Week-2

Lecture-2
2.2 Sample Spaces and Algebra Sets

The starting point for studying probability is the definition of four key terms: *experiment, sample outcome, sample space,* and *event.*

By an *experiment* we will mean any procedure that can be repeated, theoretically, an infinite number of times and has a well-defined set of possible outcomes.

Each of the potential eventualities of an experiment is referred to as a *sample outcome, s,* and their totality is called the *sample space, S.* To signify the membership of s in S, we write s ∈ S. Any designated collection of sample outcomes, including individual outcomes, the entire sample space, and the null set, constitutes an *event.* The latter is said to *occur* if the outcome of the experiment is one of the members of the event.

Example 2.2.1

Consider the experiment of flipping a coin three times. What is the sample space? Which sample outcomes make up the event A: Majority of coins show heads?

Think of each sample outcome here as an ordered triple, its components representing the outcomes of the first, second, and third tosses, respectively. Altogether, 2.2 Sample Spaces and the Algebra of Sets 19 there are eight different triples, so those eight comprise the sample space:

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

By inspection, we see that four of the sample outcomes in S constitute the event A:

$$A = \{HHH, HHT, HTH, THH\}$$

Example 2.2.1

A local TV station advertises two newscasting positions. If three women (W_1, W_2, W_3) and two men (M_1, M_2) apply, the "experiment" of hiring two coanchors generates a sample space of ten outcomes:

$S = \{(W_1, W_2), (W_1, W_3), (W_2, W_3), (W1, M1), (W_1, M_2), (W_2, M_1),$
$(W_2, M_2), (W_3, M_1), (W_3, M_2), (M_1, M_2)\}$

Does it matter here that the two positions being filled are equivalent? Yes. If the station were seeking to hire, say, a sports announcer and a weather forecaster, the number of possible outcomes would be twenty: (W_2, M_1), for example, would represent a different staffing assignment than (M_1, W_2).

Unions, Intersections, and Complements

Definition 2.2.1
Let A and B be any two events defined over the same sample space S. Then

a. The intersection of A and B, written $A \cap B$, is the event whose outcomes belong to both A and B.

b. The union of A and B, written $A \cup B$, is the event whose outcomes belong to either A or B or both.

Example 2.2.1

Let A be the set of $x's$ for which $x^2 + 2x = 8$;let B be the set for which $x^2 + x = 6$. Find $A \cap B$ and $A \cup B$.

Since the first equation factors into $(x + 4)(x - 2) = 0$, its solution set is $A = \{-4, 2\}$. Similarly, the second equation can be written $(x+3)*(x-2) = 0$, making $B = \{-3, 2\}$. Therefore, $A \cap B = \{2\}$ and $A \cup B = \{-4, -3, 2\}$

Definition 2.2.2
Events A and B defined over the same sample space are said to be *mutually exclusive* if they have no outcomes in common—that is, if $A \cap B = \varnothing$, where $\varnothing$ is the null set.

Example 2.2.9

Consider a single throw of two dice. Define A to be the event that the *sum* of the faces showing is odd. Let B be the event that the two faces themselves are odd. Then clearly, the intersection is empty, the sum of two odd numbers necessarily being even. In symbols, $A \cap B = \varnothing$. (Recall the event $B \cap C$ asked for in Example 2.2.6.)

Definition 2.2.3
Let A be any event defined on a sample space S. The *complement* of A, written A^C, is the event consisting of all the outcomes in S other than those contained in A.

Example 2.2.9

Let A be the set of (x, y)'s for which $x^2 + y^2 < 1$. Sketch the region in the x y-plane corresponding to A^C.

From analytic geometry, we recognize that $x^2 + y^2 < 1$ describes the interior of a circle of radius 1 centered at the origin. Figure 2.2.3 shows the complement—the points on the circumference of the circle and the points outside the circle.

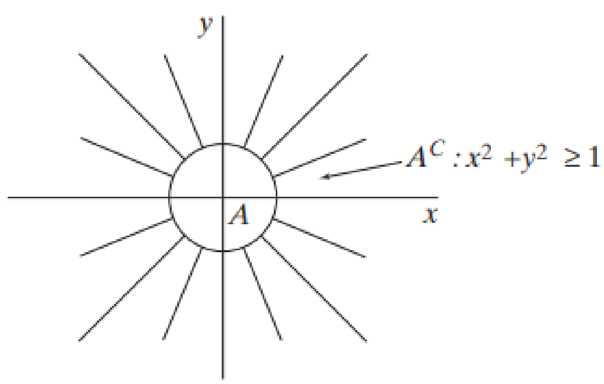


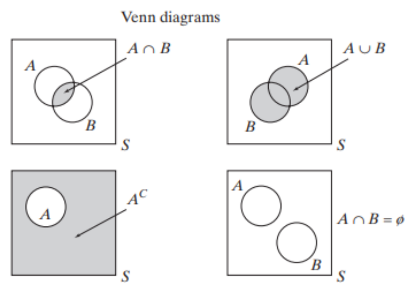
Figure 2.2.3

The notions of union and intersection can easily be extended to more than two events. For example, the expression $A_1 \cup A_2 \cup \cup A_k$ defines the set of outcomes belonging to any of the A_i's (or to any combination of the A_i's). Similarly, $A_1 \cap A_2 \cap \cap A_k$ is the set of outcomes belonging to all of the A_i's.

Expressing Events Graphically: Venn Diagrams

An alternative approach that can be highly effective is to represent the underlying events graphically in a format known as a *Venn diagram*. Figure 2.2.4 shows Venn diagrams for an intersection, a union, a complement, and two events that are mutually exclusive.

Figure 2.2.4



The shaded area in Figure 2.2.5 represents the event E that either A or B, but not both, occurs (that is, *exactly one* occurs).

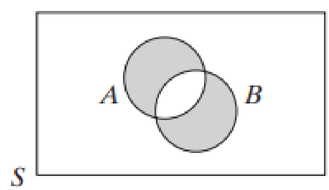


Figure 2.2.5

Figure 2.2.6 shows the event F that at most one (of the two events) occurs. Since the latter includes every outcome except those belonging to both A and B, we can write

$$F = (A \cap B)^C$$

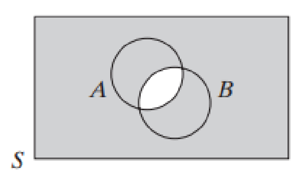


Figure 2.2.6

2.3 The Probability Function

If A is any event defined on a sample space S, the symbol P(A) will denote the *probability* of A, and we will refer to P as the *probability function.* It is, in effect, a mapping from a set (i.e., an event) to a number.

If S has a finite number of members, Kolmogorov showed that as few as three axioms are necessary and sufficient for characterizing the probability function P:

Axiom 1. *Let A be any event defined over S. Then* $P(A) \geq 0$.

Axiom 2. *P(S) = 1.*

Axiom 3. *Let A and B be any two mutually exclusive events defined over S. Then*

$$P(A \cup B) = P(A) + P(B)$$

When S has an infinite number of members, a fourth axiom is needed:

Axiom 4. *Let* $A_1, A_2, ...,$ *be events defined over S. If* $A_i \cap A_j = \varnothing$ *for each* $i \neq j$*, then*

$$p(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

From these simple statements come the general rules for manipulating the probability function-rules that apply no matter what specific mathematical forms the function may happen to take.

Some basic properties of P

Some of the immediate consequences of Kolmogorov's axioms are the results given in Theorems 2.3.1 through 2.3.6. Despite their simplicity, these properties prove to be extraordinarily useful in solving all sorts of problems.

Theorem 2.3.1 $P(A^C) = 1 - P(A)$

Proof: By Axiom 2 and Definition 2.2.3,

$$P(S) = 1 = P(A \cup A^C)$$

But A and A^C are mutually exclusive,so

$$P(A \cup A^C) = P(A) + P(A^C)$$

and the result follows.

Theorem 2.3.2 $P(\varnothing) = 0$

Proof: Since $\varnothing = S^C, P(\varnothing) = P(S^C) = 1 - P(S) = 0$

Theorem 2.3.3 If $A \subset B$, then $P(A) \leq P(B)$

Proof: Note that the event B may be written in the form

$$B = A \cup (B \cap A^C)$$

where A and $(B \cap A^C)$ are mutually exclusive. Therefore,

$$P(B) = P(A) + P(B \cap A^C)$$

which implicies that $P(B) \geq P(A)$ since $P(B \cap A^C) \geq 0$

Theorem 2.3.4 For any event A, $P(A) \leq 1$

Proof: The proof follows immediately from Theorem 2.3.3 because $A \subset S$ and $P(S) = 1$

Theorem 2.3.5 Let $A_1, A_2, ..., A_n$ be events defined over S. If $A_i \cap A_j = \varnothing$ for $i \neq j$ then

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$$

Proof: The proof is a straightforward induction argument with Axiom 3 being the straight point.

Theorem 2.3.6 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Proof: The Venn diagram for $A \cup B$ certainly suggests that the statement of the theorem is true (recall Figure 2.2.4). More formally, we have from Axioms 3 that

$$P(A) = P(A \cap B^C) + P(A \cap B)$$

and

$$P(B) = P(B \cap A^C) + P(A \cap B)$$

Adding these two equations gives

$$P(A) + P(B) = [P(A \cap B^C) + P(B \cap A^C) + P(A \cap B) + P(A \cap B)]$$

By Theorem 2.3.5, the sum in the brackets is $P(A \cup B)$. If we subtract $P(A \cap B)$ from both sides of the equation, the result follows.

The next result is a generalization of Theorem 2.3.6 that considers the probability of the union of n events. We have elected to retain the two-event case, $P(A \cup B)$, as a separate theorem simply for pedagogical reasons.

Theorem 2.3.7 Let $A_1, A_2, ..., A_n$ be any n events defined on S. Then

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) -$$

$$- \sum_{i<j<k<l} P(A_i \cap A_j \cap A_k \cap A_l) + ... + (-1)^{n+1} * P(A_1 \cap A_2 \cap ... \cap A_n)$$

Example 2.3.1

Let A and B be two events defined on a sample space S such that $P(A) = 0.3, P(B) = 0.5$, and $P(A \cup B) = 0.7$. Find
(a) $P(A \cap B)$,
(b) $P(A^C \cup B^C)$,
(c) $P(A^C \cap B)$.

a. Transposing the terms in Theorem 2.3.6 yields a general formula for the probability of an intersection:

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

Here

$$P(A \cap B) = 0.3 + 0.5 - 0.7 = 0.1$$

b. The two cross-hatched regions in Figure 2.3.1 correspond to AC and BC. The union of AC and BC consists of those regions that have cross-hatching in either or both directions. By inspection, the only portion of S not included in $A^C \cup B^C$ is the intersection, $A \cap B$. By Theorem 2.3.1, then,

$$P(A^C \cup B^C) = 1 - P(A \cap B) = 1 - 0.1 = 0.9$$

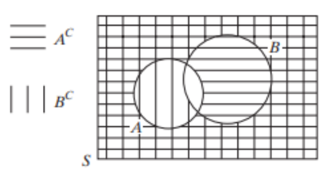


Figure 2.3.1

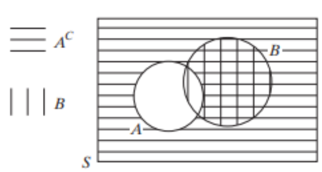


Figure 2.3.2

c. The event $A^C \cap B$ corresponds to the region in Figure 2.3.2 where the crosshatching extends in both directions—that is, everywhere in B except the intersection with A. Therefore,

$$P(A^C \cap B) = P(B) - P(A \cap B) = 0.5 - 0.1 = 0.4$$

Example 2.3.1

Having endured (and survived) the mental trauma that comes from taking two years of chemistry, a year of physics, and a year of biology, Biff decides to test the medical school waters and sends his MCATs to two colleges, X and Y . Based on how his friends have fared, he estimates that his probability of being accepted at X is 0.7, and at Y is 0.4. He also suspects there is a 75% chance that at least one of his applications will be rejected. What is the probability that he gets at least one acceptance?

Let A be the event "School X accepts him" and B the event "School Y accepts him." We are given that $P(A) = 0.7, P(B) = 0.4$, and $P(A^C \cup B^C) = 0.75$. The question is asking for $P(A \cup B)$.
From Theorem 2.3.6,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Recall from Question 2.2.32 that $A^C \cup B^C = (A \cap B)^C$, so

$$P(A \cap B) = 1 - P[(A \cap B)^C] = 1 - 0.75 = 0.25$$

It follows that Biff's prospects are not all that bleak—he has an 85% chance of getting in somewhere:

$$P(A \cup B) = 0.7 + 0.4 - 0.25 = 0.85$$

Comment Notice that $P(A \cup B)$ varies directly with $P(A^C \cup B^C)$:

$$P(A \cup B) = P(A) + P(B) - [1 - P(A^C \cup B^C)] = P(A) + P(B) - 1 + P(A^C \cup B^C)$$

If P(A) and P(B), then, are fixed, we get the curious result that Biff's chances of getting at least one acceptance increase if his chances of at least one rejection increase.

Practice-2

Exercise 2.2.1

A graduating engineer has signed up for three job interviews. She intends to categorize each one as being either a "success" or a "failure" depending on whether it leads to a plant trip. Write out the appropriate sample space. What outcomes are in the event A: Second success occurs on third interview? In B: First success never occurs? (Hint: Notice the similarity between this situation and the coin-tossing experiment described in Example 2.2.1.)

Exercise 2.3.1

According to a family-oriented lobbying group, there is too much crude language and violence on television. Forty-two percent of the programs they screened had language they found offensive, 27% were too violent, and 10% were considered excessive in both language and violence. What percentage of programs did comply with the group's standards?

Exercise 2.3.17

Let A, B, and C be three events defined on a sample space, S. Arrange the probabilities of the following events from smallest to largest:
(a) $A \cup B$
(b) $A \cap B$
(c) A
(d) S
(e) $(A \cap B) \cup (A \cap C)$

Exercise 2.2.3

An urn contains six chips numbered 1 through 6. Three are drawn out. What outcomes are in the event "Second smallest chip is a 3"? Assume that the order of the chips is irrelevant.

Exercise 2.2.19

An electronic system has four components divided into two pairs. The two components of each pair are wired in parallel; the two pairs are wired in series. Let A_ij denote the event "ith component in jth pair fails," $i = 1, 2; j = 1, 2$. Let A be the event "System fails." Write A in terms of the A_ij's.

Exercise 2.2.5

In the lingo of craps-shooters (where two dice are tossed and the underlying sample space is the matrix pictured in Figure 2.2.1) is the phrase "making a hard eight." What might that mean?

Exercise 2.3.5

Suppose that three fair dice are tossed. Let A_i be the event that a 6 shows on the ith die, $i = 1, 2, 3$. Does $P(A_1 \cup A_2 \cup A_3) = \frac{1}{2}$? Explain.

Exercise 2.2.21

Let A be the set of five-card hands dealt from a 52-card poker deck, where the denominations of the five cards are all consecutive—for example, (7 of hearts, 8 of spades, 9 of spades, 10 of hearts, jack of diamonds). Let B be the set of five-card hands where the suits of the five cards are all the same. How many outcomes are in the event $A \cap B$?

Exercise 2.2.7

Let P be the set of right triangles with a 5” hypotenuse and whose height and length are a and b, respectively. Characterize the outcomes in P.

Exercise 2.3.7

Let $A_1, A_2, ..., A_n$ be a series of events for which $A_i \cap A_j = \varnothing$ if $i \neq j$ and $A_1 \cup A_2 \cup \cup A_n = S$. Let B be any event defined on S. Express B as a union of intersections.

Exercise 2.2.23

Let A, B, and C be any three events defined on a sample space S. Show that

(a) the outcomes in $A \cup (B \cap C)$ are the same as the outcomes in $(A \cup B) \cap (A \cup C)$.

(b) the outcomes in $A \cap (B \cup C)$ are the same as the outcomes in $(A \cap B) \cup (A \cap C)$.

Exercise 2.2.9

A telemarketer is planning to set up a phone bank to bilk widows with a Ponzi scheme. His past experience (prior to his most recent incarceration) suggests that each phone will be in use half the time. For a given phone at a given time, let 0 indicate that the phone is available and let 1 indicate that a caller is on the line. Suppose that the telemarketer’s “bank” is comprised of four telephones.

(a) Write out the outcomes in the sample space.

(b) What outcomes would make up the event that exactly two phones are being used?

(c) Suppose the telemarketer had k phones. How many outcomes would allow for the possibility that at most one more call could be received?

(Hint: How many lines would have to be busy?)

Exercise 2.3.9

In the game of “odd man out” each player tosses a fair coin. If all the coins turn up the same except for one, the player tossing the different coin is declared the odd man out and is eliminated from the contest. Suppose that three people are playing. What is the probability that someone will be eliminated on the first toss?

(Hint: Use Theorem 2.3.1.)

Exercise 2.2.25

Let A, B, and C be any three events defined on a sample space S. Show that the operations of union and intersection are associative by proving that

(a) $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$

(b) $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$

Exercise 2.2.11
A woman has her purse snatched by two teenagers. She is subsequently shown a police lineup consisting of five suspects, including the two perpetrators. What is the sample space associated with the experiment "Woman picks two suspects out of lineup"? Which outcomes are in the event A: She makes at least one incorrect identification?

Exercise 2.3.11
If State's football team has a 10% chance of winning Saturday's game, a 30% chance of winning two weeks from now, and a 65% chance of losing both games, what are their chances of winning exactly once?

Exercise 2.2.13
In the game of craps, the person rolling the dice (the shooter) wins outright if his first toss is a 7 or an 11. If his first toss is a 2, 3, or 12, he loses outright. If his first roll is something else, say, a 9, that number becomes his "point" and he keeps rolling the dice until he either rolls another 9, in which case he wins, or a 7, in which case he loses. Characterize the sample outcomes contained in the event "Shooter wins with a point of 9."

Exercise 2.2.27
What must be true of events A and B,if
(a) $A \cup B = B$ (b) $A \cap B = A$

Exercise 2.3.13
Consolidated Industries has come under considerable pressure to eliminate its seemingly discriminatory hiring practices. Company officials have agreed that during the next five years, 60% of their new employees will be females and 30% will be minorities. One out of four new employees, though, will be a white male. What percentage of their new hires will be minority females?

Exercise 2.2.29
A coin is tossed four times and the resulting sequence of heads and/or tails is recorded. Define the events A, B, and C as follows:
A: exactly two heads appear
B: heads and tails alternate
C: first two tosses are heads
(a) Which events, if any, are mutually exclusive?
(b) Which events, if any, are subsets of other sets?

Exercise 2.2.15
Suppose that ten chips, numbered 1 through 10, are put into an urn at one minute to midnight, and chip number 1 is quickly removed. At one-half minute to midnight, chips numbered 11 through 20 are added to the urn, and chip

number 2 is quickly removed. Then at one-fourth minute to midnight, chips numbered 21 to 30 are added to the urn, and chip number 3 is quickly removed. If that procedure for adding chips to the urn continues, how many chips will be in the urn at midnight (148)?

Exercise 2.3.15

A coin is to be tossed four times. Define events X and Y such that
X: first and last coins have opposite faces
Y : exactly two heads appear
Assume that each of the sixteen head/tail sequences has the same probability. Evaluate
(a) $P(X^C \cap Y)$ (b) $P(X \cap Y^C)$

Exercise 2.2.17

Referring to Example 2.2.7, find $A \cap B$ and $A \cup B$ if the two equations were replaced by inequalities: $x^2 + 2x \leq 8$ and $x^2 + x \leq 6$.

Exercise 2.2.31

During orientation week, the latest Spiderman movie was shown twice at State University. Among the entering class of 6000 freshmen, 850 went to see it the first time, 690 the second time, while 4700 failed to see it either time. How many saw it twice?

Exercise 2.2.33

Let A, B, and C be any three events. Use Venn diagrams to show that
(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Exercise 2.2.35

Let A and B be any two events defined on a sample space S. Which of the following sets are necessarily subsets of which other sets?
a)A
b)B
c)$A \cup B$
d)$A \cap B$
e)$A^C \cap B$ f)$A \cap B^C$
g)$(A^C \cup B^C)^C$

Note: *In the practice sections, problems will be solved in a prescribed order. Any remaining problems that were not addressed during the session should be tackled by students independently as part of their self-study.*