Solving DSGE models

Macro II - Fluctuations - ENSAE, 2023-2024

Pablo Winant

2024-03-20

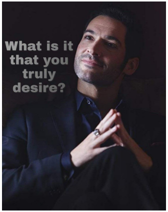
Plan for the rest of the course

- session 4:
  - DSGE: solution algorithms
- session 5:
  - open economy
- session 6:
  - heterogeneity
- session 7:
  - climate change

Introduction

What is the main specificity of economic modeling?
In (macro)economics, we *model* the behaviour of economic agents by specifying:

- their objective $\max_{c_t} \beta^t U(c_t)$, $\max \pi_t$, ...
- their constraints (budget constraint, econ. environment...)



This has important implications:

- macro models are *forward looking*
- macro models need to be **solved**

Dynare

- ~2000 Michel Juillard created a specialized software to solve DSGE models
  - DSGE: Dynamic Stochastic General Equilibrium
- It has been widely adopted:
  - early version in Gauss
  - then Matlab/Octave/Scilab
  - latest version in Julia

Figure 1: Michel Juillard

DSGE Models in institutions

- Nowadays most DSGE models built in institutions have a Dynare version (IMF/GIMF, EC/Quest, ECB/, NYFed/FRBNY)
  - they are usually based on the *midsize model* from Smets & Wouters (10 equations)
  - but have grown up a lot (»100 equations)
- Institutions are (slowly) diversifying their model
  - CGEs
  - Agent-based
  - Semi-structural models (again)

Solving a model

Model

A very concise representation of a model

$$\mathbb{E}_t \left[f(y_{t+1}, y_t, y_{t-1}, \epsilon_t) \right] = 0$$

The ingredients:

- $y_t \in \mathbb{R}^n$: the vector of endogenous variables
- $\epsilon_t \in \mathbb{R}^{n_e}$: the vector of exogenous variables
  - we assume that ϵ_t is zero-mean gaussian process
- $f : \mathbb{R}^n \to \mathbb{R}^n$: the model equations

The solution: g such that

$$\forall t, y_t = g(y_{t-1}, \epsilon_t)$$

The timing of the equations

> **Tip**
>
> In dynare the model equations are coded in the `model; ... ; end;` block.

New information arrives with the innovations ϵ_t.

At date t, the information set is spanned by $\mathcal{F}_t = \mathcal{F}(\cdots, \epsilon_{t-3}, \epsilon_{t-2}, \epsilon_{t-1}, \epsilon_t)$

By convention *an endogenous variable has a subscript t if it is known first at date t*.

Example

The timing of equations

Using Dynare's timing conventions:

- Write the production function in the RBC
- Write the law of motion for capital k, with a depreciation rate δ and investment i
  - when is capital known?
  - when is investment known?
- Add a multiplicative investment efficiency shock χ_t. Assume it is an $AR1$ driven by innovation η_t and autocorrelation ρ_χ

Example: correction

Steady-state

The deterministic steady-state satisfies:

$$f(\overline{y}, \overline{y}, \overline{y}, 0) = 0$$

Often, there is a closed-form solution.

Otherwise, one must resort to a numerical solver to solve $\overline{y} \rightarrow f(f(\overline{y}, \overline{y}, \overline{y}, 0))$

> **Tip**
>
> In dynare the steady-state values are provided in the `steadystate_model; ... ; end;` block. One can check they are correct using the `check;` statement.
> To find numerically the steady-state: `steady;`.

The implicit system

Replacing the solution

$$y_t = g(y_{t-1}, \epsilon_t)$$

in the system

$$\mathbb{E}_t \left[f(y_{t+1}, y_t, y_{t-1}, \epsilon_t) \right] = 0$$

we obtain:

$$\mathbb{E}_t \left[f(g(g(y_{t-1}, \epsilon_t), \epsilon_{t+1}), g(y_{t-1}, \epsilon_t), y_{t-1}, \epsilon_t) \right] = 0$$

It is an equation defining implicitly the function $g()$

The state-space

$$\mathbb{E}_t \left[f(g(g(y_{t-1}, \epsilon_t), \epsilon_{t+1}), g(y_{t-1}, \epsilon_t), y_{t-1}, \epsilon_t) \right] = 0$$

In this expression, y_{t-1}, ϵ_t is the state-space.

The state-space

$$\mathbb{E}_t\left[f(g(g(y_{t-1},\epsilon_t),\epsilon_{t+1}),g(y_{t-1},\epsilon_t),y_{t-1},\epsilon_t)\right]=0$$

In this expression, y_{t-1},ϵ_t is the state-space.

Dropping the time subscripts, the equation must be satisfied for any realization of (y,ϵ)

$$\forall(y,\epsilon)\ \Phi(g)(y,\epsilon)=\mathbb{E}_{\epsilon'}\left[f(g(g(y,\epsilon),\epsilon'),g(y,\epsilon),y,\epsilon)\right]=0$$

It is a functional equation $\Phi(g)=0$

Expected shocks

Assume $|y_t - \overline{y}| << 1$,$|\epsilon| << 1$,$|\epsilon'| << 1$

First we can perform a Taylor expansion with respect to future shock:

$$
\begin{aligned}
& \mathbb{E}_{\epsilon'}\left[f(g(g(y,\epsilon),\epsilon'),g(y,\epsilon),y,\epsilon)\right] && (1)\\
= \quad & \mathbb{E}_{\epsilon'}\left[f(g(g(y,\epsilon),0),g(y,\epsilon),y,\epsilon)\right] && (2)\\
& +\mathbb{E}_{\epsilon'}\left[f'_{y_{t+1}}(g(g(y,\epsilon),0),g(y,\epsilon),y,\epsilon)g'_{\epsilon}\epsilon'\right] + o(\epsilon') && (3)\\
\approx \quad & f(g(g(y,\epsilon),0),g(y,\epsilon),y,\epsilon) && (4)
\end{aligned}
$$

Expected shocks

Assume $|y_t - \overline{y}| << 1$,$|\epsilon| << 1$,$|\epsilon'| << 1$

First we can perform a Taylor expansion with respect to future shock:

$$
\begin{aligned}
& \mathbb{E}_{\epsilon'}\left[f(g(g(y,\epsilon),\epsilon'),g(y,\epsilon),y,\epsilon)\right] && (1)\\
= \quad & \mathbb{E}_{\epsilon'}\left[f(g(g(y,\epsilon),0),g(y,\epsilon),y,\epsilon)\right] && (2)\\
& +\mathbb{E}_{\epsilon'}\left[f'_{y_{t+1}}(g(g(y,\epsilon),0),g(y,\epsilon),y,\epsilon)g'_{\epsilon}\epsilon'\right] + o(\epsilon') && (3)\\
\approx \quad & f(g(g(y,\epsilon),0),g(y,\epsilon),y,\epsilon) && (4)
\end{aligned}
$$

This uses the fact that $\mathbb{E}\left[\epsilon'\right] = 0$. At first order, expected shocks play no role.

First order perturbation

We are left with the system:

$$F(y, \epsilon) = f(g(g(y, 0), \epsilon), g(y, \epsilon), y, \epsilon) = 0$$

We can now use a variant of the implicit function theorem to recover a first approximation of g as:

$$g(y, \epsilon) = \overline{y} + g'_y(y - \overline{y}) + g'_e \epsilon_t$$

We can obtain the unknown quantities g'_y, and g'_e using the methods of undeterminate coefficients.

Use these to write $F'_y(\overline{y}, 0) = 0$ and $F'_\epsilon(\overline{y}, 0) = 0$.

The transition matrix

Recall the system:

$$F(y, \epsilon) = f(g(g(y, 0), \epsilon), g(y, \epsilon), y, \epsilon) = 0$$

We have

$$F'_y(\overline{y}, 0) = f'_{y_{t+1}} g'_y g'_y + f'_{y_t} g'_y + f'_{y_{t-1}} = 0$$

Or

$$AX^2 + BX + C$$

where A, B, C and $X = g'_y$ are square matrices.

The Riccati Equation

Recall the system:

$$F(y,\epsilon) = f(g(g(y,0),\epsilon), g(y,\epsilon), y, \epsilon) = 0$$

We have

$$F'_y(\overline{y},0) = f'_{y_{t+1}} g'_y g'_y + f'_{y_t} g'_y + f'_{y_{t-1}} = 0$$

This is a specific Riccati equation

$$AX^2 + BX + C$$

where A, B, C and $X = g'_y$ are square matrices $\in \mathbb{R}^n \times \mathbb{R}^n$

First Order Deterministic Model

Let's pause a minute to observe the first order deterministic model:

$$AX^2 + BX + C$$

From our intuition in dimension 1, we know there must be multiple solutions

- how do we find them?
- how do we select the right ones?

Obviously, the dynamics of the model are given by $y_t = Xy_{t-1}$.

For the solution to the model to be stationary, the spectral radius of X should be smaller than 1.

Multiplicity of solution

It is possible to show that the system is associated with $2n$ generalized eigenvalues:

$$|\lambda_1| \leq \cdots \leq |\lambda_{2n}|$$

For each choice C of n eigenvalues ($|C| = n$), a specific fundamental solution X_C can be *constructed*. It has eigenvalues C.

This is at least $\binom{2n}{n}$ different combinations.

Then any linear combination of two solutions is also a solution...

Well defined model

A model is well defined when there is **exactly one solution that is non divergent**.

This implies:

$$|\lambda_1| \leq \cdots \leq |\lambda_n| \leq 1 < |\lambda_{n+1}| \leq \cdots \leq |\lambda_{2n}|$$

Example 1:

Forward looking inflation:

$$\pi_t = \alpha \pi_{t+1}$$

with $\alpha > 1$.

Is it well defined?

Example 1:

Forward looking inflation:

$$\pi_t = \alpha \pi_{t+1}$$

with $\alpha > 1$.

Is it well defined?

We can rewrite the system as:

$$\alpha \pi_{t+1} - \pi_t + 0\pi_{t-1} = \pi_{t+1} - (\frac{1}{\alpha} + 0)\pi_t + \left(\frac{1}{\alpha}0\right)\pi_{t-1}$$

The generalized eigenvalues are $0 \leq 1 < \frac{1}{\alpha}$. The unique solution is $\pi_t = 0\pi_{t-1}$

Example 2:

Debt accumulation equation by a rational agent:

$$b_{t+1} - (1 + \frac{1}{\beta})b_t + \frac{1}{\beta}b_{t-1} = 0$$

Is it well-defined?

Example 2:

Debt accumulation equation by a rational agent:

$$b_{t+1} - (1 + \frac{1}{\beta})b_t + \frac{1}{\beta}b_{t-1} = 0$$

Is it well-defined?

Two generalized eigenvalues $\lambda_1 = 1 < \lambda_2 = \frac{1}{\beta}$

The unique solution is $b_t = b_{t-1}$.

- it is a *unit-root*: any initial deviation in b_{t-1} has persistent effects

Example 3:

Productivity process:

$$z_t = \rho z_{t-1}$$

with $\rho < 1$: well defined

Example 3:

Productivity process:

$$z_t = \rho z_{t-1}$$

with $\rho < 1$: well defined

In that case there is a hidden infinite eigenvalue ∞ associated to z_{t+1}.

The generalized eigenvalues are $\lambda_1 = \rho \leq 1 < \lambda_2 = \infty$

More generally, any variable that does not appear in $t+1$ creates one infinite generalized eigenvalue.

A criterium for well-definedness

Looking again at the list of eigenvalues we can aside the infinite ones.

The model is well specified iff we can sort the eigenvalues as:

$$|\lambda_1| \leq \cdots \leq |\lambda_n| \leq 1 < |\lambda_{n+1}| \leq \cdots |\lambda_{n+k}| \leq \underbrace{|\lambda_{n+k+1}| \cdots \leq |\lambda_{2n}|}_{\text{infinite eigenvalues}}$$

Blanchard-Kahn criterium

The model satisfies the Blanchard-Kahn criterium if the number of eigenvalues greater than one, is exactly equal to the number of variables *appearing* in $t+1$.
In that case the model is well-defined.

Computing the solution

There are several classical methods to compute the solution to the algebraic Riccati equation:

$$AX^2 + BX + C = 0$$

- qz decomposition
  - traditionnally used in the DSGE literature since Chris Sims
  - a little bit unintuitive
- cyclic reduction
  - new default in dynare, more adequate for big models
- linear time iteration
  - conceptually very simple

Computing the shocks

Recall:

$$F(y, \epsilon) = f(g(g(y, \epsilon), 0), g(y, \epsilon), y, \epsilon) = 0$$

We have

$$F_e'(\overline{y}, 0) = f_{y_{t+1}}' g_y' g_e' + f_{y_t}' g_e' + f_{\epsilon_t}' = 0$$

Now this is easy:

$$g_e' = -(f_{y_{t+1}}' g_y' + f_{y_t}')^{-1} f_{\epsilon_t}' = 0$$

The model solution

The result of the model solution:

$$y_t = g_y y_{t-1} + g_e \epsilon_t$$

It is an AR1, driven by exogenous shock ϵ_t.

Because it is a well known structure, one can compute

- impulse response functions
- stochastic simulations
- implied moments

Also, one can easily compute the likelihood of the model given some data to perform model estimation.

Linear Time Iteration (1)

Recall the system to solve:

$$F(y, \epsilon) = f(g(g(y, \epsilon), 0), g(y, \epsilon), y, \epsilon) = 0$$

but now assume the decision rules today and tomorrow are different:

- today: $y_t = g(y_{t-1}, \epsilon_t) = \overline{y} + X y_{t-1} + g_y \epsilon_t$
- tomorrow: $y_{t+1} = \tilde{g}(y_t, \epsilon_{t+1}) = \overline{y} + \tilde{X} y_{t-1} + \tilde{g}_y \epsilon_t$

Then the Ricatti equation is written:

$$A \tilde{X} X + B X + C = 0$$

Linear Time Iteration (2)

The linear time iteration algorithm consists in solving the decision rule X today as a function of decision rule tomorrow $\tilde{X}$.

This corresponds to the simple formula:

$$X = -(A\tilde{X} + B)^{-1}C$$

And the full algorithm can be described as:

- choose X_0
- for any X_n, compute $X_{n+1} = T(X_n) = -(AX_n + B)^{-1}C$
  - repeat until convergence

Linear Time Iteration (3)

It can be shown that, started from a random initial guess, the linear time-iteration algorithm converges to the solution X with the smallest modulus:

$$\underbrace{|\lambda_1| \leq \cdots \leq |\lambda_n|}_{\text{Selected eigenvalues}} \leq |\lambda_{n+1}| \cdots \leq |\lambda_{2n}|$$

In other words, it finds the right solution when the model is well specified.

How do you check it is well specified?

- λ_n is the biggest eigenvalue of solution X
- if only we could measure λ_{n+1}...

Linear Time Iteration (4)

One can show that:

- λ_n is the spectral radius of the simulation operator $X : \mathbb{R}^n \to \mathbb{R}^n$
- $\frac{1}{\lambda_{n+1}}$ is the spectral radius of the derivative of the time-iteration operator $T' : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$

If both $\rho(X) \leq 1$ and $\rho(T') < 1$, the model is well specified.

This approach highlights the fact that for a model to be well defined, state dynamics and iterated expectations must be stable.

Example

Take a generic first order model

$$b_{t+1} - (\lambda_1 + \lambda_2)b_t + \lambda_1\lambda_2 b_{t-1} = 0$$

write decision rule today as $b_t = \lambda_n b_{t-1}$ and decision rule tomorrow as $b_{t+1} = \lambda_{n-1} b_t$.

- What is the function $\lambda_n = f(\lambda_{n-1})$?
- Can you prove that λ_n always converges to a fixed point $\overline{\lambda}$?
- Compute $f'(\overline{\lambda})$
- Prove that knowing λ and $f'(\lambda)$ is sufficient to know that the model satisfies Blanchard-Kahn conditions.

Conclusion

What can you do with the solution

- The solution of a model has an especially simple form: it is an AR1
  - $y_t = X y_{t-1} + Y \epsilon_t$
  - where the covariances Σ of ϵ_t can be chosen by the modeler
- On can thus:
  - compute (conditional and unconditional) moments
  - perform impulse response function, that is compute the effect of a shock
- Going further:
  - "estimate" the model: compute the likelihood of a solution and maximize it by choosing the right parameters
  - "identify" shocks in the data

Other Dynare features

Dynare allows to perform relatively easily

- higher order approximation
- model estimation
- ramsey plan
- discretionary policy
- ...

Other Dynare features

Dynare allows to perform relatively easily

- higher order approximation
- model estimation
- ramsey plan
- discretionary policy
- ...

You should not really be using all these features unless you know what you are doing...