Time Iteration

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Introduction

Introduction

How do we solve models? One needs to choose between several representations.

- bellman representation
- first order conditions
- other iterative schemes

Goal:

- define what is the time-iteration method for first-order models
- explain how to discretize a model and apply it in practice

First order representation of a model

Generic Value Function Representation

- All variables of the model are vectors:
  - states $s \in \mathcal{S} \subset R^{n_s}$
  - controls $x \in \mathcal{F}(\mathcal{S}, R^{n_x})$
    - we assume bounds $a(s) \leq x \leq b(s)$
  - shocks: $\epsilon \sim$ i.i.d. distrib
- Transition:

$$s_{t+1} = g(s_t, x_t, \epsilon_{t+1})$$

- Value function:

$$V(s) = E_0 \sum_{t \geq 0} \beta^t \left[U(s_t, x_t) \right]$$

- Solution is a function $V()$ (value) which is a fixed point of the Bellman-operator:

$$\forall s, V(s) = \max_{a(s) \leq x \leq b(s)} U(s, x) + \beta E \left[V(g(s, x, \epsilon)) \right]$$

- The argmax, defines a decision rule *function*: $x = \varphi(s)$

First order representation

- All variables of the model are vectors:
  - states s_t
  - controls x_t
    - we assume bounds $a(s) \leq x \leq b(s)$
  - shocks: ϵ_t (i.i.d. random law)
- Transition:

$$s_{t+1} = g(s_t, x_t, \epsilon_{t+1})$$

- Decision rule:

$$x_t = \varphi(s_t)$$

- Arbitrage:

$$E_t\left[f(s_t, x_t, s_{t+1}, x_{t+1})\right] = 0 \perp a(s_t) \leq x_t \leq b(s_t)$$

- These equations must be true for any s_t.
- Remark: time subscript are conventional. They are used to precise:
  - when expectation is taken (w.r.t ϵ_{t+1})
  - to avoid repeating $x_t = \varphi(s_t)$ and $x_{t+1} = \varphi(s_t)$
- Sometimes there are bounds on the controls
  - we encode them with complementarity constraints
  - more on it later

Example 1: neoclassical growth model

- capital accumulation:

$$k_t = (1-\delta)k_{t-1} + i_{t-1}$$

- production:

$$y_t = k_t^{\alpha}$$

- consumption:

$$c_t = (1-s_t)y_t$$

$$i_t = s_t y_t$$

- optimality:

$$\beta E_t \left[\frac{U'(c_{t+1})}{U'(c_t)} (1 - \delta + \alpha k_{t+1}^{\alpha-1} \alpha) \right] = 1$$

- states: k_t, with one transition equation
- controls: y_t, c_t, i_t, s_t, with four "arbitrage" equations
  - it is possible but not mandatory to reduce the number of variables/equations by simple susbtitutions

Example 2: consumption-savings model

Simplest consumption/savings model:

- Transition:

$$w_t = \exp(\epsilon_t) + (w_{t-1} - c_{t-1})\overline{r}$$

- Objective:

$$\max_{0 \le c_t \le w_t} E_0 \left[\sum \beta^t U(c_t) \right]$$

- First order conditions:

$$\beta E_t \left[\frac{U'(c_{t+1})}{U'(c_t)} \overline{r} \right] - 1 \perp 0 \le c_t \le w_t$$

- F.O.C. reads as:

$$\beta E_t \left[\frac{U'(c_{t+1})}{U'(c_t)} \overline{r} \right] - 1 \le 0 \perp c_t \le w_t$$

and

$$0 \le \beta E_t \left[\frac{U'(c_{t+1})}{U'(c_t)} \overline{r} \right] - 1 \perp 0 \le c_t$$

- First one reads: *if my marginal utility of consumption today is higher than expected mg. utility of cons. tomorrow, I'd like to consume more, but I can't because, consumption is bounded by income (and no-borrowing constraint).*
- Second one reads: *only way I could tolerate higher utility in the present, than in the future, would be if I want dissave more than I can, or equivalently,*

Example 3: new-keynesian with / without ZLB

Consider the following new keynesian model:

- Assume z_t is an autocorrelated shock:

$$z_t = \rho z_{t-1} + \epsilon_t$$

- New philips curve (PC):

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa y_t$$

- dynamic investment-saving equation (IS):

$$y_t = \beta \mathbb{E}_t y_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t(\pi_{t+1})) - z_t$$

- Interest Rate Setting (taylor rule):

$$i_t = \alpha_\pi \pi_t + \alpha_y y_t$$

- The model satisfies the same specification with:
  - one state z_t and one transition equation
  - three controls: π_t, y_t and i_t with three "arbitrage" equation
- These are not real first order conditions as they are not derived from a maximization program
  - unless one tries to microfound them...
- It is possible to add a zero-lower bound constraint by replacing IRS with:

$$\alpha_\pi \pi_t + \alpha_y y_t \leq i_t \perp 0 \leq i_t$$

Time iteration

Time iteration

- So we have the equation, $\forall s_t$

$$E_t\left[f(s_t, x_t, s_{t+1}, x_{t+1})\right] = 0 \perp a(s_t) \leq x_t \leq b(s_t)$$

$$E_t\left[f(s_t, x_t, s_{t+1}, x_{t+1})\right] = 0$$

where

$$s_{t+1} = g(s_t, x_t, \epsilon_{t+1})$$

$$x_t = \varphi(s_t)$$

$$x_{t+1} = \tilde{\varphi}(s_{t+1})$$

- Let's leave the complementarity conditions aside for now
- In equilibrium $\tilde{\varphi} = \varphi$

Time iteration

- We can rewrite everything as one big functional equation:

$$\forall s, \Phi(\varphi, \tilde{\varphi})(s) = E\left[f(s, \varphi(s), g(s, \varphi(s), \epsilon), \tilde{\varphi}(g(s, \varphi(s), \epsilon))\right]$$

- A solution is φ such that $\Phi(\varphi, \varphi) = 0$
- The Coleman operator $\mathcal{T}$ is defined implicitly by:

$$\Phi(\mathcal{T}(\varphi), \varphi) = 0$$

- The core of the time iteration algorithm, consists in the recursion:

$$\varphi_{n+1} = \mathcal{T}(\varphi_n)$$

- It maps future decision rules to current decision rules
  - same as "linear time iterations", remember?
- Sounds fun but how do we implement it concretely?

Practical implementation

- We need to find a way to:
  1. compute expectations
  2. represent decision rules φ and φ with a finite number of parameters

Practical implementation (2)

- Computing expectations:
  - discretize shock ϵ with finite quantization $(w_i, e_i)_{i=1:K}$
  - replace optimality condition with:

$$\forall s, \Phi(\varphi, \tilde{\varphi})(s) = \sum_i w_i f(s, \varphi(s), g(s, \varphi(s), e_i), \tilde{\varphi}(g(s, \varphi(s), e_i))$$

- ... but we still can't compute all the φ

Approximating decision rules

- We'll limit ourselves to *interpolating* functional spaces
  - We define a finite grid $\mathbf{s} = (s_1, ...s_N)$ to approximate the state space ($\mathbf{s}$ is a finite vector of points)
  - If we know the vector of values $\mathbf{x} = (x_1, ..., x_N)$ a function φ takes on $\mathbf{s}$, we approximate φ at any s using an interpolation scheme $\mathcal{I}$:

$$\varphi(s) \approx \mathcal{I}(s, \mathbf{s}, \mathbf{x})$$

- Now if we replace φ by $\mathcal{I}(s, \mathbf{s}, \mathbf{x})$ and $\tilde{\varphi}$ by $\mathcal{I}(s, \mathbf{s}, \tilde{\mathbf{x}})$ the functional equation becomes:

$$\forall s, \Phi(\varphi, \tilde{\varphi})(s) \approx F(\mathbf{x}, \tilde{\mathbf{x}})(s) = \sum_i w_i f(s, x, \tilde{s}, \tilde{x})$$

where

$$x = \mathcal{I}(s, \mathbf{s}, \mathbf{x})$$
$$\tilde{s} = g(s, x, e_i)$$
$$\tilde{x} = \mathcal{I}(s, \mathbf{s}, \tilde{\mathbf{x}})$$

Pinning down decision rules

- Note that this equation must be statisfied $\forall s$.
- In order to pin-down the N coefficients $\mathbf{x}$, it is enough to satisfy the equations at N different points.
- Hence we solve the square system:

$$\forall i \in [1, N], F(\mathbf{x}, \tilde{\mathbf{x}})(s_i) = 0$$

- In vectorized form, this is just:

$$F(\mathbf{x}, \tilde{\mathbf{x}})(\mathbf{s}) = 0$$

- Or, since grid $\mathbf{s}$ is fixed:

$$F(\mathbf{x}, \tilde{\mathbf{x}}) = 0$$

- Now the vector of decisions today, at each point of the grid, is determined as a function of the vector of decisions tomorrow, on the same grid.

Recap

- Choose a finite grid for states $\mathbf{s} = (s_1, ..., s_N)$
- For a given vector of controls tomorrow $\tilde{\mathbf{x}}$, one can compute theoptimality of a vector of controls today by computing the value of :

$$F(\mathbf{x}, \tilde{\mathbf{x}}) = \sum_i w_i f(\mathbf{s}, \mathbf{x}, \tilde{\mathbf{s}}, \tilde{\mathbf{x}})$$

$$\tilde{\mathbf{s}} = g(\mathbf{s}, \mathbf{x}, e_i)$$

$$\tilde{\mathbf{x}} = \mathcal{J}(\tilde{\mathbf{s}}; \mathbf{s}, \mathbf{x})$$

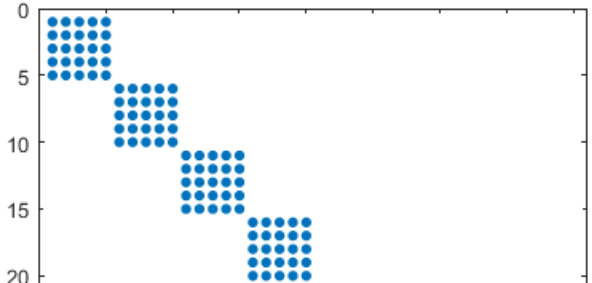
- Note that because we use interpolating approximation: $\forall i, x_i = \mathcal{J}(s, \mathbf{s}, \mathbf{x})$
- We have enough to define an approximated time-iteration operator: implicitly defined by

$$F(T(\mathbf{x}), \mathbf{x}))$$

- We can then implement time-iteration, but...
  - how do we compute $T(x)$?

Computing $T(\mathbf{x})$

- In each step, we have a guess, for decision rule tomorrow $\tilde{\mathbf{x}}$
- We can then find the decision rule today, by solving numerically for: $\mathbf{x} \mapsto F(\mathbf{x}, \tilde{\mathbf{x}})$
  - usually with some variant of a Newton method
- It is possible to solve for the values at each grid point separately...
  - for each i, find optimal controls x_i in state s_i that satisfy $F(x_i, \tilde{\mathbf{x}}) = 0$
  - all the problems are independent from each other
- ...or to solve everything as a big system
  - the jacobian is block-diagonal: finding optimal value in state i or in state j today are two independent problems



Time iteration algorithm

- Discretize state-space with grid $\mathbf{s} = (s_1, ..., s_N)$
- Choose initial values, for the vector of controls on the grid $\mathbf{x} = (x_1, ..., x_N)$
- Specify tolerance levels $\eta > 0$ and $\epsilon > 0$
- Given an intial guess $\mathbf{x_n}$
  - find the zero $\mathbf{x_{n+1}}$ of function $\mathbf{u} \mapsto F(u, \mathbf{x_n})$
    - that is, such that controls on the grid are optimal given controls tomorrow
    - nonlinear solver can use $\mathbf{x_n}$ as initial guess
  - compute norm $\eta_n = |\mathbf{x_n} - \mathbf{x_{n+1}}|$
  - if $\eta_n < \eta$, stop and return $\mathbf{x_{n+1}}$
    - else, set $\mathbf{x_n} \leftarrow \mathbf{x_{n+1}}$ and continue
- Like usual, during the iterations, it is useful to look at $\epsilon_n = |F(\mathbf{x_n}, \mathbf{x_n})|$ and $\lambda_n = \frac{\eta_n}{\eta_{n-1}}$

What about the complementarities ?

- When there aren't any occasionally binding constraint, we look for the of zero $\mathbf{x_{n+1}}$ of function $\mathbf{u} \mapsto F(u, \mathbf{x_n})$.
- If we define the vector of constraints on all grid points as $\mathbf{a} = (a(s_1), ..., a(s_N))$ and $\mathbf{b} = (b(s_1), ..., b(s_N))$, we can rewrite the system to solve as:

$$F(u) \perp \mathbf{a} \leq u \leq \mathbf{b}$$

- Then we can:
  - feed F, a and b to an NCP solver (like nlsolve.jl)
  - or transform this relation using Fisher-Burmeister function into a smooth nonlinear system

Time iteration variants

- You can check out:
  - endogenous grid points:
    - mathematically equivalent to TI,
    - much faster for models that have a particular structure (consumption saving models)
    - no need for a nonlinear solver
  - improved time iterations:
    - same as policy iterations for value function iterations
    - convergence is equivalent to that of TI
    - much faster but requires correct initial guess
  - parameterized expectations
    - requires that all controls are determined as a function of expectations