Methods

Optimization and root finding

Exercise 1 Consider the function $f(x, y) = 1 - (x - x_0)^2 - 0.5(y - y_0)^2$ with $x_0 = 0.5$ and $y_0 = 1.0$. Check the documentation for `scipy.optimize`. Use it to maximze function f.

```
import numpy as np
```

```
x_0 = 0.5
y_0 = 1.0
```

```
# one needs to define $f$ as a function of a vector
f = lambda v: 1-(v[0]-x_0)**2-0.5*(v[1]-y_0)**2
```

```
f(np.array([0.2, 0.2]))
```

```
0.59
```

```
from scipy.optimize import minimize

sol = minimize(
        lambda u: -f(u),      # note the - here because we want to maximize
        np.array([0.2, 0.2])
)
sol
```

```
  message: Optimization terminated successfully.
  success: True
   status: 0
      fun: -0.9999999999999428
        x: [ 5.000e-01  1.000e+00]
```

```
      nit: 6
      jac: [ 2.906e-07 -2.682e-07]
 hess_inv: [[ 5.041e-01 -7.380e-03]
            [-7.380e-03  1.013e+00]]
     nfev: 21
     njev: 7
```

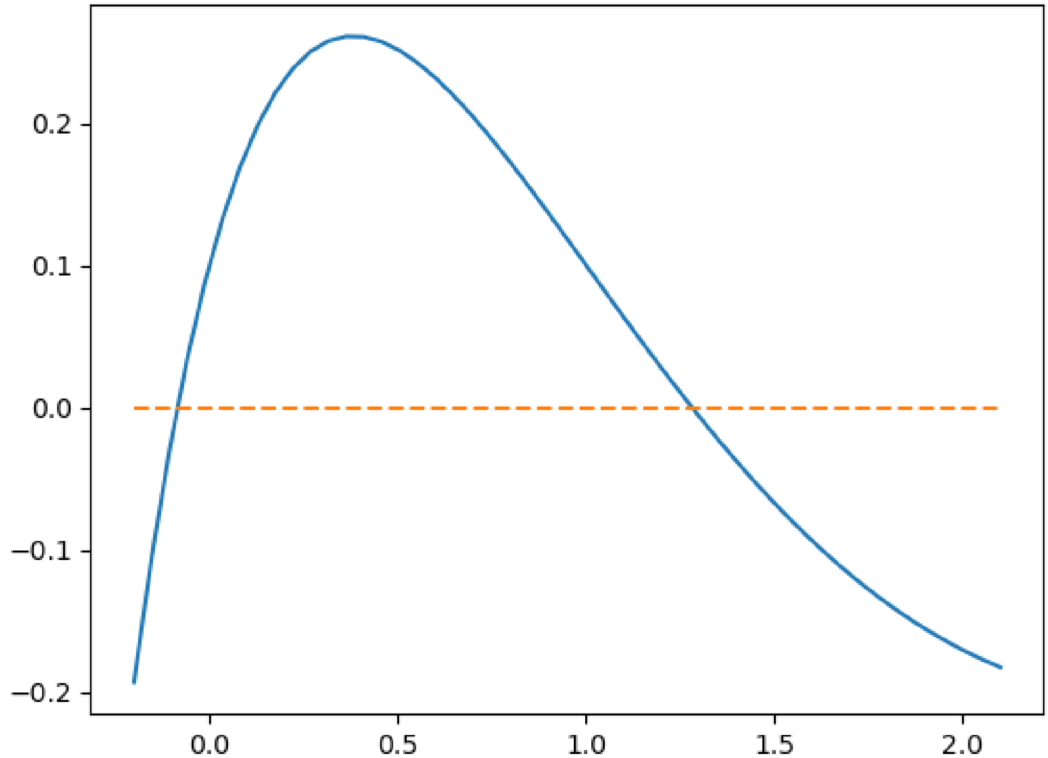
```
sol.x
```

```
array([0.50000014, 0.99999973])
```

Consider the function $g(x) = 0.1 + exp(-x)x(1-x)$ over [0,2]. Choose the scipy function and find the root of g.

```
from numpy import exp
g = lambda v: 0.1 + exp(-v)*v*(1-v)
```

```
# let's visualize the function
# we see it has a root between 0 and 2 and another one smaller than 0
from matplotlib import pyplot as plt
xvec = np.linspace(-0.2,2.1)
plt.plot(xvec, g(xvec))
plt.plot(xvec, xvec*0, linestyle='--')
```




```
from scipy.optimize import root
```

```
sol = root(g, 0.1)
sol  # converges to the wrong solution
```

```
 message: The solution converged.
 success: True
  status: 1
     fun: [ 1.388e-17]
       x: [-8.470e-02]
  method: hybr
    nfev: 12
    fjac: [[-1.000e+00]]
       r: [-1.373e+00]
     qtf: [ 1.585e-14]
```

```
sol = root(g, 1)
sol  # converges to the wrong solution
```

```
 message: The solution converged.
 success: True
  status: 1
     fun: [ 6.939e-17]
       x: [ 1.281e+00]
  method: hybr
    nfev: 9
    fjac: [[-1.000e+00]]
       r: [ 3.339e-01]
     qtf: [-9.682e-11]
```

```
g(sol.x)
```

```
array([6.9388939e-17])
```

Consider the function $h(x, y) = [0.1 + exp(-x)x(1-y), x-y]$. Choose the scipy function and find the root of g. to find the root of g?

```
g = lambda v: np.array([0.1 + exp(-v[0])*v[0]*(1-v[1]), v[0]-v[1]])
sol = root(g, np.array([1.0, 0.0]))
sol
```

```
 message: The solution converged.
 success: True
  status: 1
     fun: [ 1.388e-16  0.000e+00]
       x: [ 1.281e+00  1.281e+00]
  method: hybr
    nfev: 10
    fjac: [[-1.507e-02 -9.999e-01]
           [ 9.999e-01 -1.507e-02]]
       r: [-1.000e+00  1.005e+00 -3.338e-01]
     qtf: [-2.710e-12  1.799e-10]
```

```
sol.x
```

```
array([1.28105226, 1.28105226])
```

```
g(sol.x)
```

```
array([1.38777878e-16, 0.00000000e+00])
```

Interpolation

We consider the function $f(x) = sinc(\lambda x) = \frac{sin(\lambda x)}{\lambda x}$. Let $I = (x_i)_{i=[1,10]}$ be a regularly spaced interval between -2 and +2, containing 10 points. Call $Y = (y_i) = f(x_i)$ the values of f on this interval. Let T be a test set with 1000 regularly spaced points between -2.5 and 2.5.

The goal is to compare several ways interpolate function f on T.

Exercise 2 Define `f`, `I`, `Y`, `T` with numpy.

```
# this is a pure python
def f_pure(x, ):
    from math import sin
    if x==0:
        return 1.0
    else:
        return sin( *x)/( *x)
```

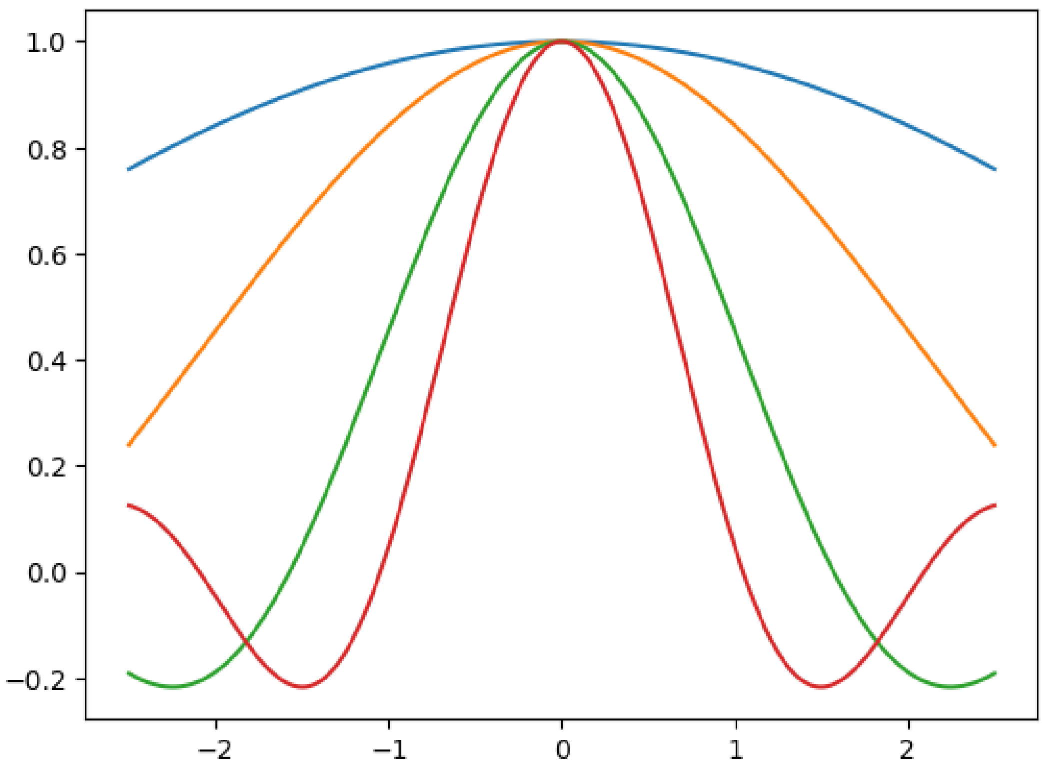
```
f_pure(2.0, 0.1)
# f(np.array([1.5, 2.0]), 0.1)
# doesn't work with vector because of:
# 1. conditional statement
# 2. math.sin does not vectorize
```

```
0.09933466539753061
```

```
# we can use numpy syntax:
I = np.linspace(-2, 2, 10)
Y = np.where(I==0, 1., np.sin(I)/I) # we choose $lambda=1$ here
```

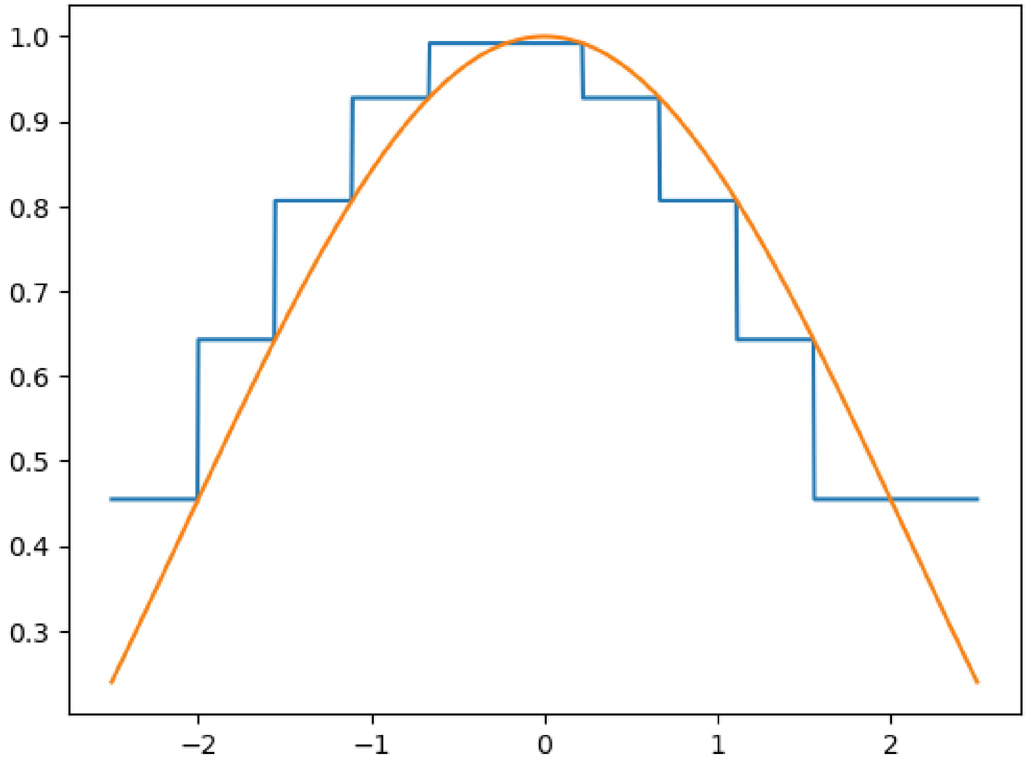
```
def f(x, ):
    return np.where(x==0, 1., np.sin(x* )/( *x))
```

```
T = np.linspace(-2.5,2.5,1000)
plt.plot(T,f(T,0.5))
plt.plot(T,f(T,1))
plt.plot(T,f(T,2))
plt.plot(T,f(T,3))
```

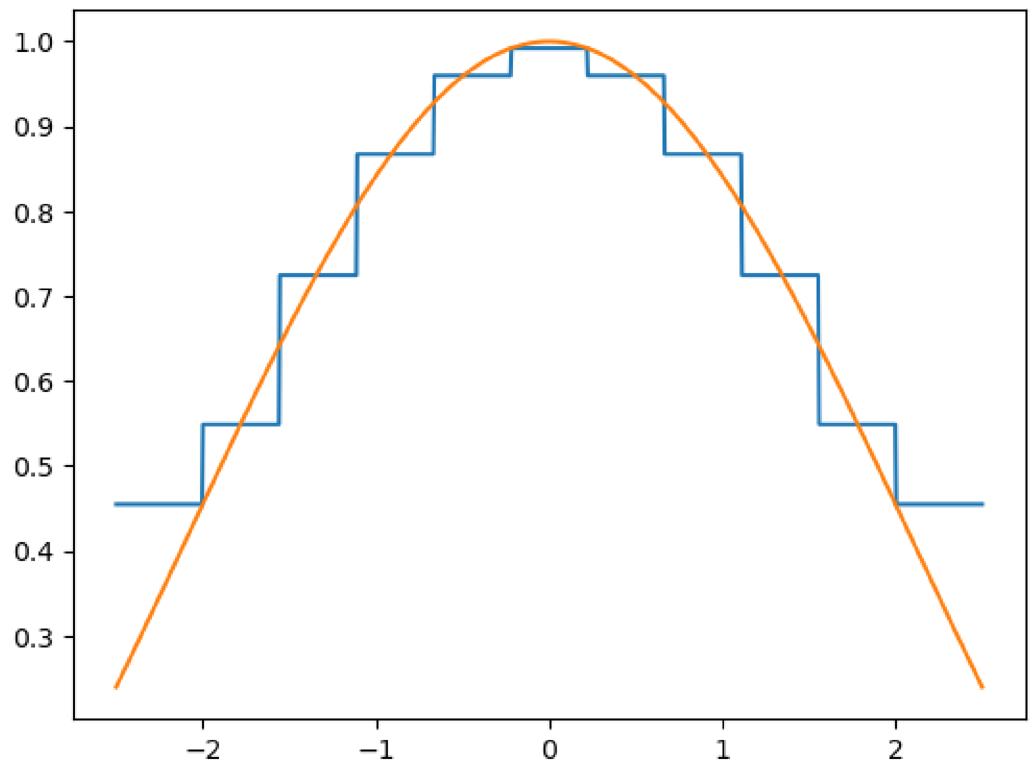


Exercise 3 Construct a stepwise approximation using numpy indexing

```
indices = np.searchsorted(I, T)
stepwise = Y[np.minimum(indices,9)]
plt.plot(T, stepwise,label="step")
plt.plot(T,f(T,1),label="true")
```




```
# We could do a better one by averaging two subsequent values
indices = np.searchsorted(I, T)-1
stepwise = (Y[np.minimum(indices+1,9)] + Y[np.maximum(indices,0)])/2
plt.plot(T, stepwise,label="step")
plt.plot(T,f(T,1),label="true")
```



Exercise 4 Plot it

```
# done above
```

Exercise 5 Construct a linear approximation using numpy

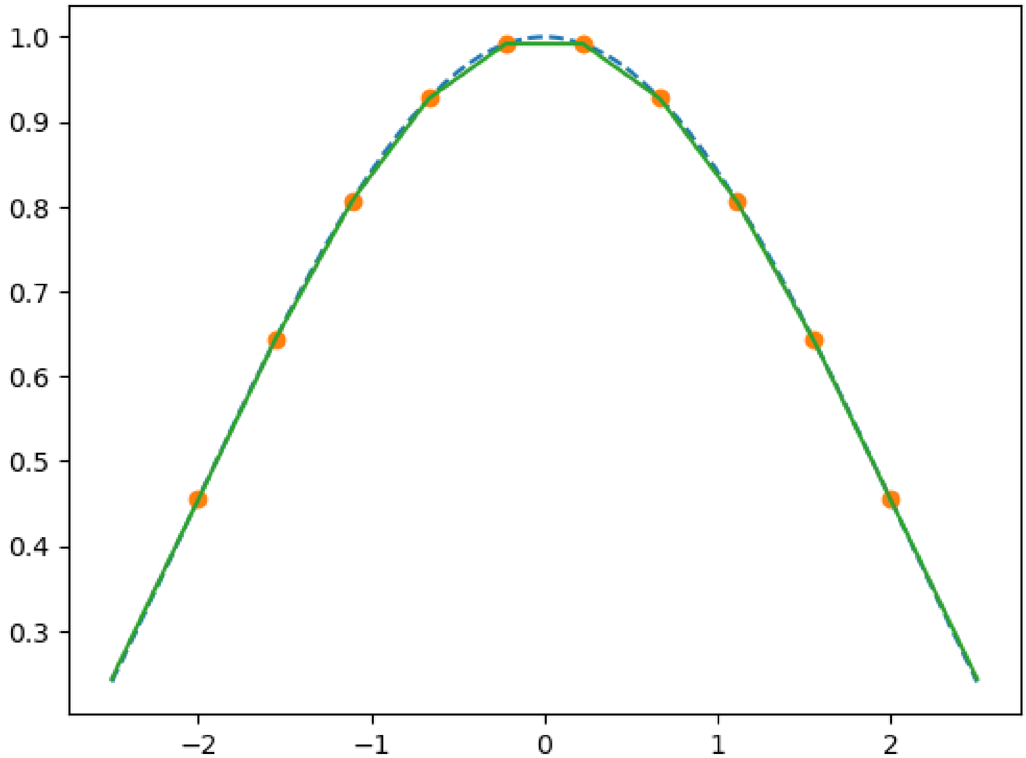
```
# instead of averageing two values, we can use a linear weighting scheme as follows:

indices = np.minimum( np.searchsorted(I, T)-1, 8 )
indices = np.maximum( indices, 0 )
x_i = I[indices]
x_ii = I[indices+1]
y_i = Y[indices]
y_ii = Y[indices+1]

lam = (T-x_i)/(x_ii-x_i)
#
V = y_i + lam*(y_ii-y_i)

# lam = np.arange(0,len(T))/100-indices
```

```
plt.plot(T,f(T,1),label="true", linestyle="--")
plt.plot(I, Y, 'o', label="data")
plt.plot(T, V, label="interpolated")
```



Exercise 6 Use scipy.interpolate to interpolate the data linearly. Compare the various extrapolation options.

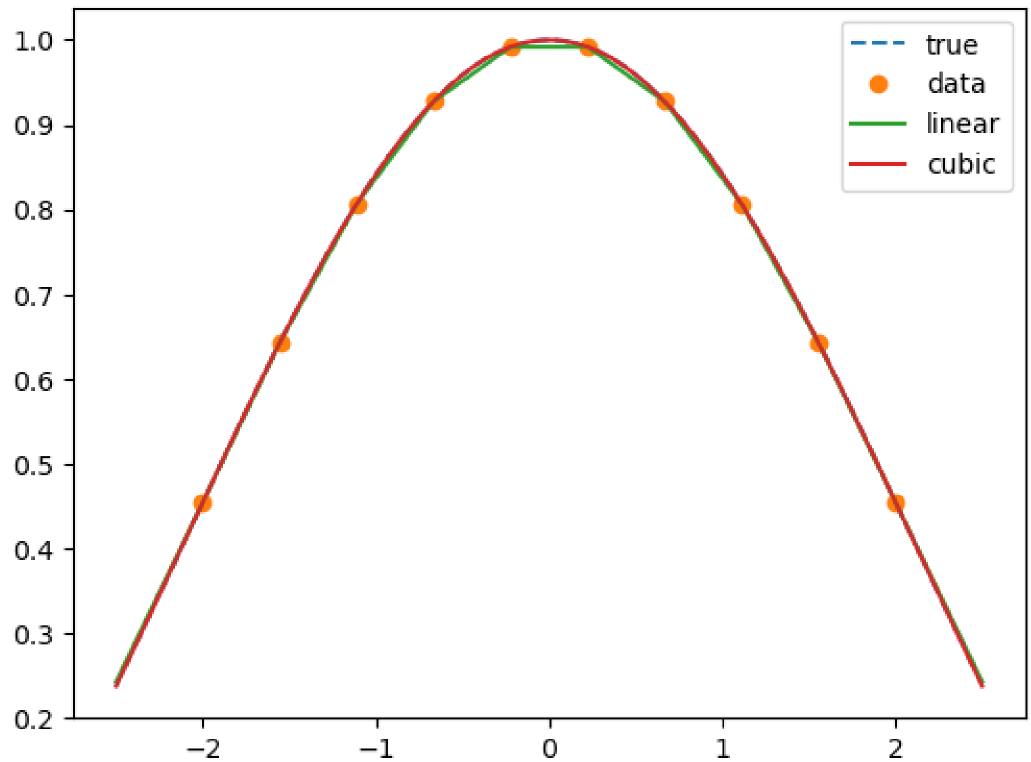
```
import scipy.interpolate
V_lin = scipy.interpolate.interp1d(I,Y, fill_value="extrapolate")(T)
```

Exercise 7 Use scipy.interpolate to interolate the data with cubic splines. Compare the various extrapolation options.

```
import scipy.interpolate
V_cub = scipy.interpolate.interp1d(I,Y, fill_value="extrapolate", kind='cubic')(T)
```

Exercise 8 Plot the results

```
# lam = np.arange(0,len(T))/100-indices
plt.plot(T,f(T,1),label="true", linestyle="--")
plt.plot(I, Y, 'o', label="data")
plt.plot(T, V_lin, label="linear")
plt.plot(T, V_cub, label="cubic")
plt.legend(loc="upper right")
```



Discretization

Exercise 9 Consider the AR1 process $y_t = \rho y_{t-1} + \epsilon_t$ where $\rho = 0.9$ and $\epsilon_t = 0.01$. Use the `quantecon` library to discretize (y_t) as a discrete markov chain.

```
import quantecon as qe
```

```
  = 0.9
_e = 0.01
```

```
# let's create a 5 states markov chain

mc = qe.rouwenhorst(5, , _e)
```

```
/tmp/ipykernel_244104/1860833655.py:3: UserWarning: The API of rouwenhorst has changed from
  mc = qe.rouwenhorst(5, , _e)
```

```
# the result is a markov chain with 5 states:
mc.state_values
```

```
array([-0.04588315, -0.02294157,  0.        ,  0.02294157,  0.04588315])
```

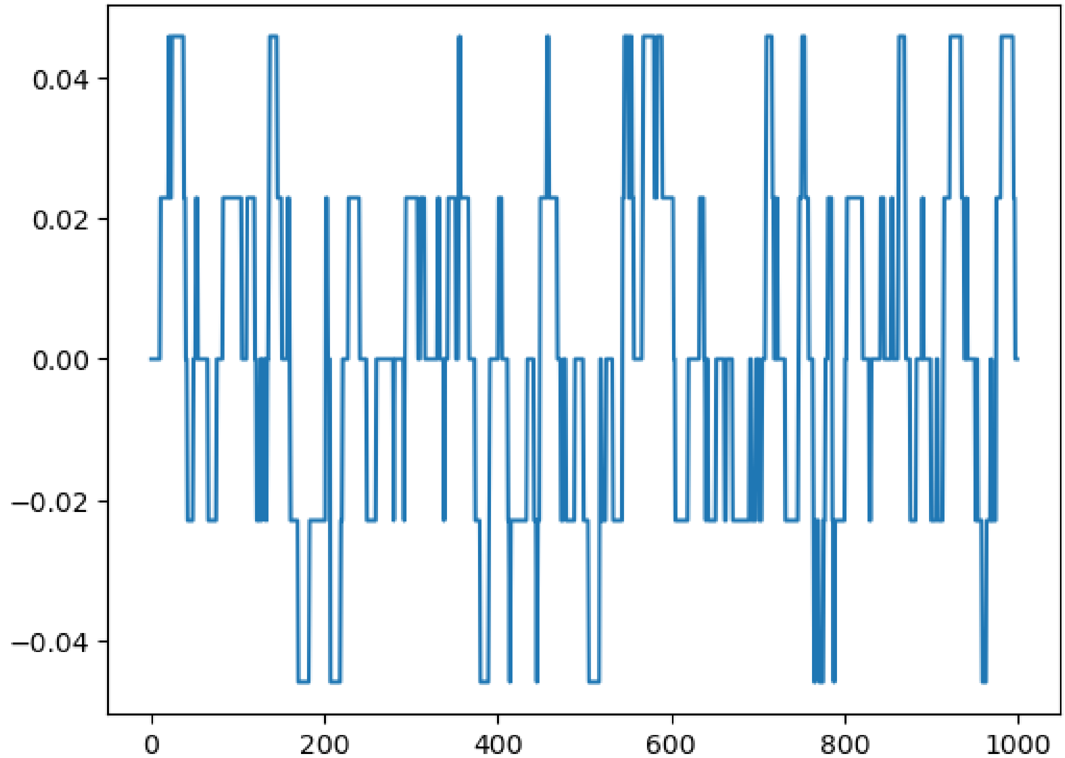
```
# and transition matrix
display(mc.P)
mc.P.sum(axis=1)
# mc.P
```

```
array([[8.1450625e-01, 1.7147500e-01, 1.3537500e-02, 4.7500000e-04,
        6.2500000e-06],
       [4.2868750e-02, 8.2127500e-01, 1.2896250e-01, 6.7750000e-03,
        1.1875000e-04],
       [2.2562500e-03, 8.5975000e-02, 8.2353750e-01, 8.5975000e-02,
        2.2562500e-03],
       [1.1875000e-04, 6.7750000e-03, 1.2896250e-01, 8.2127500e-01,
        4.2868750e-02],
       [6.2500000e-06, 4.7500000e-04, 1.3537500e-02, 1.7147500e-01,
        8.1450625e-01]])
```

```
array([1., 1., 1., 1., 1.])
```

```
# the markov chain object has a few useful methods:
sim = mc.simulate(1000)
```

```
from matplotlib import pyplot as plt
plt.plot(sim)
```




```
# Let's check the resulting standard deviation: it should be  _e/sqrt(1- ^2)
import math
(float(sim.std()),  _e/math.sqrt(1- **2))
```

```
(0.024400345122667085, 0.022941573387056182)
```

Exercise 10 Suppose ϵ follows a normal law with standard deviation $\sigma = 0.05$. Take $= 40$ and define $U(x) = (x^{-\gamma})/(-\gamma)$ We want to compute $C(\epsilon) = \mathbb{E}[U(exp(\epsilon))]$.

- Choose $N > 0$ and construct a 1d vector with N realizations of ϵ. Use it to compute the expectation.
- Estimate the standard deviation of this expectation.
- Use gauss-hermite method from numpy to compute the same expectation.
- Compare both methods.

```
  = 0.05;   = 40

from math import exp
import numpy as np
```

```
from numpy.random import normal
from matplotlib import pyplot as plt
```

```
U = lambda x: (x**(1- ))/(1- )
V = lambda e: U(exp(e))
```

```
def E_ (f, N=100):
    """Compute expectation with Monte-Carlo
    - f function to integrate
    - N number of draws
    """

    gen = (f(normal()* ) for e in range(N)) # generator


    return sum(gen)/N
```

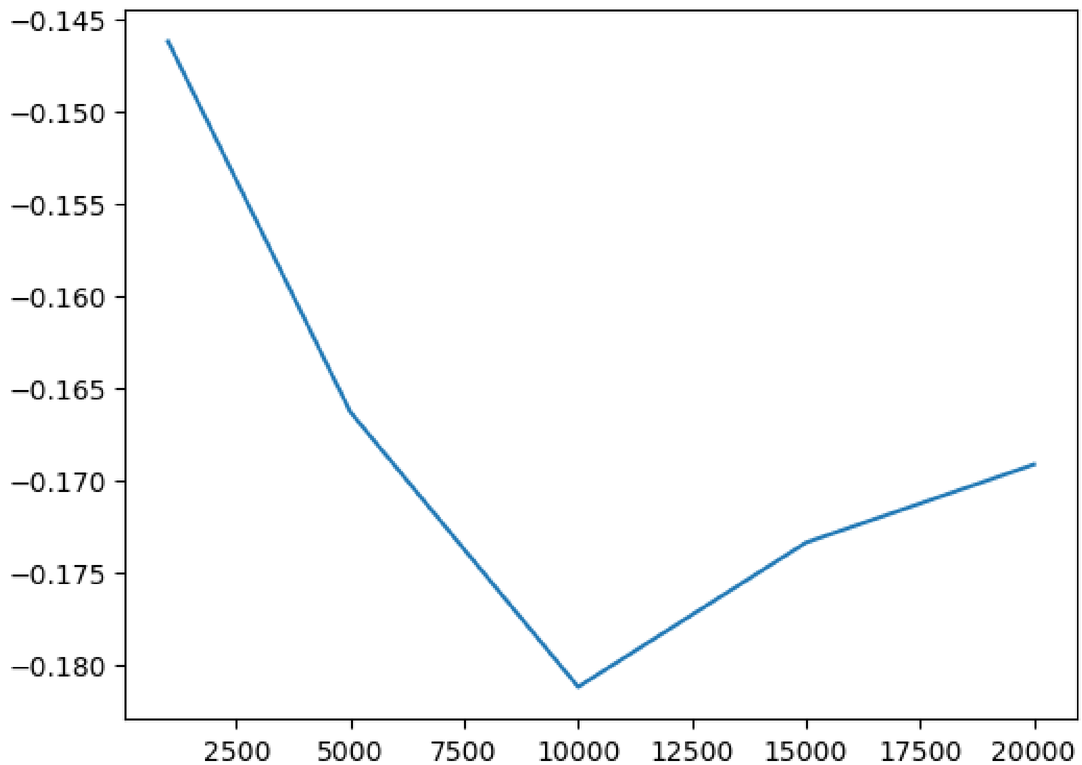
```
E_ (V)
```

```
-0.20717372707127818
```

```
# let's compare results with different number of draws
NVec = [1000, 5000, 10000, 15000, 20000]
vals = [E_ (V, N=i) for i in NVec]
```

```
from matplotlib import pyplot as plt
```

```
plt.plot(NVec, vals)
```



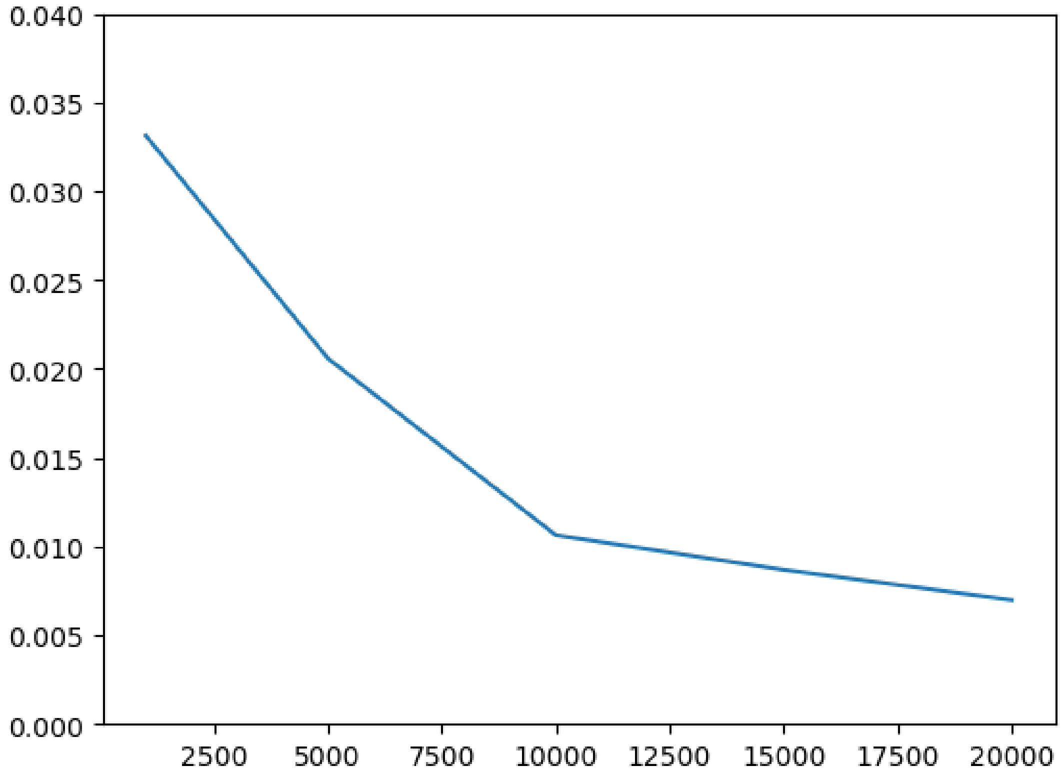

```
E_ (V, N=100)
```

```
-0.1343058368156526
```

```
# We can also use monte-carlo to compute the standard deviation of the estimator
def stdev(f, N=100, K=100):
    gen = (E_ (f,N=N) for k in range(K))
    return np.std([*gen])

sdvals = [stdev(V, N=n, K=100) for n in NVec]
```

```
plt.plot(NVec, sdvals)
plt.ylim(0,0.04)
```




```
from numpy import polynomial
from math import sqrt,pi
```

```
x, w = polynomial.hermite_e.hermegauss(8)
x = x*  # renormalize nodes
s = sum(
    w_*V(x_) for (x_,w_) in zip(x,w)
)/sqrt(pi)/sqrt(2)
# print(s)
```

```
s
```

```
-0.17164225728611746
```

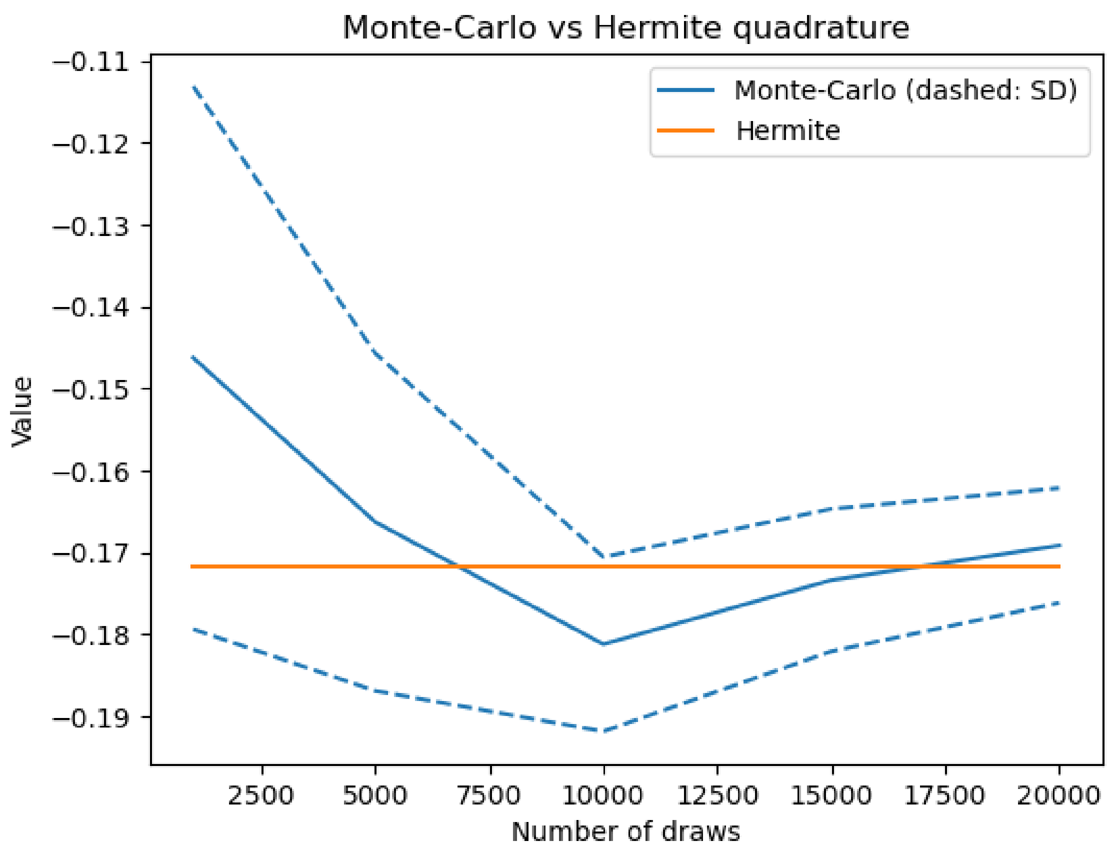
```
# very close to the value we got with monte-carlo
```

```
[-0.1462255238404466,
 -0.16630505411295535,
```

```
-0.18120122126588545,
-0.17337884704032133,
-0.1691531386839286]
```

```python
# let's plot everything together
plt.plot(NVec, vals, label="Monte-Carlo (dashed: SD)", color='C0')
plt.plot(NVec, np.array(vals)+np.array(sdvals), linestyle='--', color='C0')
plt.plot(NVec, np.array(vals)-np.array(sdvals), linestyle='--', color='C0')
plt.plot(NVec, [s]*len(NVec), color='C1', label="Hermite")
plt.xlabel("Number of draws")
plt.ylabel("Value")
plt.legend()
plt.title("Monte-Carlo vs Hermite quadrature")
```

```
Text(0.5, 1.0, 'Monte-Carlo vs Hermite quadrature')
```



```
x
```

```
array([-7.61904854, -6.51059016, -5.57873881, -4.73458133, -3.94396735,
       -3.18901482, -2.45866361, -1.74524732, -1.04294535, -0.34696416,
        0.34696416,  1.04294535,  1.74524732,  2.45866361,  3.18901482,
        3.94396735,  4.73458133,  5.57873881,  6.51059016,  7.61904854])
```