.4 Initial and Boundary Conditions

2) Boundary Conditions

- **At the interface or free surface**
- **When the two media does not penetrate, the interface does not separate.**



$$\vec{u}_{medium1} = \overleftarrow{u}_{medium2}$$

$$y = 0 \quad \mathbf{P} = \mathrm{P}_0$$
$$\mu^{\mathrm{I}}_{\mathbf{xy}} = \mu^{\mathbf{II}}_{\mathbf{xy}}$$
$$u^{\mathrm{I}}_{\mathrm{y}} = u^{\mathbf{II}}_{\mathrm{y}}$$

3.4 Initial and Boundary Conditions

2) Boundary Conditions

- **If BCs only have velocities, a modified pressure is introduced.**

$$P' = P - P_0 + \rho g z$$

$$\nabla P' = \nabla P + \rho g \Delta z = \nabla P - \rho \vec{g}$$

$$\rho \frac{D\vec{u}}{Dt} = -\nabla P' + \mu \nabla^2 \vec{u}$$

Summary Objectives

After completing study of Chapter Three, you should be able to do the following:

1. Know the processes to derivate continuity equation, Navier-Stokes equation and energy equation.
2. The physical laws beyond these basic equations.
3. The different forms of these equations at cylindrical and spherical coordinates .