# Methods

## Computational Economics @ Bundesbank

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# Introduction

# Dynamic

Dynamic programming comes in two flavours:

- Markov Discrete Problems (MDP)
  - states and controls take discrete values
- Approximate Dynamic Programming (ADP)
  - states and controls take continuous values

For ADP, objects of interest (shocks, decision rules) live in infinitely dimensional spaces.

They need be be quantized with a finite set of parameters.

This motivates the study of:

- interpolation (for the decision rule)
- discretization (for the shocks)

We will also need optimization but we will defer it to december.

# Interpolation

# Approximation

Define two continuous sets $X \in R^p$, $Y \in R^q$.

Take a dataset: $(x_i, y_i)_{i \in [1,N]} \in X \times Y$

Take $\tilde{x} \in X \setminus \{x_i\}_{i \in [1,N]}$. What should be the matching $\tilde{y}$ ?

Approximation method:

- Discover implicit relation $y = f^t(x)$ (the model) then compute $\tilde{y} = f^t(\tilde{x})$.

Concretely we choose $f$ from a family $\mathcal{F}$ of functions parameterized by a parameter $\theta$, the approximation family.

- we approximate the true $f^t(x)$ by some $f(x; \theta)$

# Interpolation vs. Regression

- *Interpolation*: $f$ is chosen such that $\forall n, y_n = f(x_n)$
- *Regression*: $f$ is chosen so as to minimize a fitness criterium such as
  - $\min_f \sum_n (y_n - f(x_n))^2$
  - or $\min_\theta \sum_n (y_n - f(x_n;\theta))^2 + \lambda||\theta||^2$ with $\lambda > 0$

- Confusing remarks:
  - sometimes one differentiate interpolation (when $\tilde{x}$) is in the convex hull of $X$ and extrapolation (when $\tilde{x}$ is outside)
  - some applied mathematicians tend interpolation for everything(i.e. interpolate=evaluate f outside of X)

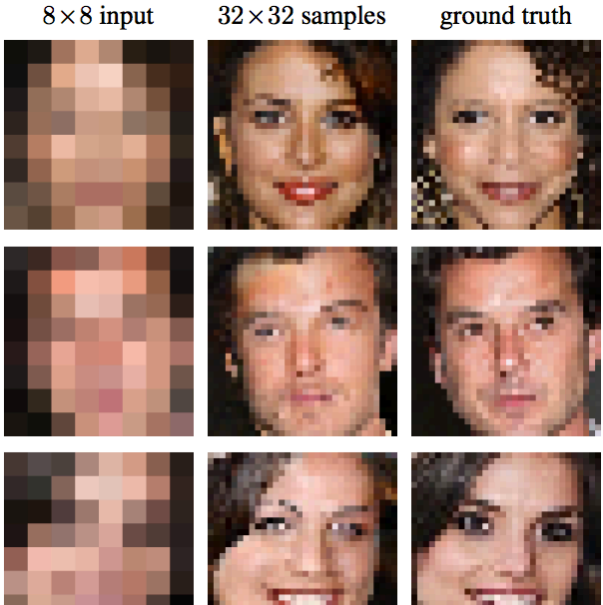
# Examples (1): Linear Interpolation

1d Graph. Join the dots. Linear/Spline

2d Graph: Regression

Conclusion: interpolate only if $f$ is known precisely on $X$

# Example (2)



# Why do we need it?

- In economics, we often solve a problem $\Phi(f) = 0$ where $f$ is a function: $\forall s, \Phi(f)(s) = 0$
- If we approximate $f$ by some element $f(;\theta) \in \mathcal{F}$ we just need to identify a finite set of parameters $\theta \in R^n$
- How do we identify $\theta$?
  - choose a finite set of $n$ criteria that must be met
    - $f$ is pinned down uniquely
    - example: **colocation**, choose $s_1, ..., s_n$. Find $f$ such that $\forall i = 1 : n, \Phi(f)(s_i) = 0$
  - choose higher number of objectives $(p > n)$ that must be minimized:
    - example: **regression**, choose $s_1, ..., s_p$. Find $f$ such that minimize $\sum_i \Phi(f)(s_i)^2 = 0$

# Several interpolation flavours

- local vs spectral:
  - local: functions in $f$ have compact support
  - spectral: noncompact support
- linear vs nonlinear:
  - $\mathcal{F}$ is a vector space: $f(x) \approx \sum_{i=1}^{N} \theta_n b_n(x)$ where $b_n$ is a base of $\mathcal{F}$
  - nonlinear: wavelets, neural networks, ....

# Linear Splines

- Take function $f$ defined on an interval $[a, b]$. Suppose the value is known at $(a = x_1, ...x_N = b)$. Denote $y_i = f(x_i)$.
- Join the dots: define a piecewise linear function as

$$\forall x \in [x_i, x_{i+1}], \tilde{f}(x) = y_i + \underbrace{\frac{x - x_i}{x_{i+1} - x_i}}_{\text{barycentric coordinate}} (y_{i+1} - y_i)$$

# Linear Splines

- Alternate view:

$$\tilde{f}(x) = \sum_{i=1}^{N} y_i B_1^i(x)$$

where $b_1^i(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}} . 1_{x \in [x_{i-1}, x_i]} + (1 - \frac{x - x_i}{x_{i+1} - x_i}) . 1_{x \in [x_i, x_{i+1}]}$

- $(B^i)$ is an interpolation basis

# Splines

- $n$-th order *spline* : piecewise polynomial function that is $n$ times differentiable except on a finite set of break points (aka *knots*), where it is $(n-1)$ times differentiable.
- in practice the data points are the breakpoints
- example: order 2
  - suppose $\tilde{f}(x_i)$ and $\tilde{f}'(x_i)$ are known, choose the coefficients for the patch $p_{i+1}(x) = a_{i+1}x^2 + b_{i+1}x + c_{i+1}$
  - Already two constraints. Condition $p_{i+1}(x_{i+1}) = \tilde{f}(x_{i+1})$ supplies another one.
  - Do it for every patch. Note that it requires to set $f'(a)$ beforehand.

# Basis Splines (much better)

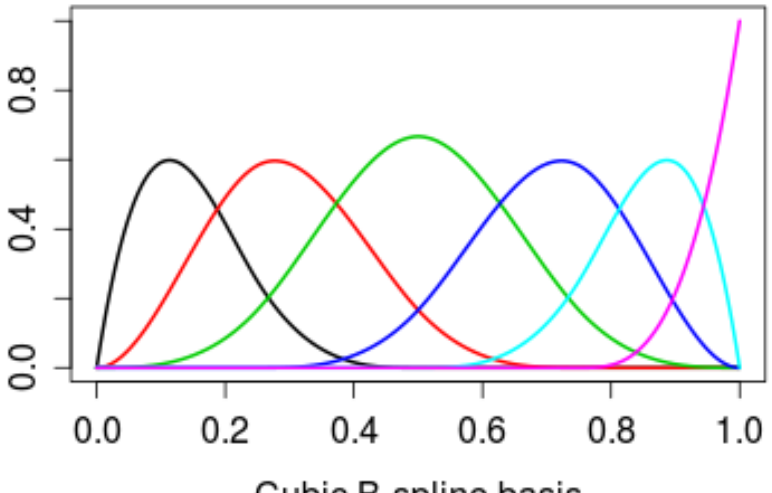
- Define

$$B_{i,1}(x) = 1_{x \in [x_i, x_{i+1}]}$$

$$B_{i,k+1}(x) = \frac{x - x_i}{x_{i+k} - x_i} B_{i,k}(x) + \frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}} B_{i+1,k}(x)$$

- Properties:
  - All basis splines have compact support.
  - If grid is regularly spaced there is $B_k$ such that $B_{i,k}(x) = B_k(x - x_i)$
- *Theorem* (de Boor): Any spline of order $k$ on the knots $(x_i)$ can be expressed as a linear combination of the basis splines $(B_{i,k})$.

# Basis splines



# Basis splines are not interpolating

- Unfortunately basis splines are not "interpolating" in the sense that in general

$$f(x_i) \neq \sum_n f(x_n) B_{n,k}(x_i)$$

- One must choose other coefficients $(c_n)$ which satisfy:

$$y_i = \sum_n c_n B_{n,k}(x_i)$$

  - there are more coefficients than data points:
    - requires boundary conditions
    - $f'' = 0$: natural spline
  - going from $y_n$ to $c_n$ is called *prefiltering*

# In practice: Interpolations

```
import numpy as np
from scipy.interpolate import RegularGridInterpolator
f = lambda x: np.log(x)
xs = np.linspace(1, 5, 10)
A = f(xs)

# linear interpolation
interp_linear = RegularGridInterpolator((xs,), A)
interp_linear([1.3]) # interpolate

# cubic spline interpolation
interp_cubic = RegularGridInterpolator((xs,), A, method="cubic")
interp_cubic([1.3]) # interpolate
```

# Mental break: matrix conditioning

- Suppose you want to solve vector equation $Ax = y$. Will a small error in $y$ affect a lot the value of $x$? (in particular round-off errors)
  - condition number: $\lim_{\epsilon \to 0} \sup_{\delta y \leq \epsilon} \frac{\delta x}{\delta y}$
  - or $\kappa(A) = ||A^{-1}||||A||$ where $||.||$ is a subordonate norm.
  - if very-large: the matrix is ill conditioned
- What makes a matrix ill-conditioned?
  - some rows/columns are very small, others are gigantic
  - rows/columns are almost colinear

# Polynomial approximation

# Fitting polynomials

- Let's approximate: $f(;\theta) = \sum_{n=0}^{K} \theta_k x^k$.
- We need $(K+1)$ points to fit a polynomial of order $K$. Let's take grid points $(x_0, ...x_K)$ and denote $y_k = f(x_k)$
- We need to solve in $(\theta_k)_{k=[0,K]}$:

$$\forall n \in [0, K], \underbrace{\sum_k \theta_k (x_n)^k}_{M\theta} = y_k$$

# Vandermonde Matrix

- $M$ has a special structure, a Vandermode matrix:

$$M = \begin{bmatrix} 1 & x_0 & x_0^2 \cdots & x_0^K \\ 1 & x_1 & x_1^2 \cdots & x_1^K \\ 1 & x_2 & x_2^2 \cdots & x_2^K \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_K & x_K^2 \cdots & x_K^K \end{bmatrix}$$

- Vandermonde matrix is ill-conditioned if points are too close or if $K$ is high.

# Orthogonal polynomials

- Define a scalar product over functions on the domain $[a, b]$ by choosing a positive weight function $w(x)$.

$$< P, Q >= \int_a^b w(x)P(x)Q(x)dx$$

- Construct an orthogonal base $(T_n)_{n=[1,K]}$.
- Approximate

$$f(x) \approx f(x;\theta) = \sum_{n=0}^{K} \theta_n T_n(x) = \sum_{n=0}^{K} < f|T_n > T_n(x)$$

  - this is optimal for the norm associated to $<>$ (projection on the orthogonal base)

# Vandermonde matrix

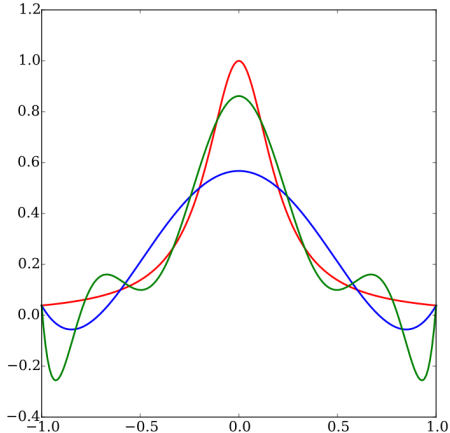
Coefficients can still be identified by inverting:

$$\forall n \in [0, K] \underbrace{\sum_k \theta_k T_k(x_n)}_{M\theta} = y_n$$

$$M = \begin{bmatrix} T_0(x_0) & T_1(x_0) & \cdots & T_K(x_0) \\ T_0(x_1) & T_1(x_1) & \cdots & T_K(x_1) \\ T_0(x_2) & T_1(x_2) & \cdots & T_K(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_K) & T_1(x_K) & \cdots & T_K(x_K) \end{bmatrix}$$

# Problem: Runge error

- Red: Runge function $f(x) = \frac{1}{1+25x^2}$
- Blue: interpolates at 6, regularly-spaced, points
- Green: interpolates at 10, regularly-spaced, points
- What happens when interpolation order increases?
  - oscillations increase.
- Does it contradict Stone-Weierstrass theorem ? No.
- Solutions:
  1. use regression method instead
  2. choose the interpolation points wisely



# Chebychev Nodes

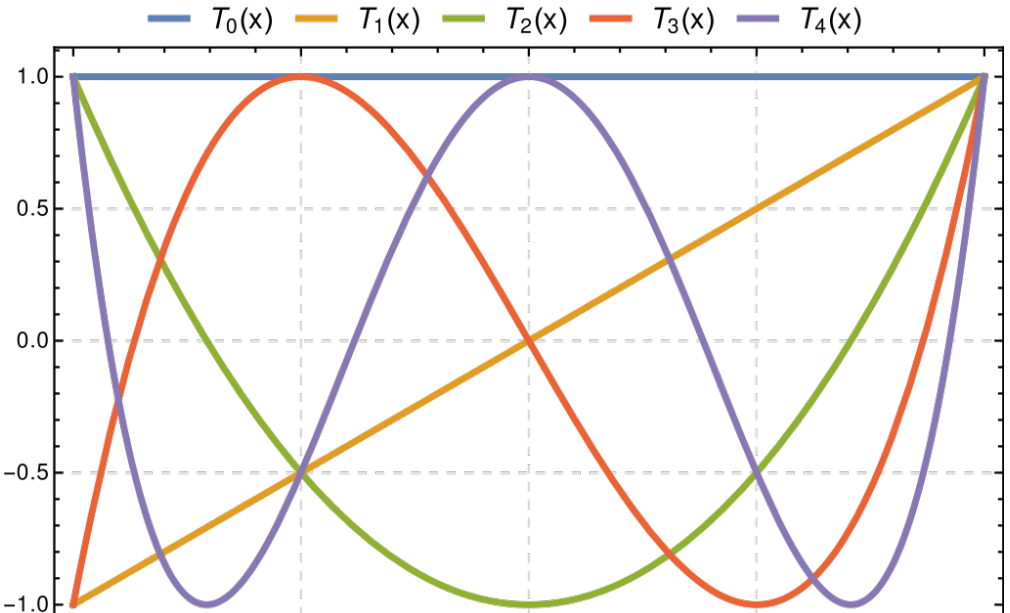
- There is an optimal way to choose the interpolation points:
  - the roots of $cos(\frac{2k-1}{2n}\pi)$ for [-1,1]
  - rescale for a finite interval [a,b]
- for the interpolating polynomial:

$$|f(x) - P_n(x)| \leq \frac{1}{2^n(n+1)!} \max_{\xi \in [-1,1]} |f^n(\xi)|$$

# Chebychev polynomials

- Chebychev polynomials (of the first kind) have their zeros on the nodes.
- Definitions:
  - $T_n(x) = \cos(n \arccos(x))$ (in [0,1])
  - recursive: $T_0(x) = 1$, $T_1(x) = x$, $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$
- Very good choice:
  - matrix $M$ is well conditioned: $\sqrt{2}$

# Chebychev Polynomial



# Multidimensional interpolation

- Consider a function $f$ defined on a space $X_1 \times X_d$
- Take $d$ grids $\mathcal{G}_1 \subset X_1, ..., \mathcal{G}_d \subset X_d$ with linear approximation bases $\mathcal{B}_1 = (b_1^1, ...b_1^{N_1}), ..., \mathcal{B}_d = (b_d^1, ...b_d^{N_d})$.
- Then $f$ can be approximated by
  $f(x_1, ...x_d; \theta) = \sum_{i_1=1}^{N_1} ... \sum_{i_d=1}^{N_d} \theta_{i_1,...i_d} \underbrace{b_{i_1}^1(x_1)...b_{i_d}^d(x_d)}_{\text{Product Base}}$
- Morality:
  - linear appoximation along each dimension induces a natural (multi)-linear in many dimensions
  - Coefficients are still the solution of a linear system:

$$M\theta = y$$

  - but $M$ has a special structure (tensor product)
- Problem: number of coefficients to determine increases exponentially with number of dimensions:
  - "Curse of Dimensionality"

# Multidimensional interpolation (2)

- Ways to mitigate the curse of dimensionality
- Remedies:
  - sparse grids
  - adaptive approximation
    - delaunay tessellation
    - adaptive sparse grid
  - neural networks
  - ...
- No black-magic theorem: *there is no solution to the curse of dimensionality*
  - .. but there are methods to adapt to problem whose intrinsic dimension is smaller than the actual number of variables

# Delaunay and sparse grid

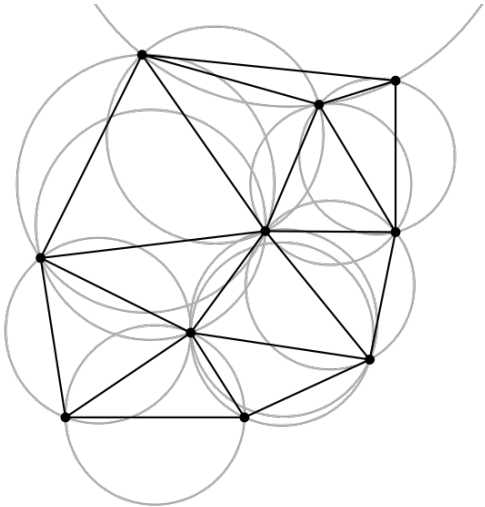


Figure 2: Delaunay Tessellation

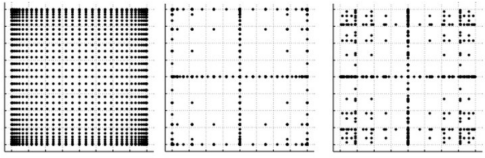



Figure 3: Sparse Grid

# Python libraries

- `scipy.interpolation` many common methods
- `interpolation.py`
  - linear and cubic splines
  - jitted with numba
  - also complete and smolyak polynomials

# Discretization

# Several kinds of Discretization

- approximate operator with a finite number of iterations:
  - compute $\int_a^b f(x)dx$
  - compute $E_\omega f(\omega)$
- represent an infinite dimensional object with a finite set of parameters:
  - $f \equiv (f(x_i))_{i=1:N}$ with $x_i = a + \frac{i-1}{N-1}(b-a)$
    - discretize arguments
  - $\omega \equiv (\mu_i, \omega_i)_{i=1:N}$ such that $E_\omega f(\omega) \approx \sum_i \mu_i f(\omega_i)$ (quantization)
- discretize continous process by a discrete one:
  - continuous markov chain to discrete markov Chain

# Discretizing an AR1

- Take $AR1$ process

$$x_t = \rho x_{t-1} + \epsilon_t$$

  - with $|\rho| < 1$ and $\epsilon \sim N(0, \sigma)$
- Can we replace $(x_t)$ by a discrete markov chain?
  - approximate version:
    - good time $x^g$ and bad time $x^b$. Probability $\pi$ of staying in the same, $1 - \pi$ of switching.
  - two systematic methods (available in *QuantEcon.py*)
    - Tauchen
    - Rouwenhorst

# AR1: Tauchen

- The unconditional distribution of an AR1 is a normal law $\mathcal{N}(0, \frac{\sigma}{\sqrt{1-\rho^2}})$
- Choose $m > 0$, typically $m = 3$
- Bound the process: $\underline{x} = -m\frac{\sigma}{\sqrt{1-\rho^2}}$ and $\overline{x} = m\frac{\sigma}{\sqrt{1-\rho^2}}$
- Define the $N$ discretized points ($i \in [1, n]$): $y_i = \underline{x} + \frac{i-1}{N-1}(\overline{x} - \underline{x})$
- Define the transitions:

$$\pi_{ij} = prob\left(y_{t+1} = y_j | y_t = y_i\right)$$

$$\pi_{ij} = prob\left(|y_{t+1} - x_j| = \inf_k |y_{t+1} - x_k| \, | y_t = y_i\right)$$

# AR1: Tauchen (2)

- Formulas $\delta = \frac{\overline{x}-\underline{x}}{N}$:
  - if $1 < k < N-1$

$$\pi_{jk} = F(\frac{y_k + \delta/2 - \rho y_j}{\sigma_\epsilon}) - F(y_k + \delta/2 - \rho y_j)$$

  - if $k = 1$

$$\pi_j = F(\frac{y_k + \delta/2 - \rho y_j}{\sigma_\epsilon})$$

  - if $k = N$

$$\pi_j = 1 - F(\frac{y_k - \delta/2 - \rho y_j}{\sigma_\epsilon})$$

# How to assess the quality of approximation ?

- compare generated stationary moments between discretized process and true AR1:
  - E(), Var(), ACor()
- by looking at the exact ergodic distribution or by doing some simulations
- not very precise when the process is very persistent $\rho \approx 1$

# Rouvenhorst method (1)

- N = 2
  - choose $y_1 = -\psi$, $y_2 = \psi$
  - define transition matrix:

$$\Theta_2 = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$$

  - choose $p$, $q$ and $\psi$ to match some moments: $E()$, $Var()$, $ACor()$
    - they can be computed analytically for AR1 and for discretized version.

# Rouvenhorst method (2)

- N >2

$$\Theta_N = p \begin{bmatrix} \Theta_{N-1} & 0 \\ 0 & 0 \end{bmatrix} + (1-p) \begin{bmatrix} 0 & \Theta_{N-1} \\ 0 & 0 \end{bmatrix} + (1-q) \begin{bmatrix} 0 & 0 \\ \Theta_{N-1} & 0 \end{bmatrix} + q \begin{bmatrix} 0 & 0 \\ 0 & \Theta_{N-1} \end{bmatrix}$$

- Normalize all lines

# Rouvenhorst method (3)

- Procedure converges to Bernouilli distribution.
- Moments can be computed in closed form:
  - $E() = \frac{(q-p)\psi}{2-(p+q)}$
  - $Var() = \psi^2 \left[1 - 4s(1-s) + \frac{4s(1-s)}{N-1}\right]$
  - $Acor() = p + q - 1$
- Rouwenhorst method performs better for highly correlated processes

# Discretizing an iid law

- Given $f$, and an iid process $\epsilon \sim N(0, \sigma^2)$, how to approximate $E_\epsilon f(\epsilon)$ ?
- Ideas:
  - draw *lots* of random $(\epsilon\_n)\_n = 1 : N$ and compute

$$\frac{1}{N} \sum_{n=1}^{N} f(\epsilon_n)$$

    - aka Monte-Carlo simulations
  - given a method to approximate integrals, compute

$$\int_{u=-\infty}^{\infty} f(u)\mu(u)du$$

    with $\mu(u) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}}$
  - *discretize* (or *quantize*) the signal $\epsilon$ as $(w_i, \epsilon_i)_{i=1:N}$ and compute:

$$\frac{1}{N} \sum_{n} w_n f(\epsilon_n)$$

# What's wrong with Monte-Carlo Simulations?

- Let's take an exemple:
  - consumption is $C(\epsilon) = U(e^{\epsilon})$
  - with $\sigma\_\epsilon = 0.05$ and $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ and $\gamma = 40$.
- Let's compute $E_{\epsilon}(C(\epsilon))$ precisely.
- Discuss value of $\gamma$: is it crazy? (risk return)

# What's wrong with Monte-Carlo Simulations?

Compute expectation

```
  = 0.05;   = 40

from math import exp
import numpy as np
from numpy.random import normal
from matplotlib import pyplot as plt

U = lambda x: (x**(- ))/(- )
C = lambda e: U(exp(e))

def E_ (f, N=100):

    gen = (f(normal()* ) for e in range(N))
    return sum(gen)/N

NVec = [1000, 5000, 10000, 15000, 20000]
```

# What's wrong with Monte-Carlo Simulations?

```
def stdev(f, N=100, K=100):
    gen = (E_ (f,N=N) for k in range(K))
    return np.std([*gen])

sdvals = [stdev(C, N=n, K=1000) for n in NVec]
```

# Quick theory (1)

- Fact: the sum of several independent gaussian variables is a gaussian variable
- So $T_N = \frac{1}{N}\sum_{n=1}^{N} \epsilon_n$ is gaussian variable. Its mean is 0 (unbiased). Let's compute its variance:

$$E(T_N^2) = \frac{1}{N^2}\sum_{n=1}^{N} E\left[\epsilon_n^2\right]$$

- The standard deviation is:

$$s_N = \sigma(T_N^2) = \frac{1}{\sqrt{N}}\sigma_\epsilon$$

- Conclusion: the precision of (basic) Monte-Carlo decreases only as a square root of the number of experiments.

# Quick theory (2)

- In the general case, the Monte-Carlo estimator is:

$$T_N^{MC} = \frac{1}{N} \sum_{n=1}^{N} f(\epsilon_n)$$

- It is unbiased:

$$E(T_N^{MC}) = E\left[f(\epsilon)\right]$$

- It's variance is

$$E(T_N^{MC}) \propto \frac{1}{\sqrt{N}}$$

  - slow
  - on the plus side: rate independent of the dimension of $\epsilon$

# Quantization using quantiles

- *Equiprobable* discretization
- Works for any distribution with pdf and cdf
- Split the space into equal $N$ quantiles:

$$(I_i = [a_i, a_{i+1}])_{i=1:N}$$

such that

$$prob(\epsilon \in I_i) = \frac{1}{N}$$

- Choose the *nodes* as the median of each interval:

$$prob(\epsilon \in [a_i, x_i]) = prob(\epsilon \in [x_i, a_{i+1}])$$

- The quantization is $(1/N, x_i)_{i=1:N}$

# Quadrature rule

Idea:

- $f \in \mathcal{F}$ a Banach space
  - $I : f \to E_\epsilon f(\epsilon)$ is a linear application
- suppose there is a dense family of polynomials $P_n$, spanning $\mathcal{F}_n$
  - $I$ restricted to $\mathcal{F}_N$ is a $N$-dimensional linear form
- take $N$ points $(a_n)_{n\in[1,N]}$. The set $\{f \to \sum_{n=1}^{N} w_n f(a_n) | w_1, ...w_N\}$ is a vectorial space.
  - one element matches exactly $I|_{\mathcal{F}}$
- so the quadrature rule $(w_n, a_n)$ is exactly accurate for polynomials of order $n < N$.
  - how to choose the points $a_n$?

# Gauss-Hermite

- $f \in \mathcal{F}$ a Banach space (or $\mathbb{R}^n$), $\epsilon$ a gaussian variable
  - $I : f \to E_\epsilon f(\epsilon)$ is a linear application
- suppose there is a dense family of polynomials $P_n$, spanning $\mathcal{F}_n$
  - $I$ restricted to $\mathcal{F}_N$ is a $N$-dimensional linear form
- Gauss quadrature magic
  - a way to choose $\epsilon_i$ **and** $w_i$ such that

$$\left( f \to \sum_{n=1}^{N} w_n f(\epsilon_i) \right) = I|_{\mathcal{F}_{2N}}$$

# Gauss-Hermite

- Very accurate if a function can be approximated by polynomials
- Bad:
  - imprecise if function $f$ has kinks or non local behaviour
    - points $\epsilon_n$ can be very far from the origin
  - not super easy to compute weights and nodes (but there are good libraries)

# Gauss-*

- Same logic can be applied to compute integration with weight function $w(x)$:

$$\int_a^b f(x)w(x)$$

- Gauss-Hermite:
  - $w(x) = \frac{e^{-x^2}}{2}$, $[a, b] = [-\infty, \infty]$
- Gauss-Legendre:
  - $w(x) = 1$
- Gauss-Chebychev:
  - $w(x) = \sqrt{1 - x^2}$, $[a, b] = [-1, 1]$
  - for periodic functions

# In practice

Beware that weight is not the density of the normal law:

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int f(x)e^{-\frac{x^2}{2\sigma^2}}dx = \frac{1}{\sqrt{2\pi}}\int f(u\sigma)e^{-\frac{u^2}{2}}du$$

$$\frac{1}{\sqrt{2\pi}}\sum_n w_n f(\epsilon_n\sigma)$$

```
x, w = polynomial.hermite_e.hermegauss(8)
x = x*  # renormalize nodes
s = sum( w_*U(exp(x_)) for (x_,w_) in zip(x,w))/sqrt(pi)/sqrt(2)
print(s)
```