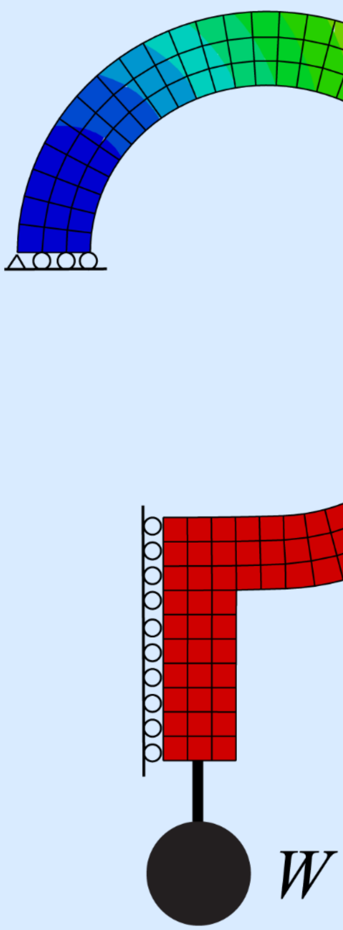
Example Problems in Mechanics of Solids



Allan F. Bower

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Preface

The best way to learn the physical sciences and mathematics is to dive in and start solving problems by yourself. The hard part is knowing what problems to solve. With this in mind, 'Example Problems in Mechanics of Solids' offers a large number of exercises that are designed to help students and practicing engineers to learn how to predict stresses and deformation in solids. The problems may also be useful for instructors who are looking for worked examples to use in lectures or assignments.

The book is intended to be a companion volume to A.F. Bower's "Applied Mechanics of Solids," 2nd ed. CRC press, Boca Raton FL (2025) (and the free e-text at https://solidmechanics.org). The chapter headings and sub-sections are identical in both books, and every topic covered in the main text has a set of relevant exercises. Where appropriate, the example problems also reference the relevant sections of Applied Mechanics of Solids.

A solution manual - A.F. Bower's "Solved Problems in Applied Mechanics of Solids," CRC press, Boca Raton FL (2026) - is in press.

There are 528 problems. The examples in each section are progressive: like piano etudes, the first few are usually deceptively easy, but the last ones can be very challenging. They cover the following topics:

- The Appendix contains a set of review problems covering vectors and matrices; tensors; index notation, and polar coordinates. Working through a few of these problems would be a good starting point for a self-study course.
- Chapter 1 contains a few general questions on applications of solid mechanics.
- Chapter 2 has a large number of problems covering kinematics; stress measures; and field equations.
- Chapter 3 covers constitutive laws, including: elasticity; plasticity; viscoelasticity; soils; and models of interfaces between solids.
- Chapter 4 has a set of boundary value problems for solids with a simple geometry and an elastic constitutive law, which can be solved analytically; and simple examples of plane wave propagation.
- Chapter 5 covers applications of the theory of linear elasticity, including plane and three-dimensional boundary value problems; waves; and the theory of dislocations in solids.
- Chapter 6 has problems that illustrate solutions to boundary value problems and limit theorems for plastically deforming solids
- Chapter 7 has a set of introductory problems related to finite element analysis, and includes a set of tutorial exercises that are designed to be solved using the 'learning edition' of the commercial finite element code ABAQUS®. This code can be downloaded at no cost from the third-party website https://www.3ds.com/edu/education/students/solutions/abaqus-le.
- The problems in Chapter 8 cover the detailed implementation of the finite element method, and include examples that involve finite element coding in MATLAB®.
- Chapter 9 has problems related to modeling failure in solids, and the theory of fracture mechanics.
- Chapter 10 has example of problems involving elastic solids with special shapes, such as cables, beams, rods, plates and shells.

A companion Github site at https://github.com/albower/Applied_Mechanics_of_Solids/tree/main offers several additional resources, including example MATLAB® codes, solutions to the finite element coding problems, and solved model databases for the problems that use ABAQUS®.

Contents

Chapter 1
Objectives and Methods of Solids Mechanics

1.1 Defining a Problem in Solid Mechanics

Problem 1.1 For each of the following applications, answer briefly:

- What would you calculate if you were asked to model the component for a design application?
- What level of detail is required in modeling the geometry of the solid?
- How would you model loading applied to the solid?
- Would you conduct a static or dynamic analysis? Is it necessary to account for thermal stresses? Is it necessary to account for temperature variation as a function of time?
- What constitutive law would you use to model the material behavior?

(a) A load cell intended to measure forces applied to a specimen in a tensile testing machine
(b) The seat-belt assembly in a vehicle
(c) The solar panels on a communications satellite.
(d) A compressor blade in a gas turbine engine
(e) A MEMS optical switch
(f) An artificial knee joint
(g) A solder joint on a printed circuit board
(h) An entire printed circuit board assembly
(i) The metal interconnects inside a microelectronic circuit

Problem 1.2 What is the difference between a linear elastic stress-strain law and a hyperelastic stress-strain law? Give examples of representative applications for both material models.

Problem 1.3 What is the difference between a rate-dependent (viscoplastic) and rate independent plastic constitutive law? Give examples of representative applications for both material models.

Chapter 2
Governing Equations

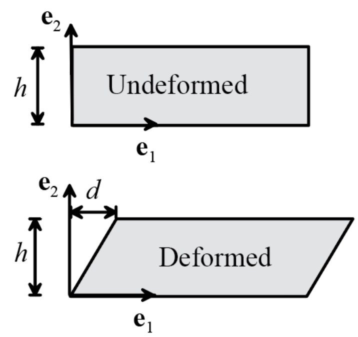
2.1 Basic Assumptions

Problem 2.1 List the common assumptions inherent to solid mechanics, and discuss their validity. You might find it helpful to come back to this problem now and again during your reading and add to the list.

Problem 2.2 What is a solid?

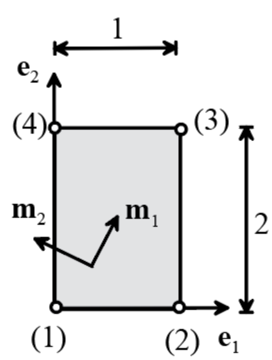
2.2 Mathematical Description of Shape Changes in Solids

Problem 2.3 A thin film of material with thickness h is deformed in simple shear during a plate impact experiment, as shown in the figure.



(a) Write down expressions for the displacement field in the film, in terms of x_1, x_2, d and h, expressing your answer as components in the basis shown.
(b) Calculate the Lagrange strain tensor associated with the deformation, expressing your answer as components in the basis shown.
(c) Calculate the infinitesimal strain tensor for the deformation, expressing your answer as components in the basis shown.
(d) Find the principal values of the infinitesimal strain tensor, in terms of d and h

Problem 2.4 The figure shows a rectangular (2D) finite element. The displacement vector (as Cartesian components in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis, in arbitrary units) at each of its corners is



$$\mathbf{u}^{(1)} = [0,0]; \quad \mathbf{u}^{(1)} = \mathbf{u}^{(2)} = \mathbf{u}^{(4)} = [0,0]; \quad \mathbf{u}^{(3)} = [0.2, 0.4];$$

The displacement at an arbitrary point inside the element is computed using a linear interpolation between values at the four corners (this only works for the rectangle shown, the general formula for an arbitrary rectangular finite element is more complicated)

$$\mathbf{u} = \frac{1}{2}(2-x_2)(1-x_1)\mathbf{u}^{(1)} + \frac{1}{2}(2-x_2)x_1\mathbf{u}^{(2)} + \frac{1}{2}x_1x_2\mathbf{u}^{(3)} + \frac{1}{2}(1-x_1)x_2\mathbf{u}^{(4)}$$

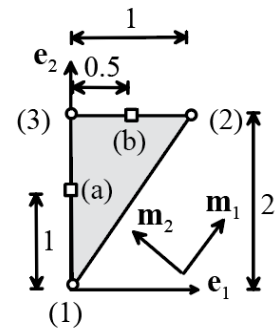
where $\mathbf{x}$ is the position of a point in the element before deformation.

(a) Find the components of $\mathbf{u}\nabla$ (give the answer as a 2x2 matrix, which will be a function of position) and hence calculate the infinitesimal strain tensor $\boldsymbol{\varepsilon} = \left\{\mathbf{u}\nabla + \left(\mathbf{u}\nabla\right)^T\right\}/2$
(b) Find the components of the deformation gradient (in 2D) $\mathbf{F}$ and hence deduce the components of the Lagrange strain tensor $\mathbf{E} = (\mathbf{F}^T\mathbf{F} - \mathbf{I})/2$
(c) Find $\int_A \nabla \cdot \mathbf{u} dA$ where A denotes the area of the square. Check your answer using the divergence theorem; i.e. calculate $\int_C \mathbf{n} \cdot \mathbf{u} ds$ around the perimeter of the square.

(d) Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be unit vectors parallel and perpendicular to a line from corner (1) to corner (2) of the element. Calculate the components of $\mathbf{u}\nabla$ in this basis. Check your answer by calculating $\nabla \cdot \mathbf{u}$ using the components in $\{\mathbf{m}_1, \mathbf{m}_2\}$.

Problem 2.5 The figure shows a triangular finite element. The displacement vector (as Cartesian components in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis, in arbitrary units) at each of its corners is

$$\mathbf{u}^{(1)} = [0,0]; \quad \mathbf{u}^{(2)} = [0.1, 0.1]; \quad \mathbf{u}^{(3)} = [0, 0.2];$$



The displacement at an arbitrary point inside the element is computed using a linear interpolation between values at the three corners (this only works for the triangle shown, it is not a general formula for all triangles)

$$\mathbf{u} = \frac{1}{2}(2 - x_2)\mathbf{u}^{(1)} + x_1\mathbf{u}^{(2)} + \frac{1}{2}(x_2 - 2x_1)\mathbf{u}^{(3)}$$

where $\mathbf{x}$ is the position of a point in the element before deformation.

(a) Find the components of $\mathbf{u}\nabla$ (give the answer as a 2x2 matrix)

(b) Use the gradient to calculate $\mathbf{u}^a - \mathbf{u}^b$, where the two points *a* and *b* are shown in the figure.

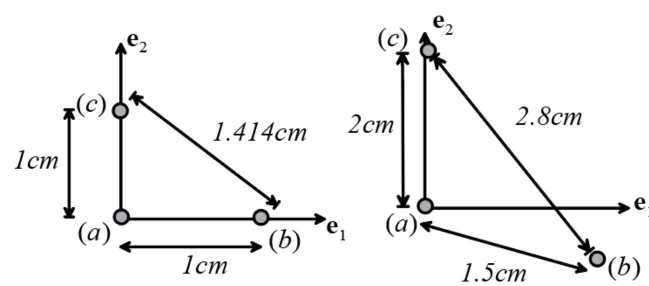
(c) Calculate the divergence of $\mathbf{u}$ (i.e. $\nabla \cdot \mathbf{u}$)

(d) Find $\int_A \nabla \cdot \mathbf{u} dA$ where *A* denotes the area of the triangle. Check your answer using the divergence theorem; i.e. calculate $\int_C \mathbf{n} \cdot \mathbf{u} ds$ around the perimeter of the triangle

(e) Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be unit vectors parallel and perpendicular to the side (1-2) of the triangle. Calculate the components of $\mathbf{u}\nabla$ in this basis. Check your answer by calculating $\nabla \cdot \mathbf{u}$ using the components in $\{\mathbf{m}_1, \mathbf{m}_2\}$.

Problem 2.6 Find a displacement field that corresponds to a uniform Lagrange strain tensor E_{ij}. Is the displacement unique? Find a formula for the most general displacement field that generates a uniform Lagrange strain.

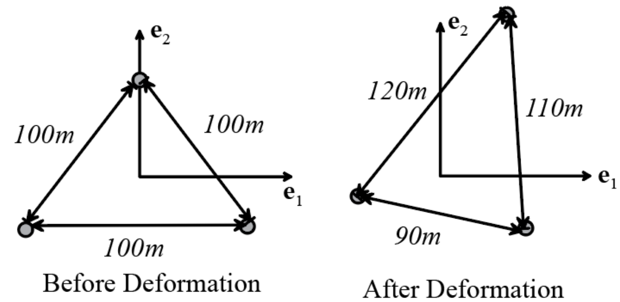
Problem 2.7 To measure the in-plane deformation of a sheet of metal during a forming process, three small hardness indentations are placed on the sheet. Using a travelling microscope, you determine that the initial lengths of the sides of the triangle formed by the three indents are 1cm, 1cm, 1.414cm, as shown in the picture below. After deformation, the sides have lengths 1.5cm, 2.0cm and 2.8cm.



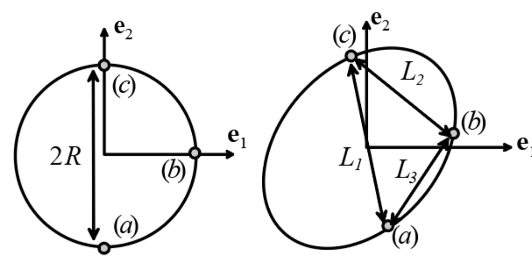
(a) Calculate the components of the Lagrange strain tensor E_{11}, E_{22}, E_{12} in the basis shown.

(b) Calculate the components of the Eulerian strain tensor $E^*_{11}, E^*_{22}, E^*_{12}$ in the basis shown.

Problem 2.8 To track the deformation in a slowly moving glacier, three survey stations are installed at the corners of an equilateral triangle, spaced 100m apart, as shown in the picture. After a suitable period of time, the spacing between the three stations is measured again, and found to be 90m, 110m and 120m, as shown in the figure. Assuming that the deformation of the glacier is homogeneous over the region spanned by the survey stations, find the components of the Lagrange strain tensor associated with this deformation, expressing your answer as components in the basis shown.



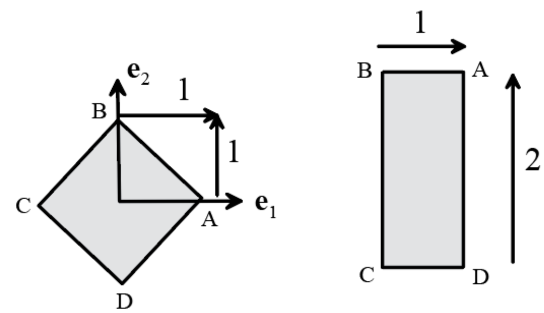
Problem 2.9 'Circle Grid Analysis' is used to estimate strains induced in sheet metal parts during stamping or drawing. A grid of circles is etched on the sheet before it is deformed, and the strain at the center of each circle is then determined by measuring how the circles change their shape. The figure shows a typical example. Suppose that the three lengths L_1, L_2, L_3 on a circle with initial radius R are measured.



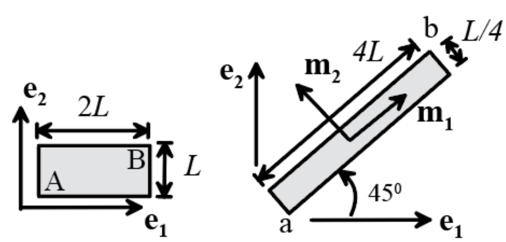
(a) Find a formula for the components of the 2D Lagrange strain tensor in terms of the lengths and R

(b) Check that you get the correct solution for a rigid rotation (i.e. put in values for L_1, L_2, L_3 consistent with a rigid rotation).

Problem 2.10 The figure shows a material element on the surface of a specimen before and after deformation. Calculate the (2D) Lagrange strain tensor (as a 2x2 matrix)



Problem 2.11 The figure shows the reference and deformed configurations for a solid. The out-of-plane dimensions are unchanged. Corners a and b on the deformed solid are the positions after deformation of corners A and B in the undeformed solid. Determine:

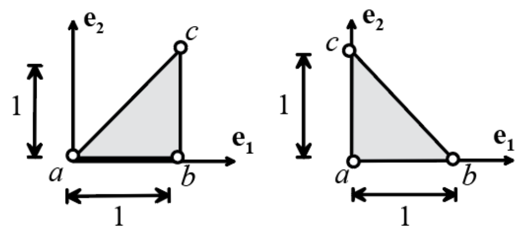


(a) The right stretch tensor $\mathbf{U}$, expressed as components in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. (A 2x2 matrix is sufficient). There is no need for lengthy calculations – you may write down the result by inspection.

(b) The rotation tensor **R** in the polar decomposition of the deformation gradient **F=RU=VR.**

(c) The deformation gradient, expressed as components in $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. Try to do this without using the basis-change formulas.

Problem 2.12 The figure shows a plane 2D triangular constant strain finite element before and after deformation. Calculate the Lagrange strain in the element.



Problem 2.13 Construct (i.e. find a displacement field) a homogeneous deformation that has the following properties:

- The volume of the solid is doubled;
- A material fiber parallel to the $\mathbf{e}_1$ direction in the undeformed solid increases its length by a factor of $\sqrt{2}$ and is oriented parallel to the $\mathbf{e}_1+\mathbf{e}_2$ direction in the deformed solid;
- A material fiber parallel to the $\mathbf{e}_2$ direction in the undeformed is oriented parallel to the $-\mathbf{e}_1+\mathbf{e}_2$ direction in the deformed solid;
- A material fiber parallel to the $\mathbf{e}_3$ direction in the undeformed solid preserves its length and orientation in the deformed solid.

Problem 2.14 A rigid body motion is a nonzero displacement field that does not distort any infinitesimal volume element within a solid. Thus, a rigid body displacement induces no strain, and hence no stress, in the solid. The deformation corresponding to a 3D rigid rotation about an axis through the origin is

$$\mathbf{y}=\mathbf{R}\mathbf{x} \quad \text{or} \quad y_i = R_{ij}x_j$$

where $\mathbf{R}$ must satisfy $\mathbf{R}\cdot\mathbf{R}^T=\mathbf{R}^T\cdot\mathbf{R}=\mathbf{I}$, det($\mathbf{R}$)>0.

(a) Show that the Lagrange strain associated with this deformation is zero.

(b) As a specific example, consider the deformation

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This is the displacement field caused by rotating a solid through an angle θ about the $\mathbf{e}_3$ axis. Find the deformation gradient for this displacement field, and show that the deformation gradient tensor is orthogonal, as predicted by part (a).

(c) Show also that the infinitesimal strain tensor for this displacement field is not generally zero, but is of order θ^2 if θ is small.

(d) If the displacements are small, we can find a simpler representation for a rigid body displacement. Consider a deformation of the form

$$y_i = x_i + \in_{ijk}\ \omega_j x_k$$

Here $\boldsymbol{\omega}$ is a vector with magnitude <<1, which represents an infinitesimal rotation about an axis parallel to $\boldsymbol{\omega}$. Show that the infinitesimal strain tensor associated with this displacement is always zero.

(e) Show further that the Lagrange strain associated with this displacement field is

$$E_{ij} = \frac{1}{2}\left(\delta_{ij}\omega_k\omega_k - \omega_i\omega_j\right)$$

This is not, in general, zero. It is small if all $\omega_k << 1$.

Problem 2.15 The formula for the deformation associated with a rotation through an angle θ about an axis parallel to a unit vector **n** that passes through the origin is

$$y_i = \left[\cos\theta \delta_{ij} + (1-\cos\theta) n_i n_j + \sin\theta \in_{ikj} n_k \right] x_j$$

(a) Calculate the components of corresponding deformation gradient
(b) Verify that the deformation gradient satisfies $F_{ik}F_{jk} = F_{ki}F_{kj} = \delta_{ij}$
(c) Find the components of the *inverse* of the deformation gradient
(d) Verify that both the Lagrange strain tensor and the Eulerian strain tensor are zero for this deformation. What does this tell you about the distorsion of the material?
(e) Calculate the Jacobian of the deformation gradient. What does this tell you about volume changes associated with the deformation?

Problem 2.16 A solid is subjected to a rigid rotation so that a unit vector **a** in the undeformed solid is rotated to a new orientation **b** in the deformed (rotated) solid. Find a rotation tensor **R** that is consistent with this deformation, in terms of the components of **a** and **b**. Is the rotation tensor unique? If not, find the most general formula for the rotation tensor.

Problem 2.17 Show that the Lagrange strain **E**, the right Cauchy-Green deformation tensor **C** and the right stretch tensor **U** have the same principal directions (eigenvectors). Similarly, show that $\mathbf{E}^*, \mathbf{B}, \mathbf{V}$ have the same principal directions.

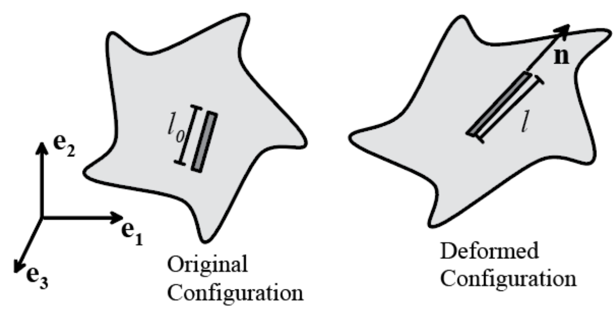
Problem 2.18 Show that the Eulerian strain tensor

$$E_{ij}^* = (\delta_{ij} - F_{ki}^{-1} F_{kj}^{-1}) / 2$$

can be used to relate the length of a material fiber in a deformable solid before and after deformation, using the formula

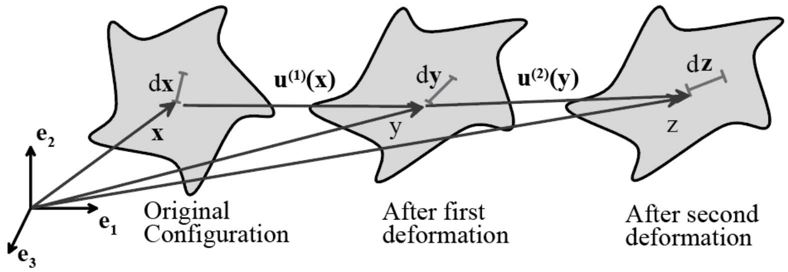
$$\frac{l^2 - l_0^2}{2l^2} = E_{ij}^* n_i n_j$$

where n_i are the components of a unit vector parallel to the material fiber after deformation, as shown in the figure.



Problem 2.19 The Lagrange strain tensor can be used to calculate the change in angle between any two material fibers in a solid as the solid is deformed. In this problem you will calculate the formula that can be used to do this. To this end, consider two infinitesimal material fibers in the undeformed solid, which are characterized by vectors with components $dx_i^{(1)} = l_1 m_i^{(1)}$ and $dx_i^{(2)} = l_2 m_i^{(2)}$, where $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$ are two unit vectors. Recall that the angle θ_0 between $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$ before deformation can be calculated from $\cos\theta_0 = m_i^{(1)} m_i^{(2)}$. Let $dy_i^{(1)}$ and $dy_i^{(2)}$ represent the two material fibers after deformation. Show that the angle between $dy_i^{(1)}$ and $dy_i^{(2)}$ can be calculated from the formula

$$\cos\theta_1 = \frac{2E_{ij} m_i^{(1)} m_j^{(2)} + \cos\theta_0}{\sqrt{1 + 2E_{ij} m_i^{(1)} m_j^{(1)}} \sqrt{1 + 2E_{ij} m_i^{(2)} m_j^{(2)}}}$$



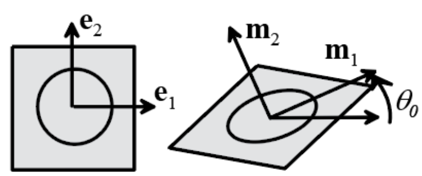
Problem 2.20 Suppose that a solid is subjected to a sequence of two homogeneous deformations (i) a rigid rotation **R**, followed by (ii) an arbitrary homogeneous deformation **A**. Taking the original configuration as reference, find formulas for the following deformation measures for the final configuration of the solid, in terms of **F** and **R**:

(a) The deformation gradient
(b) The Left and Right Cauchy-Green deformation tensors
(c) The Lagrange strain
(d) The Eulerian strain.

Problem 2.21 Repeat problem 2.20, but this time assume that the sequence of the two deformations is reversed, i.e. the solid is first subjected to an arbitrary homogeneous deformation **F,** and is subsequently subjected to a rigid rotation **R**.

Problem 2.22 A sheet of material is subjected to a two dimensional homogeneous deformation of the form

$$y_1 = A_{11}x_1 + A_{12}x_2 \qquad y_2 = A_{21}x_1 + A_{22}x_2$$



where A_{ij} are constants. Suppose that a circle of unit radius is drawn on the undeformed sheet, as shown in the figure. This circle is distorted to a smooth curve on the deformed sheet. Show that the distorted circle is an ellipse, with semi-axes that are parallel to the principal directions of the left stretch tensor **V,** and that the lengths of the semi-axes of the ellipse are equal to the principal stretches for the deformation.

Problem 2.23 The center of mass and the mass moment of inertia tensor in the reference and deformed configurations of a solid are (by definition)

$$r_i^{c0} = \frac{1}{M}\int_{V_0} x_i \rho_0 dV_0 \qquad I_{ij}^{c0} = \int_{V_0} \left(x_i - r_i^{c0}\right)\left(x_j - r_j^{c0}\right)\rho_0 dV_0$$

$$r_i^{c} = \frac{1}{M}\int_{V} y_i \rho dV \qquad I_{ij}^{c} = \int_{V} \left(y_i - r_i^{c}\right)\left(y_j - r_j^{c}\right)\rho dV$$

where superscripts (0) denote quantities associated with the reference configuration; ρ_0, ρ are the mass density of the solid in the reference and deformed configurations, $\mathbf{x}, \mathbf{y}$ are the positions of material particles in the reference and deformed configurations, and *M* is the total mass. Suppose that a solid is subjected to a homogeneous deformation

$$y_i = A_{ik}x_k + c_i$$

where A_{ij} and c_i are constants. Then:

(a) Find formulas for r_{ic}, I_{ij}^C in terms of r_i^{c0} I_{ij}^{c0}, A_{ij} and c_i .

(b) Suppose that A_{ij} is a rigid rotation (this means $A_{ik}A_{jk} = A_{ki}A_{kj} = \delta_{ij}$. Use the solution to (a) to show that the time derivative of I_{ij} can be expressed as

$$\frac{dI_{ij}^{c}}{dt} = W_{ik}I_{kj}^{c} - I_{ik}^{c}W_{kj}$$

where $W_{ik} = \dfrac{dA_{ik}}{dt}A_{jk}$ is the spin tensor.

(c) Suppose that a rigid body rotates with angular velocity ω_k and therefore has angular momentum

$$h_i = I_{ij}^{c}\omega_j$$

Use (b) to show that the time derivative of the angular momentum is

$$\frac{dh_i}{dt} = I_{ij}\frac{d\omega_j}{dt} + \in_{ijk}\ \omega_j I_{kl}\omega_l$$

Problem 2.24 Let **F** be a deformation gradient, **E** the Lagrange strain tensor and **D** the stretch rate tensor. Show that

$$\mathbf{D} = \mathbf{F}^{-T}\frac{d\mathbf{E}}{dt}\mathbf{F}^{-1}$$

Problem 2.25 Let **n** be a unit vector parallel to infinitesimal material fiber in a deforming solid., and let **D** and **W** denote the stretch rate and spin tensors. Show that

$$\frac{d\mathbf{n}}{dt} = \mathbf{Dn} + \mathbf{Wn} - (\mathbf{n}\cdot\mathbf{Dn})\mathbf{n}$$

Problem 2.26 The properties of many rubbers and foams are specified by functions of the following invariants of the left Cauchy-Green deformation tensor $B_{ij} = F_{ik}F_{jk}$

$$I_1 = \text{trace}(\mathbf{B}) = B_{kk}$$

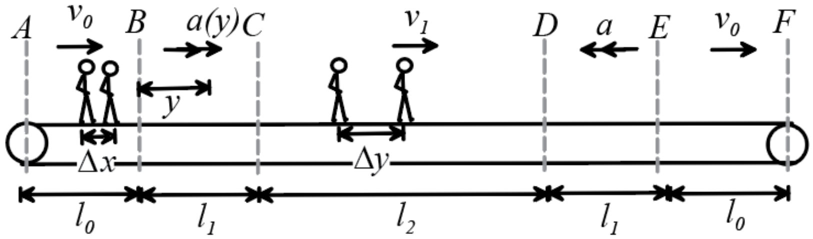
$$I_2 = \frac{1}{2}\left(I_1^2 - \mathbf{B}\cdot\cdot\mathbf{B}\right) = \frac{1}{2}\left(I_1^2 - B_{ik}B_{ki}\right)$$

$$I_3 = \det\mathbf{B} = J^2$$

(a) Verify that I_1, I_2, I_3 are invariants. The simplest way to do this is to show that I_1, I_2, I_3 are unchanged during a change of basis.

(b) In order to calculate stress-strain relations for these materials, it is necessary to evaluate derivatives of the invariants. Show that

$$\frac{\partial I_1}{\partial F_{ij}} = 2F_{ij} \qquad \frac{\partial I_2}{\partial F_{ij}} = 2\left(I_1F_{ij} - B_{ik}F_{kj}\right) \qquad \frac{\partial I_3}{\partial F_{ij}} = 2I_3F_{ji}^{-1}$$



Problem 2.27 The figure shows a design for a high-speed moving walkway (Toronto airport has a walkway of this kind). Passengers standing on the walkway passes through five regions:

(i) between A and B they move at constant speed v_0;

(ii) between B and C they accelerate (with an acceleration to be specified below);

(iii) between C and D they move with constant (high) speed v_1;

(iv) between D and E they decelerate;

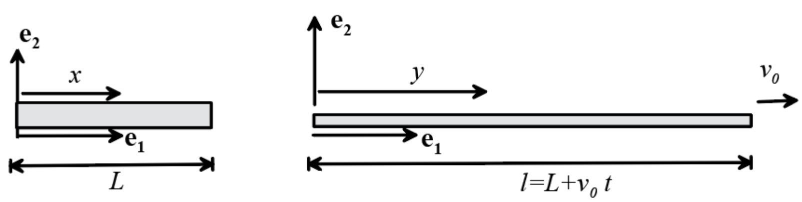
(v) between E and F they travel at speed v_0 again.

In this problem we will just focus on portion (ii) of the motion – i.e. between *B* and *C*.

(a) Suppose that the walkway is designed so that the velocity varies linearly with distance between *B* and *C*. Assume that a person walks with speed *w* relative to the moving walkway. Determine their acceleration as a function of distance *y* from *B*, and also as a function of time after passing the point *B*. Find a formula for the maximum value of the acceleration, and identify the point where it occurs.

(b) Suppose instead that the walkway is designed instead so that a person standing on the track has constant acceleration *a*. Calculate the required velocity distribution *v(y)* as a function of distance *y* from B, and determine the acceleration of the person walking along the accelerating walkway as a function of *y* and also a function of *t*.

Problem 2.28 The rubber band shown in the figure has initial length *L*. The left end of the band is held fixed. For time *t>0* the other end is pulled at constant speed v_0, so that at some time *t* it has length $l = L + v_0 t$. Following the usual convention, let *x* denote position in the reference configuration, and let *y* denote position in the deformed configuration.



(a) Write down the position *y* of a material particle as a function of its initial position *x* and time *t*.

(b) Hence, determine the (horizontal) velocity distribution as both a function of *x* and a function of *y*

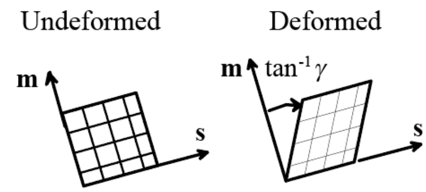
(c) Find the deformation gradient (you only need the F_{11} component)

(d) Find the velocity gradient (only one component is needed)

(e) Suppose that a fly walks along the rubber band with speed *w* relative to the band. Calculate the acceleration of the fly as a function of time and other relevant variables.

(f) Suppose that the fly is at *x=y=0* at time *t=0*. Find how long it takes for the fly to walk to the other end of the rubber band, in terms of *L*, v_0 and *w*. It is easiest to do this by calculating dx/dt for the fly.

Problem 2.29 A single crystal deforms by shearing on a single active slip system as illustrated in the figure. The crystal is loaded so that the slip direction $\mathbf{s}$ and normal to the slip plane $\mathbf{m}$ maintain a constant direction during the deformation



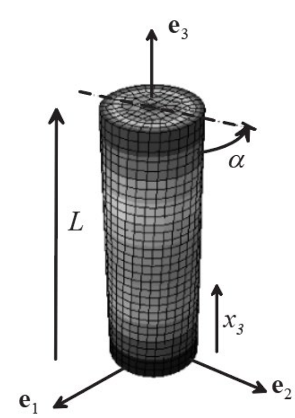
(a) Show that the deformation gradient can be expressed in terms of the components of the slip direction $\mathbf{s}$ and the normal to the slip plane $\mathbf{m}$ as $F_{ij} = \delta_{ij} + \gamma s_i m_j$ where γ denotes the shear, as illustrated in the figure.
(b) Suppose shearing proceeds at some rate $\dot{\gamma}$. At the instant when $\gamma = 0$, calculate (i) the velocity gradient tensor; (ii) the stretch rate tensor and (iii) the spin tensor associated with the deformation.
(c) Find an expression for the extension rate and angular velocity of a material fiber with length l parallel to a unit vector $\mathbf{n}$ in the deformed solid, in terms of $\dot{\gamma}, \mathbf{s}, \mathbf{m}$.

Problem 2.30 The displacement field in a homogeneous, isotropic circular shaft with radius a is twisted through angle α at one end is given by

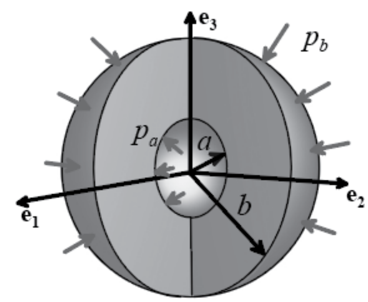
$$u_1 = x_1\left[\cos\left(\frac{\alpha x_3}{L}\right) - 1\right] - x_2 \sin\left(\frac{\alpha x_3}{L}\right)$$

$$u_2 = x_1 \sin\left(\frac{\alpha x_3}{L}\right) + x_2\left[\cos\left(\frac{\alpha x_3}{L}\right) - 1\right]$$

$$u_3 = 0$$



(a) Calculate the matrix of components of the deformation gradient tensor
(b) Calculate the matrix of components of the Lagrange strain tensor.
(c) At the point **x**=(a,0,0), find the stretch l / l_0 of material fibers that are parallel to the three basis vectors in the undeformed bar.
(d) Calculate the components of the infinitesimal strain tensor. Show that, for small values of α, the infinitesimal strain tensor is identical to the Lagrange strain tensor, but for finite rotations the two measures of deformation differ.
(e) At the point **x**=(a,0,0), use the infinitesimal strain tensor to obtain estimates for the stretch l / l_0 of material fibers that are parallel to unit vectors $\mathbf{m}_1 = (\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{2}$ and $\mathbf{m}_2 = (\mathbf{e}_2 - \mathbf{e}_3)/\sqrt{2}$ in the undeformed bar. How large can $\alpha a / L$ be before the error in this estimate reaches 1%?

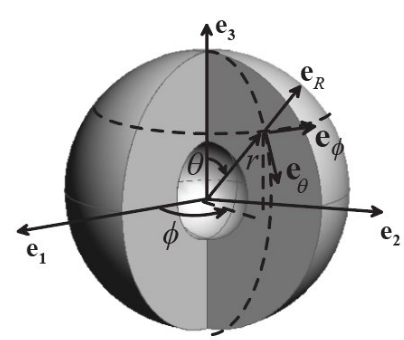


Problem 2.31 A spherical shell (see the figure) is made from an incompressible material. In its undeformed state, the inner and outer radii of the shell are A, B. After deformation, the new values are a, b. The position **y** of a material particle in the shell is related to its position **x** before deformation by the equation

$$y_i = r\frac{x_i}{R} \qquad r = \left(R^3 + a^3 - A^3\right)^{1/3} \qquad R = \sqrt{x_k x_k} \qquad r = \sqrt{y_i y_i}$$

(a) Calculate the components of the deformation gradient tensor
(b) Verify that the deformation is volume preserving
(c) Find the deformed length of an infinitesimal radial line that has initial length l_0, expressed as a function of R
(d) Find the deformed length of an infinitesimal circumferential line that has initial length l_0, expressed as a function of R
(e) Using the results of (c) and (d), write down the principal stretches for the deformation.
(f) Find the *inverse* of the deformation gradient, expressed as a function of y_i. You can do this by: inspection; by inverting (a) (not recommended!); or by working out a formula that enables you to calculate x_i in terms of y_i and $r = \sqrt{y_i y_i}$ and differentiating the result. The first is quickest!

Problem 2.32 Suppose that the spherical shell described in problem 2.31 is continuously expanding (visualize a balloon being inflated). The rate of expansion can be characterized by the velocity $v_a = da/dt$ of the surface that lies at $R=A$ in the undeformed cylinder.
(a) Calculate the velocity field $v_i = dy_i/dt$ in the sphere as a function of x_i
(b) Calculate the velocity field as a function of y_i
(c) Calculate the time derivative of the deformation gradient tensor calculated in problem (a)
(d) Calculate the components of the velocity gradient $L_{ij} = \dfrac{\partial v_i}{\partial y_j}$ by differentiating the result of (b)
(e) Calculate the components of the velocity gradient using the results of (c) and 2.31(f)
(f) Calculate the stretch rate tensor D_{ij}. Verify that the result represents a volume preserving stretch rate field.



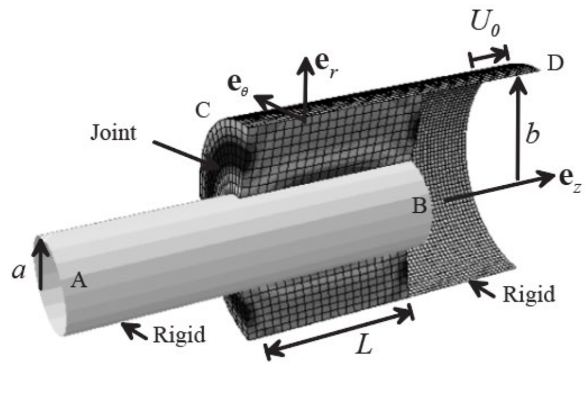
Problem 2.33 Repeat Problem 2.31(a), 2.31(f) and all of 2.32(b), 2.32(d), but this time solve the problem using spherical-polar coordinates, using the various formulas for vector and tensor operations given in Appendix D of Applied Mechanics of Solids. In this case, you may assume that a point with position $\mathbf{x} = R\mathbf{e}_R$ in the undeformed solid has position vector

$$\mathbf{y} = \left(R^3 + a^3 - A^3\right)^{1/3} \mathbf{e}_R$$

after deformation.

Problem 2.34 The figure shows a cross-section through an elastomeric joint connecting two hollow cylindrical shafts. The joint is a hollow cylinder with external radius b and internal radius a. It is bonded to the two rigid shafts AB and CD. Shaft AB is fixed (no translation or rotation), and an axial displacement $\mathbf{u} = U_0 \mathbf{e}_z$ is applied to the hollow cylinder CD. The displacement field in the joint can be shown to be approximately

$$\mathbf{u} = \left[\frac{U_0}{\log(b/a)} \log(r/a) \right] \mathbf{e}_z$$

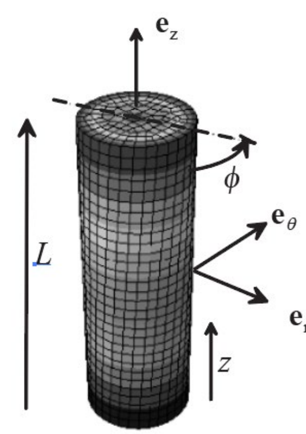


Find
(a) The displacement gradient;
(b) The deformation gradient;
(c) The Lagrange strain tensor and
(d) The infinitesimal strain tensor field in the joint, expressing your answer as components in the cylindrical-polar basis shown in the figure.

Problem 2.35 The twisted cylinder shown in the figure has a displacement field (in polar coordinates)

$$\mathbf{u} = \frac{\phi}{L} rz \mathbf{e}_\theta$$

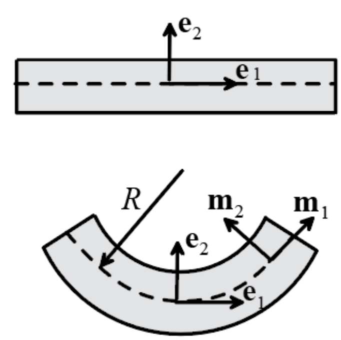
where ϕ is the angle of twist at the end of the cylinder. Calculate the length of a material fiber that lies on the outer surface of the cylinder ($r=a$) and is parallel to the $\mathbf{e}_z$ direction before the cylinder is twisted



Problem 2.36 An initially straight beam is bent into a circle with radius R as shown in the figure. Material fibers that are perpendicular to the axis of the undeformed beam are assumed to remain perpendicular to the axis after deformation, and the beam's thickness and the length of its axis are assumed to be unchanged. Under these conditions the deformation can be described as

$$y_1 = (R - x_2)\sin(x_1 / R) \qquad y_2 = R - (R - x_2)\cos(x_1 / R)$$

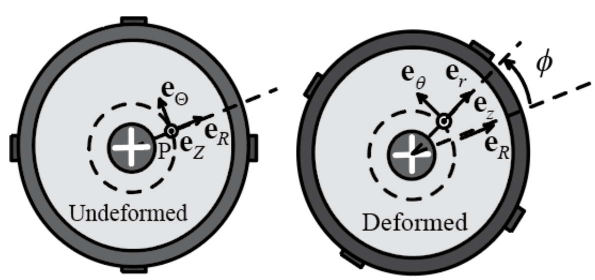
where (as usual) $\mathbf{x}$ is the position of a material particle in the undeformed beam, and $\mathbf{y}$ is the position of the same particle after deformation.



(a) Calculate the deformation gradient field in the beam, expressing your answer as a function of x_1, x_2, and as components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis shown.
(b) Calculate the Lagrange strain field in the beam.

(c) Calculate the infinitesimal strain field in the beam.
(d) Compare the values of Lagrange strain and infinitesimal strain for two points that lie at $(x_1 = 0, x_2 = h)$ and $(x_1 = L, x_2 = 0)$. Explain briefly the physical origin of the difference between the two strain measures at each point. Recommend maximum allowable values of *h/R* and *L/R* for use of the infinitesimal strain measure in modeling beam deflections.
(e) Calculate the deformed length of an infinitesimal material fiber that has length l_0 and orientation $\mathbf{e}_1$ in the undeformed beam. Express your answer as a function of x_2.
(f) Calculate the change in length of an infinitesimal material fiber that has length l_0 and orientation $\mathbf{e}_2$ in the undeformed beam.
(g) Show that the two material fibers described in (3) and (f) remain mutually perpendicular after deformation. Is this true for *all* material fibers that are mutually perpendicular in the undeformed solid?
(h) Find the components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the Left and Right stretch tensors $\mathbf{U}$ and $\mathbf{V}$ as well as the rotation tensor $\mathbf{R}$ for this deformation. You should be able to write down $\mathbf{U}$ and $\mathbf{R}$ by inspection, without needing to wade through the laborious general process. The results can then be used to calculate $\mathbf{V}$.
(i) Find the principal directions of $\mathbf{U}$ as well as the principal stretches. You should be able to write these down without doing any tedious calculations.
(j) Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be a basis in which $\mathbf{m}_1$ is parallel to the axis of the deformed beam, as shown in the figure. Write down the components of each of the unit vectors $\mathbf{m}_i$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Hence, compute the transformation matrix $Q_{ij} = \mathbf{m}_i \cdot \mathbf{e}_j$ that is used to transform tensor components from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{m}_1, \mathbf{m}_2\}$.
(k) Find the components of the deformation gradient tensor, Lagrange strain tensor, as well as $\mathbf{U}$ $\mathbf{V}$ and $\mathbf{R}$ in the basis $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. It is best to do these with a symbolic manipulation program.
(l) Find the principal directions of $\mathbf{V}$ expressed as components in the basis $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. Again, you should be able to simply write down this result.

Problem 2.37 The figure shows a test designed to measure the response of a polymer to large shear strains. The sample is a hollow cylinder with internal radius a_0 and external radius a_1. The inside diameter is bonded to a fixed rigid cylinder. The external diameter is bonded inside a rigid tube, which is rotated through an angle $\alpha(t)$. Assume that the specimen deforms as indicated in the figure, i.e. (a) cylindrical sections remain cylindrical; (b) no point in the specimen moves in the axial or radial directions; (c) that a cylindrical element of material at radius R rotates through angle $\phi(R,t)$ about the axis of the specimen. Take the undeformed configuration as reference. Let (R, Θ, Z) denote the cylindrical-polar coordinates of a material point in the reference configuration, and let $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ be cylindrical-polar basis vectors at (R, Θ, Z). Let (r, θ, z) denote the coordinates of this point in the deformed configuration, and let $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ by cylindrical-polar basis vectors located at (r, θ, z).



(a) Write down expressions for (r, θ, z) in terms of (R, Θ, Z) (this constitutes the deformation mapping)
(b) Let P denote the material point at at (R, Θ, Z) in the reference configuration. Write down the reference position vector $\mathbf{x}$ of P, expressing your answer as components in the basis $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$.

(c) Write down the deformed position vector **y** of P, expressing your answer in terms of (R,Θ,Z) and basis vectors $\{\mathbf{e}_R,\mathbf{e}_\Theta,\mathbf{e}_Z\}$.

(d) Find the components of the deformation gradient tensor **F** in $\{\mathbf{e}_R,\mathbf{e}_\Theta,\mathbf{e}_Z\}$.

(e) Show that the deformation gradient can be decomposed into a sequence $\mathbf{F}=\mathbf{RS}$ of a simple shear **S** followed by a rigid rotation through angle ϕ about the $\mathbf{e}_Z$ direction **R.** In this case the simple shear deformation will have the form

$$\mathbf{S}=\mathbf{e}_R\otimes\mathbf{e}_R+\mathbf{e}_\Theta\otimes\mathbf{e}_\Theta+\mathbf{e}_Z\otimes\mathbf{e}_Z+\alpha\mathbf{e}_\Theta\otimes\mathbf{e}_R$$

where α is to be determined.

(f) Find the components of the deformation gradient **F** in $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$.

(g) Verify that the deformation is volume preserving

(h) Find the components of the right Cauchy-Green deformation tensors in $\{\mathbf{e}_R,\mathbf{e}_\Theta,\mathbf{e}_Z\}$

(i) Find the components of the left Cauchy-Green deformation tensor in $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$

(j) Find $\mathbf{F}^{-1}$ in $\{\mathbf{e}_R,\mathbf{e}_\Theta,\mathbf{e}_Z\}$.

(k) Write down the velocity field **v** in terms of (r,θ,z) in the basis $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$

(l) Calculate the spatial velocity gradient **L** in the basis $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$

Problem 2.38 Find a displacement field corresponding to a uniform infinitesimal strain field ε_{ij}. (Don't make this hard – think about what kind of function, when differentiated, gives a constant). Is the displacement unique?

Problem 2.39 Find a formula for the most general displacement field that generates *zero* infinitesimal strain

Problem 2.40 The infinitesimal strain field in a long cylinder containing a hole at its center is given by

$$\varepsilon_{31}=-bx_2/r^2 \qquad \varepsilon_{32}=bx_1/r^2 \qquad r=\sqrt{x_1^2+x_2^2}$$

(a) Show that the strain field satisfies the equations of compatibility.

(b) Show that the strain field is consistent with a displacement field of the form $u_3=\theta$, where $\theta=2b\tan^{-1}x_2/x_1$. Note that although the strain field is compatible, the displacement field is *multiple valued* – i.e. the displacements are not equal at $\theta=2\pi$ and $\theta=0$, which supposedly represent the same point in the solid. Of course, displacement fields like this do exist in solids – they are caused by dislocations in a crystal.

Problem 2.41 Calculate the displacement field that generates the following 3D infinitesimal strain field

$$\varepsilon_{ij}=(1+\nu)(x_kx_k\delta_{ij}+2x_ix_j)-(3-\nu)\delta_{ij}$$

Problem 2.42 Consider the 2D infinitesimal strain field

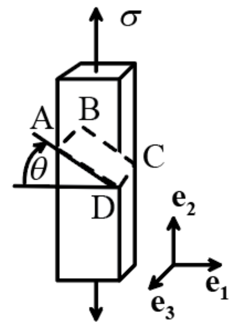
$$\varepsilon_{11}=\frac{1}{r^2}-\frac{2x_1^2}{r^4} \qquad \varepsilon_{22}=\frac{1}{r^2}-2\frac{x_2^2}{r^4} \qquad \varepsilon_{12}=-2\frac{x_1x_2}{r^4} \qquad r=\sqrt{x_1^2+x_2^2}$$

Show that the strain field is compatible, and find the corresponding displacement field.

2.3 Mathematical Description of Internal Forces in Solids

Problem 2.43 A rectangular bar is loaded in a state of uniaxial tension with magnitude σ, as shown in the figure.

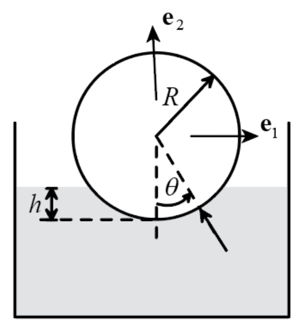
(a) Write down the components of the stress tensor in the bar, using the basis vectors shown.
(b) Calculate the components of the normal vector to the plane ABCD shown, and hence deduce the components of the traction vector acting on this plane, expressing your answer as components in the basis shown, in terms of θ
(c) Compute the normal and tangential tractions acting on the plane ABCD.



Problem 2.44 Consider a state of hydrostatic stress $\sigma_{ij} = p\delta_{ij}$. Show that the traction vector acting on any internal plane in the solid (or, more likely, fluid!) has magnitude p and direction normal to the plane.

Problem 2.45 A cylinder of radius R is partially immersed in a static fluid. Recall that the pressure at a depth d in a fluid has magnitude $\rho g d$.

(a) Write down an expression for the horizontal and vertical components of traction acting on the surface of the cylinder in terms of θ.
(b) Hence compute the resultant force (per unit out of plane distance) exerted by the fluid on the cylinder, in terms of ρ, g, h, R.



Problem 2.46 For the Cauchy stress tensor with components (in MPa)

$$\boldsymbol{\sigma} = \begin{bmatrix} 100 & 250 & 0 \\ 250 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}$$

compute:

(a) The traction vector acting on an internal material plane with normal $\mathbf{n} = (\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$
(b) The principal stresses
(c) The hydrostatic stress
(d) The deviatoric stress tensor
(e) The Von-Mises equivalent stress

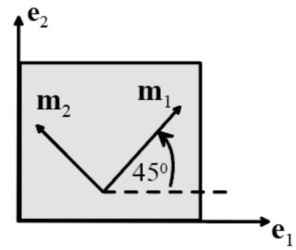
Problem 2.47 For the Cauchy stress tensor with component

$$\boldsymbol{\sigma} = \begin{bmatrix} 50 & 250 & 0 \\ 250 & 130 & -10 \\ 0 & -10 & 320 \end{bmatrix}$$

compute:

(a) The traction vector acting on an internal material plane with normal $\mathbf{n} = (\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$
(b) The principal stresses
(c) The hydrostatic stress
(d) The deviatoric stress tensor
(e) The Von-Mises equivalent stress (find the answer using the answers to both (b) and (d))

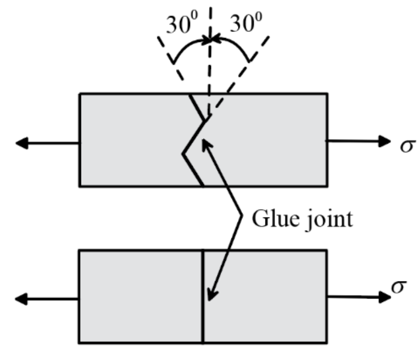
Problem 2.48 A sheet of material that lies in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ plane is subjected to a state of stress with the following properties:



- The stress is in a state of *plane stress*
- The principal stress directions are parallel to the $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ directions shown in the figure
- The hydrostatic stress is zero
- The Von-Mises stress has magnitude 250 MPa

(a) For the plane stress state, which components of stress are zero?
(b) Write down the formulas for hydrostatic stress and von-Mises stress in terms of the principal stresses. Hence, find the components of stress in the $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ basis (i.e. the principal basis). If you find more than one possible solution give them all…
(c) Hence, find the stress components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis

Problem 2.49 The figure shows two designs for a glue joint. They are subjected to a uniaxial stress σ. The glue will fail if the stress acting normal to the joint exceeds 60 MPa, or if the shear stress acting parallel to the plane of the joint exceeds 300 MPa.



(a) Calculate the normal and shear stress acting on each joint, in terms of the applied stress σ.
(b) Hence, calculate the value of σ that will cause each joint to fail.

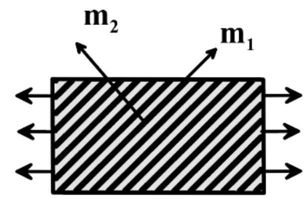
Problem 2.50 An internal surface plane that makes equal angles with each of the three principal stress directions is known as the *octahedral plane.* Show that the normal component of stress acting on this plane is $I_1/3$, where I_1 is the first invariant of the stress tensor, and that the magnitude of the shear traction acting on the plane is

$$\frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2} = \sqrt{-2I_2/3}$$

where $(\sigma_1, \sigma_2, \sigma_3)$ are the three principal stresses, and I_2' is the second invariant of the *deviatoric* stress tensor $S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$ (recall that the second invariant of S_{ij} is $I_2' = (S_{ii}S_{jj} - S_{ij}S_{ij})/2$)

Problem 2.51 Show that the hydrostatic stress σ_{kk} is invariant under a change of basis – i.e. if $\sigma_{ij}^{\mathbf{e}}$ and $\sigma_{ij}^{\mathbf{m}}$ denote the components of stress in bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$, respectively, show that $\sigma_{kk}^{\mathbf{e}} = \sigma_{kk}^{\mathbf{m}}$.

Problem 2.52 The 'Tsai-Hill' criterion is used to predict the critical loads that cause failure in brittle laminated fiber-reinforced composites and wood. A specimen of laminated composite subjected to in-plane loading is sketched in the figure. The Tsai-Hill criterion assumes that a plane stress state exists in the solid. Let $\sigma_{11}, \sigma_{22}, \sigma_{12}$ denote the nonzero components of stress, with basis vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ oriented parallel and perpendicular to the fibers in the sheet, as shown. The Tsai-Hill failure criterion is



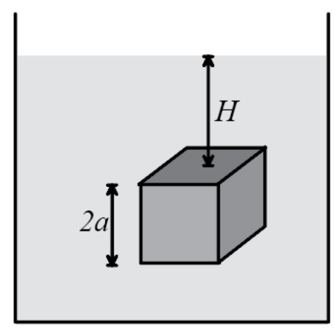
$$\left(\frac{\sigma_{11}}{\sigma_{TS1}}\right)^2 + \left(\frac{\sigma_{22}}{\sigma_{TS2}}\right)^2 - \frac{\sigma_{11}\sigma_{22}}{\sigma_{TS1}^2} + \frac{\sigma_{12}^2}{\sigma_{SS}^2} = 1$$

at failure, where σ_{TS1}, σ_{TS2} and σ_{SS} are material properties. (The material fails when the quantity on the left hand side of the equation is equal to 1). The table gives data for the properties of a graphite-epoxy CFRP.

Property	Value (MPa)
σ_{TS1}	2206
σ_{TS2}	56.5
σ_{SS}	110.3

Suppose that a specimen is loaded in uniaxial tension with the tensile axis at 45 degrees to the fiber direction, as shown in the figure. Calculate the maximum stress that the material can withstand (you will need to use the basis change formulas for a tensor).

Problem 2.53 A rigid, cubic solid is immersed in a fluid with mass density ρ. Recall that a stationary fluid exerts a compressive pressure of magnitude $\rho g h$ at depth h.

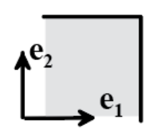


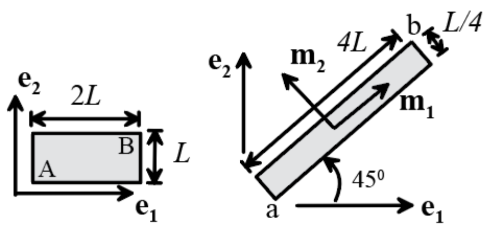
(a) Write down expressions for the traction vector exerted by the fluid on each face of the cube. You might find it convenient to take the origin for your coordinate system at the center of the cube, and take basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ perpendicular to the cube faces.

(b) Calculate the resultant force due to the tractions acting on the cube, and show that the vertical force is equal and opposite to the weight of fluid displaced by the cube.

(c) Show that the result of problem (b) applies to any arbitrarily shaped solid immersed below the surface of a fluid, i.e. prove that the resultant force acting on an immersed solid with volume V is $P_i = \rho g V \delta_{i3}$, where it is assumed that $\mathbf{e}_3$ is vertical.

Problem 2.54 A component contains a feature with a 90 degree corner as shown in the picture. The surfaces that meet at the corner are not subjected to any loading. List all the stress components that must be zero at the corner.





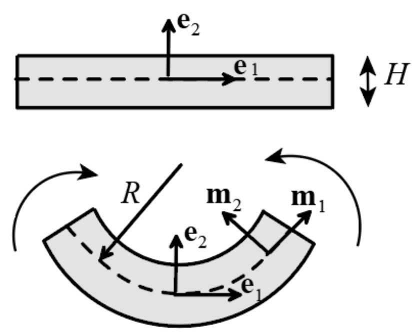
Problem 2.55 The figure shows the reference and deformed configurations for a solid. The out-of-plane dimensions are unchanged. Points a and b are the positions of points A and B after deformation. The Cauchy stress in the deformed solid is $\sigma \mathbf{m}_1 \otimes \mathbf{m}_1$. With the aid of Problem 2.11, determine:

(a) The components of Cauchy stress in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

(b) The components of nominal stress $\mathbf{S}$ in both $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$.

(c) The nominal stress in the mixed basis $S_{ij} \mathbf{e}_i \otimes \mathbf{m}_j$

(d) The components of material stress in both $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$



Problem 2.56 In this problem we consider further the beam bending calculation discussed in problem 2.36 (you should solve that problem before starting this one). Suppose that the beam is made from a material in which the material stress tensor is related to the Lagrange strain tensor by

$$\Sigma_{ij} = 2\mu E_{ij}$$

(this can be regarded as representing an elastic material with zero Poisson's ratio and shear modulus μ)

(a) Calculate the distribution of material stress in the bar, expressing your answer as components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis

(b) Calculate the distribution of nominal stress in the bar expressing your answer as components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis

(c) Calculate the distribution of Cauchy stress in the bar expressing your answer as components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis

(d) Repeat (a)-(c) but express the stresses as components in the $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ basis

(e) Calculate the distribution of traction on a surface in the beam that has normal $\mathbf{e}_1$ in the undeformed beam. Give expressions for the tractions in both $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$

(f) Show that the surfaces of the beam that have positions $x_2 = \pm H/2$ in the undeformed beam are traction free after deformation

(g) Calculate the resultant moment (per unit out of plane distance) acting on the ends of the beam.

Problem 2.57 Show that the von-Mises effective stress

$$\sigma_e = \sqrt{\frac{3}{2} \boldsymbol{\sigma}^{DEV} : \boldsymbol{\sigma}^{DEV}}$$

where $\boldsymbol{\sigma}^{DEV} = \boldsymbol{\sigma} - \text{trace}(\boldsymbol{\sigma})\mathbf{I}/3$ is the deviatoric stress, is invariant under a change of basis.

Problem 2.58 A solid is subjected to some loading that induces a Cauchy stress $\sigma_{ij}^{(0)}$ at some point in the solid. The solid and the loading frame are then rotated together so that the entire solid (as well as the loading frame) is subjected to a rigid rotation R_{ij}. This causes the components of the Cauchy stress tensor to change to new values $\sigma_{ij}^{(1)}$. The goal of this problem is to calculate a formula relating $\sigma_{ij}^{(0)}$, $\sigma_{ij}^{(1)}$ and R_{ij}.

(a) Let $n_i^{(0)}$ be a unit vector normal to an internal material plane in the solid before rotation. After rotation, this vector (which rotates with the solid) is $n_i^{(1)}$. Write down the formula relating $n_i^{(0)}$ and $n_i^{(1)}$

(b) Let $T_i^{(0)}$ be the internal traction vector that acts on a material plane with normal $n_i^{(0)}$ in the solid before application of the rigid rotation. Let $T_i^{(1)}$ be the traction acting on the same material plane after rotation. Write down the formula relating $T_i^{(0)}$ and $T_i^{(1)}$

(c) Finally, using the definition of Cauchy stress, find the relationship between $\sigma_{ij}^{(0)}$, $\sigma_{ij}^{(1)}$ and R_{ij}.

Problem 2.59 Repeat problem 2.58, but instead, calculate a relationship between the components of Nominal stress $S_{ij}^{(0)}$ and $S_{ij}^{(1)}$ before and after the rigid rotation.

Problem 2.60 Repeat problem 2.58, but instead, calculate a relationship between the components of material stress $\mathbf{\Sigma}_{ij}^{(0)}$ and $\mathbf{\Sigma}_{ij}^{(1)}$ before and after the rigid rotation.

Problem 2.61 One constitutive model for metallic glass (Anand and Su, J. Mech Phys. Solids **53** 1362 (2005)) assumes that plastic flow in the glass takes place by shearing on planes that are oriented at an angle θ to the principal stress directions, calculated as follows. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors parallel to the three principal stresses (with $\sigma_1 > \sigma_2 > \sigma_3$) and suppose that shearing takes place on a plane with normal $\mathbf{m}$, with shearing direction (tangent to the plane) $\mathbf{s}$. Then, let $\tau = \mathbf{m} \cdot \boldsymbol{\sigma} \cdot \mathbf{s}$ and $p_n = -\mathbf{m} \cdot \boldsymbol{\sigma} \cdot \mathbf{m}$ denote the resolved shear stress and (compressive) normal stress acting on the shear plane. The constitutive model assumes that shearing in the $\{\mathbf{e}_1, \mathbf{e}_3\}$ plane occurs on the plane for which

$$f(\theta) = \tau(\theta) - \mu p(\theta)$$

is a maximum with respect to θ. Here μ is a material property known as the 'internal friction coefficient.' Their paper (eq 71) states that 'it is easily shown that $f(\theta)$ is a maximum for

$$\theta = \left\{ \frac{\pi}{4} \pm \frac{\phi}{2} \right\} \qquad \phi = \tan^{-1}\mu \text{'}$$

Derive this result.

2.4 Equations of Motion and Equilibrium for Deformable Solids

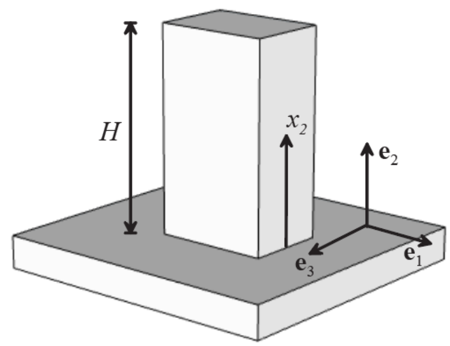
Problem 2.62 Show that the local mass balance equation

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} + \rho \frac{\partial v_i}{\partial y_i} = 0$$

can be re-written in spatial form as

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho v_i}{\partial y_i} = 0$$

Problem 2.63 A prismatic concrete column of mass density ρ supports its own weight, as shown in the figure. (Assume that the solid is subjected to a uniform gravitational body force of magnitude g per unit mass, and neglect deformation).



(a) Show that the stress distribution

$$\sigma_{22} = -\rho g (H - x_2)$$

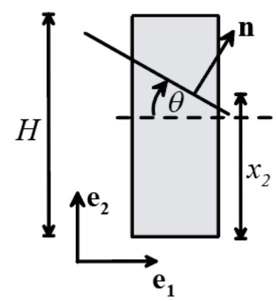
satisfies the equations of static equilibrium

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = 0$$

and also satisfies the boundary conditions $\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{0}$ on all free boundaries.

(b) Find a formula for the traction vector acting at a height x_2 on a plane with normal $\mathbf{n} = \sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_2$

(c) Find the normal and tangential tractions acting on the plane with normal $\mathbf{n} = \sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2$ at a height x_2 (see the figure)



(d) Suppose that the concrete contains a large number of randomly oriented microcracks. A crack which lies at an angle θ to the horizontal will propagate if

$$\left|\mathbf{T}_t\right| + \mu T_n > \tau$$

where μ is the friction coefficient between the faces of the crack and τ is a critical shear stress that is related to the size of the microcracks and the fracture toughness of the concrete. Assume that $\mu = 1$. Find the orientation of the microcrack that is most likely to propagate. Hence, find an expression for the maximum possible height of the column.

Problem 2.64 Is the stress field given below in static equilibrium? If not, find the acceleration or body force density required to satisfy linear momentum balance

$$\sigma_{11} = 2Cx_1x_2 \qquad \sigma_{12} = \sigma_{21} = C(a^2 - x_2^2)$$
$$\sigma_{22} = \sigma_{33} = \sigma_{23} = \sigma_{13} = 0$$

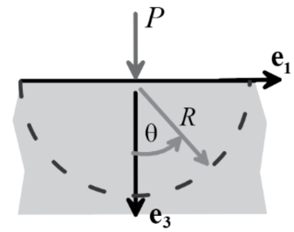
Problem 2.65 Let ϕ be a twice differentiable, scalar function of position. Derive a plane stress field from ϕ by setting

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} \qquad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} \qquad \sigma_{12} = \sigma_{21} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$$

(all other stress components are zero). Show that this stress field satisfies the equations of stress equilibrium with zero body force.

Problem 2.66 The stress field

$$\sigma_{ij} = \frac{-3Px_3x_ix_j}{2\pi R^5} \qquad R = \sqrt{x_kx_k}$$

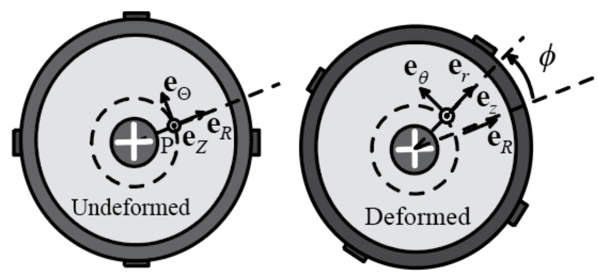


represents the stress in an infinite, incompressible linear elastic half-space that is subjected to a point force with magnitude P acting perpendicular to the surface at the origin (you can visualize a point force as a very large body force which is concentrated in a very small region).

(a) Verify that the stress field is in static equilibrium (assume small displacements, consistent with linear elasticity theory so that $\partial / \partial y_i \approx \partial / \partial x_i$)

(b) Consider a hemi-spherical region of material with radius *R* centered at the origin, as shown by the dashed line in the figure. This region is subjected to (1) the body force acting at the origin; and (2) a force exerted by the stress field on the outer surface of the sphere. Calculate the resultant force exerted on the outer surface of the sphere by the stress, and show that it is equal in magnitude and opposite in direction to the body force.

Problem 2.67 In this problem, we consider the internal forces in the polymer specimen described in Problem 2.37 (you will need to solve 2.37 before you can attempt this one). Suppose that the specimen is homogeneous, has mass density ρ in the reference configuration, and may be idealized as a viscous fluid, in which the Kirchhoff stress is related to stretch rate by



$$\boldsymbol{\tau} = 2\mu\mathbf{D} + p\mathbf{I}$$

where *p* is an indeterminate hydrostatic pressure, **D** is the stretch rate, and μ is the viscosity.

(a) Find expressions for the Cauchy stress tensor as functions of radial position *r* and time *t*, expressing your answer as components in the $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ basis

(b) Assuming quasi-static deformation (neglect accelerations), express the equations of equilibrium in terms of $\phi(R,t)$

(c) Solve the governing equation to calculate $\phi(r,t)$

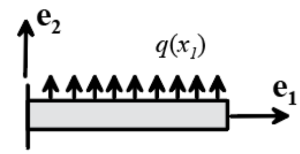
(d) Find the torque necessary to rotate the external cylinder

(e) Calculate the acceleration of a material particle in the fluid

(f) Estimate the rotation rate $\dot{\alpha}$ where inertia begins to play a significant role in determining the state of stress in the fluid

2.5 Work Done by Stresses; Principle of Virtual Work

Problem 2.68 The figure shows a cantilever beam that is subjected to surface loading $q(x_1)$ per unit length. The state of stress in the beam can be approximated by $\sigma_{11} = M(x_1)x_2 / I$, where $I = \int_A x_2^2 dA$ is the area moment of inertia of the beam's cross section and $M(x_1)$ is an arbitrary function (all other stress components are zero).



(a) By considering a virtual velocity field of the form

$$\delta v_1 = -\frac{dw(x_1)}{dx_1}x_2 \qquad \delta v_2 = w(x_1)$$

where $w(x_1)$ is an arbitrary function satisfying $w = 0$ at $x_1 = 0$, show that the beam is in static equilibrium if

$$\int_0^L M(x_1)\frac{d^2 w}{dx_1^2}dx_1 + \int_0^L q(x_1)w dx_1 = 0$$

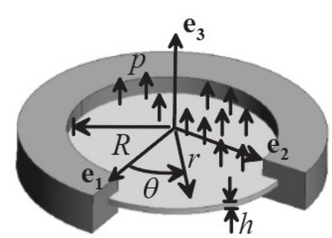
(b) By integrating the first integral expression by parts twice, show that the equilibrium equation and boundary conditions for $M(x_1)$ are

$$\frac{d^2 M}{dx_1^2} + q(x_1) = 0 \qquad M(x_1) = \frac{dM(x_1)}{dx_1} = 0 \qquad x_1 = L$$

Problem 2.69 The figure shows a plate with a clamped edge that is subjected to a pressure $p(x_1, x_2)$ on its surface. The state of stress in the plate can be approximated by

$$\sigma_{\alpha\beta} = 12M_{\alpha\beta}(x_1, x_2)x_3 / h^3 \qquad \sigma_{33} = \sigma_{3\alpha} = 0$$

where the subscripts α, β can have values 1 or 2, and $M_{\alpha\beta}(x_1, x_2)$ is a tensor valued function, and x_3 is the height above the mid-plane of the plate.



(a) By considering a virtual velocity of the form

$$\delta v_\alpha = -\frac{\partial w}{\partial x_\alpha}x_3 \qquad \delta v_3 = w(x_1, x_2)$$

where $w(x_1, x_2)$ is an arbitrary function satisfying $w = 0$ on the edge of the plate, show that the plate is in static equilibrium if

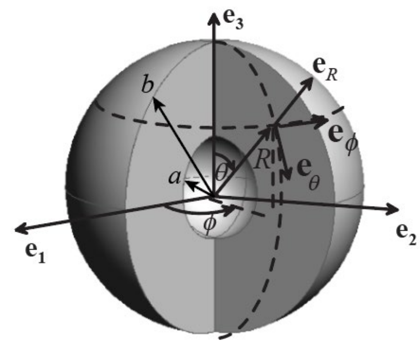
$$\int_A M_{\alpha\beta}(x_1)\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta}dA + \int_A p(x_1, x_2)w dA = 0$$

(b) By applying the divergence theorem appropriately, show that the governing equation for $M_{\alpha\beta}(x_1, x_2)$ is

$$\frac{\partial^2 M_{\alpha\beta}}{\partial x_\alpha \partial x_\beta} + p = 0$$

Problem 2.70 The shell shown in the figure is subjected to a radial body force $\mathbf{b} = \rho b(R)\mathbf{e}_R$ per unit volume, and a radial pressure p_a, p_b acting on the surfaces at $R = a$ and $R = b$. The loading induces a spherically symmetric state of stress in the shell, which can be expressed in terms of its components in a spherical-polar coordinate system as

$$\boldsymbol{\sigma} = \sigma_{RR}\mathbf{e}_R \otimes \mathbf{e}_R + \sigma_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_{\phi\phi}\mathbf{e}_\phi \otimes \mathbf{e}_\phi .$$



(a) By considering a virtual velocity of the form $\delta\mathbf{v} = w(R)\mathbf{e}_R$, show that the stress state is in static equilibrium if

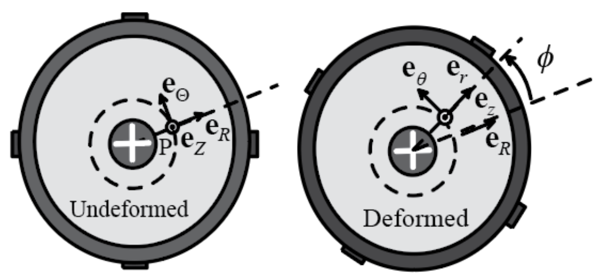
$$\int_a^b \left\{ \sigma_{RR}\frac{dw}{dR} + \left(\sigma_{\theta\theta} + \sigma_{\phi\phi}\right)\frac{w}{R} \right\} 4\pi R^2 dR - \int_a^b \rho b(R) w(R) 4\pi R^2 dR - 4\pi a^2 p_a w(a) + 4\pi b^2 p_b w(b) = 0$$

for all *w(R)*.

(b) Hence, show that the stress state must satisfy

$$\frac{d\sigma_{RR}}{dR} + \frac{1}{R}\left(2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) + b = 0 \qquad \sigma_{RR} = -p_a \ \left(R = a\right) \quad \sigma_{RR} = -p_b \ \left(R = b\right)$$

Problem 2.71 In this problem, we consider the internal dissipation in the polymer specimen described in Problem 2.37 and 2.67 (you will need to solve these two problems before you can attempt this one). Suppose that the specimen is homogeneous, has mass density ρ in the reference configuration, and may be idealized as a viscous fluid, in which the Kirchhoff stress is related to stretch rate by



$$\boldsymbol{\tau} = \mu\mathbf{D} + p\mathbf{I}$$

where p is an indeterminate hydrostatic stress and μ is the viscosity.

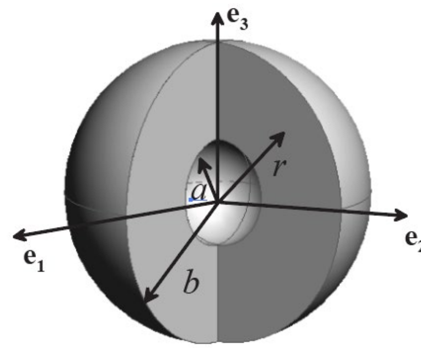
(a) Calculate the rate of external work done by the torque acting on the rotating exterior cylinder

(b) Calculate the rate of internal dissipation in the solid as a function of *r*.

(c) Show that the total internal dissipation is equal to the external work done on the specimen.

Problem 2.72 A solid with volume V is subjected to a distribution of traction t_i on its surface. Assume that the solid is in static equilibrium (this requires that t_i exerts no resultant force or moment on the boundary). By considering a virtual velocity of the form $\delta v_i = A_{ij} y_j$, where A_{ij} is a constant tensor, use the principle of virtual work to show that the average stress in a solid can be computed from the shape of the solid and the tractions acting on its surface using the expression

$$\frac{1}{V}\int_V \sigma_{ij} dV = \frac{1}{V}\int_S \frac{1}{2}\left(t_i y_j + t_j y_i\right) dA$$



Problem 2.73 A thick-walled spherical shell is made from an incompressible linear viscous material, in which the Cauchy stress is related to the stretch rate **D** by

$$\boldsymbol{\sigma} = 2\mu \mathbf{D} + p\mathbf{I}$$

where p is a hydrostatic stress to be determined, and μ is a material property (viscosity). The solid is subjected to a radial gravitational body force

$$\rho \mathbf{b} = -\rho B_0 \frac{r-a}{b-a}\mathbf{e}_R$$

where $r = \sqrt{y_k y_k}$ is the radial coordinate of a material particle in the deformed shell.

(a) Assume that the velocity field in the shell is radial $\mathbf{v} = v(r)\mathbf{e}_r$. Calculate the velocity gradient and stretch rate.

(b) Show that the incompressibility condition implies that

$$\frac{\partial v}{\partial r} + \frac{2}{r}v = 0$$

and hence find an expression for *v(r)* in terms of $v(a) = \dot{a}$.

(c) Hence, find an expression for the particle acceleration $\left.\frac{dv}{dt}\right|_{\mathbf{x}}$, in terms of r, $\dot{a}$ and $\ddot{a}$ Be careful with this – it is not just the partial time derivative of $v(r)$

(d) Find an expression for the (total) rate of work done on the shell by gravity.

(e) Find an expression for the total kinetic energy of the shell, in terms of $\dot{a}$

(f) Calculate the time derivative of kinetic energy. Note that b is not constant.

(g) Find the total internal stress power, in terms of $\dot{a}$, μ

(h) Use the principle of virtual work to show that the stress state must satisfy

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) - \rho B_0 \frac{r-a}{b-a} = \rho \left.\frac{dv}{dt}\right|_{\mathbf{x}}$$

(i) Write down the boundary conditions for σ_{rr} at $r = a, b$.

(j) Hence, show that *a(t)* satisfies the differential equation

$$\frac{d^2a}{dt^2} = -\frac{1}{2}B_0\frac{b}{a} - 4\frac{\mu}{\rho}\frac{da}{dt}\frac{(a^2+ab+b^2)}{a^2b^2} - \frac{1}{2}\left(\frac{da}{dt}\right)^2\frac{(b-a)(a^2+2ab+3b^3)}{ab^3}$$

(k) Verify the result of (j) using energy methods

(l) Show that the differential equation for *a(t)* can be expressed in dimensionless form as

$$\frac{d^2\alpha}{d\tau^2} = -\frac{a(0)^3\rho^2B_0}{\mu^2}\frac{\beta}{2\alpha} - 4\frac{d\alpha}{d\tau}\frac{\alpha^2+\beta^2+\alpha\beta}{\alpha^2\beta^2} - \left(\frac{d\alpha}{d\tau}\right)^2\frac{(\beta-\alpha)(\alpha^2+2\alpha\beta+3\beta^2)}{2\alpha\beta^3}$$

where $\tau = t\mu/(a(0)^2\rho)$ $\alpha = a(t)/a(0)$ $\beta = b(t)/a(0)$. Hence, plot $\alpha(\tau)$ with initial conditions $\beta(0) = 5$, $\dot{\alpha}(0) = 0$ for $B_0 a(0)^3\rho^2/\mu^2 = 0.5, 1, 1.5, 3.0$ (you will need to solve the differential equation numerically).

2.6 The Laws of Thermodynamics for Deformable Solids

Problem 2.74 Define the *expended power* of external forces acting on a deformable solid that occupies a region R surrounded by a surface S (which could be a sub-volume within a larger body) by

$$W_{\exp} = \int_S \mathbf{t}\cdot\mathbf{v} + \int_R \rho\mathbf{b}\cdot\mathbf{v} - \frac{d}{dt}\int_R \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}$$

Show that the expended power is zero for any rigid velocity field of the form

$$\mathbf{v}(\mathbf{y},t) = \mathbf{v}_0(t) + \boldsymbol{\omega}(t)\times(\mathbf{y}-\mathbf{y}_0)$$

where $\mathbf{v}_0(t)$, $\boldsymbol{\omega}(t)$ are vector valued functions of time (but independent of position).

Problem 2.75 It is helpful to have versions of the first and second laws of thermodynamics in terms of quantities defined on the reference configuration. To this end, define the following variables:

- Deformation mapping $y_i(x_j)$
- Temperature θ
- Reference mass density ρ_0
- Specific internal energy ε
- Specific Helmholtz free ehergy ψ
- Specific entropy s
- External heat supply per unit deformed volume q
- Nominal stress and deformation gradient $\mathbf{S}$, $\mathbf{F}$
- Jacobian $J = \det(\mathbf{F})$
- Material stress and Lagrange strain rate $\boldsymbol{\Sigma}$, $\mathbf{E}$
- Referential heat flux $\mathbf{Q} = J\mathbf{F}^{-1}\mathbf{q}$

With these definitions, show the following identities:

$$\rho_0 \left.\frac{\partial\varepsilon}{\partial t}\right|_{\mathbf{x}=const} = S_{ij}\frac{dF_{ji}}{dt} - \frac{\partial Q_i}{\partial x_i} + Jq$$

$$\rho_0 \left.\frac{\partial\varepsilon}{\partial t}\right|_{\mathbf{x}=const} = \boldsymbol{\Sigma}_{ij}\frac{dE_{ij}}{dt} - \frac{\partial Q_i}{\partial x_i} + Jq$$

$$S_{ij}\frac{dF_{ji}}{dt} - \frac{1}{\theta}Q_i\frac{\partial\theta}{\partial x_i} - \rho_0\left(\frac{\partial\psi}{\partial t} + s\frac{\partial\theta}{\partial t}\right) \ge 0$$

$$\boldsymbol{\Sigma}_{ij}\frac{dE_{ij}}{dt} - \frac{1}{\theta}Q_i\frac{\partial\theta}{\partial x_i} - \rho_0\left(\frac{\partial\psi}{\partial t} + s\frac{\partial\theta}{\partial t}\right) \ge 0$$

Problem 2.76 Starting with the local form of the second law of thermodynamics and mass conservation

$$\rho\left.\frac{\partial s}{\partial t}\right|_{\mathbf{x}=const} + \frac{\partial(q_i/\theta)}{\partial y_i} - \frac{q}{\theta} \ge 0 \qquad \left.\frac{\partial\rho}{\partial t}\right|_{\mathbf{y}} + \frac{\partial\rho v_i}{\partial y_i} = 0$$

(the symbols have their usual meaning), derive the statement of the second law for a control volume

$$\frac{\partial}{\partial t}\int_R \rho s dV + \int_B \rho s(\mathbf{v}\cdot\mathbf{n})dA + \int_B \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}dA - \int_R \frac{q}{\theta}dV \ge 0$$

2.7 Transformation of Variables under Changes of Reference Frame

Problem 2.77 Determine how the following quantities transform under a change of observer

(a) The spatial heat flux vector $\mathbf{q}$ (recall that $\mathbf{q}\cdot\mathbf{n}dA$ gives the heat flux across a surface element with area dA and normal $\mathbf{n}$, and note that $\mathbf{n}$ is an objective vector, and that the all observers must see the same heat flux….)

(b) The referential heat flux vector $\mathbf{\Theta} = J\mathbf{F}^{-1}\mathbf{q}$ (we normally use $\mathbf{Q}$ to denote referential heat flux but in this section that symbol is already used to denote the rotation of the observer's frame).

(c) The spatial gradient of a scalar function of position in a deformed solid $\mathbf{g} = \nabla_{\mathbf{y}}\phi(\mathbf{y})$

(d) The material gradient of a scalar function of particle position in a solid $\mathbf{G} = \nabla\phi(\mathbf{x})$

Problem 2.78 The expended power of external forces acting on a deformable solid was defined in Problem 2.76 as

$$W_{\exp}(\mathbf{t},\mathbf{b},\frac{d\mathbf{y}}{dt}) = \int_S \mathbf{t}\cdot\frac{d\mathbf{y}}{dt} + \int_R \rho\mathbf{b}\cdot\frac{d\mathbf{y}}{dt} - \frac{d}{dt}\int_R \frac{1}{2}\rho\frac{d\mathbf{y}}{dt}\cdot\frac{d\mathbf{y}}{dt}$$

Show that the external power is invariant to all changes of observer if and only if linear momentum and angular momentum are conserved, i.e.

$$W_{\exp}(\mathbf{t}^*,\mathbf{b}^*,\frac{d\mathbf{y}^*}{dt}) = W_{\exp}(\mathbf{t},\mathbf{b},\frac{d\mathbf{y}}{dt}) \qquad \forall \qquad \mathbf{y}^* = \mathbf{y}_0^*(t) + \mathbf{Q}(t)(\mathbf{y}-\mathbf{y}_0) \;\;)$$

$$\Leftrightarrow \int_A \mathbf{t}dA + \int_R \rho\mathbf{b}dV = \frac{d}{dt}\int_R \rho\frac{d\mathbf{y}}{dt}dV \qquad\qquad \int_A \mathbf{y}\times\mathbf{t}dA + \int_R \mathbf{y}\times\rho\mathbf{b}dV = \frac{d}{dt}\int_R \mathbf{y}\times\rho\mathbf{v}dV$$

(Suggestion: It is best to re-write the kinetic energy term in terms of accelerations first. Then note that

$$\mathbf{Q}^T\frac{d\mathbf{y}_0^*}{dt} + \mathbf{Q}^T\frac{d\mathbf{Q}}{dt}(\mathbf{y}-\mathbf{y}_0)$$

can be interpreted as the velocity field associated with a rigid body motion. You can use many of the steps from Problem 2.76)

Chapter 3
Constitutive Models:
Relations between Stress and Strain

3.1 General Requirements for Constitutive Equations

Problem 3.1. Consider a linear viscous constitutive equation of the form

$$\sigma_{ij} = -p(\rho,\theta)\delta_{ij} + \tau_{ij}^{0} + 2\mu D_{ij}$$

with τ_{ij}^{0} and μ constants, ρ,θ the density and temperature, p a positive function and D_{ij} the stretch rate. Show that this constitutive law violates the second law of thermodynamics.

Problem 3.2. Suppose that the deformation of a viscoelastic material is modeled by representing the deformation gradient **F** of a material element as a sequence of an irreversible deformation $\mathbf{F}^{p}$, followed by a reversible (elastic) deformation $\mathbf{F}^{e}$, so that $\mathbf{F} = \mathbf{F}^{e}\mathbf{F}^{p}$. The Helmholtz free energy $\psi(\mathbf{F}^{e},\theta)$ of the material is assumed to be a function of $\mathbf{F}^{e}$ and temperature θ only.

(a) Show that the velocity gradient **L** can be decomposed into elastic and plastic parts as

$$\mathbf{L} = \mathbf{L}^{e} + \mathbf{L}^{p} \qquad \mathbf{L}^{e} = \frac{d\mathbf{F}^{e}}{dt}\mathbf{F}^{e-1} \qquad \mathbf{L}^{p} = \mathbf{F}^{e}\frac{d\mathbf{F}^{p}}{dt}\mathbf{F}^{p-1}\mathbf{F}^{e-1}$$

(b) Show that the dissipation inequality

$$\sigma_{ij}D_{ij} - \frac{1}{\theta}q_i\frac{\partial\theta}{\partial y_i} - \rho\left(\frac{\partial\psi}{\partial t} + s\frac{\partial\theta}{\partial t}\right) \ge 0$$

requires that the Cauchy stress is related to the free energy by

$$JF_{kj}^{e-1}\sigma_{ji} = \rho_0\frac{\partial\psi}{\partial F_{ik}^{e}}$$

(where ρ_0 is the mass per unit reference volume) and that the plastic part of the velocity gradient must satisfy

$$\sigma_{ij}L_{ij}^{p} \ge 0$$

(c) Assume that $\mathbf{F}^{e}$ and $\mathbf{F}^{p}$ transform under a change of observer according to $\mathbf{F}^{e*} = \mathbf{Q}\mathbf{F}^{e}$, $\mathbf{F}^{p*} = \mathbf{F}^{p}$. Verify that the transformation is consistent with the transformation of deformation gradient **F** under an observer change, and determine expressions for $\mathbf{L}^{e*},\mathbf{L}^{p*}$ in terms of $\mathbf{Q}$ and $\boldsymbol{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^{T}$.

(d) Consider a constitutive relation in which the plastic velocity gradient is given by

$$L_{ij}^{p} = \eta\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)$$

Show that if $\det(\mathbf{F}^{p}) = 1$ at time *t=0*, then $\det(\mathbf{F}^{p}) = 1$ for all *t>0*. (Hint: consider L_{kk}^{p})

(e) Show that the constitutive relation in (d) satisfies both frame indifference and the dissipation inequality (assume $\eta > 0$).

3.2 Linear Elastic Material Behavior

Problem 3.3. Using the table of values given in Section 3.2.4 of Applied Mechanics of Solids, find values of bulk modulus, Lame modulus, and shear modulus for mild steel, aluminum and rubber (calculate a range for the latter two).

Problem 3.4. Show that, at constant temperature, the isotropic linear elastic stress-strain relation can be re-written as two separate equations

$$S_{ij} = 2\mu e_{ij} \qquad p = K e_v$$

where μ, K are the shear and bulk modulus,

$$e_v = \varepsilon_{kk} \qquad e_{ij} = \varepsilon_{ij} - e_v \delta_{ij} / 3$$

are the volumetric and deviatoric strains, while

$$S_{ij} = \sigma_{ij} - p\delta_{ij} \qquad p = \sigma_{kk} / 3$$

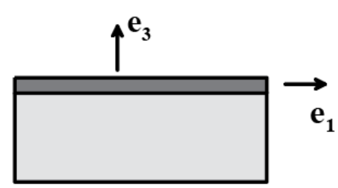
are the deviatoric and hydrostatic stress.

Problem 3.5. A specimen of material is placed inside a rigid box that prevents the material from stretching in any direction. This means that the strains in the specimen are zero. The specimen is then heated to increase its temperature by ΔT.

(a) Find a formula for the stress in the specimen.
(b) Find a formula for the strain energy density.
(c) How much strain energy would be stored in a 1cm^3 sample of steel if its temperature were increased by 100C? Compare the strain energy with the heat required to change the temperature by 100C – the specific heat capacity of steel is about 470 J/(kg-K).

Problem 3.6. A specimen of an isotropic, linear elastic solid is free of stress, and is heated to increase its temperature by ΔT. Find expressions for the strain and displacement fields in the solid.

Problem 3.7. A thin isotropic, linear elastic thin film with Young's modulus E, Poisson's ratio ν and thermal expansion coefficient α is bonded to a stiff substrate with zero thermal expansion coefficient. The film is stress free at some initial temperature, and then heated to increase its temperature by T. The substrate prevents the film from stretching in its own plane, so that $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{12} = 0$, while the surface is traction free, so that the film deforms in a state of plane stress. Calculate the stresses in the film in terms of material properties and temperature, and deduce an expression for the strain energy density in the film.



Problem 3.8. Suppose that the stress-strain relation for a linear elastic solid is expressed in matrix form as $\underline{\sigma} = [C]\underline{\varepsilon}$, where $\underline{\sigma}$, $\underline{\varepsilon}$ and $[C]$ represent the stress and strain vectors and the matrix of elastic constants defined in Section 3.2.8. Show that the material has a positive definite strain energy density ($U > 0 \ \forall [\varepsilon] \neq 0$) if and only if the eigenvalues of $[C]$ are all positive.

Problem 3.9. Write down an expression for the increment in stress resulting from an increment in strain applied to a linear elastic material, in terms of the matrix of elastic constants $[C]$. Hence show that, for a linear elastic material to be stable in the sense of Drucker, the eigenvalues of the matrix of elastic constants $[C]$ must all be positive or zero.

Problem 3.10. Let $\underline{\sigma}$, $\underline{\varepsilon}$ and $[C]$ represent the stress and strain vectors and the matrix of elastic constants in the isotropic linear elastic constitutive equation

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}-\frac{E\alpha\Delta T}{1-2\nu}\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$$

(a) Calculate the eigenvalues of the stiffness matrix $[C]$ for an isotropic solid in terms of Young's modulus and Poisson's ratio. Hence, show that the eigenvalues are positive (a necessary requirement for the material to be stable – see problem 3.9) if and only if $-1<\nu<1/2$ and $E>0$.

(b) Find the eigenvectors of $[C]$ and briefly describe the deformations associated with these eigenvectors.

Problem 3.11. A cubic material may be characterized either by its moduli as

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{bmatrix}=\begin{bmatrix}c_{11} & c_{12} & c_{12} & 0 & 0 & 0\\ & c_{11} & c_{12} & 0 & 0 & 0\\ & & c_{11} & 0 & 0 & 0\\ & sym & & c_{44} & 0 & 0\\ & & & 0 & c_{44} & 0\\ & & & 0 & 0 & c_{44}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{13}\\ 2\varepsilon_{23}\end{bmatrix}$$

or by the engineering constants

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{13}\\ 2\varepsilon_{23}\end{bmatrix}=\begin{bmatrix}1/E & -\nu/E & -\nu/E & 0 & 0 & 0\\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0\\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0\\ 0 & 0 & 0 & 1/\mu & 0 & 0\\ 0 & 0 & 0 & 0 & 1/\mu & 0\\ 0 & 0 & 0 & 0 & 0 & 1/\mu\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{bmatrix}$$

Calculate formulas relating c_{ij} to E,ν and μ, and deduce an expression for the anisotropy factor $A=2\mu(1+\nu)/E$ in terms of c_{ij}

Problem 3.12. Let $\underline{\sigma}$, $\underline{\varepsilon}$ and $[C]$ represent the stress and strain vectors and the matrix of elastic constants in the isotropic linear elastic constitutive equation for a cubic crystal

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{12}\\ \sigma_{13}\end{bmatrix}=\begin{bmatrix}c_{11} & c_{12} & c_{12} & 0 & 0 & 0\\ & c_{11} & c_{12} & 0 & 0 & 0\\ & & c_{11} & 0 & 0 & 0\\ & sym & & c_{44} & 0 & 0\\ & & & 0 & c_{44} & 0\\ & & & 0 & 0 & c_{44}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{12}\\ 2\varepsilon_{13}\end{bmatrix}$$

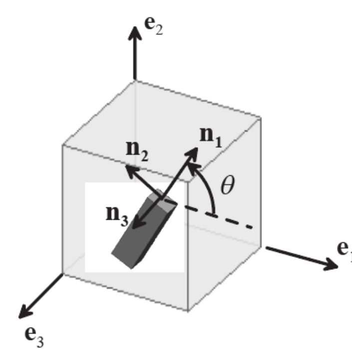
Calculate the eigenvalues of the stiffness matrix $[C]$ and hence find expressions for the admissible ranges of c_{11}, c_{12}, c_{44} for the eigenvalues to be positive.

Problem 3.13. Let $C^{\mathbf{e}}_{ijkl}$ denote the components of the elasticity tensor in a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Let $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ be a second basis, and define $\mathbf{\Omega}_{ij} = \mathbf{m}_i \cdot \mathbf{e}_j$. Recall that the components of the stress and strain tensor in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ are related by $\sigma^{(\mathbf{m})}_{ij} = \mathbf{\Omega}_{ik}\sigma^{(\mathbf{e})}_{kl}\mathbf{\Omega}_{jl}$ $\quad \varepsilon^{(\mathbf{m})}_{ij} = \mathbf{\Omega}_{ik}\varepsilon^{(\mathbf{e})}_{kl}\mathbf{\Omega}_{jl}$. Use this result, together with the elastic constitutive equation, to show that the components of the elasticity tensor in $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ can be calculated from

$$C^{(\mathbf{m})}_{ijkl} = \mathbf{\Omega}_{ip}\mathbf{\Omega}_{jq}C^{(\mathbf{e})}_{pqrs}\mathbf{\Omega}_{kr}\mathbf{\Omega}_{ls}$$

Problem 3.14. Consider a cube-shaped specimen of an *anisotropic*, linear elastic material. The solid is placed inside a rigid box that prevents the material from stretching in any direction. This means that the strains in the specimen are zero. The tensor of elastic moduli and the thermal expansion coefficient for the solid (expressed as components in a basis aligned with the edges of the box) are C_{ijkl}, α_{ij}. The specimen is heated to increase its temperature by ΔT. Find a formula for the strain energy density in the material, and show that the result is independent of the orientation of the material with respect to the box.

Problem 3.15. The figure shows a cubic crystal. Basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are aligned perpendicular to the faces of the cubic unit cell. A tensile specimen is cut from the cube – the axis of the specimen lies in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ plane and is oriented at an angle θ to the $\mathbf{e}_1$ direction. The specimen is then loaded in uniaxial tension σ_{nn} parallel to its axis. This means that the stress in the basis $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ shown in the picture is $\sigma_{nn}\mathbf{n}_1 \otimes \mathbf{n}_1$



(a) Use the basis change formulas for tensors to calculate the components of stress in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis in terms of θ.
(b) Use the stress-strain equations in Section 3.1.16 of Applied Mechanics of Solids to find the strain components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis, in terms of the engineering constants E, ν, μ Use the basis change formulas for tensors to calculate the components of stress in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis in terms of θ.

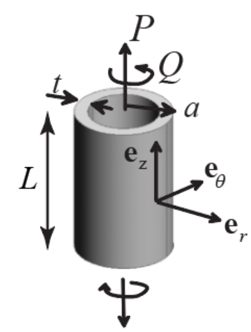
(c) Use the basis change formulas again to calculate the strain components in the $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ basis oriented with the specimen. Again, you need only calculate $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$. Check your answer by setting $\mu = E / \left[2(1+\nu)\right]$ - this makes the crystal isotropic, and you should recover the isotropic solution. It is easiest to do this calculation on the computer.

(d) Define the effective axial Young's modulus of the tensile specimen as $E(\theta) = \sigma_{nn} / \varepsilon_{nn}$, where $\varepsilon_{nn} = \mathbf{n}_1 \cdot \boldsymbol{\varepsilon} \mathbf{n}_1$ is the strain component parallel to the $\mathbf{n}_1$ direction. Find a formula for $E(\theta)$ in terms of E, ν, μ.

(e) Using data for copper from Section 3.1.17 of Applied Mechanics of Solids, plot a graph of $E(\theta)$ against θ. For copper, what is the orientation that maximizes the longitudinal stiffness of the specimen? Which orientation minimizes the stiffness?

3.3 Hypoelasticity

Problem 3.16. A thin-walled tube with radius *a*, length *L* and wall thickness *t* can be idealized using the hypoelastic constitutive equation described in Section 3.3 of Applied Mechanics of Solids. You may assume that the axial load induces a uniaxial stress $\sigma_{zz} = P / (2\pi a t)$ while the torque induces a shear stress $\sigma_{z\theta} = Q / (2\pi a^2 t)$. The shear strains are related to the rotation of the top of the tube ϕ by $\varepsilon_{z\theta} = \phi / (2L)$, while the axial strains are related to the extension Δ of the tube by $\varepsilon_{zz} = \Delta / L$.



(a) Calculate a relationship between the axial load *P* and the extension Δ for a tube subjected only to axial loading

(b) Calculate a relationship between the torque *Q* and the twist ϕ for a tube subjected only to torsional loading

(c) Calculate a relationship between *P, Q* and Δ, ϕ for a tube subjected to combined axial and torsional loading.

Problem 3.17. Consider a material with the hypoelastic constitutive equation described in Section 3.3 of Applied Mechanics of Solids. Calculate an expression for the tangent stiffness $C_{ijkl} = \left(\partial\sigma_{ij} / \partial\varepsilon_{kl} + \partial\sigma_{ij} / \partial\varepsilon_{lk}\right) / 2$. (The two terms in the derivative may look surprising – to see where this comes from, note that the derivative of a symmetric tensor with respect to another symmetric tensor has an indeterminate skew part. It is helpful to choose the indeterminate part so that the tangent stiffness has the expected symmetry). Hence, show that the material is stable in the sense of Drucker as long as $K > 0$, $\sigma_0 / \varepsilon_0 > 0$ *n>0.*

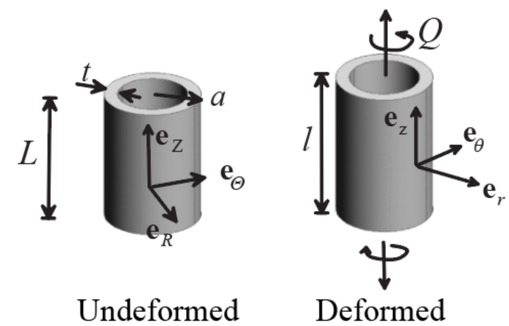
3.4 Generalized Hooke's Law: Materials subjected to small strains and large rotations

Problem 3.18. A uniaxial tensile specimen with initial length L and cross-sectional area A is idealized with a constitutive law that relates the material stress Σ_{ij} to the Lagrange strain E_{ij} by

$$\Sigma_{ij} = \frac{E}{1+\nu}\left\{ E_{ij} + \frac{\nu}{1-2\nu} E_{kk}\delta_{ij} \right\}$$

where E and ν are elastic constants. The specimen is subjected to a uniaxial force P which induces an extension δ. Calculate the relationship between P and δ, and compare the results with the predictions of a linear elastic constitutive equation.

Problem 3.19. A thin walled tube with initial length L, radius a and wall thickness $t \ll a$ is subjected to a torque Q, with no axial force. The tube can be idealized using the constitutive equation described in the preceding problem. Assume that, during deformation,



(i) Plane cross- sections of the tube remain plane;
(ii) Cross sections of the tube rotate through an angle $\alpha(z) = \phi z / l$, where z is the height of a cross section above the base; l is the length of the deformed tube, and ϕ is the rotation of the cross-section at the top end of the tube;
(iii) The radial, circumferential and axial stretches in the tube are $\lambda_R, \lambda_\Theta, \lambda_Z$ (which need to be calculated as part of the solution).
(iv) The only nonzero components of Cauchy stress are $\sigma_{z\theta}$, $\sigma_{z\theta} = \sigma_{\theta z} = Q/(2\pi a^2 t)$

Let $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ be a cylindrical polar coordinate basis located at a material particle in the undeformed tube, and let $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ denote the cylindrical-polar basis located at the position of the same material particle in the deformed tube.

(a) Find formulas for $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ in terms of $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$, α
(b) Assume that the deformation of the tube can be constructed as (i) a radial, circumferential and axial stretch $\lambda_R, \lambda_\Theta, \lambda_Z$ parallel to the $\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z$ direction (you can think of the stretches as changing the wall thickness; the radius, and the length of the tube, respectively), followed by (ii) A twist, in which cross-sections remain plane, do not change their height or shape, and rotate through the angle $\alpha(z) = \phi z / l = \phi z / (\lambda_Z L)$. Write down the deformation gradient associated with these two deformations, and hence find a formula for the Lagrange strain. Express all tensors in the $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ basis, and use the thin-walled approximation to simplify your expressions at the end.
(c) Assume that the Cauchy stress in the tube is $\boldsymbol{\sigma} = \sigma_{z\theta}(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r)$. Find a formula for the Cauchy stress in the $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ basis, and hence find a formula for the material stress in this basis, in terms of $\sigma_{z\theta}$, $\lambda_R, \lambda_\Theta, \lambda_Z$ and other relevant variables.
(d) Write down a formula for the nonzero Cauchy stress component $\sigma_{z\theta}$, in terms of $\lambda_R, \lambda_\Theta, Q, a, t$ (assume $t \ll a$).
(e) Hence, deduce an expression relating the torque Q and the stretches $\lambda_R, \lambda_\Theta, \lambda_Z$ to the tube's twist ϕ, the stress free tube radius a and length L, in terms of relevant material properties.
(f) Compare the result with the predictions of a simple linear elastic constitutive equation

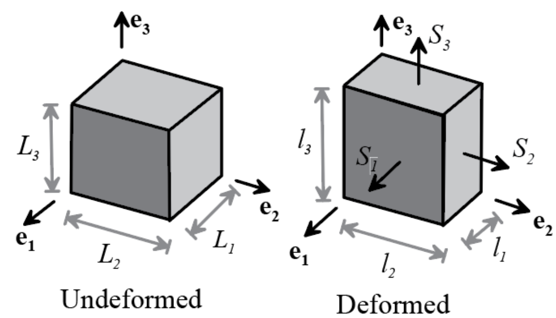
3.5 Hyperelasticity

Problem 3.20. Derive the stress-strain relations for an incompressible, Neo-Hookean material subjected to

(i) Uniaxial tension

(ii) Equibiaxial tension

(iii) Pure shear



Derive expressions for the Cauchy stress, the Nominal stress, and the Material stress tensors (the solutions for nominal stress are listed in the table in Section 3.5.6 of Applied Mechanics of Solids). You should use the following procedure: (i) assume that the specimen experiences the length changes listed in 3.5.6 of Applied Mechanics of Solids; (ii) use the formulas in Section 3.5.5 of Applied Mechanics of Solids to compute the Cauchy stress, leaving the hydrostatic part of the stress p as an unknown; (iii) Determine the hydrostatic stress from the boundary conditions (e.g. for uniaxial tensile parallel to $\mathbf{e}_1$ you know $\sigma_{22} = \sigma_{33} = 0$; for equibiaxial tension or pure shear in the $\mathbf{e}_1, \mathbf{e}_2$ plane you know that $\sigma_{33} = 0$)

Problem 3.21. Repeat problem 3.20 for an incompressible Mooney-Rivlin material.

Problem 3.22. Repeat problem 3.20 for an incompressible Gent material.

Problem 3.23. Repeat problem 3.20 for an incompressible Arruda-Boyce material.

Problem 3.24. Repeat problem 3.20 for an incompressible Ogden material.

Problem 3.25. Using the results listed in the table in Section 3.5.6 and the material properties listed in Section 3.5.7 of Applied Mechanics of Solids, plot graphs showing the nominal stress as a function of stretch ratio λ for each of (i) a Neo-Hookean material; (ii) a Mooney-Rivlin material; (iii) the Arruda-Boyce material and (iv) the Ogden material when subjected to (a) uniaxial tension, (b) biaxial tension, and (c) pure shear (for the latter case, plot the largest tensile stress S_1).

Problem 3.26. A foam specimen is idealized as an Ogden-Storakers foam with strain energy density

$$\tilde{U} = \frac{\mu}{\alpha}\left(\lambda_1^{\alpha} + \lambda_2^{\alpha} + \lambda_3^{\alpha} - 3 + \frac{1}{\beta}(J^{-\alpha\beta} - 1) \right)$$

where μ, α and β are material properties. Calculate:

(a) The Cauchy stress in a specimen subjected to a pure volume change with principal stretches $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$

(b) The Cauchy stress in a specimen subjected to volume preserving uniaxial extension

(c) The Cauchy stress in a specimen subjected to uniaxial tension, as a function of the tensile stretch ratio λ_1. (To solve this problem you will need to assume that the solid is subjected to principal stretches λ_1 parallel to $\mathbf{e}_1$, and stretches λ_2 parallel to $\mathbf{e}_2$ and $\mathbf{e}_3$. You will need to determine λ_2 from the condition that $\sigma_{22} = \sigma_{33} = 0$ in a uniaxial tensile test).

Problem 3.27. Suppose that a hyperelastic solid is characterized by a strain energy density $\bar{U}(\bar{I}_1,\bar{I}_2,J)$ (which must satisfy $\partial\bar{U}/\partial J=0$ at J=1 for the undeformed solid to be stress free), where

$$\bar{I}_1=\frac{B_{kk}}{J^{2/3}} \quad \bar{I}_2=\frac{1}{2}\left(\bar{I}_1^2-\frac{B_{ik}B_{ki}}{J^{4/3}}\right) \quad J=\sqrt{\det\mathbf{B}}$$

are invariants of the Left Cauchy-Green deformation tensor $B_{ij}=F_{ik}F_{jk}$. Suppose that the solid is subjected to an infinitesimal strain, so that $\mathbf{B}$ can be approximated as $B_{ij}\approx\delta_{ij}+2\varepsilon_{ij}$, where ε_{ij} is a symmetric infinitesimal strain tensor. Linearize the constitutive equations for $\varepsilon_{ij}<<1$, and show that the relationship between Cauchy stress σ_{ij} and infinitesimal strain ε_{ij} is equivalent to the isotropic linear elastic constitutive equation. Give formulas for the bulk modulus and shear modulus for the equivalent solid in terms of the derivatives of $\bar{U}$.

Problem 3.28. The constitutive law for a hyperelastic solid is derived from a strain energy potential $U(I_1,I_2,I_3)$, where

$$I_1=\text{trace}(\mathbf{B})=B_{kk} \quad I_2=\frac{1}{2}\left(I_1^2-\mathbf{B}\cdot\cdot\mathbf{B}\right)=\frac{1}{2}\left(I_1^2-B_{ik}B_{ki}\right) \quad I_3=\det\mathbf{B}=J^2$$

are the invariants of the Left Cauchy-Green deformation tensor $\mathbf{B}=\mathbf{F}\mathbf{F}^T$.

(a) Calculate the Cauchy stress induced in the solid when it is subjected to a rigid rotation, followed by an arbitrary homogeneous deformation. Hence, demonstrate that the constitutive law is isotropic.
(b) Demonstrate that the constitutive law is frame indifferent.

Problem 3.29. A compressible, neo-Hookean solid has a stress-Left C-G strain relation given by

$$\sigma_{ij}=\frac{1}{J}\left\{\frac{\mu_1}{J^{2/3}}\left(B_{ij}-\frac{1}{3}B_{kk}\delta_{ij}\right)+K_1J\left(J-1\right)\delta_{ij}\right\}$$

Suppose that a solid consisting of such a material is first subjected to a deformation characterized by $F_{ij}^0, J_0 B_{ij}^0$, inducing a stress σ_{ij}^0. This deformation maps a material particle at position x_i in the reference configuration to position y_i in the deformed solid. The solid is then subjected to a further small deformation that induces an additional displacement distribution Δu_i in the material. Let

$$\Delta\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial\Delta u_i}{\partial y_j}+\frac{\partial\Delta u_j}{\partial y_i}\right) \qquad \Delta w_{ij}=\frac{1}{2}\left(\frac{\partial\Delta u_i}{\partial y_j}-\frac{\partial\Delta u_j}{\partial y_i}\right)$$

denote the increment of infinitesimal strain and rotation associated with this displacement. Show that the stress can be expanded as a Taylor as

$$\sigma_{ij}=\sigma_{ij}^0(1-\Delta\varepsilon_{kk})+\sigma_{kj}^0\Delta w_{ik}+\sigma_{ki}^0\Delta w_{jk}+\frac{1}{J_0}C_{ijkl}\Delta\varepsilon_{kl}+\mathbf{O}(\Delta u_{kl})^2$$

and find an expression for C_{ijkl}.

Problem 3.30. The strain energy density of a hyperelastic solid is sometimes specified as a function of the *right* Cauchy-Green deformation tensor $C_{ij} = F_{ki}F_{kj}$, instead of B_{ij} as described in Section 3.5 of Applied Mechanics of Solids. (This procedure must be used if the material is anisotropic, for example)

(a) Suppose that the strain energy density has the general form $W(C_{ij})$. Derive formulas for the Material stress, Nominal stress and Cauchy stress in the solid as functions of F_{ij}, C_{ij} and $\partial W / \partial C_{ij}$

(b) Demonstrate that the constitutive law is frame indifferent

(c) Calculate the Cauchy stress induced in the solid when it is subjected to a rigid rotation, followed by an arbitrary homogeneous deformation. Hence, demonstrate that the constitutive law is not, in general, isotropic.

(d) Suppose that the constitutive law is simplified further by writing the strain energy density as a function of the *invariants* of **C,** i.e. $W(C_{ij}) = U(I_1, I_2, I_3)$, where

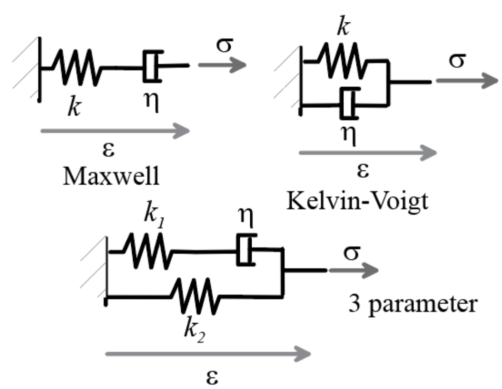
$$I_1 = C_{kk} \quad I_2 = \frac{1}{2}\left(I_1^2 - C_{ik}C_{ki}\right) \quad I_3 = \det \mathbf{C} = J^2$$

Derive expressions relating the material stress components to $\partial U / \partial I_j$.

(e) Demonstrate that the simplified constitutive law described in (d) characterizes an isotropic solid.

3.6 Viscoelasticity

Problem 3.31. Suppose that uniaxial tensile stress-strain behavior of a viscoelastic material is idealized using the spring-damper systems illustrated in the figure, as discussed in Section 3.6.3 of Applied Mechanics of Solids.



(a) Derive the differential equations relating stress to strain for each system.

(b) Calculate expressions for the relaxation modulus for the Maxwell material and the 3 parameter model.

(c) Calculate expressions for the creep compliance of all three materials

(d) Calculate expressions for the complex modulus for all three materials.

(e) Calculate expressions for the complex compliance for all three materials.

Problem 3.32. The shear modulus of a viscoelastic material can be approximated by a Prony series given by $G(t) = G_\infty + G_1 e^{-t/t_1}$.

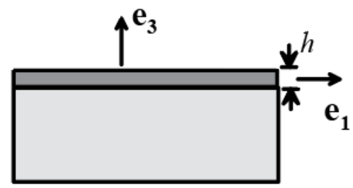
(a) Find the creep shear compliance of the material

(b) Find the complex shear modulus of the material

(c) Find the complex shear compliance of the material

Problem 3.33. A uniaxial tensile specimen is made from a viscoelastic material with time independent bulk modulus *K,* and has a shear modulus that can be approximated by the Prony series $G(t) = G_\infty + G_1 e^{-t/t_1}$. The specimen is subjected to step increase in uniaxial stress, so that $\sigma_{11} = \sigma_0 \quad t > 0$ with all other stress components zero. Find an expression for the history of strain the specimen. It is easiest to solve this problem by first calculating the creep shear compliance for the material.

Problem 3.34. A floor is covered with a pad with thickness h of viscoelastic material, as shown in the figure. The pad is perfectly bonded to the floor, so that $\varepsilon_{11} = \varepsilon_{22} = 0$. The pad can be idealized as a viscoelastic solid with time independent bulk modulus K, and has a shear modulus that can be approximated by the Prony series $G(t) = G_\infty + G_1 e^{-t/t_1}$. The surface of the pad is subjected to a history of displacement $\mathbf{u} = u(t)\mathbf{e}_3$.



(a) Calculate the history of stress induced in the pad by $u(t) = 0 \;\; t < 0 \quad u(t) = u_0 \;\; t > 0$

(b) Calculate the history of stress induced in the pad by $u(t) = 0 \;\; t < 0 \quad u(t) = u_0 \sin \omega t \quad t > 0$.

(c) Assume that the pad is subjected to a displacement $u(t) = u_0 \sin \omega t$ for long enough for the cycles of stress and strain to settle to steady state. Calculate the total energy dissipated per unit area of the pad during a cycle of loading. What frequency maximizes the energy dissipation?

Problem 3.35. The general expression for the stress in a material with a viscoelastic shear modulus and elastic bulk modulus is

$$\sigma_{ij}(t) = \int_0^t 2G\left((t-\xi)/A(T;T_1),T_1\right)\left[\dot{\varepsilon}_{ij}(\xi) - \frac{1}{3}\dot{\varepsilon}_{kk}(\xi)\delta_{ij}\right]d\xi + K\varepsilon_{kk}\delta_{ij}$$

Take the Laplace transform of both sides of this expression (a symbolic manipulation program makes this painless) to show that

$$\tilde{\sigma}_{ij} = 2\tilde{G}(s/A(T;T_1))s\left\{\tilde{e}_{ij} - e_{ij}(0)\right\} + K\tilde{e}_v\delta_{ij}$$

where

$$e_v = \varepsilon_{kk}, \qquad e_{ij} = \varepsilon_{ij} - e_v\delta_{ij}/3$$

are the volumetric and deviatoric strains, and $\tilde{x}$ denotes the Laplace transform of x

Problem 3.36. A uniaxial tensile specimen has a shear modulus that can be approximated by the Prony series $G(t) = G_\infty + G_1 e^{-t/t_1}$, and a bulk modulus K. It is subjected to a step increase in strain $\varepsilon_{11} = \varepsilon_0$ *t>0* inducing a uniaxial stress $\sigma_{11}(t)$. Using the solution to problem 3.35, show that the stress in the specimen can be expressed as

$$\sigma_{11}(t) = E(t)\varepsilon_0$$

$$E(t) = E_\infty + E_1 \exp\left(-\frac{t}{t_1^e}\right)$$

and give expressions for E_∞, E_1, t_1^e in terms of G_∞, G_1, t_1 and K.

27 ^{0}C		35 ^{0}C		45 ^{0}C		55 ^{0}C		65 ^{0}C	
Time (Sec)	Modulus (GPa)	Time (Sec)	Modulus (GPa)	Time (Sec)	Modulus (GPa)	Time (Sec)	Modulus GPa	Time (Sec)	Modulus GPa
9.00	24.69	13.00	19.79	17.00	8.90	25.00	5.98	8.50	4.98
17.00	24.36	23.00	18.96	31.00	8.81	40.00	5.60	15.00	4.99
29.00	24.89	41.00	18.53	55.00	8.39	100.00	5.12	25.00	4.94
52.00	24.65	73.00	17.63	98.00	7.91	175.00	5.02	45.00	4.67
93.00	24.27	131.00	16.33	174.00	7.53	250.00	4.82	75.00	4.42
166.00	24.47	233.00	14.80	310.00	6.57	350.00	4.78	150.00	4.05
295.00	24.20	415.00	12.69	752.00	5.89	650.00	4.51	310.00	3.57
525.00	23.10	738.00	10.86	1200.00	5.27	850.00	4.11	660.00	2.94
933.00	22.15	1312.00	9.59	2500.00	4.98	2700.00	3.49	1000.00	2.69
1660.00	21.35	2334.00	9.21			6000.00	2.95	2000.00	2.57
2952.00	20.55	4151.00	8.89					5508.00	2.55
5250.00	19.55	7381.00	8.34					8000.00	2.61
9337.00	18.81	13126.00	7.92						
16600.00	18.70								

Problem 3.37. The table above lists (fake!) measured relaxation (shear) modulus for a (fictitious) polymer at various temperatures. The polymer has a glass transition temperature of 30 ^{0}C.

(a) Plot a graph of the modulus as a function of time for each temperature, using a log scale for both axes.

(b) Hence, show that the data for various temperatures can be collapsed onto a single master-curve by scaling the times in each experiment by a temperature dependent factor $A(T,T_1)$, as described in Section 3.6.1 of Applied Mechanics of Solids. Plot the master-curve for 27 ^{0}C.

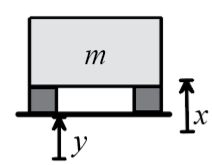
(c) Plot a graph of $\log(A(T,T_1))$ as a function of temperature, and show that the data can be fit by a function of the form $\log[A(T;T_1)] = -C_1(T-T_1)/\left\{C_2+(T-T_1)\right\}$. Determine the values of C_1, C_2 that best fit the data.

(d) Hence, determine the constants C_1^g, C_2^g for the material by shifting the reference temperature to the glass transition temperature.

(e) Hence, scale the times in the experimental data to plot the relaxation modulus at the glass transition temperature $G(t,T_g)$.

(f) Find a Prony series fit to $G(t,T_g)$. Check your solution by plotting the predicted curve on the same graph as part (e).

Problem 3.38. An instrument with mass *m=10kg* is mounted a set of four rubber pads as shown in the figure. Each pad has cross sectional area A=5 cm^2 and height *h=3cm*. The pads are made from polyisobutylene, with properties listed in Section 3.6.6 of Applied Mechanics of Solids. The base vibrates harmonically displacement $y(t) = Y_0 \exp(i\omega t)$ and (angular) frequency ω, causing the instrument to vibrate (also harmonically) with a phase-shifted displacement $x(t) = X_0 \exp(i\{\omega t + \phi\})$.



(a) Assume that the pads are all subjected to a uniaxial state of stress and are incompressible. Show that X_0, ϕ, Y_0 are related to to the harmonic modulus $G^*(\omega)$ of the material and *m, A, h* and ω by

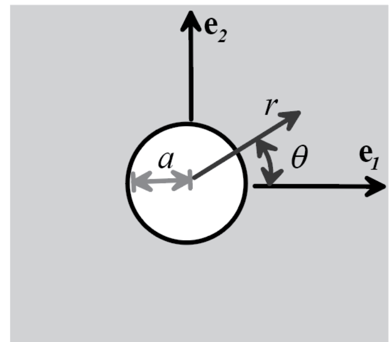
$$\left(\frac{12A}{h}G^*(\omega) - m\omega^2\right)X_0 \exp(i\phi) = \frac{12A}{h}G^*(\omega)Y_0$$

(b) Find an expression for the harmonic modulus $G(\omega)$ in terms of the material properties G_∞ and G_i, t_i.

(c) Hence, plot a graph showing the variation of X_0 / Y_0 as a function of frequency, for temperatures $0^0 C,\ 25^0 C,\ 40^0 C$

3.7 Small Strain, Rate Independent Plasticity

Problem 3.39. The stress state induced by stretching a large, thick plate containing a cylindrical hole of radius a at the origin is given by



$$\sigma_{11} = \sigma_0 \left(1 + \left(\frac{3a^4}{2r^4} - \frac{a^2}{r^2} \right) \cos 4\theta - \frac{3a^2}{2r^2} \cos 2\theta \right)$$

$$\sigma_{22} = \sigma_0 \left(\left(\frac{a^2}{r^2} - \frac{3a^4}{2r^4} \right) \cos 4\theta - \frac{a^2}{2r^2} \cos 2\theta \right)$$

$$\sigma_{12} = \sigma_0 \left(\left(\frac{3a^4}{2r^4} - \frac{a^2}{r^2} \right) \sin 4\theta - \frac{a^2}{2r^2} \sin 2\theta \right)$$

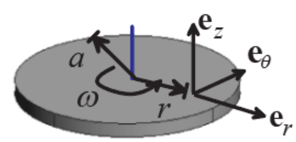
$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

Here, σ_0 is the horizontal uniaxial stress in the plate far from the hole.

(a) Plot contours of von-Mises equivalent stress (normalized by σ_0) as a function of r/a and θ, for a material with $\nu = 0.3$ Hence identify the point in the solid that first reaches yield.

(b) Assume that the material has a yield stress Y. Calculate the critical value of σ_0 / Y that will just cause the plate to reach yield.

Problem 3.40. The stress state (expressed in cylindrical-polar coordinates) in a thin disk with radius a, mass density ρ_0 and Poisson's ratio ν that spins with angular velocity ω can be shown to be



$$\sigma_{rr} = (3+\nu) \frac{\rho_0 \omega^2}{8} \left\{ a^2 - r^2 \right\}$$

$$\sigma_{\theta\theta} = \frac{\rho_0 \omega^2}{8} \left\{ (3+\nu) a^2 - (3\nu + 1) r^2 \right\}$$

Assume that the disk is made from an elastic-plastic material with yield stress Y and $\nu = 1/3$.

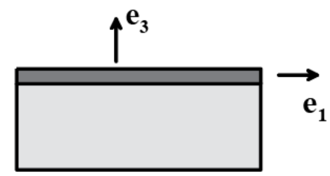
(a) Find a formula for the critical angular velocity that will cause the disk to yield, assuming Von-Mises yield criterion. Where is the critical point in the disk where plastic flow first starts?

(b) Find a formula for the critical angular velocity that will cause the disk to yield, using the Tresca yield criterion. Where is the critical point in the disk where plastic flow first starts?

(c) Using parameters representative of steel, estimate how much kinetic energy can be stored in a disk with a 0.5m radius and 0.1m thickness without exceeding yield in the disk. (In practice, kinetic energy storage systems use long cylinders rather than thin disks. This is an example in problem 4.14).

(d) Recommend the best choice of material for the flywheel in a flywheel energy storage system.

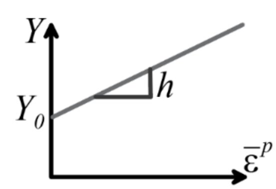
Problem 3.41. An isotropic, elastic-perfectly plastic thin film with Young's Modulus E, Poisson's ratio ν, yield stress in uniaxial tension Y and thermal expansion coefficient α is bonded to a stiff substrate. It is stress free at some initial temperature and then heated. The substrate prevents the film from stretching in its own plane, so that $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{12} = 0$, while the surface is traction free, so that the film deforms in a state of plane stress. Calculate the critical temperature change ΔT_y that will cause the film to yield, using (a) the Von Mises yield criterion and (b) the Tresca yield criterion.



Problem 3.42. Assume that the thin film described in problem 3.41 shows so little strain hardening behavior that it can be idealized as an elastic-perfectly plastic solid, with uniaxial tensile yield stress Y. Suppose the film is stress free at some initial temperature, and then heated to a temperature $\beta\Delta T_y$, where ΔT_y is the yield temperature calculated in problem 3.41, and $\beta > 1$.

(a) Find the stress in the film at this temperature.
(b) The film is then cooled back to its original temperature. Find the stress in the film after cooling.

Problem 3.43. Suppose that the thin film described in problem 3.41 is made from an elastic, isotropically hardening plastic material with a Mises yield surface, and yield stress-v-plastic strain as shown in the figure. The film is initially stress free, and then heated to a temperature $\beta\Delta T_y$, where ΔT_y is the yield temperature calculated in problem 3.41, and $\beta > 1$.



(a) Find a formula for the stress in the film at this temperature.
(b) The film is then cooled back to its original temperature. Find the stress in the film after cooling.
(c) The film is cooled further by a temperature change $\Delta T < 0$. Calculate the critical value of ΔT that will cause the film to reach yield again.

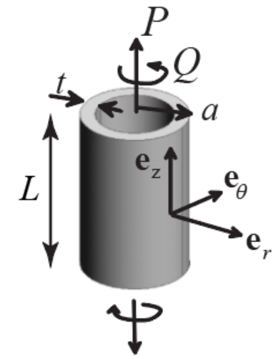
Problem 3.44. Suppose that the thin film described in problem 3.41 is made from an elastic, linear kinematically hardening plastic material with a Mises yield surface, with a yield stress Y_0 and linear hardening slope c. The film is initially stress free, and then heated to a temperature $\beta\Delta T_y$, where ΔT_y is the yield temperature calculated in problem 3.41, and $\beta > 1$.

(a) Find a formula for the stress in the film at this temperature.
(b) The film is then cooled back to its original temperature. Find the stress in the film after cooling.
(c) The film is cooled further by a temperature change $\Delta T < 0$. Calculate the critical value of ΔT that will cause the film to reach yield again.

Problem 3.45. A thin-walled tube of mean radius *a* and wall thickness *t*<<*a* is subjected to an axial load *P* which exceeds the initial yield load by 10% (i.e. $P = 1.1P_Y$). The axial load is then removed, and a torque *Q* is applied to the tube. You may assume that the axial load induces a uniaxial stress $\sigma_{zz} = P/(2\pi a t)$ while the torque induces a shear stress $\sigma_{z\theta} = Q/(2\pi a^2 t)$. Find the magnitude of *Q* to cause further plastic flow, assuming that the solid is

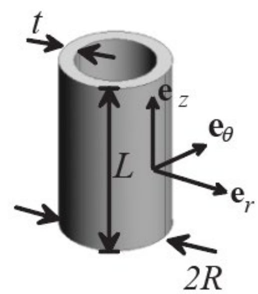
(a) an isotropically hardening solid with a Mises yield surface;
(b) a linear kinematically hardening solid with a Mises yield surface.

Express your answer in terms of P_Y and appropriate geometrical terms, and assume infinitesimal deformation.

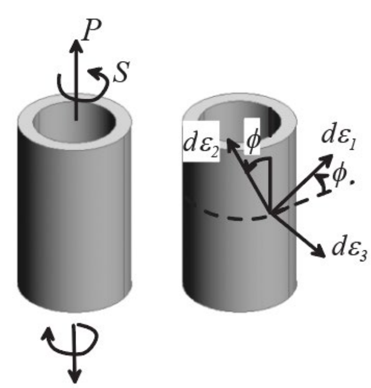


Problem 3.46. A cylindrical, thin-walled pressure vessel with initial radius *R*, length *L* and wall thickness *t*<<*R* is subjected to internal pressure *p*. The vessel is made from an isotropic elastic-plastic solid with Young's modulus *E*, Poisson's ratio ν, and its yield stress varies with accumulated plastic strain ε_e as $Y = Y_0 + h\varepsilon_e$. Recall that the stresses in a thin-walled pressurized tube are related to the internal pressure by $\sigma_{zz} = pR/(2t)$, $\sigma_{\theta\theta} = pR/t$

(a) Calculate the critical value of internal pressure required to initiate yield in the solid
(b) Find a formula for the strain increment $d\varepsilon_{rr}, d\varepsilon_{\theta\theta}, d\varepsilon_{zz}$ resulting from an increment in pressure dp
(c) Suppose that the pressure is increased 10% above the initial yield value. Find a formula for the change in radius, length and wall thickness of the vessel. Assume small strains.



Problem 3.47. In a classic paper, Taylor, G. I., and Quinney, I., 1932, "The Plastic Distortion of Metals," Philos. Trans. R. Soc. London, Ser. A, 230, pp. 323–362 described a series of experiments designed to investigate the plastic deformation of various ductile metals. Among other things, they compared their experimental measurements with the predictions of the von-Mises and Tresca yield criteria and their associated flow rules. They used the apparatus shown in the figure. Thin walled cylindrical tubes were first subjected to an axial stress $\sigma_{zz} = \sigma_0$. The stress was sufficient to extend the tubes plastically. The axial stress was then reduced to a magnitude $m\sigma_0$, with $0 < m < 1$, and a progressively increasing torque was applied to the tube so as to induce a shear stress $\sigma_{z\theta} = \tau$ in the solid. The twist, extension and internal volume of the tube were recorded as the torque was applied. In this problem you will compare their experimental results with the predictions of plasticity theory. Assume that the material is made from an isotropically hardening *rigid* plastic solid, with a Von Mises yield surface, and yield stress-v-plastic strain given by $Y = Y_0 + h\bar{\varepsilon}^p$.



(a) In one set of experiments, Taylor and Quinney measured the normalized shear stress τ/σ_0 required to initiate yield in the tube during torsional loading as a function of *m*. Show that theory predicts that $\tau/\sigma_0 = \sqrt{(1-m^2)}/\sqrt{3}$. Compare your predictions with the experimental data (for copper – you can find data for many other materials in the publication if you are curious) given in the table below.

(b) Compute the magnitudes of the principal stresses $(\sigma_1, \sigma_2, \sigma_3)$ at the point of yielding under combined axial and torsional loads in terms of σ_0 and *m*. Choose $(\sigma_1 > \sigma_2 > \sigma_3)$

(c) Suppose that, for a given axial stress $\sigma_{zz} = m\sigma_0$, the shear stress τ is first brought to the critical value required to initiate yield in the solid, and is then increased by an infinitesimal increment $d\tau$. Deduce expressions for the magnitudes of the principal strains increments $(d\varepsilon_1, d\varepsilon_2, d\varepsilon_3)$ resulting from the stress increment $d\tau$. Choose $(d\varepsilon_1 > d\varepsilon_2 > d\varepsilon_3)$

m	τ / σ_0	μ	ν
0.025	0.56	-0.021	-0.011
0.28	0.541	-0.242	-0.171
0.515	0.48	-0.471	-0.393
0.65	0.436	-0.597	-0.502
0.7	0.405	-0.652	-0.554
0.8	0.329	-0.773	-0.659
0.9	0.247	-0.874	-0.775
0.95	0.149	-0.954	-0.876

(d) Using the results of (b) and (c), calculate the so-called "Lode parameters," defined as

$$\nu = 2\frac{(d\varepsilon_2 - d\varepsilon_3)}{(d\varepsilon_1 - d\varepsilon_3)} - 1 \qquad \mu = 2\frac{(\sigma_2 - \sigma_3)}{(\sigma_1 - \sigma_3)} - 1$$

(with $(d\varepsilon_1 > d\varepsilon_2 > d\varepsilon_3)$ and $(\sigma_1 > \sigma_2 > \sigma_3)$), and show that the theory predicts $\nu = \mu$. Compare the prediction with the Taylor-Quinney data.

Problem 3.48. The Taylor/Quinney experiments show that the constitutive equations for an isotropically hardening Von-Mises solid predict behavior that matches reasonably well with experimental observations, but there is a clear systematic error between theory and experiment. In this problem, you will compare the predictions of a *linear kinematic hardening* law with experiment. Assume that the solid has a yield function and hardening law given by

$$f = \sqrt{\frac{3}{2}(S_{ij} - \alpha_{ij})(S_{ij} - \alpha_{ij})} - Y_0 = 0 \qquad d\alpha_{ij} = \frac{2}{3}c\,d\varepsilon_{ij}^p$$

(a) Assume that during the initial tensile test, the axial stress $\sigma_{zz} = \sigma_0$ in the specimens reached a magnitude $\sigma_0 = \beta Y_0$, where Y_0 is the initial tensile yield stress of the solid and $\beta > 1$ is a scalar multiplier. Assume that the axial stress was then reduced to $m\sigma_0$ and a progressively increasing shear stress was applied to the solid. Show that the critical value of τ / σ_0 at which plastic deformation begins is given by

$$\frac{\tau}{\sigma_0} = \frac{1}{\beta\sqrt{3}}\left\{1 - A^2\right\}^{1/2} \qquad A = \beta(m-1) + 1$$

Plot τ / σ_0 against *m* for various values of β.

(b) Suppose that, for a given axial stress $\sigma_{zz} = m\sigma_0$, the shear stress is first brought to the critical value required to initiate yield in the solid, and is then increased by an infinitesimal increment $d\tau$. Find expressions for the resulting plastic strain increments, in terms of *m*, β, *c* Y_0 and $d\tau$.

(c) Hence, deduce the magnitudes of the principal strains in the specimen $(d\varepsilon_1, d\varepsilon_2, d\varepsilon_3)$. Choose the order for the principal strains so that $(d\varepsilon_1 > d\varepsilon_2 > d\varepsilon_3)$ with $\beta(m-1) + 1 > 0$

(d) Compute the magnitudes of the principal stresses $(\sigma_1, \sigma_2, \sigma_3)$ at the point of yielding under combined axial and torsional loads in terms of *m*, β, and Y_0. Take $(\sigma_1 > \sigma_2 > \sigma_3)$

(e) Finally, find expressions for Lode's parameters

$$\nu = 2\frac{(d\varepsilon_2 - d\varepsilon_3)}{(d\varepsilon_1 - d\varepsilon_3)} - 1 \qquad \mu = 2\frac{(\sigma_2 - \sigma_3)}{(\sigma_1 - \sigma_3)} - 1$$

(with $(d\varepsilon_1 > d\varepsilon_2 > d\varepsilon_3)$ and $(\sigma_1 > \sigma_2 > \sigma_3)$. Plot ν versus μ for various values of β, and compare your predictions with Taylor and Quinney's measurements.

Problem 3.49. An elastic-nonlinear kinematic hardening solid has Young's modulus E, Poisson's ratio ν, a Von-Mises yield surface

$$f(\sigma_{ij},\alpha_{ij})=\sqrt{\frac{3}{2}\left(S_{ij}-\alpha_{ij}\right)\left(S_{ij}-\alpha_{ij}\right)}-Y=0$$

where Y is the initial yield stress of the solid, and a hardening law given by

$$d\alpha_{ij}=\frac{2}{3}cd\varepsilon_{ij}^{p}-\gamma\alpha_{ij}d\overline{\varepsilon}^{p}$$

where c and γ are material properties and $d\overline{\varepsilon}^{p}=2d\varepsilon_{ij}^{p}d\varepsilon_{ij}^{p}/3$. In the undeformed solid, $\alpha_{ij}=0$, which (see problem 3.50) implies that $\alpha_{kk}=0$. Calculate the formulas relating the total strain increment $d\varepsilon_{ij}$ to the state of stress σ_{ij}, the state variables α_{ij} and the increment in stress $d\sigma_{ij}$ applied to the solid.

Problem 3.50. Consider a *rigid* nonlinear kinematic hardening solid, with yield surface and hardening law described in the preceding problem.

(a) Show that the constitutive law implies that $\alpha_{kk}=0$.

(b) Show that under uniaxial loading with $\sigma_{11}=\sigma$, $\alpha_{22}=\alpha_{33}=-\alpha_{11}/2$.

(c) Suppose the material is subjected to a monotonically increasing uniaxial tensile stress $\sigma_{11}=\sigma$. Show that the uniaxial stress-strain curve has the form $\sigma=Y+(c/\gamma)[1-\exp(-\gamma\varepsilon)]$ (it is simplest to calculate α_{ij} as a function of the strain and then use the yield criterion to find the stress).

Problem 3.51. Suppose that a solid contains a large number of randomly oriented slip planes, so that it begins to yield when the resolved shear stress on *any* plane in the solid reaches a critical magnitude *k*.

(a) Suppose that the material is subjected to principal stresses $\sigma_1,\sigma_2,\sigma_3$. Find a formula for the maximum resolved shear stress in the solid, and by means of appropriate sketches, identify the planes that will begin to slip depending on the relative magnitudes of the principal stresses.

(b) Draw the yield locus for this material in principal stress space.

Problem 3.52. Consider a rate independent plastic material with yield criterion $f(\sigma_{ij})=0$. Assume that (i) the constitutive law for the material has an associated flow rule, so that the plastic strain increment is related to the yield criterion by $d\varepsilon_{ij}^{p}=d\overline{\varepsilon}^{p}\partial f/\partial\sigma_{ij}$; and (ii) the yield surface is convex, so that

$$f\left[\sigma_{ij}^{*}+\beta(\sigma_{ij}-\sigma_{ij}^{*})\right]-f\left[\sigma_{ij}^{*}\right]\geq 0$$

for all stress states σ_{ij} and σ_{ij}^{*} satisfying $f(\sigma_{ij})=0$ and $f(\sigma_{ij}^{*})\leq 0$ and $0\leq\beta\leq 1$. Show that the material obeys the principle of maximum plastic resistance.

Problem 3.53. The yield strength of a frictional material (such as sand) depends on hydrostatic pressure. A simple model of yield and plastic flow in such a material is proposed as follows:

Yield criterion $F(\sigma_{ij})=f(\sigma_{ij})-\mu\sigma_{kk}=0$ $\qquad f(\sigma_{ij})=\sqrt{\frac{3}{2}S_{ij}S_{ij}}$

Flow rule $d\varepsilon_{ij}^{p}=d\overline{\varepsilon}^{p}\dfrac{\partial f}{\partial\sigma_{ij}}$

where μ is a material constant (some measure of the friction between the sand grains).

(a) Sketch the yield surface for this material in principal stress space (note that the material looks like a Mises solid whose yield stress increases with hydrostatic compression. You will need to sketch the full 3D surface, not just the projection that is used for pressure independent surfaces)
(b) Sketch a vector indicating the direction of plastic flow for some point on the yield surface drawn in part (a)
(c) By finding a counter-example, demonstrate that this material does not satisfy the principle of maximum plastic resistance

$$(\sigma_{ij} - \sigma_{ij}^{*})d\varepsilon_{ij}^{p} \geq 0$$

(you can do this graphically, or by finding two specific stress states that violate the condition)
(d) Demonstrate that the material is not stable in the sense of Drucker – i.e. find a cycle of loading for which the work done by the traction increment through the displacement increment is non-zero.
(e) What modification would be required to the constitutive law to make it satisfy the PMPR and Drucker stability? How does the physical response of the stable material differ from the original model (think about compaction under combined shear and pressure).

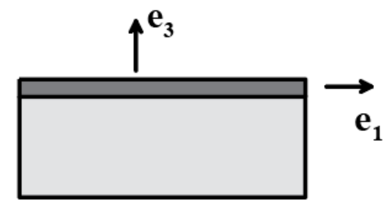
3.8 Small Strain Viscoplasticity: Creep and High Rate Deformation of Metals

Problem 3.54. Suppose that a uniaxial tensile specimen with length made from Aluminum can be characterized by a viscoplastic constitutive law with properties listed in Section 3.8.4 of Applied Mechanics of Solids. Plot a graph showing the strain rate of the specimen as a function of stress. Use log scales for both axes, with a stress range between 5 and 60 MPa, and show data for room temperature; 100^0C, 200^0C, 300^0C, 400^0C and 500^0C. Would you trust the predictions of the constitutive equation outside this range of temperature and stress? Give reasons for your answer.

Problem 3.55. A uniaxial tensile specimen can be idealized as an elastic-viscoplastic solid, with Young's modulus *E*, and a flow potential given by

$$g(\sigma_e) = \dot{\varepsilon}_0 \left(\frac{\sigma_e}{Y} \right)^m$$

where Y, $\dot{\varepsilon}_0$ and m are material properties. The specimen is stress free at time $t=0$, and is then stretched at a constant (total) strain rate $\dot{\eta}$.
(a) Show that the equation governing the axial stress in the specimen can be expressed in dimensionless form as $(d\tilde{\sigma} / d\tilde{t}) + \tilde{\sigma}^m = 1$, where $\tilde{\sigma} = (\sigma / Y)(\dot{\varepsilon}_0 / \dot{\eta})^{1/m}$ and $\tilde{t} = \left(E\dot{\eta}t / Y\right)\left(\dot{\varepsilon}_0 / \dot{\eta}\right)^{1/m}$ are dimensionless measures of stress and time.
(b) Hence, deduce that the normalized stress $\tilde{\sigma}$ is a function only of the material parameter m and the normalized strain $\tilde{\varepsilon} = \varepsilon(E / Y)(\dot{\varepsilon}_0 / \dot{\eta})^{1/m}$.
(c) Show that during steady state creep $\tilde{\sigma} = 1$.
(d) Obtain an analytical solution relating $\tilde{\sigma}$ to $\tilde{\varepsilon}$ for m=1.
(e) Obtain an analytical solution relating $\tilde{\sigma}$ to $\tilde{\varepsilon}$ for very large m (note that, in this limit the material behaves like an elastic-perfectly plastic, rate independent solid, with yield stress Y).
(f) By integrating the governing equation for $\tilde{\sigma}$ numerically, plot graphs relating $\tilde{\sigma}$ to $\tilde{\varepsilon}$ for a few values of m between $m=1$ and $m=100$.
(g) Estimate the time, and strain, required for a tensile specimen of Aluminum to reach steady state creep at a temperature of 400^0C, when deformed at a strain rate of $10^{-3}s^{-1}$



Problem 3.56. The figure shows a thin polycrystalline film on a rigid substrate. The film can be idealized as an elastic-viscoplastic solid with Young's modulus E, Poisson's ratio ν and thermal expansion coefficient α, and a uniaxial plastic strain rate, stress temperature relation given by $\dot{\varepsilon} = \dot{\varepsilon}_0 \exp(-Q/kT)\left(\sigma/Y\right)^m$, where $\dot{\varepsilon}_0$, Q and Y are material constants, and k is the Boltzmann constant. The stress in the film is uniform with $\sigma_{11} = \sigma_{22} = \sigma$ (with all other components zero) and the substrate prevents the film from stretching in-plane so $\varepsilon_{11} = \varepsilon_{22} = 0$. The film is heated to induce a time dependent temperature $T(t)$.

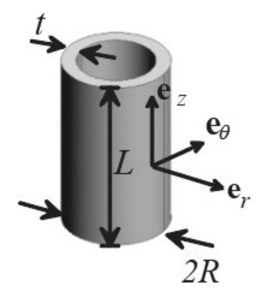
(a) Show that the stress state in the film satisfies the dimensionless differential equation

$$\frac{d\tilde{\sigma}}{d\tilde{t}} + \frac{1}{2}\exp(-\tilde{Q}/\tilde{T})\left(|\tilde{\sigma}|\right)^m \operatorname{sign}(\tilde{\sigma}) + \frac{d\tilde{T}}{d\tilde{t}} = 0$$

where

$$\tilde{\sigma} = \frac{\sigma}{Y} \quad \tilde{t} = t\frac{E\dot{\varepsilon}_0}{(1-\nu)Y} \quad \tilde{T} = \frac{E\alpha}{(1-\nu)Y}T \qquad \tilde{Q} = \frac{E\alpha}{(1-\nu)kY}Q$$

(b) Using material properties for Aluminum alloy, plot the variation of dimensionless stress as a function of dimensionless time, for a film that is heated from 25C to 400C at constant rate. Use heating rates of 1, 4, and 16 C/min.

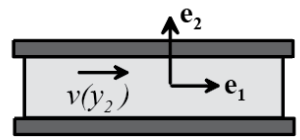


Problem 3.57. A cylindrical, thin-walled pressure vessel with initial radius R, length L and wall thickness $t \ll R$ is subjected to internal pressure p. The vessel is made from an elastic-power-law viscoplastic solid with Young's modulus E, Poisson's ratio ν, and a flow potential given by

$$g(\sigma_e) = \dot{\varepsilon}_0\left(\frac{\sigma_e}{Y}\right)^m$$

where σ_e is the Von-Mises eequivalent stress. Recall that the stresses in a thin-walled pressurized tube are related to the internal pressure by $\sigma_{zz} = pR/(2t)$, $\sigma_{\theta\theta} = pR/t$, $\sigma_{rr} \approx 0$. Calculate the steady-state strain rate in the vessel, as a function of pressure and relevant geometric and material properties. Hence, calculate expressions for the rate of change of the vessel's length, radius and wall thickness as a function of time.

3.9 Large Strain, Rate Dependent Plasticity



Problem 3.58. The figure shows a thin film of material that is deformed plastically during a pressure-shear plate impact experiment. The goal of this problem is to derive the equations governing the velocity and stress fields in the specimen. Assume that:

- The film deforms in simple shear, with velocity $\mathbf{v} = v(y_2, t)\mathbf{e}_1$ and that the Kirchoff stress can be approximated as
  $$\boldsymbol{\tau} = q(y_2,t)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + p(y_2,t)(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) + s(y_2,t)(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2)$$
- The material has mass density ρ and isotropic elastic response, with Young's modulus E and Poisson's ratio ν
- The film can be idealized as a finite strain viscoplastic solid with power-law Mises flow potential, as described in Section 3.9 of Applied Mechanics of Solids. Assume that the plastic spin is zero.

(a) Calculate the velocity gradient tensor **L**, the stretch rate tensor **D** and spin tensor **W** for the deformation, expressing your answer as components in the $\mathbf{e}_1, \mathbf{e}_2$ basis shown in the figure

(b) Find an expression for the plastic stretch rate, in terms of the stress and material properties

(c) Use the elastic stress rate-stretch rate relation $\overset{\nabla}{\tau}_{ij} = C_{ijkl} D^e_{kl}$ to obtain expressions for the time derivatives of p,q,s in terms of $v(x_2)$, p,q,s (i.e. the equations will be coupled), and appropriate material properties. (Assume the plastic spin is zero)

(d) Write down the linear momentum balance equation in terms of $\boldsymbol{\tau}$ and v

(e) How would the results of (c) change if $\mathbf{W}^p = \mathbf{W}$?

3.10 Large Strain Viscoelasticity

Problem 3.59. A cylindrical specimen is made from a material that can be idealized using the finite-strain viscoelasticity model described in Section 3.10 of Applied Mechanics of Solids. The specimen may be approximated as incompressible. It is loaded in uniaxial tension parallel to its axis.

(a) Let L denote the length of the deformed specimen, and L_0 denote the initial length of the specimen. Write down the deformation gradient in the specimen in terms of $\lambda = L / L_0$ (use a cylindrical-polar basis with $\mathbf{e}_z$ parallel to the axis of the specimen)

(b) Let $\lambda = \lambda_e \lambda_p$ denote the decomposition of stretch in to elastic and plastic parts (both are volume preserving) . Write down the elastic and plastic parts of the deformation gradient in terms of λ_e, λ_p and find expressions for the elastic and plastic parts of the stretch rate in terms of $\dot{\lambda}_e, \dot{\lambda}_p$

(c) Assume that the material can be idealized using Arruda-Boyce potentials
$$U_\infty = \mu_\infty \left\{ \frac{1}{2}(\bar{I}_1 \ -3) + \frac{1}{20\beta_\infty^2}(\bar{I}_1^2 - 9) + \frac{11}{1050\beta_\infty^4}(\bar{I}_1^3 - 27) + ... \right\} + \frac{K}{2}(J-1)^2$$
$$U_T = \mu_T \left\{ \frac{1}{2}(\bar{I}_1^e - 3) + \frac{1}{20\beta_T^2}(\bar{I}_1^{e2} - 9) + \frac{11}{1050\beta_T^4}(\bar{I}_1^{e3} - 27) + ... \right\}$$

Obtain an expression for the stress in the specimen in terms of λ_e, λ_p, using only the first two term in the expansion for simplicity. Since the material is incompressible you will need to calculate the hydrostatic part of the stress from the boundary conditions.

(d) Calculate the deviatoric stress measure

$$\tau'_{ij} = 2\left[\frac{1}{J_e^{2/3}} \left(\frac{\partial U_T}{\partial \bar{I}_1^e} + \bar{I}_1^e \frac{\partial U_T}{\partial \bar{I}_2^e} \right) B_{ij}^e - \frac{\bar{I}_1^e}{3} \frac{\partial U_T}{\partial \bar{I}_1^e} \delta_{ij} - \frac{1}{J_e^{4/3}} \frac{\partial U_T}{\partial \bar{I}_2^e} B_{ik}^e B_{kj}^e \right]$$

in terms of λ_e, and hence find an expression for $\dot{\lambda}_p$ in terms of λ_e. Use the Bergstrom-Boyce model for stress relaxation in Section 3.10.4 of Applied Mechanics of Solids to calculate the plastic stretch rate.

(e) Suppose that the specimen is subjected to a harmonic cycle of nominal strain such that $L = L_0\left(1 + \alpha \sin \omega t\right)$. Use the results of (b) and (d) to obtain a nonlinear differential equation for λ_e

(f) Use the material data given in Section 3.10.5 to calculate the variation of Cauchy stress induced in the solid by cyclic straining. Plot the results as a curve of Cauchy stress as a function of true strain $\log(\lambda)$. Try frequencies of 0.1Hz, 8Hz and 100Hz at a stretch amplitude $\alpha = 0.8$, and stretch amplitudes $\alpha = 0.1$, $\alpha = 0.5$, $\alpha = 0.9$ at a frequency of 1Hz. Note that the differential equation in € must be solved with a numerical method that can handle stiff differential equations.

3.11 Critical State Models for Soils

Problem 3.60. A drained specimen of a soil can be idealized as Cam-clay, using the constitutive equations listed in Section 3.11 of Applied Mechanics of Solids. At time *t=0* the soil has a strength a_0. The specimen is subjected to a monotonically increasing hydrostatic stress *p*, and the volumetric strain $\Delta V / V = \varepsilon_{kk}$ is measured. Calculate a relationship between the pressure and volumetric strain, in terms of the initial strength of the soil a_0 and the hardening rate *c*.

Problem 3.61. An undrained specimen of a soil can be idealized as Cam-clay, using the constitutive equations listed in Section 3.11 of Applied Mechanics of Solids.

- The elastic constants of the soil are characterized by its bulk modulus *K* and Poisson's ratio ν, while its plastic properties are characterized by *M* and *c*.
- The fluid has a bulk modulus K_w.
- At time $t = 0$ the soil has a cavity volume fraction n_0 and strength a_0, and $p_s = p_w = 0$.

The specimen is subjected to a monotonically increasing hydrostatic pressure *p*, and is then unloaded. The volumetric strain $\Delta V / V = \varepsilon_{kk}$ is measured. Assume that both elastic and plastic strains are small. Show that the relationship between the normalized pressure p / a_0 and normalized volumetric strain $K\varepsilon_{kk} / a_0$ is a function of only three dimensionless material properties: $\alpha = K_w / a_0$, $\beta = K / a_0 c$ and $\gamma = n_0 c$. Plot the dimensionless pressure-volume curves (showing both the elastic and plastic parts of the loading cycle for a few representative values of α, β and γ. Rough values of material properties are listed in the table below.

Bulk modulus K	Bulk modulus K_w	Hardening rate c	Initial strength a_0	Initial porosity n_0
2GPa	2.2GPa	5	0.2MPa	0.2

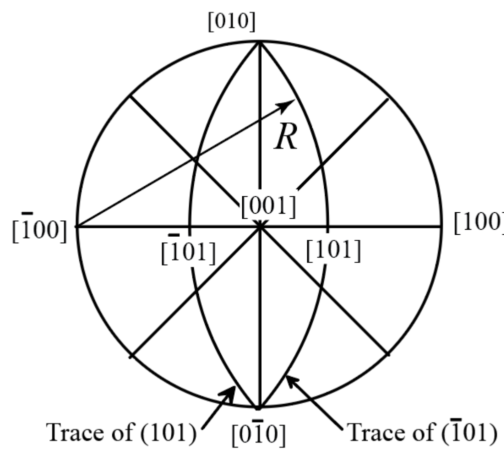
Problem 3.62. A drained specimen of Cam-clay is first subjected to a monotonically increasing confining pressure *p*, with maximum value $p > a_0$. The confining pressure is then held constant, and the specimen is subjected to a monotonically increasing shear stress $\sigma_{12} = \sigma_{21} = q$. Find expressions relating the normalized volumetric strain increment $2c\Delta\varepsilon_{kk}^{p}$ and the normalized engineering shear strain $Mc\gamma/\left(2\sqrt{3}\right)$ (where $\gamma = 2\varepsilon_{12}^{p}$) to the normalized shear stress $\sqrt{3}q/(Mp)$ during the shear loading, and plot a graph showing $\sqrt{3}q/(Mp)$ and $2c\Delta\varepsilon_{kk}^{p}$ as functions of $Mc\gamma/\left(2\sqrt{3}\right)$.

3.12 Crystal Plasticity

Problem 3.63. Draw an inverse pole figure for an fcc single crystal, with [100], [010] and [001] directions parallel to the {**i,j,k**} directions, showing the following:

(a) The trace of the (010), (100), (110) and $(\overline{1}10)$ planes
(b) The trace of the $\{111\}$ planes
(c) The twelve slip directions, labeled according to the convention given in Section 3.12.2 of Applied Mechanics of Solids (i.e. a_1, a_2, a_3, etc)

Problem 3.64. Consider the inverse pole figure for an fcc crystal with [100], [010] and [001] directions parallel to the {**i,j,k**} directions, as shown in the figure. Show that the circle corresponding to the trace of the $(\overline{1}01)$ plane has radius $R = \sqrt{2}$, and is centered at the point corresponding to the $[\overline{1}00]$ as shown in the figure. The simplest approach to this problem is to note that the [010] direction and the [101] direction both lie on the circle.

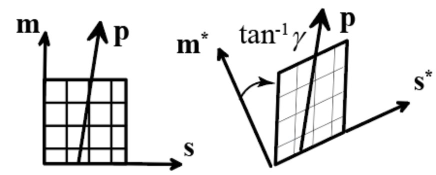


Problem 3.65. Plot a contour map on the standard triangle of the inverse pole figure for an fcc single crystal, showing the magnitude of the resolved shear stress induced by uniaxial tensile stress on the critical (*d1*) slip system (plot the ratio of the resolved shear stress to the magnitude of the tensile stress). Find the direction of the tensile axis that gives the largest resolved shear stress (express your answer as a unit vector in a basis with (**i,j,k**) parallel to the [100], [010] and [001] directions)

Problem 3.66. A rigid plastic fcc single crystal deforms by shearing at rate $\dot{\gamma}$ on the d1 $(1\overline{1}1)$ $[011]$ and $-\dot{\gamma}$ on the c2 $(\overline{1}11)$ $[\overline{1}0\overline{1}]$ slip systems.

(a) Show that the material fiber parallel to the [112] direction has zero angular velocity;
(b) Calculate the rate of stretching of the material fiber parallel to the [112] direction.

Problem 3.67. A single crystal is loaded in uniaxial tension. The direction of the loading axis, specified by a unit vector **p** remains fixed during straining. The crystal deforms by slip on a single system, with slip direction $\mathbf{s}^{\alpha}$ and slip plane normal $\mathbf{m}^{\alpha}$. The deformation gradient resulting from a shear strain γ^{α} is



$$F_{ij} = R_{ik}\left(\delta_{kj} + \gamma^{\alpha} s_k^{\alpha} m_j^{\alpha}\right)$$

where R_{ij} is a proper orthogonal tensor (i.e. detI=1, $R_{ik}R_{jk} = \delta_{ij}$), representing a rigid rotation. Assume that, during deformation, the material fiber parallel to **p** not rotate, and that material fibers normal to the plane of **p** and **s** do not change their orientation. Show that:

$$R_{ij} = \delta_{ij}\cos\theta + (1-\cos\theta)n_i n_j + \sin\theta \in_{ikj} n_k$$

where

$$\cos\theta = (1+\gamma^{\alpha} p_i s_i^{\alpha} p_k m_k^{\alpha})/C\,,\ \sin\theta = \gamma^{\alpha}(p_i m_i^{\alpha})\sqrt{\left(1-(p_i s_i^{\alpha})^2\right)}/C\,,\ n_i = \in_{ijk} s_j^{\alpha} p_k \Big/ \sqrt{\left(1-(p_i s_i^{\alpha})^2\right)}$$

and $C = \sqrt{1+\gamma^{\alpha 2}(p_i m_i^{\alpha})^2 + 2\gamma^{\alpha} p_i s_i^{\alpha} p_k m_k^{\alpha}}$.

Problem 3.68. Consider the single crystal loaded in uniaxial tension described in the preceding problem. Calculate

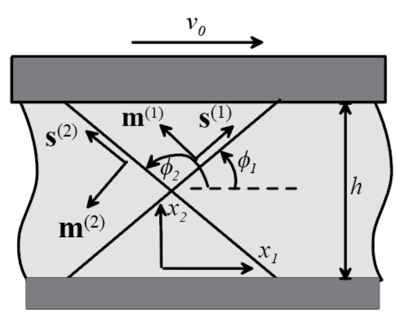
(a) The angular velocity vector that describes the angular velocity of the slip direction and slip plane normal (it is easiest to do this by first assuming that the slip direction and slip plane normal are fixed, and calculating the angular velocity of the material fiber parallel to **p**, and hence deducing the angular velocity that must be imposed on the crystal to keep **p** pointing in the same direction)

(b) The spin tensor **W** associated with the angular velocity calculated in (a)

Problem 3.69. The resolved shear stress on a slip system in a crystal is related to the Kirchhoff stress $\boldsymbol{\tau}$ by $\tau^{\alpha} = \mathbf{s}^{*\alpha}\cdot\boldsymbol{\tau}\cdot\mathbf{m}^{*\alpha}$. Show that the rate of change of resolved shear stress is related to the elastic Jaumann rate of Kirchhoff stress by

$$\frac{d\tau^{\alpha}}{dt} = s_i^{*\alpha}\left(\overset{\nabla e}{\tau_{ij}} - \tau_{ik}D_{kj}^{e} + D_{ik}^{e}\tau_{kj}\right)m_j^{*\alpha}$$

Problem 3.70. A rigid –perfectly plastic single crystal contains two slip systems, oriented at angles ϕ_1 and ϕ_2 as illustrated in the figure. The solid is deformed in simple shear as indicated



(a) Suppose that $\phi_1 = 60^0$, $\phi_2 = 120^0$. Sketch the yield locus (in σ_{11},σ_{22} space) for the crystal.

(b) Write down the velocity gradient **L** in the strip, and compute an expression the deformation rate **D**. Hence, show that (at the instant shown) the slip rates on the two slip systems are given by

$$\dot{\gamma}^{(1)} = \dot{\gamma}\sin 2\phi_2 / \sin 2(\phi_2-\phi_1) \qquad \dot{\gamma}^{(2)} = \dot{\gamma}\sin 2\phi_1 / \sin 2(\phi_1-\phi_2)$$

and give an expression for $\dot{\gamma}$ in terms of v_0 and *h*.

(c) Assume that $\sigma_{11}+\sigma_{22}=0$, and that the slip systems have critical resolved shear stress τ_c. Show that the stress in the crystal is

$$\sigma_{11} = -\sigma_{22} = \tau_c(\sin 2\phi_2 - \sin 2\phi_1)/\sin 2(\phi_2-\phi_1)$$
$$\sigma_{12} = \tau_c(\cos 2\phi_2 - \cos 2\phi_1)/\sin 2(\phi_2-\phi_1)$$

Problem 3.71. A copper single crystal is idealized using the constitutive equation given in Section 3.12 of Applied Mechanics of Solids. The table lists parameter values determined by Wu, Neale and Van der Giessen, Int J plasticity, **12**, p.1199, 1996. To simplify this problem, elastic deformation may be neglected. Assume that the crystal is deformed at a constant stretch rate of $0.1 s^{-1}$ in uniaxial tension, and take *m=15*.

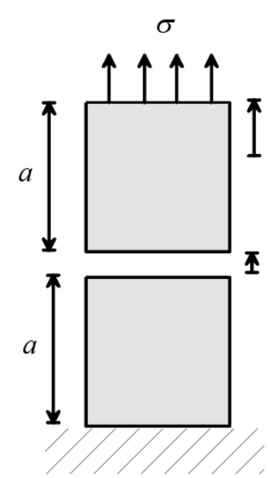
$\dot{\gamma}_0$	$0.001\ s^{-1}$
m	10-20
g_0	16 MPa
g_s	70.4 MPa
h_0	132 MPa
h_s	8 MPa
q	1.4

(a) Plot a graph showing the uniaxial true stress-v-true strain curve for a crystal that is loaded parallel to the [112] direction.
(b) Plot a graph showing the uniaxial true stress-v-true strain curve for a crystal loaded parallel to the [111] direction
(c) Plot a graph showing the uniaxial true stress-v-true strain curve for a crystal loaded parallel to the [001] direction.

Problem 3.72. A rate independent, rigid perfectly plastic fcc single crystal is loaded in uniaxial tension, with tensile axis parallel to the [102] crystallographic direction. Assume that the crystal rotates to maintain the material fiber parallel to the [102] direction aligned with the tensile axis.
(a) Assuming the magnitude of the shearing rate is $\dot{\gamma}$ on all active slip systems, calculate the velocity gradient in the crystal, expressing your answer as components in a basis aligned with the {100} directions.
(b) Hence, show that the [102] loading direction is not a stable orientation – i.e. the tensile axis rotates with respect to the crystallographic directions.
(c) Calculate the instantaneous axial stretch rate of the tensile axis as a function of $\dot{\gamma}$.
(d) Deduce the angular velocity vector that characterizes the instantaneous rotation of the crystal relative to the tensile axis.
(e) Without calculations, predict the eventual steady-state orientation of the tensile axis with respect to the loading axis.

3.13 Constitutive Laws for Contacting Surfaces and Interfaces in Solids

Problem 3.73. The figure shows two elastic blocks with Young's modulus E that are bonded together at an interface. The interface can be characterized using the reversible constitutive law described in Section 3.13.1 of Applied Mechanics of Solids. The top block is subjected to tractions which induce a uniaxial stress $\sigma_{22} = \sigma$ in the blocks, a separation Δ_n at the interface, and a displacement U at the surface of the upper block.



(a) Show that the stress and displacement can be expressed in dimensionless form as

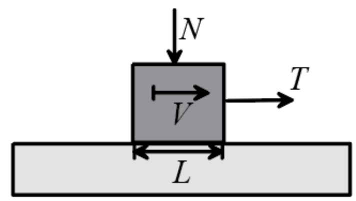
$$\frac{\sigma}{\sigma_{\max}} = \frac{\Delta_n}{\delta_n}\exp\left(1-\frac{\Delta_n}{\delta_n}\right) \qquad \frac{U}{\delta_n} = \frac{\Delta_n}{\delta_n} + \frac{\sigma}{\sigma_{\max}}\Lambda$$

where $\Lambda = a\sigma_{\max}/(\delta_n E)$.

(b) Plot graphs of $\sigma/\sigma_{\max}$ and Δ_n/δ_n as functions of U/δ_n for various values of Λ. Hence, show that if Λ is less than a critical value, the interface separates smoothly under monotonically increasing U/δ_n. In contrast, if Λ exceeds the critical value, the interface snaps apart (with a sudden drop in stress) at a critical value of U/δ_n. (Under decreasing U/δ_n the interface re-adheres, with a similar transition from smooth attachment to sudden snapping at a critical Λ). Give an expression for the critical value of Λ.
(c) Plot a graph showing the critical displacement U/Δ_n at separation and attachment as a function of $1/\Lambda$.

Problem 3.74. Two rigid surfaces slide against one another under an applied shear stress *T* and a normal pressure *N*. The interface may be characterized using the rate and state dependent friction law described in Section 3.13.2 of Applied Mechanics of Solids. The blocks slide at relative speed V until the friction force reaches its steady state value. The sliding speed is then increased instantaneously to a new value V_2. Calculate an expression for the variation of the friction force *T* as a function of the distance slid *s*.

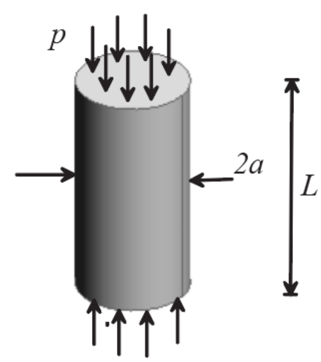
Problem 3.75. A rigid block with length *L* slides over a flat rigid surface at constant speed *V* under an applied shear force *T* and a normal force *N*. The contacting surfaces may be idealized using the rate and state dependent friction law described in Section 3.13.2 of Applied Mechanics of Solids. The state variables $p = \omega = 0$ for any point on the stationary surface that lies ahead of the rigid block. Calculate the steady-state shear force *T* as a function of the length of the block, the sliding speed, and relevant material properties. Assume that the normal stress is constant and uniform, and take the elastic stiffness $k_e \to \infty$ for the tangential constitutive equation. You will need to leave your answer in terms of an integral which cannot be evaluated in closed form.



Chapter 4
Simple Boundary Value and Initial Value Problems for Elastic Solids

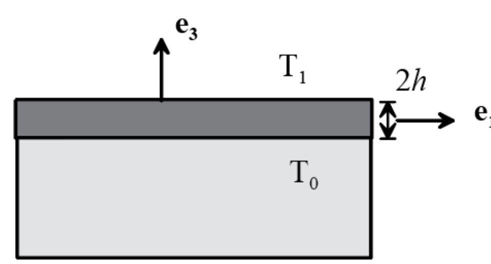
4.1 Axially and Spherically Symmetric Problems for Elastic Solids

Problem 4.1 A solid cylindrical bar with radius a and length L is subjected to a uniform pressure p on its ends. The bar is made from a linear elastic solid with Young's modulus E and Poisson's ratio ν.



(a) Write down an equation for the stress in the bar. Show that the stress satisfies the equation of static equilibrium, and the boundary conditions $\sigma_{ij}n_i = t_j$ on all its surfaces. Express your answer as components in a Cartesian basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_1$ parallel to the axis of the cylinder.
(b) Find the strain in the bar (neglect temperature changes)
(c) Find the displacement field in the bar
(d) Calculate a formula for the change in length of the bar
(e) Find a formula for the stiffness of the bar (stiffness = force/extension)
(f) Find the change in volume of the bar
(g) Calculate the total strain energy in the bar.

Problem 4.2 A thermal barrier coating is idealized as a linear elastic thin film with thickness $2h$, Young's modulus E, Poisson's ratio ν and thermal expansion coefficient α. It is bonded to a substrate (a turbine blade, eg) with dimensions much greater than the film thickness, and elastic properties E_s, ν_s, α_s. The film and substrate are stress free at some initial temperature. The top surface of the film is then exposed to combustion gases at temperature T_1 (above the initial temperature), while the substrate is kept at a lower temperature T_0 (again, above the initial temperature). The steady-state temperature distribution in the film is

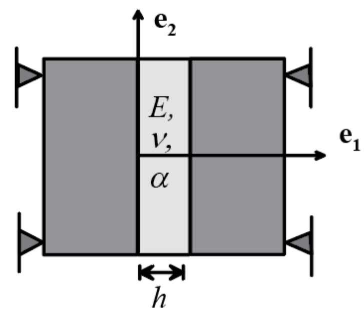


$$T(x_3) = T_1(1 + x_3 / h) / 2 + T_0(1 - x_3 / h) / 2$$

The substrate remains stress free.
(a) Write down the boundary conditions for the stress state at the film surface, and the boundary conditions for displacement and stress at the interface between film and substrate. Use the displacement boundary conditions to show that $\varepsilon_{11}^{film} = \varepsilon_{11}^{substrate}$ $\varepsilon_{22}^{film} = \varepsilon_{22}^{substrate}$ at the film/substrate interface
(b) Assume that the substrate is stress free, and the stresses in the film vary only in the x_3 direction. Find the stress and strain distribution in the film – show that your solution satisfies: (i) The equilibrium equations for stress; (ii) the stress-strain-temperature relations; (iii) the strain equations of compatibility.
(c) Find a formula for the strain energy per unit area of the film.

Problem 4.3 A specimen of material is placed inside a rigid box that prevents the material from stretching in any direction. This means that the strains in the specimen are zero. The specimen is then heated to increase its temperature by ΔT. Find a formula for the stress in the specimen. Find a formula for the strain energy density. How much strain energy would be stored in a $1cm^3$ sample of steel if its temperature were increased by 100C? Compare the strain energy with the heat required to change the temperature by 100C – the specific heat capacity of steel is about 470 J/(kg-C)

Problem 4.4 The figure shows a thin elastic film that is bonded between two large rigid solids, which are held fixed. The film has Young's modulus *E*, Poisson's ratio ν and thermal expansion coefficient α. The rigid solids have zero thermal expansion coefficient .The film is stress free at some reference temperature. The solid on the right is heated, which causes the temperature of the film to increase by



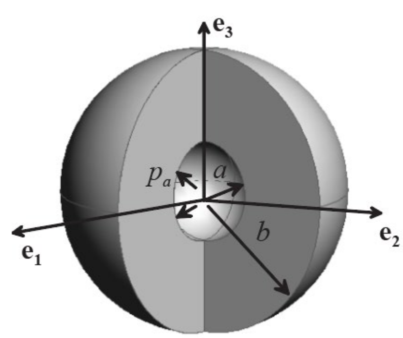
$$\Delta T = \lambda x_1$$

where λ is a constant. The goal of this problem is to calculate the displacement and stress fields in the film. You can assume that the displacement in the film is only in the $\mathbf{e}_1$ direction, and is only a function of x_1 i.e.

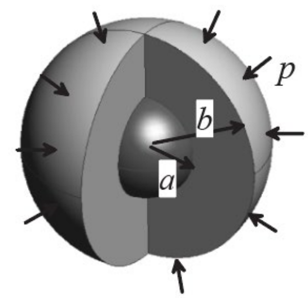
$$\mathbf{u} = u(x_1)\mathbf{e}_1$$

(a) Find a formula the 3x3 infinitesimal strain tensor (matrix) in the film, in terms of derivatives of the unknown displacement u.
(b) Hence, find a formula for the stress components in the film in terms of u (and its derivatives) and an appropriate sub-set of $(E,\nu,\alpha,\lambda,x_1)$.
(c) Write down the three equations of static equilibrium in the film. Show that two of them are satisfied trivially, and express the third one in terms of u (and its derivatives) and $(E,\nu,\alpha,\lambda,x_1)$.
(d) Write down the boundary conditions for u at $x_1 = 0 \quad x_1 = h$.
(e) Hence, calculate $u(x_1)$
(f) Hence, find a formula for the stresses in the film, in terms of E,ν,α,λ

Problem 4.5 Elementary calculations predict that the stresses in an internally pressurized thin-walled sphere with radius *R* and wall thickness *t*<<*R* are $\sigma_{\theta\theta} \approx \sigma_{\phi\phi} \approx pR/2t \quad \sigma_{rr} \approx p/2$. Compare this estimate with the exact solution in Section 4.1.4 of Applied Mechanics of Solids. To do this, set $a = R[1-t/(2R)] \quad b = R[1+t/(2R)]$ and expand the formulas for the stresses as a Taylor series in *t/R*. Suggest an appropriate range of *t/R* for the thin-walled approximation to be accurate.

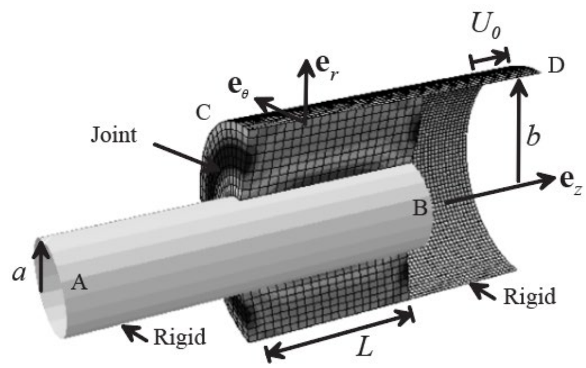


Problem 4.6 A baseball can be idealized as a small rubber core with radius *a*, surrounded by a shell of yarn with outer radius *b*. As a first approximation, assume that the yarn can be idealized as a linear elastic solid with Young's modulus E_s and Poisson's ration ν_s, while the core can be idealized as an incompressible material. Suppose that ball is subjected to a uniform pressure *p* on its outer surface. Note that, if the core is incompressible, its outer radius cannot change, and therefore the radial displacement $u_R = 0$ at $R = a$. Calculate the full displacement and stress fields in the yarn.

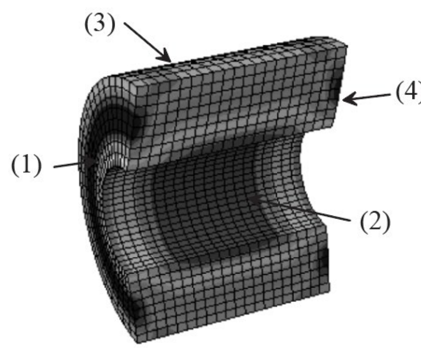


Problem 4.7 Reconsider problem 4.6, but this time assume that the core is to be idealized as a linear elastic solid with Young's modulus E_c and Poisson's ration ν_c. Give expressions for the displacement and stress fields in both the core and the outer shell.

Problem 4.8 The figure shows a cross-section through a joint connecting two hollow cylindrical shafts. The joint is a hollow cylinder with external radius *b* and internal radius *a*. It is bonded to the two rigid shafts AB and CD. Shaft AB is fixed (no translation or rotation), and an axial displacement $\mathbf{u} = U_0 \mathbf{e}_z$ is applied to the hollow cylinder CD. The goal of this problem is to estimate the axial force necessary to displace the shaft CD, and hence determine the stiffness of the joint.



(a) Assume that the displacement field in the joint can be approximated by $\mathbf{u} = u(r)\mathbf{e}_z$, where $u(r)$ is a function to be determined. Calculate the infinitesimal strain tensor in the joint in terms of $u(r)$ and its derivatives.



(b) Assume that the joint can be idealized as an isotropic, linear elastic material with Youngs modulus *E* and Poisson's ratio ν. Find a formula for the stress in the joint *a<r<b,* in terms of derivatives of $u(r)$ and the material properties.

(c) Use the equation of static equilibrium $\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}$ to show that $u(r)$ must satisfy

$$\left\{ \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} \right\} = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr} \right) = 0$$

(d) Write down the boundary conditions for displacements and/or stresses on the four external surfaces of the joint.

(e) Find a solution for $u(r)$ that satisfies I and boundary conditions on *r=a* and *r=b*.

(f) Find a formula for the axial force F_z that must be applied to shaft CD to cause the necessary axial displacement U_0. Hence, find a formula for the stiffness of the joint.

Problem 4.9 A solid, spherical nuclear fuel pellet with outer radius a is subjected to a uniform internal distribution of heat due to a nuclear reaction. The heating induces a steady-state temperature field

$$T(r) = (T_a - T_0)\frac{r^2}{a^2} + T_0$$

where T_0 and T_a are the temperatures at the center and outer surface of the pellet, respectively. Assume that the pellet can be idealized as a linear elastic solid with Young's modulus E, Poisson's ratio ν and thermal expansion coefficient α. Calculate the distribution of stress in the pellet.

Problem 4.10 Suppose that an elastic sphere, with outer radius $a+\Delta$, and with Young's modulus E and Poisson's ratio ν is inserted into a spherical shell with identical elastic properties, but with inner radius a and outer radius b. Assume that $\Delta << a$ so that the deformation can be analyzed using linear elasticity theory. Calculate the stress and displacement fields in both the core and the outer shell.

Problem 4.11 A spherical planet with outer radius a has a radial variation in its density that can be described as

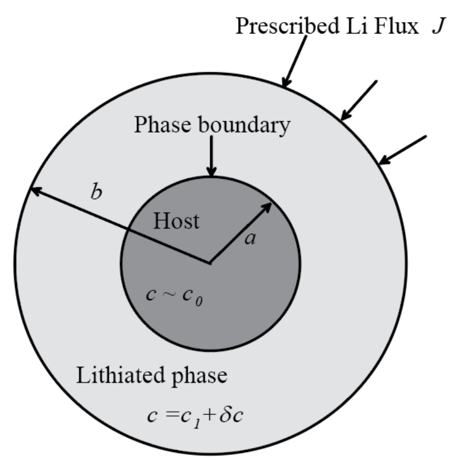
$$\rho(R) = \rho_0\left(1 + \frac{3}{5}\left[1 - \frac{R^2}{a^2}\right]\right)$$

As a result, the interior of the solid is subjected to a radial body force field (per unit mass)

$$\mathbf{b} = -\frac{gR}{31a^3}\left(40a^2 - 9R^2\right)\mathbf{e}_r$$

where g is the acceleration due to gravity at the surface of the sphere. Assume that the planet can be idealized as a linear elastic solid with Young's modulus E and Poisson's ratio ν. Plot the variation of normalized stress $\sigma_{RR}/(g\rho_0 a)$ and $\sigma_{\theta\theta}/(g\rho_0 a)$ in the sphere as a function of *R/a*, for $\nu = 1/3$

Problem 4.12 The figure shows a spherical Li-ion battery particle. The host material has a negligible Li concentration, and Li is inserted into the particle through an electrochemical reaction at the particle surface. The material phase separates with equilibrium concentrations c_0, c_1 so that at some representative instant during Li insertion the particle consists of a spherical core with radius a containing a low uniform Li concentration c_0 surrounded by an outer shell with higher, nonuniform Li concentration $c_1 + \delta c$, where δc is to be determined.



The material is elastic with (concentration independent) Young's modulus E and Poisson ratio ν. When lithiated, the material experiences a compositional strain $\varepsilon_{ij}^c = \beta c \delta_{ij}/3$, where β is a constant (quantifying the volumetric strain caused by Li insertion, analogous to a thermal expansion coefficient), so the total strain in the sphere (taking the un-lithiated material as reference configuration) is

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{\beta c}{3}\delta_{ij}$$

The deviation δc of Li concentration from its equilibrium value in the sphere satisfies (approximately) the diffusion equation

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{d\mu}{dR}\right)=0 \qquad a<R<b$$

where μ is the stress and concentration dependent chemical potential of Li in the host, given by

$$\mu=\mu_0+\boldsymbol{\Gamma}\delta c-\frac{1}{3}\beta\sigma_{kk}$$

Here, $\mu_0,\boldsymbol{\Gamma}$ are constants. The boundary conditions for chemical potential and concentration are

$$\delta c=0 \quad (r=a), \qquad D\frac{\partial\mu}{\partial r}=J^* \quad (r=b)$$

where D is the diffusion coefficient for Li transport through the outer shell.

(a) Assume that the state of stress in the core region $0<r<a$ is a state of uniform hydrostatic stress $\sigma_{rr}=\sigma_{\theta\theta}=p$, where p is to be determined. Calculate the displacement field (relative to a sphere with zero Li concentration) in the core region, in terms of p and other relevant variables.

(b) Show that the displacement field in $a<r<b$ must satisfy

$$\left.\begin{aligned}&\frac{du_R}{dR}+\frac{2u_R}{R}=\beta c_1+\beta\delta c+\frac{1-2\nu}{E}(\sigma_{RR}+2\sigma_{\theta\theta})\\&\frac{du_R}{dR}-\frac{u_R}{R}=\left(\frac{1+\nu}{E}(\sigma_{rr}-\sigma_{\theta\theta})\right)\end{aligned}\right\} \qquad a<R<b$$

(c) Show (use a symbolic manipulation program to do the algebra) that the equation of equilibrium can be expressed in the form

$$\frac{d^2u_R}{dR^2}+\frac{2}{R}\frac{du_R}{dR}-\frac{2}{R^2}u_R=\frac{d}{dR}\left\{\frac{1}{R^2}\frac{d}{dR}\left(R^2u_R\right)\right\}=\frac{\beta(1+\nu)}{3(1-\nu)}\frac{dc}{dR}$$

(d) Hence, show that the displacement field is related to the concentration by

$$u_R=AR-\frac{1+\nu}{3(1-\nu)}\frac{\beta}{R}\int_0^R\xi^2c(\xi)d\xi$$

where A is a constant, and hence show that the diffusion equation can be expressed in the form

$$\left(\frac{2E\beta^2}{9(1-\nu)}+\boldsymbol{\Gamma}\right)\frac{d\delta c}{dr}=\frac{J^*b^2}{DR^2}$$

(e) Hence, show that the Li concentration in $a<r<b$ is

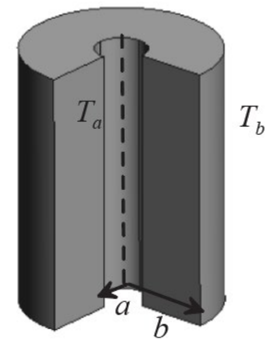
$$c=c_1+\frac{Jb^2}{D\boldsymbol{\Lambda}}\frac{R-a}{aR} \qquad \boldsymbol{\Lambda}=\frac{2E\beta^2}{9(1-\nu)}+\boldsymbol{\Gamma}$$

(f) Finally, show that the stress field in the particle is

$$\sigma_{RR}=\frac{2E\beta(c_1-c_0)}{9(1-\nu)}\frac{a^3(b^3-R^3)}{b^3R^3}-\frac{E\beta J^*}{9D(1-\nu)\boldsymbol{\Lambda}}\frac{(b-R)}{bR^3}(a^2b^2+a^2bR+a^2R^2-3b^2R^2)$$

$$\sigma_{\theta\theta}=\frac{2E\beta(c_1-c_0)}{9(1-\nu)}\frac{a^3(b^3+2R^3)}{b^3R^3}+\frac{E\beta J^*}{18D(1-\nu)\boldsymbol{\Lambda}}\frac{(a^2b^3+2a^2R^3+3b^3R^2-6b^2R^3)}{bR^3}$$

Problem 4.13 A 'hydrocarbon cracker tube' is a long cylindrical pipe with inner radius a and outer radius b, which has hot gases with temperature T_a flowing through it and is heated to a higher temperature T_b at its outer surface. The inner and outer surfaces of the pipe are traction free. Assume that planes transverse to the pipe's axis remain plane, but the tube is free to expand along its length, so that the axial strain ε_{zz} can be determined from the condition that no axial force acts on the pipe. In addition, assume that the temperature distribution in the pipe is given by the steady state solution to the heat equation



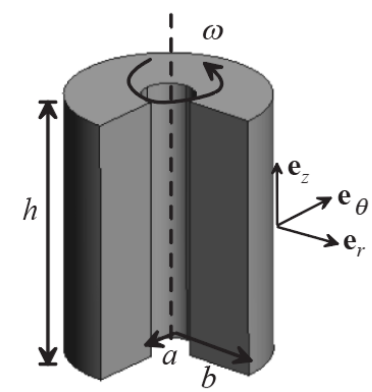
$$T(r)=T_b+(T_a-T_b)\frac{\log(r/b)}{\log(a/b)}$$

(a) Calculate the stress components $\sigma_{rr},\sigma_{\theta\theta},\sigma_{zz}$ in the pipe.

(b) Plot the variation of normalized Von-Mises stress $(1-\nu)\sigma_e/\{E\alpha(T_b-T_a)\}$ as a function of $\xi-(r-a)/(b-a)$ for a few representative values of a/b. Hence, find where the maximum von-Mises stress occurs and find a formula for the maximum von-Mises stress.

(c) A typical cracker tube has inner radius 42cm, outer radius 50cm, a Young's modulus of 210GPa and thermal expansion coefficient $11.5\times10^{-6}K^{-1}$, and is heated to 1050C on its outer wall and to 850C on its inner wall. For comparison, the yield stress of a typical steel used in these applications is of order 50MPa or lower at the relevant temperatures.

Problem 4.14 The goal of this problem is to work through some rough design calculations for a flywheel energy storage system. Idealize the flywheel as a hollow cylinder with mass density ρ that spins at constant angular speed ω.



(a) Calculate the stress and radial displacement in the flywheel, using the following boundary conditions:
   - The radial displacement at $r=a$ is zero,
   - The radial stress at $r=b$ is zero.
   - The axial force $F_z=\int_a^b 2\pi\sigma_{zz}r\,dr$ is zero.

   Simplify your final expression by substituting $a=0$ (to produce a solid cylinder) . Hence, plot graphs showing the normalized stresses $\sigma_{rr}/(\rho\omega^2b^2),\sigma_{\theta\theta}/(\rho\omega^2b^2),\sigma_{zz}/(\rho\omega^2b^2)$, as well as the normalized von-Mises stress $\sigma_e/(\rho\omega^2b^2)$ as functions of normalized radial position r/b in the disk, with Poisson's ratio $\nu=0.3$.

(b) Suppose that the material in the disk is made from a metal, so that the device must be designed and operated to ensure that the Von-Mises stress σ_e stays below a critical value Y (the yield stress of the metal). Show that the maximum kinetic energy that can be stored in the disk is

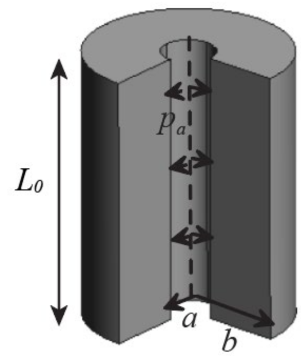
$$U_{\max}=\beta YV$$

where V is the volume of the cylinder and β is a constant that depends only on Poisson's ratio. Find the formula for β

(c) A typical disk is made from steel, and has the following specifications: Weight: 10000lb; Energy stored: 32 kWhr. Steel has a density of 8050kg/m^3. Find the minimum allowable value of the yield stress.

4.2 Axially and Spherically Symmetric Solutions to Large Strain Elasticity Problems

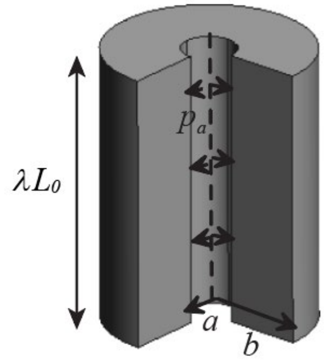
Problem 4.15 The goal of this problem is to calculate the deformation and stress in an internally pressurized hollow rubber cylinder deforming in plane strain, as shown in the picture. Assume that



- Before deformation, the cylinder has inner radius A and outer radius B
- After deformation, the cylinder has inner radius a and outer radius b
- The solid is made from an incompressible Mooney-Rivlin solid, with strain energy potential
$$U = \frac{\mu_1}{2}(I_1 - 3) + \frac{\mu_2}{2}(I_2 - 3)$$
where I_1, I_2 are the first and second invariants of the right Cauchy Green deformation tensor $B_{ij} = F_{ik}F_{jk}$
- No body forces act on the cylinder; the inner surface *r*=*a* is subjected to pressure p_a; while the outer surface *r*=*b* is free of stress.
- Assume plane strain deformation (so the length of the cylinder L_0 remains fixed).

To proceed, suppose that a material particle that has radial position *R* before deformation moves to a position *r=f(R)* after the cylinder is loaded. This problem should be solved using cylindrical-polar coordinates.

(a) Find an expression for the deformation gradient **F** in terms of *f(R)* and *R*
(b) Express the incompressibility condition det(**F**)=1 in terms of *f(R)*
(c) Integrate the incompressibility condition to calculate *r* in terms of *R*,*A* and *a*, and also calculate the inverse expression that relates *R* to *A*, *a* and *r*.
(d) Calculate the components of the left Cauchy-Green deformation tensor $B_{rr}, B_{\theta\theta}$
(e) Find an expression for the Cauchy stress components $\sigma_{rr}, \sigma_{\theta\theta}$ in the cylinder in terms of $B_{rr}, B_{\theta\theta}, B_{zz}$ and an indeterminate hydrostatic stress *p*.
(f) Use (c), (d) and the equilibrium equation to derive a differential equation for σ_{rr}. Integrate this equation and use the boundary conditions to find a relationship between the applied pressure and $\alpha = a/A$, $\beta = b/B$.
(g) Plot a graph showing the variation of normalized pressure $p_a B^2 / \left\{(\mu_1 + \mu_2)(B^2 - A^2)\right\}$ as a function of the normalized displacement of the inner bore of the cylinder $a/A - 1$, for a few values of B/A Compare the nonlinear elastic solution with the equivalent linear elastic solution. Find a formula for the maximum possible pressure that can be applied to the cylinder (at the max pressure the deformed radii are infinite)
(h) Plot the variation of normalized stress $\sigma_{rr}/(\mu_1 + \mu_2)$, $\sigma_{\theta\theta}/(\mu_1 + \mu_2)$ as a function of normalized position $\xi = (r-a)/(b-a)$ in the deformed cylinder, for various values of $\alpha = a/A$ and $\rho = B/A = 3$.



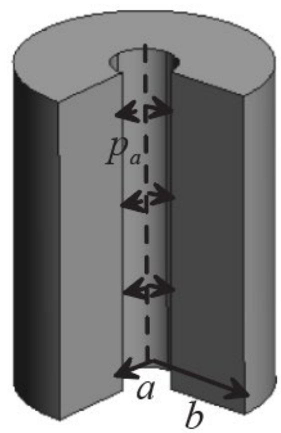
Problem 4.16 The goal of this problem is to calculate the deformation and stress in an internally pressurized hollow rubber cylinder that is stretched axially by an amount λ before it is pressurized. Assume that

- Before deformation, the cylinder has inner radius *A* and outer radius *B*
- After deformation, the cylinder has inner radius *a* and outer radius *b*
- The solid is made from an incompressible heo-Hookean solid, with strain energy potential

$$U = \frac{\mu_1}{2}(I_1 - 3)$$

- No body forces act on the cylinder; the inner surface *r=a* is subjected to pressure p_a; while the outer surface *r=b* is free of stress.
- Plane sections transverse to the cylinder's axis remain plane, but the deformed cylinder has a length $L = \lambda L_0$, where L_0 is its unstretched length.

Before solving the problem, you could try to guess whether stretching it will make it easier or harder to inflate. Then, follow steps (a)-(g) in Problem 4.15 to plot a graph showing the variation of normalized pressure $p_a B^2 / \left\{ \mu_1 (B^2 - A^2) \right\}$ as a function of the normalized displacement of the inner bore of $a\sqrt{\lambda} / A - 1$, for a few values of λ, for a tube with diameter ratio $B^2 / A^2 = 3$



Problem 4.17 The goal of this problem is to calculate the deformation and stress in an internally pressurized hollow rubber cylinder that is free of axial force, as shown in the picture. Assume that

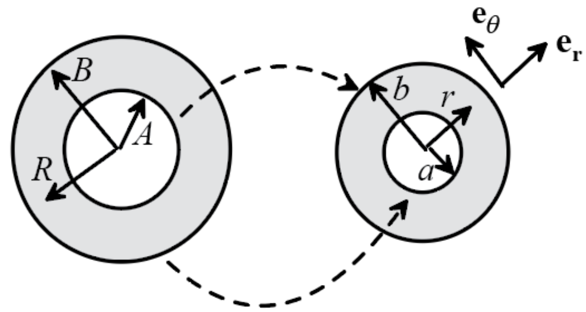
- Before deformation, the cylinder has inner radius *A* and outer radius *B*
- After deformation, the cylinder has inner radius *a* and outer radius *b*
- The solid is made from an incompressible heo-Hookean solid, with strain energy potential

$$U = \frac{\mu_1}{2}(I_1 - 3)$$

- No body forces act on the cylinder; the inner surface *r=a* is subjected to pressure p_a; while the outer surface *r=b* is free of stress.
- Plane sections transverse to the cylinder's axis remain plane, but no axial force acts on the cylinder.
- The cylinder stretches axially by an amount λ to be determined .

You can solve the problem using the procedure outlined in Problme 4.16, except that you will need to calculate the axial stress σ_{zz} in the cylinder, and use the result to find a formula for the axial force on the cylinder in terms of λ. Setting the force to zero will give a nonlinear equation for λ that can be solved numerically. Use your solution to plot a graph showing the variation of normalized pressure $p_a B^2 / \left\{ \mu_1 (B^2 - A^2) \right\}$ as a function of the normalized displacement of the inner bore of the cylinder $a / A - 1$, for a few values of B / A.

Problem 4.18 A rubber tube with internal radius A and external radius B is turned inside out, so that the surfaces at $R=A$, $R=B$ now lie at deformed radius $r=b$, $r=a$, respectively. These surfaces are free of traction. Assume that plane cross-sections of the tube remain plane, ***and that the length of the cylinder does not change***. Note that a cross section at distance Z along the axis of the undeformed tube moves to a new position $z=-Z$ after deformation. Idealize the tube as an incompressible, neo-Hookean material with stress-strain relation



$$\sigma_{ij} = \mu B_{ij} + p(r)\delta_{ij}$$

where μ is a constant, B_{ij} is the Left Cauchy-Green deformation tensor and p is a pressure that must be determined from the boundary conditions.

(a) By considering the areas of material in the annular regions between r and a, and between R and B, find an expression for the radial position r of a material particle that starts at radius R in the tube before deformation.

(b) Find the deformation gradient **F** in terms of r and R.

(c) Hence find an expression for the stress in the tube as a function of r, R and p.

(d) Solve the equilibrium equation for the radial stress, and hence show that the radii of the tube after deformation must satisfy the equations

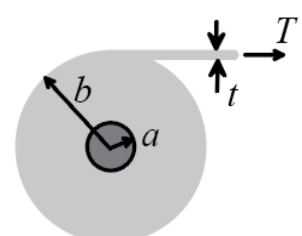
$$\log\left(\frac{Bb}{Aa}\right) + \frac{A^2}{2b^2} - \frac{B^2}{2a^2} = 0$$

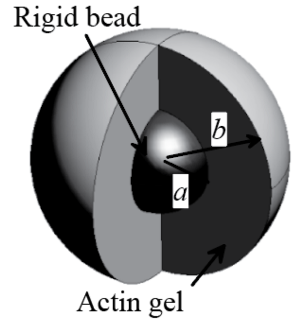
$$b^2 - a^2 = B^2 - A^2$$

(e) Plot the variations of b/B and a/A with B/A. Does the cylinder get fatter or thinner after the inversion?

(f) Plot graphs showing the variations of σ_{rr}/μ and $\sigma_{\theta\theta}/\mu$ as functions of $(r-a)/(b-a)$ for a few representative values of ρ

Problem 4.19 A rubber sheet is wrapped around a rigid cylindrical shaft with radius a. The sheet can be idealized as an incompressible neo-Hookean solid. A constant tension T per unit out-of-plane distance is applied to the sheet during the wrapping process. The stretched sheet has thickness t. After wrapping, the end of the sheet is glued to the surface of the roll, and the tension is removed, to produce a cylindrical roll that contains a self-equilibriating internal stress field. Calculate the Cauchy stress components $\sigma_{rr}, \sigma_{\theta\theta}$ in the cylinder of rubber, and find an expression for the radial pressure acting on the shaft. Assume that $t/b \rightarrow 0$, and that no slip occurs at the interfaces where the rubber sheet contacts itself.





Problem 4.20 In a model experiment intended to duplicate the propulsion mechanism of the lysteria bacterium, a spherical bead with radius a is coated with an enzyme known as an "Arp2/3 activator." (See, e.g. L.A. Cameron, *et al*, Proc. Natl. Acad. Sci. U.S.A. 96 (9) 4908-4913, 1999). When suspended in a solution of actin, the enzyme causes the actin to polymerize at the surface of the bead. The polymerization reaction causes a spherical gel of a dense actin network to form around the bead. New gel is continuously formed at the bead/gel interface, forcing the rest of the gel to expand radially around the bead. The actin gel is a long-chain polymer and consequently can be idealized as a rubber-like incompressible neo-Hookean material. Experiments show that after reaching a critical radius the actin gel loses spherical symmetry and occasionally will fracture. Stresses in the actin network are believed to drive both processes. In this problem you will calculate the stress state in the growing, spherical, actin gel.

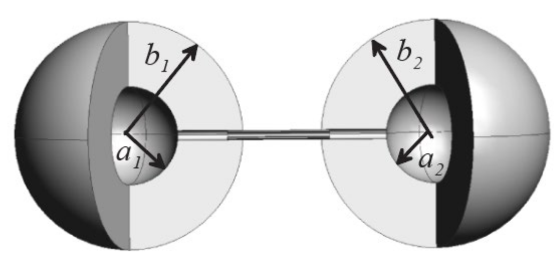
(a) Note that this is an unusual boundary value problem in solid mechanics, because a compatible reference configuration cannot be identified for the gel. Nevertheless, it is possible to write down a deformation gradient field that characterizes the change in shape of infinitesimal volume elements in the gel. To this end: (i) write down the length of a circumferential line at the surface of the bead; (ii) write down the length of a circumferential line at radius r in the gel; (iii) use these results, together with the incompressibility condition, to write down the deformation gradient characterizing the shape change of a material element that has been displaced from $r=a$ to a general position r. Assume that the bead is rigid, and that the deformation is spherically symmetric.

(b) Suppose that new actin polymer is generated at volumetric rate $\dot{V}$. Use the incompressibility condition to write down the velocity field in the actin gel in terms of $\dot{V}$, a and r.

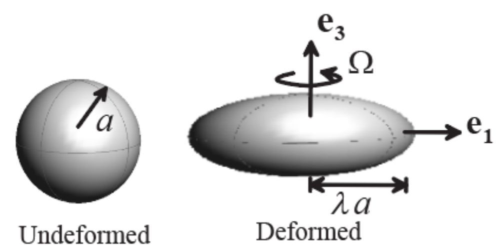
(c) Calculate the velocity gradient $\mathbf{v}\nabla$ in the gel (i) by direct differentiation of (b) and (ii) by using the results of (a). Show that the results are consistent.

(d) Calculate the components of the left Cauchy-Green deformation tensor field $\mathbf{B}$ and hence write down an expression for the Cauchy stress field in the solid, in terms of an indeterminate hydrostatic pressure and the shear modulus. Assume that the stress is hydrostatic when $\mathbf{B}=\mathbf{I}$. (This is an arbitrary assumption – the material could be in any state of stress when it is first formed)

(e) Use the equilibrium equations and boundary condition to calculate the full Cauchy stress distribution in the bead. Assume that the outer surface of the gel (at $r=b$) is traction free.



Problem 4.21 Two spherical, hyperelastic shells are connected by a thin tube, as shown in the picture. When stress free, both spheres have internal radius A and external radius B. The material in each sphere can be idealized as an incompressible, neo-Hookean solid, with material constant μ_1. Suppose that the two spheres together contain a volume $V \ge 8\pi A^3/3$ of an incompressible fluid. As a result, the two spheres have deformed internal and external radii (a_1, b_1), (a_2, b_2) as shown in the picture. Use the solution for an internally pressurized hyperelastic shell in Section 4.2.3 of Applied Mechanics of Solids to investigate the possible equilibrium configurations for the system, as functions of the dimensionless fluid volume $\omega^3 = V/(8\pi A^3/3)$ and B/A. To display your results, plot a graph showing the equilibrium values of $\alpha_1 = a_1/A$ as a function of ω, for various values of B/A. You should find that for small values of ω there is only a single stable equilibrium configuration. For ω exceeding a critical value, there are three possible equilibrium configurations: two in which one sphere is larger than the other (these are stable), and a third in which the two spheres have the same size (this is unstable).



Problem 4.22 An incompressible, neo-Hookean sphere with radius a and shear modulus μ and mass density ρ spins at constant angular rate $\mathbf{\Omega}$ about the $\mathbf{e}_3$ axis. The exterior surface is traction free. As a result, the sphere tends to elongate in directions perpendicular to the spin axis, while contracting parallel to the spin axis. The goal of this problem is to estimate the shape change.

(a) Assume that the deformed shape can be approximated as an ellipsoid, with semi axes $(\lambda a, \lambda a, a/\lambda_3)$. Use the incompressibility condition to write down λ_3 in terms of λ

(b) The deformation gradient can be expressed in the form $\mathbf{VR}$ where $\mathbf{R}$ represents a rigid rotation about the $\mathbf{e}_3$ axis and $\mathbf{V}$ is a stretch. Express $\mathbf{V}$ in terms of λ and hence find an expression for the Cauchy stress in the ellipsoid, in terms of λ, μ, and the unknown hydrostatic stress p. (You can assume $\boldsymbol{\sigma} = \mu\mathbf{B} + p\mathbf{I}$). Express your answer as components in a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ which rotates with the solid.

(c) Show that the velocity gradient $\mathbf{L}$ has the form $\mathbf{L} = \mathbf{D} + \mathbf{W}$, where $\mathbf{D}$ is a symmetric tensor that depends only on λ and $\dot{\lambda}$, and $\mathbf{W}$ is a skew tensor that depends only on $\mathbf{\Omega}$. Give expressions for $\mathbf{D}$ and $\mathbf{W}$ as matrices (or in dyadic notation if you prefer).

(d) Noting that the velocity gradient is independent of position, write down the Eulerian velocity field in the sphere, in terms of $\mathbf{D}, \mathbf{W}$ and position vector $\mathbf{y}$.

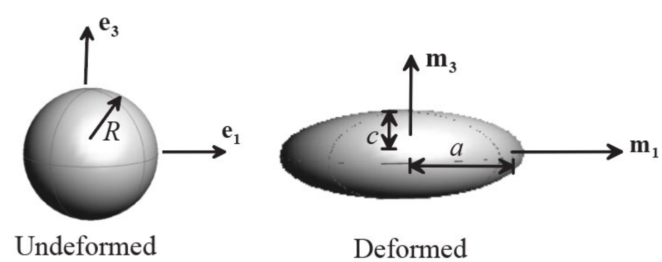
(e) For steady rotation with constant λ , write down the acceleration vector of a material particle at position $\mathbf{y} = y_i \mathbf{e}_i$ in the deformed sphere (no lengthy derivations are necessary – material particles are in circular motion about the $\mathbf{e}_3$ axis at constant speed).

(f) By considering a volume preserving virtual velocity $\delta\mathbf{v}$ resulting from a stretch variation $\delta\dot{\lambda}$, show that the principle of virtual work requires that

$$\frac{8\pi}{3} a^3 \mu(\lambda - \frac{1}{\lambda^5})\delta\dot{\lambda} = \mathbf{\Omega}^2 \frac{\delta\dot{\lambda}}{\lambda} I_{33}$$

where $I_{33} = 8\pi\rho\lambda^2 a^5 / 15$ is the mass moment of inertia of the deformed sphere about the $\mathbf{e}_3$ axis (you don't need to do the integral to derive the value of I_{33}, it is sufficient to identify the mass moment of inertia term in the virtual work principle).

(g) Hence, calculate an expression for λ in terms of $\mathbf{\Omega}$ and other relevant parameters.



Problem 4.23 An incompressible neo-Hookean sphere with radius R, modulus μ and mass density ρ is deformed into an ellipsoid with semi-axes $a = \lambda_1 R \quad b = \lambda_1 R \quad c = \lambda_2 R$.

(a) Use the incompressibility condition to express λ_2 in terms of λ_1

(b) Find an expression for the total elastic strain energy in the solid, in terms of λ_1

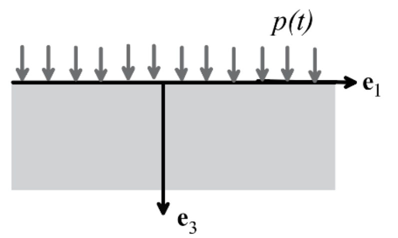
(c) Find an expression for the total kinetic energy of the solid, in terms of λ_1 and its time derivatives. Assume the center is stationary.

(d) Hence, estimate the natural frequency of small amplitude oscillations of this (approximate) vibration mode

4.3 Solutions to Simple Dynamic Problems Involving Linear Elastic Solids

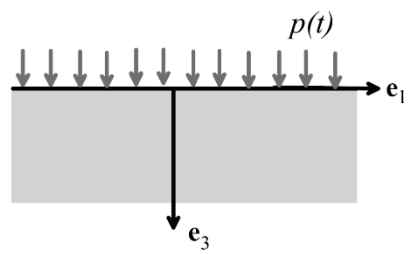
Problem 4.24 Calculate longitudinal and shear wave speeds in (a) Silicon Carbide; (b) Steel; (c) Aluminum and (d) Rubber.

Problem 4.25 A linear elastic half-space with mass density ρ, Young's modulus E and Poisson's ratio ν is stress free and stationary at time *t=0*¸ is then subjected to a constant pressure p_0 on its surface for *t>0*.



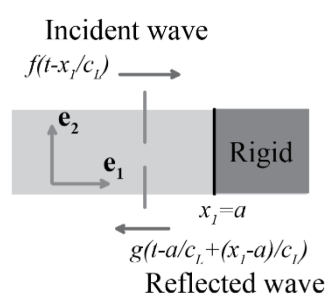
(a) Calculate the stress, displacement and velocity in the solid as a function of time (you can express your answer in terms of p_0, t, x_3, ρ and the wave speed c_L)
(b) Calculate the total kinetic energy (per unit area of the surface) of the half-space as a function of time
(c) Calculate the total strain energy of the half-space (per unit area of surface) as a function of time
(d) Verify that the sum of the strain energy and kinetic energy is equal to the work done by the tractions acting on the surface of the half-space.

Problem 4.26 The figure shows an infinite linear elastic half-space with mass density ρ, Young's modulus E and Poisson's ratio ν. It is at rest for time *t<0*, and is subjected to a harmonic pressure on its surface, given by $p(t) = p_0 \sin \omega t$ *t>0*, with *p=0* for *t<0*.



(a) What are the distributions of stress, displacement and velocity in the solid? (you can express your answer in terms of $p_0, \omega, t, x_3, \rho, \nu$ and the wave speed c_L)
(b) Calculate the total work done by the applied pressure in one cycle of loading and hence determine the rate of work done by the pressure. This energy is radiated in kinetic energy away from the surface (ie the power expended by whatever is applying the pressure). Does the rate of work depend on the frequency?

Problem 4.27 A linear elastic solid with Young's modulus E Poisson's ratio ν and density ρ is bonded to a rigid solid at $x_1 = a$. Suppose that a plane wave with displacement and stress field

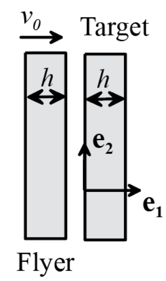


$$u_2(x_1,t) = \begin{cases} \dfrac{(1-2\nu)(1+\nu)}{(1-\nu)} \dfrac{\sigma_0}{E}(c_L t - x_1) & x_1 < c_L t \\ 0 & x_1 > c_L t \end{cases}$$

$$\sigma_{11} = \begin{cases} -\sigma_0 & x_1 < c_L t \\ 0 & x_1 > c_L t \end{cases}$$

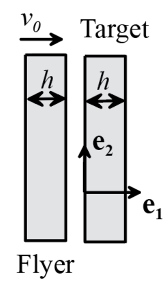
is induced in the solid, and at time $t = a / c_L$ is reflected off the interface. Find the reflected wave, and sketch the variation of stress and velocity in the elastic solid just before and just after the reflection occurs.

Problem 4.28 The figure shows a plate impact experiment with a collision in the normal direction between the target and flyer plates. The plates have identical thickness *h,* which is much smaller than their lateral dimensions. Both target and plate have mass density ρ and longitudinal wave speed c_L At time *t=0* (the instant of collision) the target is stationary, and the flyer has a uniform horizontal velocity v_0.

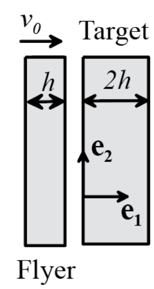


(a) Draw graphs showing the stress and velocity at the (i) the impact face; (ii) the rear face and (iii) the mid-plane of the flyer plate as a function of time.
(b) Draw a graph showing the total strain energy and total kinetic energy of the flyer and target as a function of time. Verify that total energy is conserved.

Problem 4.29 In a plate impact experiment, two identical elastic plates with thickness *h,* Young's modulus *E*, Poisson's ratio ν , density ρ and longitudinal wave speed c_L are caused to collide, as shown in the picture. Just prior to impact, the projectile has a uniform velocity v_0. Draw the (*x,t*) diagram for the waves in the target and flyer plates.

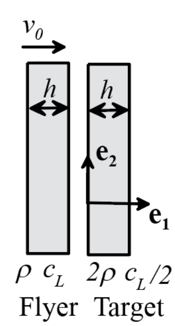


Problem 4.30 In a plate impact experiment, an elastic plate with thickness *h,* Young's modulus *E*, Poisson's ratio ν , density ρ and longitudinal wave speed c_L impacts a second plate with identical properties, but thickness *2h*, as shown in the picture. Just prior to impact, the flyer has a uniform velocity v_0.

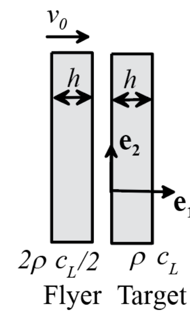


(a) Draw the (*x,t*) diagram for the two solids after impact.
(b) Find the velocity of the center of mass of the target plate after the impact. What is the apparent coefficient of restitution for the collision, in the terminology of rigid body collisions? Where did the missing translational kinetic energy go?

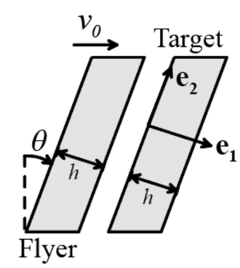
Problem 4.31 In a plate impact experiment, two plates with identical thickness *h* are caused to collide, as shown in the picture. The flyer plate has mass density ρ and wave longitudinal wave speed c_L, while the target has mass density 2ρ and wave speed $c_L/2$. Find the stress and velocity behind the waves generated by the impact in both target and flyer plate. Hence, draw the (*x,t*) diagram for the two solids after impact.



Problem 4.32 In a plate impact experiment, two plates with identical thickness *h,* Young's modulus *E*, Poisson's ratio ν, and density ρ are caused to collide, as shown in the picture. The target plate has mass density ρ and wave longitudinal wave speed c_L, while the flyer has mass density 2ρ and wave speed $c_L/2$. Find the stress and velocity behind the waves generated by the impact in both target and flyer plate. Hence, draw the (*x,t*) diagram for the two solids after impact.



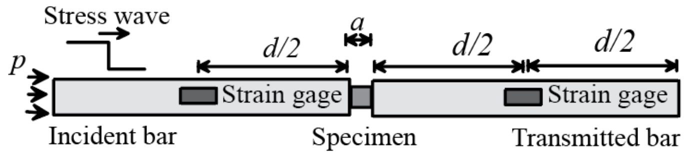
Problem 4.33 The figure shows a *pressure-shear* plate impact experiment. A flyer plate with speed v_0 impacts a stationary target. Both solids have identical thickness *h,* Young's modulus *E*, Poisson's ratio ν, density ρ and longitudinal and shear wave speeds c_L and c_S. The faces of the plates are inclined at an angle θ to the initial velocity, as shown in the figure. Both pressure and shear waves are generated by the impact. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ denote unit vectors Let $\sigma_{11} = \sigma_0$ denote the (uniform) stress behind the propagating pressure wave in both solids just after impact, and $\sigma_{12} = \tau_0$ denote the shear stress behind the shear wave-front. Similarly, let $\Delta v_1^F, \Delta v_2^F$ denote the change in longitudinal and transverse velocity in the flier across the pressure and shear wave fronts, and let $\Delta v_1^T, \Delta v_2^T$ denote the corresponding velocity changes in the target plate. Assume that the interface does not slip after impact, so that both velocity and stress must be equal in both flier and target plate at the interface just after impact.

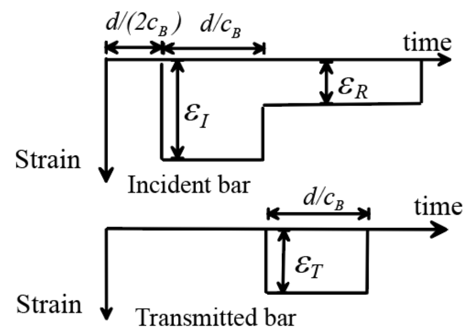


(a) Find expressions for σ_0, τ_0, $\Delta v_1^F, \Delta v_2^F$, $\Delta v_1^T, \Delta v_2^T$ in terms of v_0, θ and relevant material properties.

(b) Draw the full (*x,t*) diagram for the pressure-shear configuration. Assume that the interface remains perfectly bonded until it separates under the application of a tensile stress When you draw the diagram, assume that that the shear wave speed is half the pressure wave speed. Note that you will have to show (*x,t*) diagrams associated with both shear and pressure waves: these can be displayed on separate figures.

Problem 4.34 A " Split-Hopkinson bar" or "Kolsky bar" is an apparatus that is used to measure plastic flow in materials at high rates of strain (of order 1000/s). The apparatus is sketched in the figure. A small specimen of the material of interest, with length $a<<d$, is placed between two long slender bars with length d. Strain gages are attached near the mid-point of each bar. At time $t=0$ the system is stress free and at rest. At time $t=0$, constant pressure p is applied to the end of the incident bar, sending a plane wave down the bar. This wave eventually reaches the specimen. At this point, part of the wave is reflected back up the incident bar, and part of it travels through the specimen and into the second bar (known as the 'transmission bar'). The history of stress and strain in the specimen can be deduced from the history of strain measured by the two strain gages. For example, if the specimen behaves as rigid-perfectly plastic solid, the incindent and reflected gages would record the data shown in the figure. The goal of this problem is to calculate a relationship between the measured strains and the stress and strain rate in the specimen. Assume that the bars are linear elastic with Young's modulus E and density ρ, and wave speed $c_B \approx \sqrt{E/\rho}$, and that the bars deform in uniaxial compression.





(a) Write down the stress, strain and velocity field in the incident bar as a function of time and distance down the bar in terms of the applied pressure p and relevant material and geometric parameters, for $t < d/c_B$.

(b) Assume that the waves reflected from, and transmitted through, the specimen are both plane waves. Let ε_R and ε_T denote the compressive strains in the regions behind the reflected and transmitted wave fronts, respectively (these depend on the behavior of the specimen, and may be functions of time). Write down expressions for the stress and velocity behind the wave fronts in both incident and transmitted bars in terms of ε_R and ε_T, for $2d/c_B > t > d/c_B$

(c) The axial force in the bar behind the reflected and transmitted waves must equal the axial force in the specimen. In addition, the strain rate in the specimen can be calculated from the relative velocity of the incident and transmitted bars where they touch the specimen. Show that the strain rate in the specimen (with compressive strain positive) can be calculated from the measured strains as

$$\frac{d\varepsilon_s}{dt} = \frac{c_B}{a}(2\varepsilon_I - \varepsilon_R - \varepsilon_T) = \frac{2c_B}{a}(\varepsilon_I - \varepsilon_R)$$

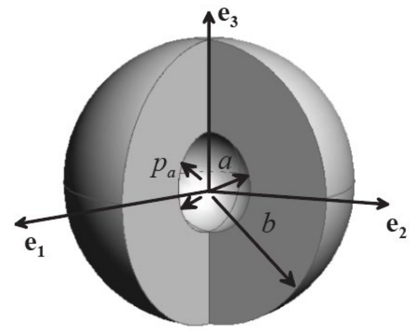
while the axial stress in the specimen can be calculated from

$$\sigma_{11} = -\frac{A_S}{A_B}E\varepsilon_T = -E\frac{A_S}{A_B}\varepsilon_R .$$

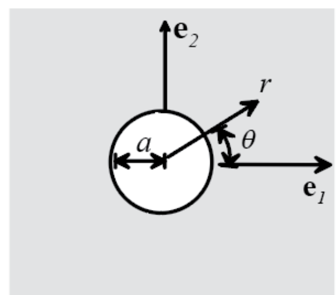
where A_S, A_B denote the cross-sectional areas of the specimen and bar, respectively.

Chapter 5
Solutions for Linear Elastic Solids

5.1 General Principles



Problem 5.1 A spherical shell is simultaneously subjected to internal pressure p, and is heated internally to raise its temperature at $r=a$ to a temperature T_a, while at $r=b$ its surface is traction free, and temperature is T_b. Use the principle of superposition, together with the solutions given in Chapter 4 of Applied Mechanics of Solids, to determine the stress field in the sphere.



Problem 5.2 The stress field around a cylindrical hole in an infinite solid, which is subjected to uniaxial tension $\sigma_{11}=\sigma_0$ far from the hole, is given by

$$\sigma_{11}=\sigma_0\left(1+\left(\frac{3a^4}{2r^4}-\frac{a^2}{r^2}\right)\cos 4\theta-\frac{3a^2}{2r^2}\cos 2\theta\right)$$

$$\sigma_{22}=\sigma_0\left(\left(\frac{a^2}{r^2}-\frac{3a^4}{2r^4}\right)\cos 4\theta-\frac{a^2}{2r^2}\cos 2\theta\right)$$

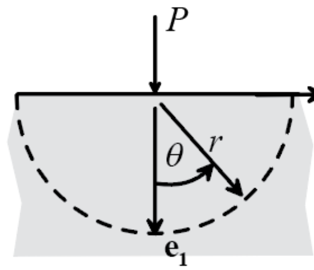
$$\sigma_{12}=\sigma_0\left(\left(\frac{3a^4}{2r^4}-\frac{a^2}{r^2}\right)\sin 4\theta-\frac{a^2}{2r^2}\sin 2\theta\right)$$

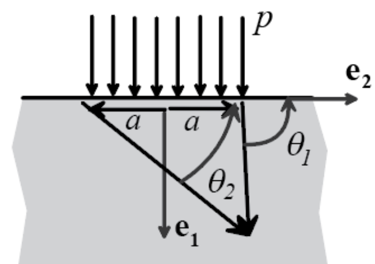
Using the principle of superposition, calculate the stresses near a hole in a solid which is subjected to shear stress $\sigma_{12}=\tau_0$ at infinity.

Problem 5.3 The stress field in an infinite solid that contains a spherical cavity with radius a at the origin, and is subjected to a uniform uniaxial stress $\sigma_{33}=\sigma_0$ far from the sphere is given by

$$\frac{\sigma_{ij}}{\sigma_0}=\frac{3a^3}{2(7-5\nu)R^3}\left(3-5\nu-4\frac{a^2}{R^2}\right)\delta_{ij}+\frac{3a^3x_ix_j}{2(7-5\nu)R^5}\left(6-5\nu-5\frac{a^2}{R^2}+10\frac{x_3^2}{R^2}\right)$$
$$+\frac{\delta_{i3}\delta_{j3}}{(7-5\nu)}\left((7-5\nu)+5(1-2\nu)\frac{a^3}{R^3}+3\frac{a^5}{R^5}\right)-\frac{15a^3x_3(x_j\delta_{i3}+x_i\delta_{j3})}{(7-5\nu)R^5}\left(\frac{a^2}{R^2}-\nu\right)$$

Show that the hole only influences the stress field in a region close to the hole.





Problem 5.4 The stress field due to a concentrated line load, with force per unit out-of-plane distance P acting on the surface of a large flat elastic solid are given by

$$\sigma_{11}=-\frac{2P}{\pi}\frac{x_1^3}{\left(x_1^2+x_2^2\right)^2} \qquad \sigma_{22}=-\frac{2P}{\pi}\frac{x_1x_2^2}{\left(x_1^2+x_2^2\right)^2} \qquad \sigma_{12}=-\frac{2P}{\pi}\frac{x_1^2x_2}{\left(x_1^2+x_2^2\right)^2}$$

The stress field due to a uniform pressure distribution acting on a strip with width $2a$ is

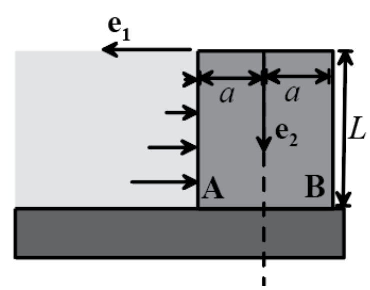
$$\sigma_{22}=-\frac{p}{2\pi}\left(2\left(\theta_1-\theta_2\right)+\left(\sin 2\theta_1-\sin 2\theta_2\right)\right)$$

$$\sigma_{11}=-\frac{p}{2\pi}\left(2\left(\theta_1-\theta_2\right)-\left(\sin 2\theta_1-\sin 2\theta_2\right)\right)$$

$$\sigma_{12}=\frac{p}{2\pi}\left(\cos 2\theta_1-\cos 2\theta_2\right)$$

where $0\le\theta_\alpha\le\pi$ and $\theta_1=\tan^{-1}x_1/(x_2-a)$ $\theta_2=\tan^{-1}x_1/(x_2+a)$

Show that, for $\sqrt{x_1^2+x_2^2}>>a$ the stresses due to the uniform pressure become equal to the stresses induced by the line force (you can do this graphically, or analytically).



Problem 5.5 A rectangular dam is subjected to pressure $p(x_2)=\rho_w x_2$ on one face, where ρ_w is the weight density of water. The dam is made from concrete, with weight density ρ_c (and is therefore subjected to a body force $\rho_c\mathbf{e_2}$ per unit volume). The goal is to calculate formulas for a and L to avoid failure.

(a) Write down the boundary conditions on all four sides of the dam.
(b) Consider the following approximate state of stress in the dam

$$\sigma_{22}=\frac{\rho_w x_2^3 x_1}{4a^3}+\frac{\rho_w x_2 x_1}{20a^3}\left(-10x_1^2+6a^2\right)-\rho_c x_2$$

$$\sigma_{11}=-\frac{\rho_w x_2}{2}+\frac{\rho_w x_2 x_1}{4a^3}\left(x_1^2-3a^2\right)$$

$$\sigma_{12}=\frac{3\rho_w x_2^2}{8a^3}(a^2-x_1^2)-\frac{\rho_w}{8a^3}\left(a^4-x_1^4\right)+\frac{3\rho_w}{20a}(a^2-x_1^2)$$

Show that (i) The stress state satisfies the equilibrium equations (ii) the stress state exactly satisfies boundary conditions on the sides $x_1=\pm a$, (iii) The stress does not satisfy the boundary condition on $x_2=0$ exactly.

(c) Show, however, that the resultant *force* acting on $x_2 = 0$ is zero, so by Saint Venant's principle the stress state will be accurate away from the top of the dam.

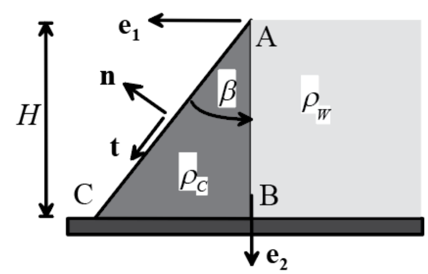
(d) The concrete cannot withstand any tension. Assuming that the greatest principal tensile stress is located at point A $(x_1 = a, x_2 = L)$, show that if $\rho_c / \rho_w < 3/10$ (which is unlikely!) the dam width must satisfy

$$\frac{a}{L} < \frac{10\rho_w}{3\rho_w - 10\rho_c}$$

(e) The concrete fails by crushing when the minimum principal stress reaches $\sigma_{1\min} = -\sigma_c$. Assuming the greatest principal compressive stress is located at point B, $(x_1 = -a, x_2 = L)$ show that the height of the dam cannot exceed

$$L < \frac{\sigma_c}{\rho_w}\left(\frac{L^2}{4a^2} + \frac{\rho_c}{\rho_2} - \frac{1}{5}\right)^{-1}$$

Problem 5.6 The figure shows a simple design for a dam. The wedge-shaped dam is made from a material with weight density ρ_C. It is loaded on its vertical face by pressure $p = -\rho_W x_2$, where ρ_W is the weight density of the fluid.



(a) Write down formulas for unit vectors **t, n** tangent and normal to the back face of the dam, in terms of β

(b) Write down the boundary conditions on the two faces AB, AC of the dam, in terms of the 2D stress components $\sigma_{11}, \sigma_{22}, \sigma_{12}$.

(c) Consider the stress state

$$\sigma_{11} = -\rho_W x_2$$
$$\sigma_{22} = \rho_C\left(x_1\cot(\beta) - x_2\right) - \rho_W\cot^2\beta(2x_1\cot(\beta) - x_2)$$
$$\sigma_{12} = -\rho_W x_1\cot^2\beta$$

Show that the stress state satisfies equilibrium in the dam (inside the wedge) and also satisfies the boundary conditions on the faces AB and AC. Note that on face AC $x_1 = x_2\tan\beta$.

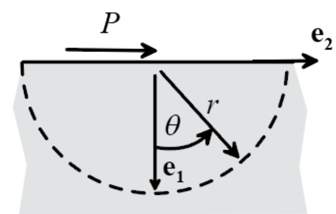
(d) Suppose that the dam is made from concrete, which cannot safely withstand tensile stress. Assuming that the maximum principal tensile stress occurs at B, find a formula for the minimum allowable value for the angle β , in terms of ρ_C, ρ_W

(e) Assume that the concrete fails by crushing if the minimum principal stress reaches $\sigma_3 = -\sigma_c$. Assuming that the minimum principal stress occurs at point C, and that β has the minimum value calculated in (d), find an expression for the maximum height of the dam, in terms of ρ_C, ρ_W . You may find the trig identity $1 + \tan^2\beta = 1/\cos^2\beta$ helpful.

5.2 Airy Function Solutions to Plane Stress and Strain Static Linear Elastic Problems

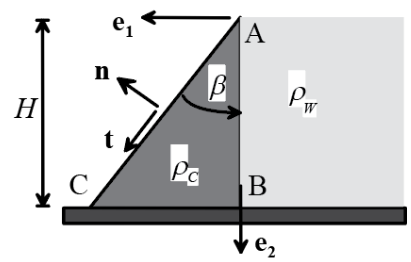
Problem 5.7 The stress due to a line load magnitude P per unit out-of-plane length acting tangent to the surface of a homogeneous, isotropic half-space can be generated from the Airy function

$$\phi = -\frac{P}{\pi} r\theta\cos\theta$$



(a) Find the stress field in the solid (use polar coordinates).
(b) Find the strain field in the solid (assume plane strain deformation)
(c) Calculate the displacement field in the solid.

Problem 5.8 The figure shows a simple design for a dam. The fluid has weight density ρ_w. For simplicity, assume that the weight density of the dam $\rho_c = 0$.



(a) Write down an expression for the hydrostatic pressure in the fluid at a depth x_2 below the surface, in terms of the weight density ρ_w of water.
(b) Hence, write down an expression for the traction vector acting on face AB of the dam.
(c) Write down an expression for the traction acting on face AC
(d) Write down the components of the unit vector normal to face AC in the basis shown
(e) Hence write down the boundary conditions for the stress state in the dam on faces AB and AC
(f) Consider the candidate Airy function

$$\phi = \frac{C_1}{6}x_1^3 + \frac{C_2}{2}x_1^2 x_2 + \frac{C_3}{2}x_1 x_2^2 + \frac{C_4}{6}x_2^3$$

Show that this is a valid Airy function for a solid with no body forces.
(g) Calculate the stresses generated by the Airy function given in (f)
(h) Use the boundary conditions to find values for the coefficients in the Airy function, and hence show that the stress field in the dam is

$$\sigma_{11} = -\rho_w x_2 \qquad \sigma_{22} = \frac{-2\rho_w}{\tan^3\beta}x_1 + \frac{\rho_w}{\tan^2\beta}x_2 \qquad \sigma_{12} = \frac{-\rho_w}{\tan^2\beta}x_1$$

Problem 5.9 The goal of this problem is to show that the function

$$\phi = \frac{\sigma_0}{4}\left(-2a^2\log(r) + r^2\right) + \frac{\sigma_0}{4}\left(+2a^2 - r^2 - \frac{a^4}{r^2}\right)\cos 2\theta$$

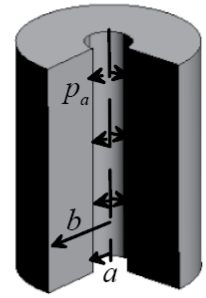
generates a stress state that represents the solution to a large plate containing a circular hole with radius a at the origin, which is loaded by a tensile stress σ_0 acting parallel to the $\mathbf{e}_1$ direction.

(a) Show that ϕ is a valid Airy function for a solid that is subjected to zero body forces.
(b) Calculate the stresses generated by ϕ
(c) Show that the surface of the hole is traction free – i.e. $\sigma_{rr} = \sigma_{r\theta} = 0$ on $r=a$
(d) Show that the stress at $r/a \to \infty$ is

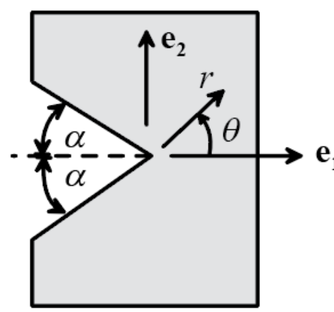
$$\sigma_{rr} = \sigma_0(1+\cos(2\theta))/2 \qquad \sigma_{\theta\theta} = \sigma_0(1-\cos(2\theta))/2 \qquad \sigma_{r\theta} = -\sigma_0\sin(2\theta)/2\,.$$

(e) Show that the stresses in (b) are equivalent to a stress $\sigma_{11} = \sigma_0, \sigma_{22} = \sigma_{12} = 0$.
(f) What is the maximum value of hoop stress $\sigma_{\theta\theta}/\sigma_0$ at $r = a$, and where does it occur?

Problem 5.10 Find the Airy function that generates the solution for an internally pressurized cylinder (with plane strain end conditions and traction free external wall), as shown in the figure. Calculate the corresponding stress field.



Problem 5.11 The goal of this problem is to calculate the asymptotic fields near the tip of a wedge in an elastic solid, illustrated in the figure.



(a) Consider an Airy function of the form $\phi = r^{n+1} f(\theta)$. Obtain a governing equation for $f(\theta)$ and show that it has solution

$$f(\theta) = A_1 \sin(n+1)\theta + A_2 \cos(n+1)\theta + A_3 \sin(n+1)\theta + A_4 \cos(n+1)\theta$$

(b) Use the boundary conditions to show that n and A_i must satisfy

$$\begin{bmatrix} \cos(n_1+1)\beta & \cos(n_1-1)\beta \\ (n_1+1)\sin(n+1)\beta & (n_1-1)\sin(n_1-1)\beta \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} (n_2+1)\cos(n+1)\beta & (n_2-1)\cos(n_2-1)\beta \\ \sin(n_2+1)\beta & \sin(n_2-1)\beta \end{bmatrix} \begin{bmatrix} A_2 \\ A_4 \end{bmatrix} = 0$$

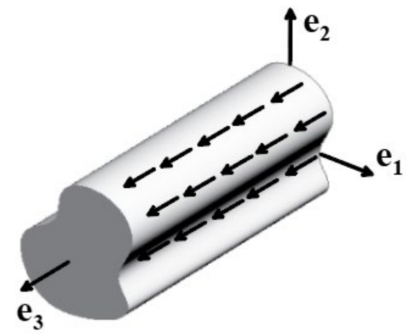
where $\beta = \pi - \alpha$, and hence deduce that n_1, n_2 n must satisfy

$$n_1 \sin 2\beta + \sin 2n_1\beta = 0$$

$$n_2 \sin 2\beta - \sin 2n_2\beta = 0$$

(c) Find the values of n that satisfy these equations, and also (i) have bounded strain energy; and (ii) give the largest possible stress the tip of the wedge. Plot a graph showing the values of n as a function of the wedge angle α

5.3 Complex Variable Solution to Static Linear Elastic Problems



Problem 5.12 A long cylinder is made from an isotropic, linear elastic solid with shear modulus μ. The solid is loaded so that

(i) the resultant forces and moments acting on the ends of the cylinder are zero;
(ii) the body force $\mathbf{b} = b(x_1, x_2)\mathbf{e}_3$ in the interior of the solid acts parallel to the axis of the cylinder; and
(iii) Tractions or displacements imposed on the sides of the cylinder have the form
$\mathbf{u}^* = u^*(x_1, x_2)\mathbf{e}_3 \ \ (\mathbf{x} \in S_1) \qquad \mathbf{t} = t(x_1, x_2)\mathbf{e}_3 \ \ (\mathbf{x} \in S_2)$

Under these conditions, the displacement field at a point far from the ends of the cylinder has the form $\mathbf{u} = u(x_1, x_2)\mathbf{e}_3$, and the solid is said to deform in a state of ***anti-plane shear***. The goal of this problem is to devise a simplified complex variable method for calculating the stress and deformation in a solid that is loaded in this way.

(a) Calculate the strain field in the solid in terms of $\underline{u}$.
(b) Find an expression for the nonzero stress components in the solid, in terms of u and the shear modulus.
(c) Find the equations of equilibrium for the nonzero stress components
(d) Write down boundary conditions for stress and displacement on the side of the cylinder
(e) Hence, show that the governing equations and boundary conditions for u reduce to

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + b = 0 \qquad \left\{ u(\mathbf{x}) = u^*(\mathbf{x}) \ \ \mathbf{x} \in S_1 \right\} \qquad \left\{ \mu\left(\frac{\partial u(\mathbf{x})}{\partial x_1} n_1(\mathbf{x}) + \frac{\partial u(\mathbf{x})}{\partial x_2} n_2(\mathbf{x}) \right) = t(\mathbf{x}) \ \ \mathbf{x} \in S_2 \right\}$$

(f) Let $v(x_1, x_2)$, $w(x_1, x_2)$ denote the real and imaginary parts of $\boldsymbol{\Theta}(z)$. Since $\boldsymbol{\Theta}(z)$ is analytic, the real and imaginary parts must satisfy the Cauchy Riemann conditions

$$\frac{\partial v}{\partial x_1} = \frac{\partial w}{\partial x_2} \qquad \frac{\partial v}{\partial x_2} = -\frac{\partial w}{\partial x_1}$$

Show that this implies that

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} = 0 \qquad \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} = 0$$

(g) Deduce that the displacement and stress in a solid that is free of body force, and loaded on its boundary so as to induce a state anti-plane shear can be derived an analytic function $\boldsymbol{\Theta}(z)$, using the representation

$$2\mu u(x_1, x_2) = \boldsymbol{\Theta}(z) + \overline{\boldsymbol{\Theta}(z)}$$
$$\sigma_{31} - i\sigma_{32} = \boldsymbol{\Theta}'(z)$$

Problem 5.13 Calculate the displacements and stresses generated when the complex potential

$$\boldsymbol{\Theta}(z) = -\frac{P}{2\pi}\log(z)$$

(where P is a real valued constant) is substituted into the representation described in Problem 5.12 (use polar coordinates, with $z = r\exp(i\theta)$). Show that the solution represents the displacement and stress in an infinite solid due to a line force with magnitude P per unit out of plane distance acting in the $\mathbf{e}_3$ direction at the origin.

Problem 5.14 Calculate the displacements and stresses generated when the complex potential

$$\Theta(z) = -\frac{i\mu b}{2\pi}\log(z)$$

is substituted into the representation described in Problem 5.12. Show that the solution represents the displacement and stress due to a screw dislocation at the origin in an infinite solid, with burgers vector and line direction parallel to $\mathbf{e}_3$. (To do this, you need to show that (a) the displacement field has the correct character; and (b) the resultant force acting on a circular arc surrounding the dislocation is zero)

Problem 5.15 Calculate (in terms of Cartesian coordinates (x_1, x_2)) the displacements and stresses generated when the complex potential

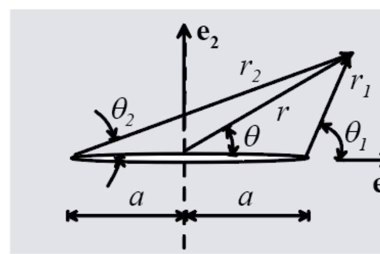
$$\Theta(z) = \tau_0\left(z + \frac{a^2}{z}\right)$$

where a, τ_0 are real valued constants, is substituted into the representation described in Problem 5.12. Show that the solution represents the displacement and stress in an infinite solid, which contains a hole with radius a at the origin, and is subjected to a constant anti-plane shear stress at infinity.

Problem 5.16 Calculate the displacements and stresses generated when the complex potential

$$\Theta(z) = -i\tau_0\sqrt{z^2 - a^2}$$

is substituted into the representation described in Problem 5.12. Show that the solution represents the displacement and stress in an infinite solid, which contains a crack with length a at the origin, and is subjected to a prescribed anti-plane shear stress at infinity. Use the procedure given in Section 5.3.6 of Applied Mechanics of Solids to calculate $\sqrt{z^2 - a^2}$



Problem 5.17 Consider complex potentials $\Omega(z) = az + b, \quad \omega(z) = cz + d$, where *a, b, c, d* are complex numbers. Let

$$2\mu D = (3-4\nu)\Omega(z) - z\,\overline{\Omega'(z)} \;-\overline{\omega(z)}$$

$$\sigma_{11} + \sigma_{22} = 2\left(\Omega'(z) + \overline{\Omega'(z)}\right) \qquad\qquad \sigma_{11} - \sigma_{22} + 2i\sigma_{12} = -2\left(z\overline{\Omega''(z)} + \overline{\omega'(z)}\right)$$

be a displacement and stress field (derived from these potentials) in an elastic solid with shear modulus μ and Poisson's ratio ν.

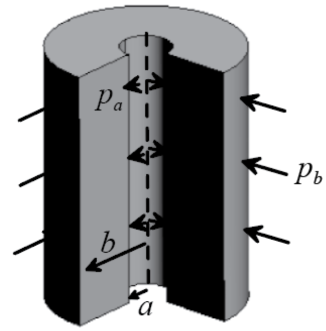
(a) Find values of *a,b,c,d* that represent a rigid displacement $u_1 = w_1 + \alpha x_2$, $u_2 = w_2 - \alpha x_1$ where w_1, w_2 are (real) constants representing a translation, and α is a real constant representing an infinitesimal rotation.

(b) Find values of *a,b,c,d* that correspond to a state of uniform stress.

Problem 5.18 Show that the complex potentials

$$\boldsymbol{\Omega}(z)=\frac{p_a a^2-p_b b^2}{2(b^2-a^2)}z \qquad \omega(z)=\frac{(p_a-p_b)a^2b^2}{(b^2-a^2)z}$$

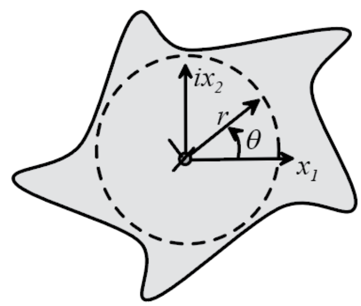
give the stress and displacement field in a pressurized circular cylinder which deforms in plane strain (You don't need to prove that the complex variable representation satisfies the equilibrium equations – just check the boundary conditions. It is best to solve this problem using polar coordinates)



Problem 5.19 The complex potentials

$$\boldsymbol{\Omega}(z)=-i\frac{E\left(b_1+ib_2\right)}{8\pi(1-\nu^2)}\log(z) \qquad \omega(z)=i\frac{E(b_1-ib_2)}{8\pi(1-\nu^2)}\log(z)$$

generate the plane strain solution to an edge dislocation at the origin of an infinite solid. Work through the algebra necessary to determine the stresses (you can check your answer using the solution given in Section 5.3.4 of Applied Mechanics of Solids). It is best to solve this problem in polar coordinates.



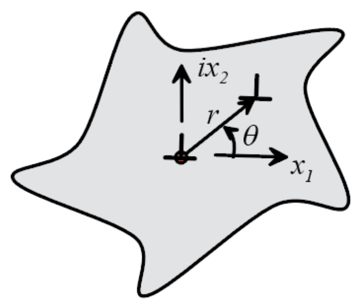
Problem 5.20 When a stress field acts on a dislocation, the dislocation tends to move through the solid. Formulas for these forces are derived in Section 5.8.5 of Applied Mechanics of Solids. For the particular case of a straight edge dislocation, with burgers vector b_i, the force can be calculated as follows:

(i) Let $\sigma_{ij}^{D\infty}$ denote the stress field in an infinite solid containing the dislocation (calculated using the formulas in Section 5.3.4 of Applied Mechanics of Solids)

(ii) Let σ_{ij} denote the actual stress field in the solid (including the effects of the dislocation itself, as well as corrections due to boundaries in the solid, or externally applied fields)

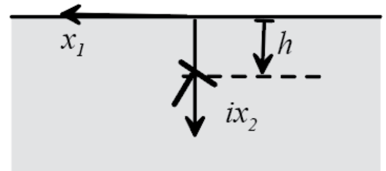
(iii) Define $\Delta\sigma_{ij}=\sigma_{ij}-\sigma_{ij}^{D\infty}$ to denote the difference between these quantities.

The force can then be calculated as $F_i=\in_{ij3}\Delta\sigma_{jk}b_k$, where $\in_{ijk}$ is the permutation symbol.

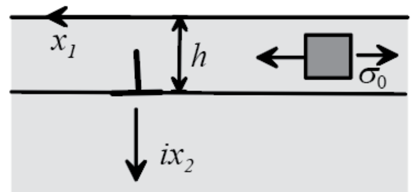


Consider two edge dislocations in an infinite solid, each with burgers vector $b_1=b,\ b_2=0$. One dislocation is located at the origin, the other is at position (x_1,x_2). Plot contours of the horizontal component of force acting on the second dislocation due to the stress field of the dislocation at the origin. (normalize the force as $8\pi(1-\nu^2)rF_1/\left(Eb^2\right)$).

Problem 5.21 The figure shows an edge dislocation below the surface of an elastic solid. Use the solution given in Section 5.3.12 of Applied Mechanics of Solids, together with the formula in Problem 5.20, to calculate an expression for the force acting on the dislocation.

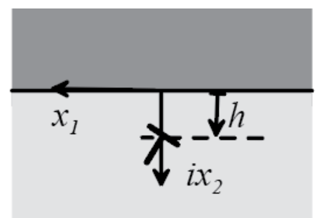


Problem 5.22 The figure shows an edge dislocation with burgers vector $b_1 = -b, b_2 = 0$ that lies in a strained elastic film with thickness h. The film and substrate have the same elastic moduli. The stress in the film consists of the stress due to the dislocation, together with a tensile stress $\sigma_{11} = \sigma_0$. Calculate the force acting on the dislocation, and hence find the film thickness below which the dislocation will be attracted to the free surface and escape from the film. You will need to use the formula given in problem 9 to calculate the force on the dislocation. You can also use the results of problem 5.21.



Problem 5.23 The figure shows a dislocation in an elastic solid with Young's modulus E and Poisson's ratio ν, which is bonded to a rigid solid. The solution can be generated from complex potentials

$$\mathbf{\Omega}(z) = \mathbf{\Omega}_0(z) + \mathbf{\Omega}_1(z) \qquad \omega(z) = \omega_0(z) + \omega_1(z)$$



where

$$\mathbf{\Omega}_0(z) = -i\frac{E(b_1 + ib_2)}{8\pi(1-\nu^2)}\log(z - ih)$$

$$\omega_0(z) = i\frac{E(b_1 - ib_2)}{8\pi(1-\nu^2)}\log(z - ih) + \frac{E(b_1 + ib_2)}{8\pi(1-\nu^2)}\frac{h}{z - ih}$$

is the solution for a dislocation at position $z_0 = ih$ in an infinite solid, and

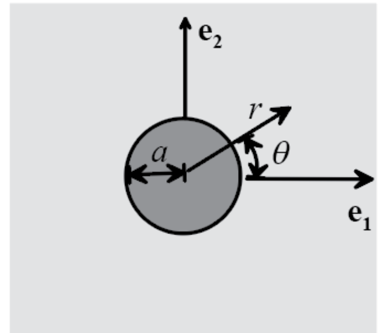
$$\mathbf{\Omega}_1(z) = \left(z\overline{\mathbf{\Omega}_0'(\bar{z})} + \overline{\omega_0(\bar{z})}\right)/\left(3 - 4\nu\right)$$

$$\omega_1(z) = (3-4\nu)\overline{\mathbf{\Omega}_0(\bar{z})} - z\left(\overline{\mathbf{\Omega}_0'(\bar{z})} + z\overline{\mathbf{\Omega}_0''(\bar{z})} + \overline{\omega_0'(\bar{z})}\right)/(3-4\nu)$$

corrects the solution to satisfy the zero displacement boundary condition at the interface.

(a) Verify that the solution satisfies the zero displacement boundary condition at the interface.
(b) Calculate the force acting on the dislocation, in terms of h and relevant material properties. You will need to use the formula from problem 5.20 to calculate the force on the dislocation.
(c) Calculate the distribution of stress along the interface between the elastic and rigid solids, in terms of h and x_1

Problem 5.24 The figure shows a rigid cylindrical inclusion with radius a embedded in an isotropic elastic matrix. The solid is subjected to a uniform uniaxial stress $\sigma_{11}=\sigma_0$ at infinity. The goal of this problem is to calculate the stress fields in the matrix.



(a) Write down the boundary conditions on the displacement field at $r=a$.

(b) Consider complex potentials of the form

$$\boldsymbol{\Omega}(z)=\frac{\sigma_0}{4}\left(z+\frac{\alpha a^2}{z}\right) \qquad \omega(z)=-\frac{\sigma_0}{2}\left(z+\frac{\beta a^2}{z}+\frac{\lambda a^4}{z^3}\right)$$

Show that the boundary condition in (a) requires that

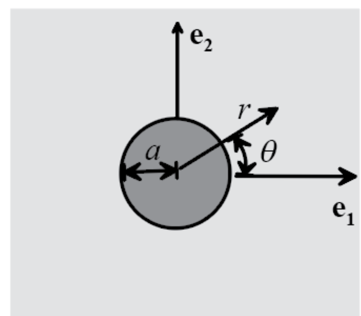
$$(3-4\nu)\frac{\sigma_0}{4}\left(z+\frac{\alpha a^2}{z}\right)-z\frac{\sigma_0}{4}\left(1-\frac{\alpha z^2}{a^2}\right)+\frac{\sigma_0}{2}\left(\frac{a^2}{z}+\beta z+\frac{\lambda z^3}{a^2}\right)=0 \qquad \text{on } r^2=z\bar{z}=a^2$$

(c) Hence, determine values for the real-valued coefficients α,β,λ that will satisfy the boundary conditions (compare the coefficients of powers of z in the answer to (b).

(d) Find an expression for the stresses acting at the inclusion/matrix boundary (use polar coordinates)

(e) Suppose that the interface between inclusion and matrix fails when the normal stress acting on the interface reaches a critical stress σ_{crit}. Find an expression for the maximum tensile stress that can be applied to the material without causing failure.

Problem 5.25 The figure shows a cylindrical inclusion with radius a and Young's modulus and Poisson's ratio E_p,ν_p embedded in an isotropic elastic matrix with elastic constants E,ν. The solid is subjected to a uniform uniaxial stress $\sigma_{11}=\sigma_0$ at infinity. The goal of this problem is to calculate the stress fields in both the particle and the matrix. The analysis can be simplified greatly by assuming *a priori* that the stress in the particle is uniform (this is not obvious, but can be checked after the full solution has been obtained). Assume, therefore, that the stress in the inclusion is $\sigma_{11}=p_{11}$, $\sigma_{22}=p_{22}$ $s_{12}=0$, where p_{11},p_{22} are to be determined. The solution inside the particle can therefore be derived from complex potentials



$$\boldsymbol{\Omega}_p(z)=\frac{p_{11}+p_{22}}{4}z \qquad \omega_p(z)=\frac{p_{22}-p_{11}}{2}z$$

In addition, assume that the solution in the matrix can be derived from complex potentials of the form

$$\boldsymbol{\Omega}(z)=\frac{\sigma_0}{4}\left(z+\frac{\alpha a^2}{z}\right) \qquad \omega(z)=-\frac{\sigma_0}{2}\left(z+\frac{\beta a^2}{z}+\frac{\lambda a^4}{z^3}\right)$$

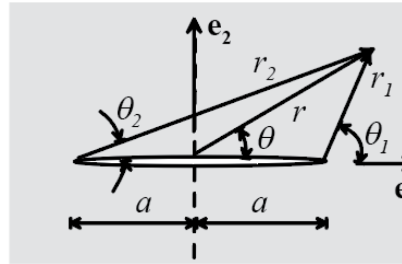
where α,β,λ are three real valued coefficients whose values you will need to determine.

(a) Write down the boundary conditions on the displacement field $r=a$. Express the boundary condition as an equation relating $\boldsymbol{\Omega}_p,\omega_p$ to $\boldsymbol{\Omega},\omega$.

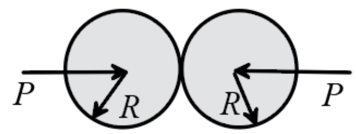
(b) Write down boundary conditions on the stress components $\sigma_{rr},\sigma_{r\theta}$ at $r=a$. Express this boundary condition as an equation relating $\boldsymbol{\Omega}_p,\omega_p$ to $\boldsymbol{\Omega},\omega$.

(c) Use the method in problem 13 to calculate expressions for p_{11},p_{22}, α,β,λ in terms of σ_0 and material properties.

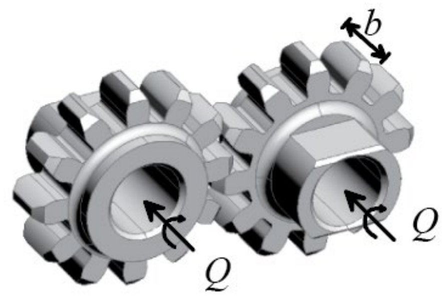
Problem 5.26 The figure shows a slit crack in an infinite solid. Using the solution given in Section 5.3.6 of Applied Mechanics of Solids, calculate the stress field very near the right hand crack tip (i.e. find the stresses in the limit as $r_1 / a \to 0$). Show that the results are consistent with the asymptotic crack tip field given in Section 5.2.9 of Applied Mechanics of Solids, and deduce an expression for the crack tip stress intensity factors in terms of $\sigma_{22}^{\infty}, \sigma_{12}^{\infty}$ and a.



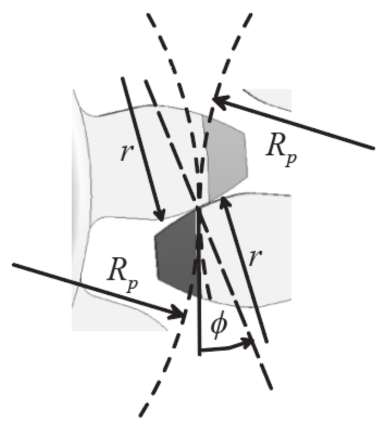
Problem 5.27 Two identical cylindrical roller bearings with radius 1cm are pressed into contact by a force P per unit out of plane length as indicated in the figure. The bearings are made from 52100 steel with a uniaxial tensile yield stress of 2.8GPa. Calculate the force (per unit length) that will just initiate yield in the bearings, and calculate the width of the contact strip between the bearings at this load.



Problem 5.28 The figure shows a pair of identical involute spur gears. The contact between the two gears can be idealized as a line contact between two cylindrical surfaces. The goal of this problem is to find an expression for the maximum torque Q that can be transmitted through the gears. The gears can be idealized as isotropic, linear elastic solids with Young's modulus E and Poisson's ratio ν.



As a representative configuration, consider the instant when a single pair of gear teeth make contact exactly at the pitch point. At this time, the geometry can be idealized as contact between two cylinders, with radius $r = R_p \sin\phi$, where R_p is the pitch circle radius of the gears and ϕ is the pressure angle. The cylinders are pressed into contact by a force per unit length $P = Q / (bR_p \cos\phi)$.



(a) Find a formula for the area contact area between the two gear teeth, in terms of Q, R_p, ϕ, b and relevant material properties.
(b) Find a formula for the maximum contact pressure acting on the contact area, in terms of Q, R_p, ϕ, b and representative material properties.
(c) Suppose that the gears have uniaxial tensile yield stress Y. Find a formula for the critical value of Q required to initiate yield in the gears.

5.4 Solutions to 3D Static Problems in Linear Elasticity

Problem 5.29 Consider the Papkovich-Neuber potentials

$$\Psi_i = \frac{(1-\nu)\sigma_0}{(1+\nu)} x_3 \delta_{i3} \qquad \phi = \frac{\nu(1-\nu)\sigma_0}{(1+\nu)}(3x_3^2 - R^2)$$

where $R = \sqrt{x_k x_k}$.

(a) Verify that the potentials satisfy the governing equations for Papkovich-Neuber potentials with zero body force.
(b) Show that the fields generated from the potentials correspond to a state of uniaxial stress, with magnitude σ_0 acting parallel to the $\mathbf{e}_3$ direction of an infinite solid

Problem 5.30 Consider the fields derived from the Papkovich-Neuber potentials

$$\Psi_i = \frac{(1-\nu)p}{(1+\nu)} x_i \qquad \phi = 0$$

(a) Verify that the potentials satisfy the governing equations for Papkovich-Neuber potentials with zero body force.
(b) Show that the fields generated from the potentials correspond to a state of hydrostatic tension $\sigma_{ij} = p\delta_{ij}$

Problem 5.31 The goal of this problem is to find Papkovich-Neuber potentials that generate the solution to an internally pressurized spherical shell. To this end, consider the Papkovich-Neuber potentials

$$\Psi_i = \alpha x_i + \beta \frac{x_i}{R^3} \qquad \phi = \frac{\beta}{R}$$

(a) Verify that the potentials satisfy the governing equations for Papkovich-Neuber potentials with zero body force (don't worry about the singularity at the origin, since this is outside the solid of interest).
(b) Show that the potentials generate a spherically symmetric displacement field.
(c) Calculate values of α and β that generate the solution to an internally pressurized spherical shell, with pressure p acting on the interior surface at $R=a$, and with the exterior surface at $R=b$ traction free.

Problem 5.32 Verify that the Papkovich-Neuber potentials

$$\Psi_i = \frac{P_i}{4\pi R} \qquad \phi = 0$$

generate the fields for a point force $\mathbf{P} = P_1\mathbf{e}_1 + P_2\mathbf{e}_2 + P_3\mathbf{e}_3$ acting at the origin of a large (infinite) elastic solid with Young's modulus E and Poisson's ratio ν . To this end:

(a) Show that the potential satisfies the governing equation

$$\frac{\partial^2 \Psi_i}{\partial x_j \partial x_j} = 0$$

everywhere except at the origin.

(b) Show that

$$\int_V \frac{\partial^2 \Psi_i}{\partial x_j \partial x_j} = -P_i$$

for any volume V that encloses the origin (evaluate the integral over a spherical region with radius R surrounding the origin. Note that you can convert the volume integral into a surface integral using the divergence theorem).

(c) What can you conclude about the body force distribution that is associated with the potential?
(d) As an alternative approach, consider a spherical region with radius R surrounding the origin. Calculate the resultant force exerted by the stress on the outer surface of this sphere, and show that they are in equilibrium with a force **P**.

Problem 5.33 Consider an infinite, isotropic, linear elastic solid with Young's modulus E and Poisson's ratio ν. Suppose that the solid contains a rigid spherical particle (an inclusion) with radius a and center at the origin. The solid is subjected to a uniaxial tensile stress $\sigma_{33} = \sigma_0$ at infinity. The objective of this problem is to calculate the displacement field in the matrix outside the inclusion.
(a) Write down the strain field induced by the uniaxial stress, and hence write down the displacement field associated with the uniaxial stress (take the displacement and rotation to be zero at the origin).
(b) Consider a spherical Eshelby problem with transformation stress $p_{ij} = A\delta_{ij} + B\delta_{i3}\delta_{j3}$. Calculate the displacement at the surface of the transformed region.
(c) Hence, superpose the two solutions and find the values of A and B required to satisfy the boundary conditions at the edge of the particle.
(d) Hence, determine the displacement field.

Problem 5.34 Consider an infinite, isotropic, linear elastic solid with Young's modulus E and Poisson's ratio ν. Suppose that the solid contains a spherical particle (an inclusion) with radius a and center at the origin. The particle has Young's modulus E_p and Poisson's ratio ν_p, and is perfectly bonded to the matrix, so that the displacement and radial stress are equal in both particle and matrix at the particle/matrix interface. The solid is subjected to a uniaxial tensile stress $\sigma_{33} = \sigma_0$ at infinity. The objective of this problem is to calculate the stress field in the elastic inclusion. You will find it helpful to solve problems 5.29 and 5.33 before attempting this one.
(a) Assume that the stress field inside the inclusion is given by $\sigma_{ij} = A\sigma_0\delta_{ij} + B\sigma_0\delta_{i3}\delta_{j3}$. Calculate the displacement field in the inclusion (assume that the displacement and rotation of the solid vanish at the origin).
(b) Find formulas for traction normal to the surface $R=a$ in terms of A, B, σ_0 and material properties.
(c) The stress field outside the inclusion can be generated from Papkovich-Neuber potentials

$$\Psi_i = \frac{(1-\nu)\sigma_0}{(1+\nu)} x_3\delta_{i3} + \frac{a^3 p_{ik}^T x_k}{3R^3} \qquad \phi = \frac{\nu(1-\nu)\sigma_0}{(1+\nu)}(3x_3^2 - R^2) + \frac{a^3 p_{ij}^T}{15R^3}\left((5R^2 - a^2)\delta_{ij} + 3a^2 \frac{x_i x_j}{R^2} \right)$$

where $p_{ij}^T = C\sigma_0\delta_{ij} + D\sigma_0\delta_{i3}\delta_{j3}$, and C and D are constants to be determined. Find formulas for the displacement and traction normal to the surface $R=a$ in terms of C, D, σ_0 and material properties.
(d) Use the conditions at $r=a$ to find four equations that can be solved for A, B, C, D.
(e) Hence, find the stress field inside the inclusion, for the special case where $\nu = \nu_p$ (the formulas for the general case can be found without difficulty, but are too long to write out).

Problem 5.35 The goal of this problem is to find a simple formula for the energy of the Eshelby inclusion described in Section 5.4.6 of Applied Mechanics of Solids. As discussed, the solid of interest consists of an infinite homogeneous, linear elastic with Young's modulus E and Poisson's ratio ν. The solid is initially stress free. An inelastic strain distribution ε_{ij}^T is introduced into an ellipsoidal region of the solid B (e.g. due to thermal expansion, or a phase transformation). Let u_i denote the displacement field, $\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^T$ denote the total strain distribution, and let σ_{ij} denote the stress field in the solid.

(a) Write down an expression for the total strain energy Φ_I within the ellipsoidal region, in terms of σ_{ij}, ε_{ij} and ε_{ij}^T (leave the answer as a volume integral)

(b) Write down an expression for the total strain energy outside the ellipsoidal region, expressing your answer as a volume integral in terms of ε_{ij} and σ_{ij}. Using the divergence theorem, show that the result can be re-written as

$$\Phi_O = -\frac{1}{2}\int_S \sigma_{ij} n_j u_i dA$$

where S denotes the surface of the ellipsoid, and n_j are the components of an outward unit vector normal to B. Note that, when applying the divergence theorem to the region outside the inclusion, you need to show that the integral taken over the (arbitrary) boundary of the solid at infinity does not contribute to the energy – you can do this by using the asymptotic formula given in Applied Mechanics of Solids for the displacements far from an Eshelby inclusion.

(c) The Eshelby solution shows that the strain $\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^T$ inside B is uniform. Write down the displacement field inside the ellipsoidal region, in terms of ε_{ij} (take the displacement and rotation of the solid at the origin to be zero). Hence, show that the result of b can be re-written as

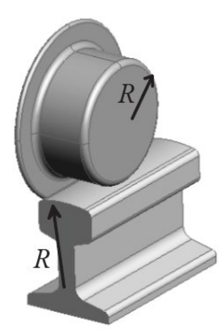
$$\Phi_O = -\frac{1}{2}\int_S \sigma_{ij}\varepsilon_{ik} x_k n_j dA$$

(d) Finally, use the results of a and c, together with the divergence theorem, to show that the total strain energy of the solid can be calculated as

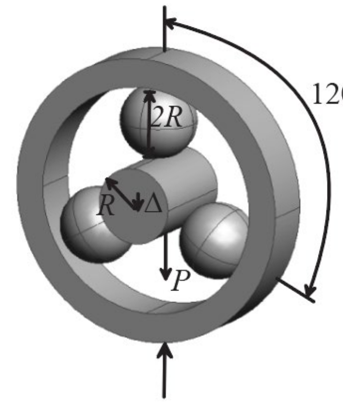
$$\Phi = \Phi_O + \Phi_I = -\frac{1}{2}\int_B \sigma_{ij}\varepsilon_{ij}^T \; dV$$

Problem 5.36 A steel ball-bearing with radius 1cm is pushed into a flat steel surface by a force P. Neglect friction between the contacting surfaces. Typical ball-bearing steels have uniaxial tensile yield stress of order 2.8 GPa. Calculate the maximum load that the ball-bearing can withstand without causing yield, and calculate the radius of contact and maximum contact pressure at this load.

Problem 5.37 The contact between the wheel of a locomotive and the head of a rail may be approximated as the (frictionless) contact between two cylinders, with identical radius R as illustrated in the figure. The rail and wheel can be idealized as elastic-perfectly plastic solids with Young's modulus E, Poisson's ratio ν and yield stress Y. Find expressions for the radius of the contact patch, the contact area, and the maximum contact pressure as a function of the load acting on the wheel and relevant geometric and material properties. By estimating values for relevant quantities, calculate the maximum load that can be applied to the wheel without causing the rail to yield.

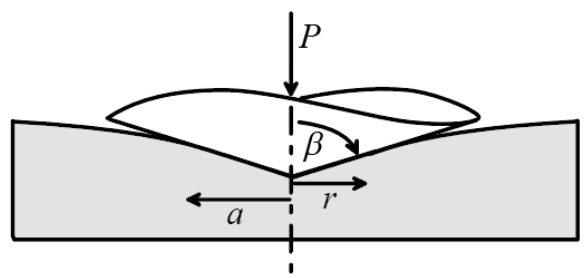


Problem 5.38 The figure shows a rolling element bearing. It consists of a cylindrical shaft with radius R, supported by three spherical ball bearings with radius R, spaced at 120 degree intervals, as shown in the figure. The housing is a hollow cylinder with internal radius $3R$. All components have Young's modulus E and Poisson's ratio ν. The assembly is stress free when unloaded. The shaft is loaded by a force P acting in the direction shown in the figure, while the housing is held fixed. As a result, the shaft is displaced vertically by a distance Δ. Note that only the bottom two balls are loaded: the top ball is held in place by a cage, which is not shown.



(a) Find an expression for the area of contact A_1 between the shaft and one of the two loaded balls, in terms of P and relevant material and geometric properties.
(b) Find an expression for the area of contact A_2 between the housing and one of the two loaded balls in terms of P and relevant geometric and material properties
(c) Find an expression for the deflection Δ, in terms of P and relevant geometric and material properties.

Problem 5.39 A rigid, conical indenter with apex angle 2β is pressed into the surface of an isotropic, linear elastic solid with Young's modulus E and Poisson's ratio ν.



(a) Write down the initial gap between the two surfaces *g(r)*.
(b) Find the relationship between the depth of penetration h of the indenter and the radius of contact a
(c) Find the relationship between the force applied to the contact and the radius of contact, and hence deduce the relationship between penetration depth and force. Verify that the contact stiffness is given by $\frac{dP}{dh} = 2E^* a$
(d) Calculate the distribution of contact pressure that acts between the contacting surfaces.

Problem 5.40 A sphere, which has radius R, is dropped through a vertical distance H onto the flat surface of a large solid. The sphere has mass density ρ, and both the sphere and the surface can be idealized as linear elastic solids, with Young's modulus E and Poisson's ratio ν. As a rough approximation, the impact can be idealized as a quasi-static elastic indentation.
(a) Write down the relationship between the force P acting on the sphere and the displacement of the center of the sphere below its height $x_2 = R$ at the instant of first contact.
(b) Show that the maximum vertical displacement $h_{\max}$ of the sphere below the point of initial contact satisfies

$$\frac{4}{3}\pi R^4 \rho g H = \frac{4}{15}\frac{E}{1-\nu^2}\sqrt{R h_{\max}^5} - \frac{4}{3}\pi R^4 \rho g h_{\max}$$

where g is the gravitational acceleration.
(c) Suppose that the two solids have yield stress in uniaxial tension Y. Find an expression for the critical value of h which will cause the solids to yield
(d) Calculate a value of the maximum value of drop height H for the materials to remain elastic, assuming that the materials are ball bearing steel with a 2.8GPa yield stress, and the sphere has a 1 cm radius.

5.5 Solutions to Dynamic Problems for Isotropic Linear Elastic Solids

Problem 5.41 Consider the Love potentials $\Psi_i = 0$, $\phi = A\sin(p_i x_i - c_L t)$, where p_i is a constant unit vector, A is a constant, and c_L is the speed of longitudinal waves.

(a) Verify that the potentials satisfy the appropriate governing equations
(b) Calculate the stresses and displacements generated from these potentials.
(c) Briefly, interpret the wave motion represented by this solution.

Problem 5.42 Consider the Love potentials $\Psi_i = A_i \sin(p_k x_k - c_s t)$, $\phi = 0$, where p_i is a constant unit vector, A_i is a constant vector, and c_s is the speed of shear waves.

(a) Find a condition relating U_i and p_i that must be satisfied for this to be a solution to the governing equations
(b) Calculate the stresses and displacements generated from these potentials.
(c) Briefly, interpret the wave motion represented by this solution.

Problem 5.43 Consider the Love potentials $\Psi_i = 0, \phi = \frac{1}{R} f(t - R / c_L)$, where $R = \sqrt{x_i x_i}$

(a) Show that the potentials satisfy the appropriate governing equations.
(b) Find expressions for the displacement and stress fields generated by the potentials.

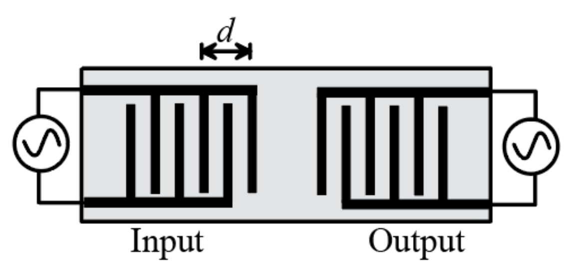
Problem 5.44 Consider Love potentials

$$\Psi_i = 0 \qquad \phi = \frac{A}{R}\sin\left\{\omega\left(t - (R-a)/c_L\right) + \gamma\right\}$$

where A, ω and γ are three scalar constants, and $R = \sqrt{x_i x_i}$.

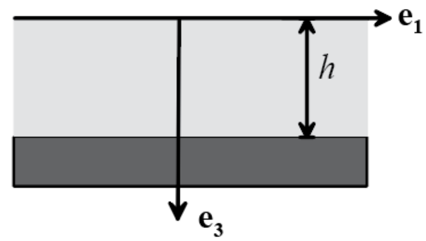
(a) Calculate the displacement and stress fields generated by the potentials
(b) Calculate the traction acting on the surface at $r=a$. Hence, find the Love potential that generates the fields around a spherical cavity with radius a, which is subjected to a harmonic pressure $p(t) = p_0 \sin \omega t$.
(c) Plot the amplitude of the normalized radial displacement $u_r 2E / [(1+\nu) a p_0]$ at $r=a$ as a function of $\omega a / c_L$ (you can try a few representative values for Poisson's ratio)

Problem 5.45 The figure shows a surface-acoustic-wave device that is intended to act as a narrow band-pass filter. A piezoelectric substrate has two transducers attached to its surface – one acts as an "input" transducer and the other as "output." The transducers are electrodes: a charge can be applied to the input transducer; or detected on the output. Applying a charge to the input transducer induces a strain on the surface of the substrate: at an appropriate frequency, this will excite a Rayleigh wave in the solid. The wave propagates to the "output" electrodes, and the resulting deformation of the substrate induces a charge that can be detected. If the electrodes have spacing d, find a formula for the frequency at which the surface will be excited, in terms of the Rayleigh wave speed (the wavelength is equal to the electrode spacing). Estimate the spacing required for a 1GHz filter made from Quartz with density 2320 kg m^{-3}, Young's modulus 71.7 GPa and Poisson's ratio 0.17



Problem 5.46 The energy per unit area of a Rayleigh wave with displacement amplitude U_0 and wave number *k* traveling along the surface of a half-space with mass density ρ, Poisson's ratio ν and Rayleigh wave speed c_R can be expressed as $\Phi = \rho c_R^2 U_0^2 k F(\nu)$. Plot a graph of the normalized energy $F(\nu)$ as a function of ν (it is easiest to plot the graph by calculating $F(\nu)$ for a range of values of ν by evaluating the various quantities in the velocity and stress, rather than to find an explicit formula for $F(\nu)$)

Problem 5.47 The figure shows a design for a solid mounted resonator. It consists of a thin piezoelectric film that is deposited on the surface of a 'Bragg reflector' – which can be idealized as a rigid boundary. The goal of this problem is to calculate a formula for the frequency of the 5th vibration mode of the resonator (through-thickness vibrations). Assume that the resonator is a thin film with thickness *h*, Youngs modulus E, Poisson's ratio ν, and mass density ρ; and that the displacement in the plate has the form $u_3 = u(x_3, t)$, with all other components zero.



(a) Show that the Navier equation for the displacement field reduces to

$$\frac{\partial^2 u}{\partial x_3^2} = \frac{1}{c_L^2}\frac{\partial^2 u}{\partial t^2}$$

and give a formula for c_L

(b) Assume that the top surface of the plate is stress free, and the bottom is fixed. Write down the boundary condition for *u* at $x_3 = 0, x_3 = h$.

(c) Consider solutions to the equation of motion of the form $u = \cos(\omega t + \phi) f(x_3)$. Use (a) to find an ODE for $f(x_3)$. Find the general solution for *f* along with the formula relating wave number *k* to frequency ω (the dispersion relation…)

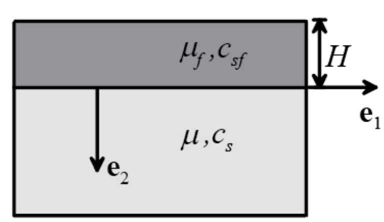
(d) Show that the boundary conditions can be expressed in matrix form as

$$\begin{bmatrix} \cos kh & -\sin kh \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

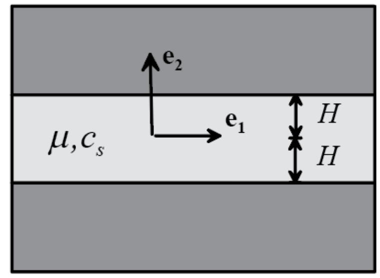
Hence, find a formula for the resonant wave numbers *k* and the corresponding resonant frequencies ω

(e) Calculate the thickness of a resonator made from PZT with Young's modulus 81GPa and Poissons ratio 0.39 and mass density 7320 kg/m^3 with a resonant frequency of 10 MHz.

Problem 5.48 The figure shows an elastic layer with thickness *H*, shear modulus μ_f and shear wave speed c_{sf} bonded to a substrate with shear modulus μ and shear wave speed c_s. The system supports 'Love waves,' with wave number *k,* which propagate through the film/substrate system with phase velocity c and group velocity c_g. Use the solution in Section 5.6.4 of Applied Mechanics of Solids to plot graphs of the phase and group velocity of the lowest propagation mode (both normalized by the shear wave speed in the substrate) as a function of *kH*, for the special case $\mu = \mu_f$, $c_s / c_{sf} = 2$. (It is not hard to run other cases, but the formulas for the group velocity becomes very complicated for arbitrary μ / μ_f , c_s / c_{sf})



Problem 5.49 The figure shows a thin elastic strip, which is bonded to immobile, rigid solids on both its surfaces. The strip has shear modulus μ and wave speed c_s, and acts as a wave-guide. The goal of this problem is to calculate the displacement field associated with transverse wave propagation down the strip.



(a) Assume that the displacement has the form

$$u_3 = f(x_2)\exp(-ik(x_1 - ct))$$

By substituting into the Navier equation, show that

$$\frac{d^2}{dx_2^2} f(x_2) + k^2\left(\frac{c^2}{c_s^2} - 1\right) f(x_2) = 0$$

Hence, write down the general solution for $f(x_2)$

(b) Show that the boundary conditions admit solutions of the form

$$f(x_2) = \begin{cases} A\sin(n\pi x_2 / H) \\ B\cos((\pi/2 + n\pi)x_2 / H) \end{cases}$$

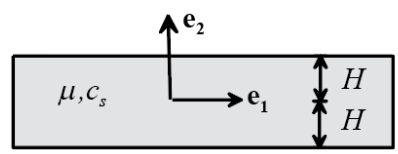
where n is an integer, so that the family of solutions can be expressed in the form

$$u_3 = U_n\left\{\cos^2(\pi n/2)\sin(n\pi x_2 / 2H) + \sin^2(\pi n/2)\cos(n\pi x_2 / 2H)\right\}\exp(ik(x_1 - ct))$$

Also, find an equation relating the phase velocity of the wave c to kH.

(c) Calculate the dispersion relation for the wave and hence deduce an expression for the group velocity.

Problem 5.50 The goal of this problem is to calculate the speed at which kinetic and strain energy travels down a simple wave guide. Consider an isotropic, linear elastic strip, with thickness $2H$, shear modulus μ and wave speed c_s as indicated in the figure. The solution for a wave propagating in the $\mathbf{e}_1$ direction, with particle velocity $\mathbf{u} = u_3\mathbf{e}_3$ is given in Section 5.6.6 of Applied Mechanics of Solids.



(a) The flux of energy per unit area crossing a plane transverse to the $\mathbf{e}_1$ direction as a result of wave propagation through the wave-guide can be computed from the rate of work done by the tractions acting on an internal material surface. The average rate of work done per unit area is given by

$$\langle P\rangle = \frac{1}{2TH}\int_0^T\int_{-H}^{H} \sigma_{ij} n_j \frac{du_i}{dt} dx_2 dt$$

where T is the period of oscillation and $n_j = -\delta_{j1}$ is a unit vector normal to an internal plane perpendicular to the direction of wave propagation. Calculate $\langle P\rangle$ for the nth wave propagation mode.

(b) The average kinetic energy per unit area in a generic cross-section of the wave-guide can be calculated from

$$\langle K\rangle = \frac{1}{2TH}\int_0^T\int_{-H}^{H} \frac{\rho}{2}\frac{du_i}{dt}\frac{du_i}{dt} dx_2 dt$$

Find $\langle K\rangle$ for nth wave propagation mode.

(c) The average potential energy per unit area in a generic cross-section of the wave-guide can be calculated from

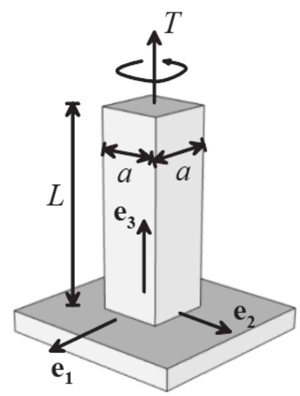
$$\langle\Phi\rangle = \frac{1}{2TH}\int_0^T\int_{-H}^{H}\frac{1}{2}\sigma_{ij}\varepsilon_{ij}dx_2 dt$$

Find $\langle\Phi\rangle$ for *n*th wave propagation mode, and show that $\langle K\rangle = \langle\Phi\rangle$.

(d) If energy travels down the wave-guide with average speed c_e, the rate of energy flux crossing unit area is given by $c_e\left(\langle K\rangle + \langle\Phi\rangle\right)$. The rate of energy flux must equal the rate of work done by the tractions acting on the plane $\langle P\rangle$. Use this condition to find c_e for the *n*th propagation mode, and compare the solution with the expression for the group velocity of the wave

$$c_g = \frac{d\omega}{dk} = \frac{c_s kH}{\sqrt{(n\pi/2)^2 + k^2H^2}}$$

5.6 Energy methods for solving linear elasticity problems



Problem 5.51 A shaft with length L and square cross section is fixed at one end, and subjected to a twisting moment T at the other. The shaft is made from a linear elastic solid with Young's modulus E and Poisson's ratio ν. The torque causes the top end of the shaft to rotate through an angle ϕ.

(a) Consider the following displacement field

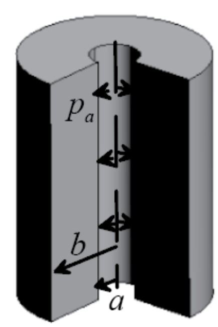
$$v_1 = -\frac{\phi}{L}x_1x_3 \qquad v_2 = \frac{\phi}{L}x_2x_3 \qquad v_3 = 0$$

Show that this is a kinematically admissible displacement field for the twisted shaft.

(b) Calculate the strains associated with this kinematically admissible displacement field

(c) Hence, find a formula for the potential energy of the shaft. You may assume that the potential energy of the torsional load is $-T\phi$

(d) Find the value of ϕ that minimizes the potential energy, and hence estimate the torsional stiffness of the shaft.



Problem 5.52 In this problem you will use the principle of minimum potential energy to find an approximate solution to the displacement in a pressurized cylinder. Assume that the cylinder is an isotropic, linear elastic solid with Young's modulus E and Poisson's ratio ν, and subjected to internal pressure p at $r=a$.

(a) Approximate the radial displacement field as

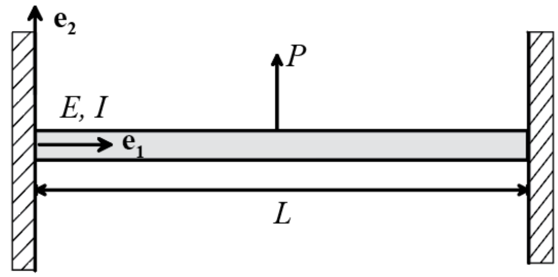
$$u_r = \frac{C_1}{r} + C_2 + C_3 r$$

where C_1, C_2, C_3 are constants to be determined. Assume all other components of displacement are zero. Calculate the strains $\varepsilon_{rr}, \varepsilon_{\theta\theta}$ in the solid

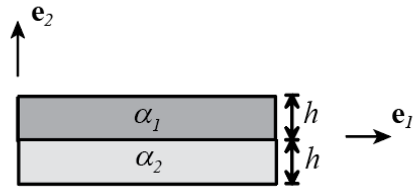
(b) Find an expression for the total strain energy of the cylinder per unit length, in terms of C_1, C_2, C_3 and relevant geometric and material parameters

(c) Hence, write down the potential energy (per unit length) of the cylinder.

(d) Find the values of C_1, C_2, C_3 that minimize the potential energy, and hence find the displacement field. You can check your answer using the exact solution.



Problem 5.53 By guessing the deflected shape, estimate the stiffness of a clamped—clamped beam subjected to a point force at mid-span. Note that your guess for the deflected shape must satisfy $w(x_1) = dw/dx_1 = 0$ on the ends of the beam, so you can't assume that it bends into a circular shape as done in the example in Section 4.6.5 of Applied Mechanics of Solids. Instead, try a deflection of the form $w(x_1) = \Delta\left(1-\cos\left(2\pi x_1 / L\right)\right)$, or a similar function of your choice (you could try a suitable polynomial, for example). If you try more than one guess and want to know which one gives the best result, remember that energy minimization always overestimates stiffness. The best guess is the one that gives the lowest stiffness.

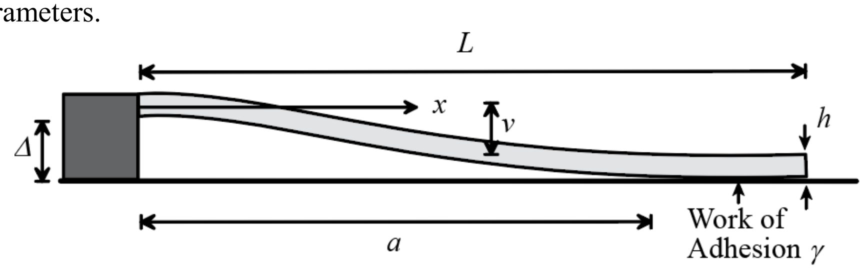


Problem 5.54 A bi-metallic strip is made by welding together two materials with identical Young's modulus and Poisson's ratio E,ν, but with different thermal expansion coefficients α_1,α_2, as shown in the picture. At some arbitrary temperature the strip is straight and free of stress. The temperature is then increased by ΔT, causing the strip to bend. Assume that, after heating, the displacement field in the strip can be approximated as

$$u_1 = \lambda x_1 + \kappa x_1 x_2 \qquad u_2 = -\kappa x_1^2/2 + \frac{-\nu x_2(2\lambda + \kappa x_2/2) + (1+\nu)\alpha\Delta T x_2}{1-\nu} \qquad u_3 = \lambda x_3$$

where $\alpha = \alpha_1 \;\; x_2 > 0 \qquad \alpha = \alpha_2 \;\; x_2 < 0$, and λ,κ are constants to be determined.

(a) Briefly explain the physical significance of the shape changes associated with λ,κ
(b) Calculate the distribution of (infinitesimal) strain associated with the kinematically admissible displacement field
(c) Hence, find a formula for the potential energy (per unit out of plane distance) of the system
(d) Minimize the potential energy to determine values for λ,κ in terms of relevant geometric and material parameters.



Problem 5.55 The figure shows a MEMS cantilever beam with Young's modulus *E,* length *L*, and rectangular cross-section with height *h* and base *b.* When straight, the base of the beam is a height Δ above a surface. An attractive short-range force with work of adhesion γ per unit area acts between the beam and the surface. As a result, if the end of the beam contacts the surface, a portion $a<x<L$ of the beam adheres to the surface, as indicated in the figure. The goal of this problem is to estimate the length *a* of the cantilever that is not in contact.

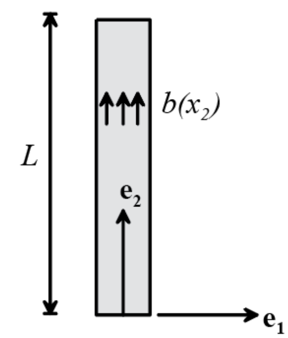
(a) Show that

$$v = \begin{cases} \Delta(1-\cos(\pi x/a))/2 & 0<x<a \\ \Delta & a<x<L \end{cases}$$

is a kinematically admissible (downward) deflection of the beam

(b) Find a formula for the potential energy of the system (be sure to include the contribution to the potential energy from work of adhesion).
(c) Hence, find a formula for a

Problem 5.56 A slender rod with length L and cross sectional area A is subjected to an axial body force per unit volume $\mathbf{b} = b(x_2)\mathbf{e}_2$. Our objective is to determine an approximate solution to the displacement field in the rod.



(a) Assume that the displacement field has the form

$$u_2 = w(x_2) \quad u_1 = u_3 = 0$$

where the function w is to be determined (this is not a very good approximation, but keeps the problem simple). Find an expression for the strains in terms of w and hence deduce the strain energy density.
(b) Show that the potential energy of the rod is

$$V(w) = EA\frac{1-\nu}{(1-2\nu)(1+\nu)}\int_0^L \frac{1}{2}\left\{\frac{dw}{dx_2}\right\}^2 dx_2 - A\int_0^L b(x_2)w(x_2)dx_2$$

(c) To minimize the potential energy, suppose that w is perturbed from the value that minimizes V to a value $w+\delta w$. Assume that δw is kinematically admissible, which requires that $\delta w = 0$ at any point on the bar where the value of w is prescribed. Calculate the potential energy $V(w+\delta w)$ and show that it can be expressed in the form

$$V(w+\delta w) = V(w) + \delta V + \frac{1}{2}\delta^2 V$$

where V is a function of w only, δV is a function of w and δw, and $\delta^2 V$ is a function of δw only. Give expressions for $V(w), \delta V, \delta^2 V$
(d) If V is stationary at $\delta w = 0$, then $\delta V = 0$. Show that, to satisfy $\delta V = 0$, we must choose w to satisfy

$$\frac{EA(1-\nu)}{(1-2\nu)(1+\nu)}\int_0^L \frac{dw}{dx_2}\frac{d\delta w}{dx_2}dx_2 - A\int_0^L b\delta w dx_2 = 0$$

(e) Integrate the first term by parts to deduce that, to minimize, V, w must satisfy

$$\frac{E(1-\nu)}{(1-2\nu)(1+\nu)}\frac{d^2 w}{dx_2^2} + b(x_2) = 0$$

Show that this is equivalent to the equilibrium condition

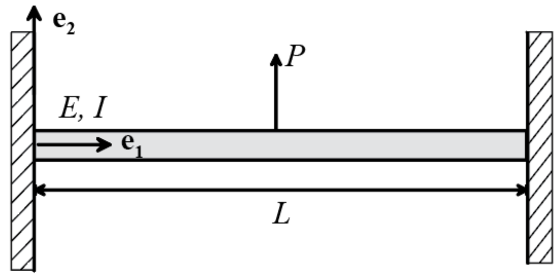
$$\frac{d\sigma_{22}}{dx_2} + b = 0$$

Furthermore, deduce that if w is not prescribed at either $x_2 = 0$, $x_2 = L$ or both, then the boundary conditions on the end(s) of the rod must be

$$\frac{dw}{dx_2} = 0$$

Show that this corresponds to the condition that $\sigma_{22} = 0$ at a free end.
(f) Use your results in ® to estimate the displacement field in a bar with mass density ρ, which is attached to a rigid wall at $x_2 = L$, is free at $x_2 = 0$, and subjected to the force of gravity (acting vertically downwards…)

Problem 5.57 The figure shows a beam with clamped ends subjected to a point force at its center. Its potential energy is



$$\Pi = \int_0^L \frac{1}{2} EI \left(\frac{d^2 v(x)}{dx^2} \right)^2 dx - Pv(L/2)$$

where v is the deflection of the beam.

(a) Start by re-writing the potential energy in dimensionless form, by defining dimensionless measures of deflection and position as follows

$$\hat{v} = \frac{vEI}{PL^3} \qquad \hat{x} = \frac{x}{L} \qquad \hat{\Pi} = \frac{\Pi EI}{P^2 L^3}$$

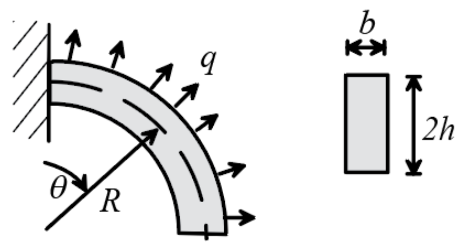
Show that the dimensionless form for potential energy is independent of loading, geometry, or material properties.

(b) Use a displacement field of the form

$$\hat{v}(x) = \sum_{n=1}^{N} a_n \hat{x}^{n-1}$$

to obtain an approximation to the deflected shape of the beam. Use a suitable symbolic manipulation program to write a script that calculates $\hat{v}(\hat{x})$ for an arbitrary number of terms in the approximation. Plot a graph of the solution with 5, 10 and 15 terms in the series.

Problem 5.58 The figure shows a beam, with rectangular cross section $b \times 2h$. In its unstressed state the beam has a constant radius of curvature $R >> h$ (measured at the mid-plane of the beam). It is clamped at $\theta = 0$, free at $\theta = \theta_0$, and is subjected to a transverse pressure $q(\theta)$ on one surface. The nonzero strain components in the beam can be approximated as



$$\varepsilon_{\theta\theta} = \frac{1}{R}\left(\frac{du_\theta}{d\theta} + u_r(\theta) \right) - \frac{(r-R)}{R^2}\left(\frac{d^2 u_r}{d\theta^2} - \frac{du_\theta}{d\theta} \right)$$

$$\varepsilon_{rr} = \varepsilon_{zz} = -\nu \varepsilon_{\theta\theta}$$

where $u_r(\theta), u_\theta(\theta)$ are the displacement components at the mid-section of the beam.

(a) Show that the potential energy for this strain field is

$$V = \frac{Ebh}{R^2}\int_0^{\theta_0} \left(\frac{du_\theta}{d\theta} + u_r(\theta) \right)^2 R d\theta + \frac{Ebh^3}{3R^4}\int_0^{\theta_0} \left[\left(\frac{d^2 u_r}{d\theta^2} - \frac{du_\theta}{d\theta} \right)^2 - 2\left(\frac{d^2 u_r}{d\theta^2} - \frac{du_\theta}{d\theta} \right)\left(\frac{du_\theta}{d\theta} + u_r(\theta) \right) \right] R d\theta$$

$$- \int_0^{\theta_0} bqu_r (R+h) d\theta$$

(b) Hence, show that the governing equations for $u_r(\theta), u_\theta(\theta)$ are

$$\frac{d}{d\theta}\left(\frac{du_\theta}{d\theta} + u_r(\theta) \right) - \frac{2h^3}{3R^2 + h^2}\frac{d}{d\theta}\left(\frac{d^2 u_r}{d\theta^2} - \frac{du_\theta}{d\theta} \right) = 0$$

$$\frac{2Eh}{R^2}\left(\frac{du_\theta}{d\theta} + u_r \right) + \frac{2Eh^3}{3R^4}\left[\frac{d^2}{d\theta^2}\left(\frac{d^2 u_r}{d\theta^2} + u_r \right) - \left(\frac{d^2 u_r}{d\theta^2} - \frac{du_\theta}{d\theta} \right) \right] - q\left(1 + \frac{h}{R} \right) = 0$$

(The term of order h^3 in the first equation is usually neglected in practical calculations), while the boundary conditions are

$$\left(\frac{du_\theta}{d\theta}+u_r(\theta)\right)-\frac{2h^3}{3R^2+h^2}\left(\frac{d^2u_r}{d\theta^2}-\frac{du_\theta}{d\theta}\right)=0 \qquad \theta=\theta_0$$

$$\left(\frac{d^2u_r}{d\theta^2}+u_r\right)=0 \qquad \theta=\theta_0$$

$$\frac{d}{d\theta}\left(\frac{d^2u_r}{d\theta^2}+u_r\right)=0 \qquad \theta=\theta_0$$

(Again, the the term of order h^3 in the first equation is usually neglected)

5.7 The Reciprocal Theorem and its Applications

Problem 5.59 A planet that deforms under its own gravitational force can be idealized as a homogeneous linear elastic sphere with radius *a*, density ρ_0 ,Young's modulus *E* and Poisson's ratio ν that is subjected to a radial gravitational force (per unit mass) $\mathbf{b}=-(gR/a)\mathbf{e}_R$, where *g* is the acceleration due to gravity at the surface of the sphere, and *R* is the radial coordinate. Use the reciprocal theorem, together with a hydrostatic stress distribution $\sigma_{ij}=p\delta_{ij}$ as the reference solution, to calculate the radial displacement of its surface.

Problem 5.60 Consider an isotropic, linear elastic solid with Young's modulus *E*, mass density ρ, and Poisson's ratio ν, which is subjected to a body force distribution b_i per unit mass, and tractions t_i on its exterior surface. By using the reciprocal theorem, together with a state of uniform stress σ_{ij}^* as the reference solution, show that the average strains in the solid can be calculated from

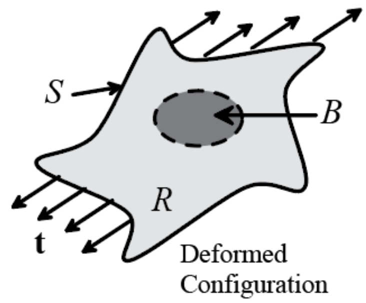
$$\frac{1}{V}\int_V \varepsilon_{ij}dV=\frac{1+\nu}{EV}\int_A x_j t_i dA-\frac{\nu}{EV}\int_A \delta_{ij}x_k t_k dA+\frac{1+\nu}{EV}\int_V x_j\rho b_i dV-\frac{\nu}{EV}\int_V \delta_{ij}x_k\rho b_k dV$$

Problem 5.61 A right non-circular cylinder with arbitrary cross-section rests with one end on a frictionless flat surface, and is subjected to a vertical gravitational body force per unit mass $\mathbf{b}=-g\mathbf{e}_3$, where $\mathbf{e}_3$ is a unit vector normal to the surface and *g* is the gravitational acceleration. The cylinder is a linear elastic solid with Young's modulus *E*, mass density ρ_0, and Poisson's ratio ν. Define the change in length of the cylinder as

$$\delta L=\frac{1}{A}\int_A u_3(x_1,x_2,L)dA$$

where $u_3(x_1,x_2,L)$ denotes the displacement of the end $x_3=L$ of the cylinder. Show that $\delta L=-WL/(2EA)$, where *W* is the weight of the cylinder, and *A* its cross-sectional area.

Problem 5.62 In this problem, you will calculate an expression for the change in potential energy that occurs when an inelastic strain ε_{ij}^T is introduced into some part *B* of an elastic solid. The inelastic strain can be visualized as a generalized version of the Eshelby inclusion problem – it could occur as a result of thermal expansion, a phase transformation in the solid, or plastic flow. Note that *B* need not be ellipsoidal.



The figure illustrates the solid of interest. Assume that:

- The solid has elastic constants C_{ijkl}
- No body forces act on the solid (for simplicity)
- Part of the surface of the solid S_1 is subjected to a prescribed displacement u_i^*
- The remainder of the surface of the solid S_2 is subjected to a prescribed traction t_i^*

Let $u_i^0, \varepsilon_{ij}^0, \sigma_{ij}^0$ denote the displacement, strain, and stress in the solid before the inelastic strain is introduced. Let Π_0 denote the potential energy of the solid in this state.

Next, with the external loading held fixed, suppose that some external process introduces an inelastic strain ε_{ij}^T into some part *B* of the solid. Let $\Delta u_i, \Delta\varepsilon_{ij}, \Delta\sigma_{ij}$ denote the displacement, strain and stress in the solid resulting after the inelastic strain has been introduced. Note that these fields satisfy

- The strain-displacement relation $\Delta\varepsilon_{ij} = (\partial \Delta u_i / \partial x_j + \partial \Delta u_i / \partial x_j)/2$
- The stress-strain law $\Delta\sigma_{ij} = C_{ijkl}(\Delta\varepsilon_{kl} - \varepsilon_{kl}^T)$ in *B*, and $\Delta\sigma_{ij} = C_{ijkl}\Delta\varepsilon_{kl}$ in (*V-B)* (i.e. outside *B*)
- Boundary conditions $\Delta u_i = 0$ on S_1, and $\Delta\sigma_{ij}n_j = 0$ on S_2.

(a) Write down an expressions for Π_0 in terms of $u_i^0, \varepsilon_{ij}^0, \sigma_{ij}^0$ and the loads applied to the solid

(b) Suppose that $u_i^0, \varepsilon_{ij}^0, \sigma_{ij}^0$ are all zero (i.e. the solid is initially stress free). Write down the potential energy Π_S resulting from the state $\Delta u_i, \Delta\varepsilon_{ij}, \Delta\sigma_{ij}$. This is called the "self energy" of the eigenstrain – the energy cost of introducing the eigenstrain ε_{ij}^T into a stress-free solid.

(c) Show that the expression for the self-energy in part (b) can be simplified to

$$\Pi_S = -\frac{1}{2}\int_B \Delta\sigma_{ij}\varepsilon_{ij}^T dV$$

(d) Now suppose that $u_i^0, \varepsilon_{ij}^0, \sigma_{ij}^0$ are all nonzero when the transformation strain is introduced. Write down the total potential energy of the system Π_{TOT}, in terms of $u_i^0, \varepsilon_{ij}^0, \sigma_{ij}^0$ and $\Delta u_i, \Delta\varepsilon_{ij}, \Delta\sigma_{ij}$.

(e) Finally, show that the total potential energy of the system can be expressed as

$$\Pi_{TOT} = \Pi_0 + \Pi_S - \int_B \sigma_{ij}^0 \varepsilon_{ij}^T dV$$

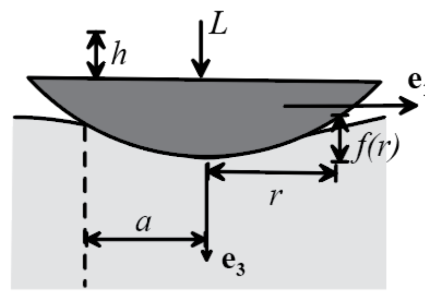
Here, the last term is called the "interaction energy" of the eigenstrain with the applied load. The steps in this derivation are very similar to the derivation of the reciprocal theorem.

Problem 5.63 An infinite, isotropic, linear elastic solid with Young's modulus E and Poisson's ratio ν is subjected to a uniaxial tensile stress σ_0. As a result of a phase transformation, a uniform dilatational strain $\varepsilon_{ij}^T = \beta\delta_{ij}$ is then induced in a spherical region of the solid with radius a.

(a) Using the solution to problem 5.62, and the Eshelby solution, find an expression for the change in potential energy of the solid caused by nucleating the dilatational strain in the sphere, in terms of β, σ_0 and relevant geometric and material parameters.

(b) Assume that the interface between the transformed material and the matrix has an energy per unit area γ. Find an expression for the critical value of σ_0 at which the total energy of the system (elastic potential energy + interface energy) is decreased as a result of the transformation

Problem 5.64 The figure shows a half-space with shear modulus μ and Poisson's ratio ν that is indented by a rigid, axisymmetric frictionless punch with profile *f(r)*. The punch is pushed into the half-space by a distance *h* under a force *L*. The area of contact is circular, with radius *a,* and is subjected to a pressure distribution $-\sigma_{33} = p(r)$.



(a) Write down the boundary conditions governing the elastic fields in the half-space.

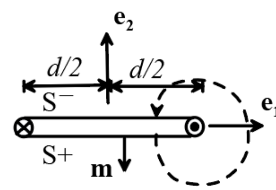
(b) Apply the reciprocal theorem selecting as auxiliary solution in Section 5.4.9 of Applied Mechanics of Solids for the fields induced a rigid flat ended circular cylindrical punch indenting a half-space to show that the total force is related to the punch profile by

$$L = \frac{4\mu}{(1-\nu)}\left(ah - \int_0^a \frac{rf(r)dr}{\sqrt{a^2 - r^2}} \right)$$

Problem 5.65 Calculate the stress induced by a straight screw dislocation in an infinite solid using the formula in Section 5.7.4 of Applied Mechanics of Solids. Compare the solution with the result of the calculation in Problem 5.14

Problem 5.66 Calculate the stress components induced by an edge dislocation in an infinite solid using the formula in Section 5.7.4 of Applied Mechanics of Solids. Compare the solution with the result given in 5.3.4

5.8 Energetics of Dislocations in Elastic Solids



Problem 5.67 The figure shows two nearby straight screw dislocations in an infinite solid, with line direction The screw dislocations can be introduced into the solid by cutting the plane between the dislocations and displacing the upper of the surfaces created by the cut (S^-) by $b\mathbf{e}_3/2$ and the lower (S^+) by $-b\mathbf{e}_3/2$, and re-connecting the surfaces. The solid deforms in anti-plane shear, with a displacement field of the form $\mathbf{u} = u(x_1, x_2)\mathbf{e}_3$

(a) Identify which components of stress and strain in the solid are nonzero.

(b) Show that the total strain energy of the solid (per unit out of plane distance) can be expressed as

$$U = \frac{1}{2}\int_A \sigma_{3\alpha} \frac{\partial u}{\partial x_\alpha} dA$$

where the integral is taken over the $\{\mathbf{e}_1, \mathbf{e}_2\}$ plane

(c) Show that the potential energy can be re-written as

$$U = -\frac{1}{2}\int_{-d/2}^{d/2} \sigma_{32}(x_1) b\, dx_1$$

where the integral is taken along the line $x_2 = 0$.

(d) Use the solution for a screw dislocation given in Problem 5.65 to show that the energy can be calculated (formally) as

$$U = \frac{\mu b^2}{4\pi}\int_{-d/2}^{d/2}\left(\frac{1}{x_1 + d/2} + \frac{1}{d/2 - x_1}\right) dx_1$$

(e) Note that the integral is unbounded, as expected. Calculate a bounded expression by truncating the integral at $d/2 - \rho$ and $-d/2 + \rho$.

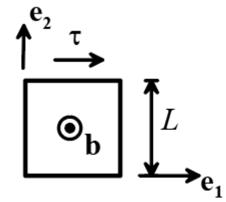
(f) Calculate the force exerted on one dislocation by the other by differentiating the expression for the energy. Is the force attractive or repulsive?

(g) Check your answer to (f) using the Peach-Koehler formula.

Problem 5.68 Calculate the nonsingular stress $\sigma_{ij}^{(\rho)}$ induced by a screw dislocation in an infinite solid using the nonsingular dislocation theory described in Section 5.8.2 of Applied Mechanics of Solids. Compare the solution with the result of the calculation in Problem 5.65.

Problem 5.69 Calculate the nonsingular self- energy per unit length of a straight dislocation segment with length $2L$ that lies along $-L < x_3 < L$ in an infinite elastic solid with shear modulus μ and Poisson's ratio ν, using the non-singular dislocation theory discussed in Section 5.8.2 of Applied Mechanics of Solids. (Of course, a straight dislocation segment can't actually exist, but we can assume that there is a surface at each end of the dislocation). Approximate the solution by expanding to first order in ρ/L.

Problem 5.70 Calculate the self-energy of the square prismatic dislocation loop with side length L shown in the figure ($\boldsymbol{\tau}$ denotes the tangent vector to the loop). Use nonsingular dislocation theory, and give your answer to zeroth order in the parameter ρ



Problem 5.71 Suppose that the dislocation loop described in the preceding problem is subjected to a uniaxial tensile stress $\sigma_{ij} = \sigma_0 \delta_{i3} \delta_{j3}$ acting perpendicular to the plane of the loop.

(a) Show that the total potential energy of the system can be expressed as

$$V_D = \frac{\mu b^2 L}{\pi(1-\nu)} \left(\log \frac{2\left(1+\sqrt{2}\right)L}{\rho} + 1 - 2\sqrt{2} \right) - \sigma_0 b L^2 + V^*$$

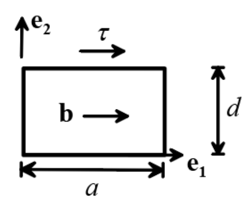
where V^* is the potential energy of a stressed solid that contains no dislocations.

(b) Display your result as a graph of normalized potential energy $(V^D - V^*)(1-\nu)/(\mu b^3)$ as a function of *L/b,* for various values of $\sigma_0(1-\nu)/\mu$. Take $\rho = b/4$ as a representative value.

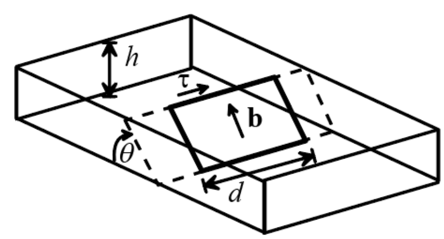
(c) Plot a graph of the critical size required for a pre-existing dislocation loop to grow spontaneously (i.e. the point where the energy starts to decrease as the loop grows), as a function of $\sigma_0(1-\nu)/\mu$.

(d) Plot a graph of the activation energy required for homogeneous nucleation of a prismatic dislocation loop, as a function of $\sigma_0(1-\nu)/\mu$ (the activation energy is the energy at the peak)

Problem 5.72 Calculate the self-energy of a rectangular glide dislocation loop with burgers vector $\mathbf{b} = b\mathbf{e}_1$ and side lengths *a,d*. Use nonsingular dislocation theory and give your answer to zeroth order in the parameter ρ



Problem 5.73 A composite material is made by sandwiching thin layers of a ductile metal between layers of a hard ceramic. Both the metal and the ceramic have identical Young's modulus E and Poisson's ratio ν. The figure shows one of the metal layers, which contains a glide dislocation loop on an inclined slip-plane. The solid is subjected to a uniaxial tensile stress σ_0 perpendicular to the layers.

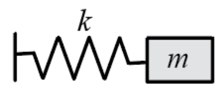


(a) Calculate the total energy of the dislocation loop, in terms of the applied stress and relevant geometric and material parameters. Use non-singular dislocation theory to calculate the self-energy of the loop. You can use the solution to problem 5.72 for the self-energy of the dislocation, and use V^* to denote the potential energy of a dislocation-free composite

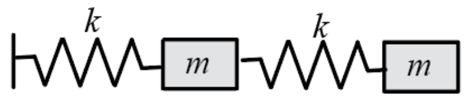
(b) Suppose that the layer contains a large number of dislocation loops with initial width $d_0 = \sqrt{2}h$ (assume that the spacing between loops is large enough to neglect their interactions). The layer starts to deform plastically if the stress is large enough to cause the loops to expand in the plane of the film (by increasing the loop dimension *d*). Find a formula for the yield stress of the composite, for the special case $\theta = 45$ degrees.

5.9 Rayleigh-Ritz Method for Estimating Natural Frequency of an Elastic Solid

Problem 5.74 Use the Rayleigh-Ritz method to obtain a formula for the natural frequency of horizontal vibration of the spring-mass system shown (the displacement associated with the vibration mode is trivial)

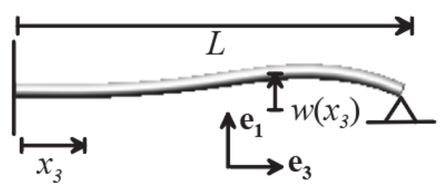


Problem 5.75 Use the Rayleigh-Ritz method to obtain a formula for the lowest natural frequency of vibration of the spring-mass system shown. You should be able to obtain an exact solution, by describing the mode shape in terms of a single parameter, and the minimizing the estimate for the frequency.

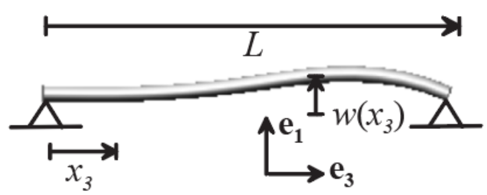


Problem 5.76 Reconsider problem 5.75. Try to find the *second* frequency of vibration for the system by selecting another approximation to the mode shape, which is (by construction) orthogonal to the first.

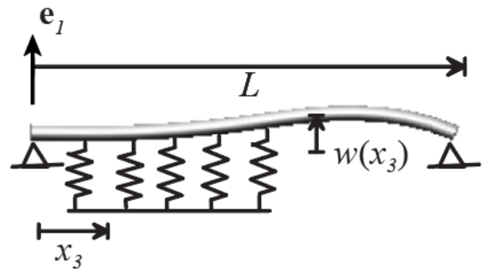
Problem 5.77 Use the Rayleigh-Ritz method to estimate the fundamental frequency of the clamped-pinned beam illustrated in the figure. Assume that the beam has Young's modulus E and mass density ρ, and its cross-section has area A and moment of area I.



Problem 5.78 Use the Rayleigh-Ritz method to estimate the fundamental frequency of the pinned-pinned beam illustrated in the figure. Assume that the beam has Young's modulus E and mass density ρ, and its cross-section has area A and moment of area I.



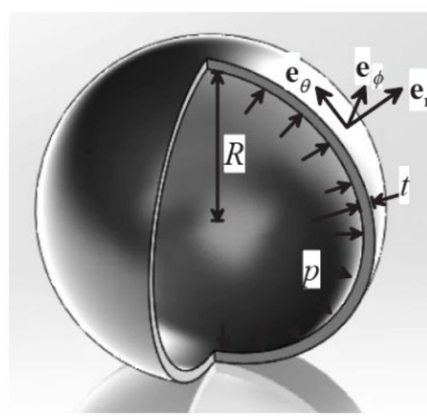
Problem 5.79 A beam with length L Young's modulus E and mass density ρ, and its cross-section has area A and moment of area I is bonded to an elastic foundation, which exerts a restoring force per unit length $p = -kw(x_3)\mathbf{e}_1$ on the beam. The beam is pinned at both ends. Use the Rayleigh-Ritz method to estimate the natural frequency of vibration of the beam.



Chapter 6
Solutions for Plastic Solids

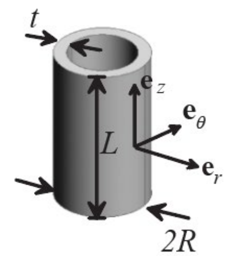
6.1 Axially and Spherically Symmetric Solutions for Elastic-Plastic Solids

Problem 6.1 A thin-walled sphere with radius R and wall thickness t is made from an elastic-plastic material with Youngs modulus E, Poissons ratio ν and a linear hardening relation $Y = Y_0 + h\varepsilon_e$. The sphere is subjected to monotonically increasing internal pressure p (with $dp/dt > 0$), which generates a stress state (in spherical-polar coordinates) $\sigma_{rr} \approx 0$, $\sigma_{\phi\phi} = \sigma_{\theta\theta} = pR/(2t)$ (note that these are principal stresses)



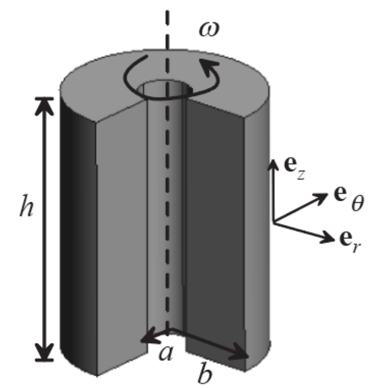
(a) Find a formula for the Von-Mises stress in the sphere wall, in terms of p,R and t. Hence, calculate the pressure that will first cause yield in the sphere wall.
(b) Find the hydrostatic and deviatoric stresses in the sphere wall.
(c) Hence, find a formula for the Von Mises plastic strain rate $d\varepsilon_e/dt$ in the sphere wall, in terms of $dp/dt, h, R, t$
(d) Hence, find a formula for the total strain rates $d\varepsilon_{rr}/dt, d\varepsilon_{\theta\theta}/dt$ (include both elastic and plastic strain rates, and give solutions for pressure both below and above yield) in the shell.
(e) Find the total hoop strains $\varepsilon_{\theta\theta}, \varepsilon_{\phi\phi}$ when the pressure reaches a value $p = 4tY_0/R$
(f) Find a formula for the change in radius of the sphere when the pressure reaches a value $p = 4tY_0/R$

Problem 6.2 A cylindrical, thin-walled pressure vessel with close ends, initial radius R, length L and wall thickness $t<<R$ is subjected to internal pressure p. The vessel is made from an isotropic elastic-plastic solid with Young's modulus E, Poisson's ratio ν, and its yield stress varies with accumulated plastic strain ε_e as $Y = Y_0 + h\varepsilon_e$. Recall that the stresses in a thin-walled pressurized tube are related to the internal pressure by $\sigma_{zz} = pR/(2t)$, $\sigma_{\theta\theta} = pR/t$ $\sigma_{rr} \approx 0$



(a) Calculate the Von-Mises stress in the tube
(b) Hence, find the critical value of internal pressure required to initiate yield in the solid (use the Von-Mises criterion)
(c) Find a formula for the strain increment $d\varepsilon_{rr}, d\varepsilon_{\theta\theta}, d\varepsilon_{zz}$ resulting from an increment in pressure dp (neglect changes in the tube geometry)
(d) Suppose that the pressure is increased 10% above the initial yield value. Find a formula for the change in radius, length and wall thickness of the vessel. Assume small strains.

Problem 6.3 The figure shows a long hollow cylindrical shaft with inner radius a and outer radius b, which spins with angular speed ω about its axis. Assume that the disk is made from an elastic-perfectly plastic material with yield stress Y and density ρ. The goal of this problem is to calculate the critical angular speed that will cause the cylinder to collapse (the point of plastic collapse occurs when the entire cylinder reaches yield).

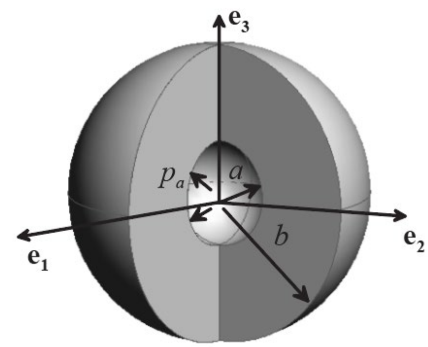


(a) Using the cylindrical-polar basis shown, list any stress or strain components that must be zero. Assume plane strain deformation.
(b) Write down the boundary conditions that the stress field must satisfy at $r=a$ and $r=b$
(c) Write down the linear momentum balance equation in terms of the stress components, the angular velocity and the disk's density. Use polar coordinates and assume axial symmetry.
(d) Using the plastic flow rule, show that $\sigma_{zz} = (\sigma_{rr} + \sigma_{\theta\theta})/2$ if the cylinder deforms plastically under plane strain conditions
(e) Using Von-Mises yield criterion, show that the radial and hoop stress must satisfy $|\sigma_{\theta\theta} - \sigma_{rr}| = 2Y/\sqrt{3}$
(f) Hence, show that the radial stress must satisfy the equation

$$\frac{d\sigma_{rr}}{dr} = -\rho r\omega^2 + \frac{2}{\sqrt{3}}\frac{Y}{r}$$

(g) Finally, calculate the critical angular speed that will cause plastic collapse.

Problem 6.4 Consider an elastic-perfectly plastic spherical pressure vessel with inner radius a and outer radius b, which is subjected to cyclic internal pressure, as described in Section 6.1.4 of the main text. Show that a cyclic plastic zone can only develop in the vessel if b/a exceeds a critical magnitude. Give a formula for the critical value of b/a, and find a (numerical, if necessary) solution for b/a.



Problem 6.5 A spherical pressure vessel is subjected to internal pressure p_a and is free of traction on its outer surface. The vessel deforms by creep, and may be idealized as an elastic-power law viscoplastic solid with flow potential

$$g(\sigma_e) = \dot{\varepsilon}_0 \left(\frac{\sigma_e}{Y} \right)^m$$

where $Y, m, \dot{\varepsilon}_0$ are material properties and σ_e is the Von-Mises equivalent stress. The goal of this problem is to calculate the steady-state stress and strain rate fields in the solid, and deduce a formula for the rate of expansion of the inner bore of the vessel.

(a) Note that at steady state, the stress is constant, and so the elastic strain rate must vanish. Use the incompressibility condition to show that the radial velocity at a distance *r* from the origin is related to the rate of expansion of the inner bore $\dot{a}$ by

$$v_r = \frac{\dot{a} a^2}{r^2}$$

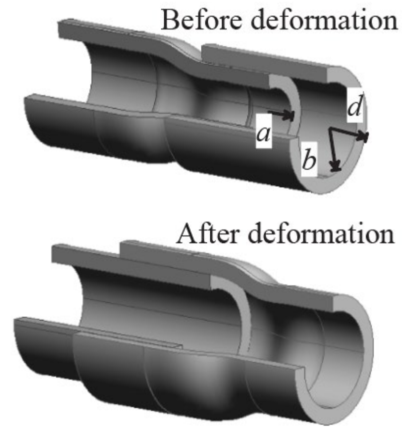
(b) Hence, find a formula for the plastic strain rates in terms of $\dot{a}$, *a* and *r,* and determine the Von Mises effective plastic strain rate

(c) Use the plastic flow potential to show that the stresses are related to the rate of expansion of the inner bore by

$$\sigma_{\theta\theta} - \sigma_{rr} = Y \left(\frac{2\dot{a} a^2}{\dot{\varepsilon}_0 r^3} \right)^{1/m}$$

(d) Hence, use the equilibrium equation and boundary conditions to show that the rate of expansion of the inner bore is related to the internal pressure by

$$\frac{\dot{a}}{a} = \frac{\dot{\varepsilon}_0}{2} \left(\frac{3 p_a}{2mY} \right)^m \left(1 - \frac{a^{3/m}}{b^{3/m}} \right)^{-m}$$



Problem 6.6 The figure illustrates a technique that is sometimes used to connect tubular components down oil wells. As manufactured, the left hand tube has inner and outer radii (a,b), while the tube on the right has inner and outer radii (b,d), so that the end of the smaller tube can simply be inserted into the larger tube. An over-sized die is then pulled through the bore of the inner of the two tubes. The radius of the die is chosen so that both cylinders are fully plastically deformed as the die passes through the region where the two cylinders overlap. As a result, a state of residual stress is developed at the coupling, which clamps the two tubes together.

The goals of this problem are (i) to determine the size of the die that is required to ensure that both cylinders are plastically deformed as it is pulled through the pipe; and (ii) To find a formula for the force required to pull the tubes apart after the process has been completed.

To simplify the problem, assume that the tubes are elastic-perfectly plastic solids with Young's modulus *E*, Poisson's ratio ν and yield stress in uniaxial tension *Y*, and assume plane strain axisymmetric deformation. As a further approximation assume that the axial elastic and plastic strains satisfy $\varepsilon_{zz}^{p}=\varepsilon_{zz}^{e}=0$

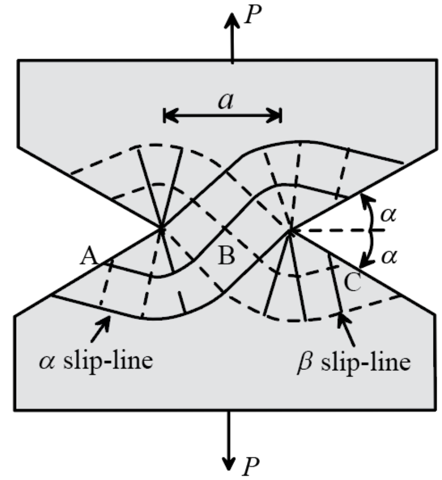
(a) Use the solution in 6.1.6 of Applied Mechanics of Solids for an internally pressurized elastic-plastic cylinder to calculate the displacement of the inner bore (at *r=a)* when the elastic-plastic boundary just reaches the outer surface of the external pipe (at *r=d)*. Hence, find a formula for the minimum die radius required to ensure that both tubes yield throughout their thickness.

(b) The die effectively subjects to the inner bore of the smaller tube to a cycle of pressure. Use the solutions given in Sections 6.1.6 and 4.1.9 of Applied Mechanics of Solids to calculate the residual stress distribution in the region where the two tubes overlap (assume that the unloading is elastic, neglect end effects and assume plane strain axisymmetric deformation).

(c) Find a formula for the force required to pull the pipes apart after the connection is complete. Assume that the pipes overlap over a length *L*, and have a friction coefficient μ (neglect end effects)

(d) For $d/a=1.5$, calculate an expression for the value of *b/a* that gives the strongest coupling, and find a formula for the force required to separate the optimal coupling.

6.2 Slip Line Fields

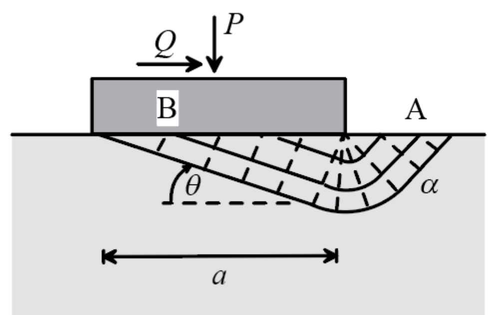


Problem 6.7 The figure shows the slip-line field for a rigid plastic double-notched bar deforming under uniaxial tensile loading. The material has yield stress in shear k

(a) Draw the Mohr's circle representing the state of stress at A. Write down (i) the value of ϕ at this point, and (ii) the magnitude of the hydrostatic stress $\bar{\sigma}$ at this point.

(b) Calculate the value of ϕ at point B, and deduce the magnitude of $\bar{\sigma}$. Draw the Mohr's circle of stress at point B, and calculate the horizontal and vertical components of stress

(c) Find an expression for the force P that causes plastic collapse in the bar.



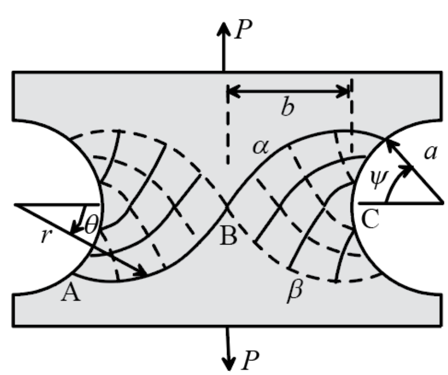
Problem 6.8 The figure shows a slip-line field for oblique indentation of a rigid-plastic solid with yield stress in shear k by a rigid flat punch

(a) Draw the Mohr's circle representing the state of stress at A. Write down (i) the value of ϕ at this point, and (ii) the magnitude of the hydrostatic stress $\bar{\sigma}$ at this point.

(b) Calculate the value of ϕ at point B, and deduce the magnitude of the hydrostatic stress $\bar{\sigma}$

(c) Draw the Mohr's circle representation for the stress state at B, and hence calculate the stress state at B, as a function of k and θ.

(d) Calculate expressions for P and Q in terms of k, a and θ, and find an expression for Q/P. What is the maximum possible value of Q/P?.



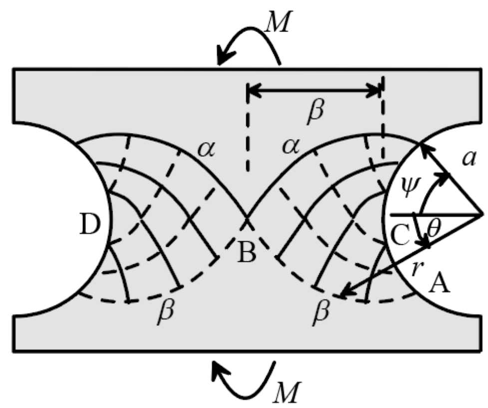
Problem 6.9 The figure shows the slip-line field for a rigid plastic double-notched bar under uniaxial tension. The material has yield stress in shear k. The slip-lines are logarithmic spirals, as discussed in Section 6.2.3 of Applied Mechanics of Solids.

(a) Write down a relationship between the angle ψ, the notch radius a and the bar width $2b$.

(b) Draw the Mohr's circle representing the state of stress at A. Write down (i) the value of ϕ at this point, and (ii) the magnitude of the hydrostatic stress $\bar{\sigma}$ at this point.

(c) Determine the value of ϕ at point B on the slip-line AB, and hence find the hydrostatic stress. Deduce formulas for the horizontal and vertical stresses at B.

(d) Hence, deduce the distribution of vertical stress along the line BC, and calculate the force P in terms of k, a and b.



Problem 6.10 The figure shows the slip-line field for a rigid plastic double-notched bar subjected to a bending moment. The slip-lines are logarithmic spirals.

(a) Write down a relationship between the angle ψ, the notch radius *a* and the bar width *b*.

(b) Draw the Mohr's circle representing the state of stress at A. Write down (i) the value of ϕ at this point, and (ii) the magnitude of the hydrostatic stress $\bar{\sigma}$ at this point.

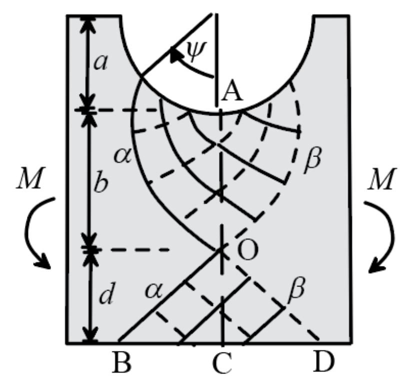
(c) Determine the value of ϕ and the hydrostatic stress just to the right of point B. Hence, deduce the horizontal and vertical components of stress at this point.

(d) Hence, deduce the distribution of vertical stress along the line BC

(e) Without calculations, write down the variation of stress along the line BD. What happens to the stress at point B?

(f) Hence, calculate the value of the bending moment *M* in terms of *b*, *a*, and *k*.

(g) Show that the slip-line field is valid only for *b* less than a critical value, and give an expression for the maximum allowable value for *b/a*.



Problem 6.11 The figure shows a slip-line field for a notched, rigid plastic bar deforming under pure bending (the field is valid only under restricted conditions, to be determined). The solid has yield stress in shear *k*.

(a) Write down the distribution of stress in the triangular region OBD

(b) Find the stress distribution along the line OA

(c) Calculate the resultant horizontal force exerted by tractions on the line AOC, and hence show that *(b/a)* can be determined from the equation

$$\frac{h}{a} = \left(1 + \frac{b}{a}\right)\left\{1 + \log\left(1 + \frac{b}{a}\right)\right\}$$

where $h=a+b+d$ is the total thickness of the bar.

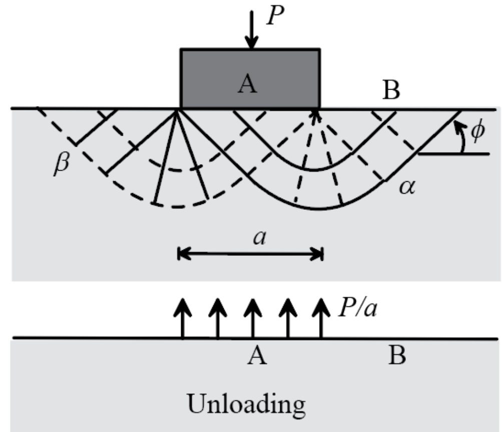
(d) Finally, calculate the resultant moment of the tractions about *O*, and show that *M* can be expressed as

$$M = kh\left(h - a\left(1 + \frac{b}{a}\right)\right) - a^2 \frac{k}{2}\frac{b}{a}\left(2 + \frac{b}{a}\right)$$

(e) Show that the slip-line field is valid only for $h/a=1+(b+d)/a$ less than a critical value, and determine an expression for the maximum allowable value for *h/a*.

Problem 6.12 Consider the notched bar described in problem 6.11. Propose a slip-line field that is valid for bars with thickness exceeding the critical value calculated in 6.11(e), and use it to find a formula for the collapse moment in terms of relevant material and geometric parameters.

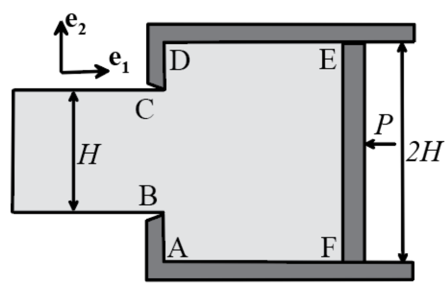
Problem 6.13 A rigid flat punch is pressed into the surface of an elastic-perfectly plastic half-space, with Young's modulus E, Poisson's ratio ν and shear yield stress k. The punch is then withdrawn.



(a) At maximum load the stress state under the punch can be estimated using the rigid-plastic slip-line field solution (the solution is accurate as long as plastic strains are much greater than elastic strains). Calculate the stress state in this condition (i) just under the contact (point A in the figure), and (ii) at the surface just outside the contact (point B in the figure).

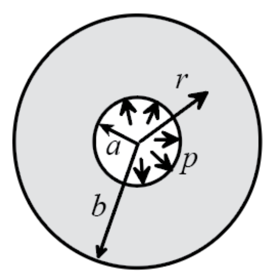
(b) The unloading process can be assumed to be elastic – this means that the *change* in stress during unloading can be calculated using the solution to an elastic half-space subjected to uniform pressure on its surface. Calculate the change in stress (i) just under the contact, and (ii) just outside the contact, using the solution given in Section 5.2.8 of Applied Mechanics of Solids.

Problem 6.14 The figure shows a strip with thickness H that is extruded from a die with height $2H$. Suggest a possible slip-line field for the flow mechanism, and use it to obtain a formula for the force per unit out-of-plane thickness P required to drive the extrusion, in terms of H and the shear yield stress of the solid k. Neglect friction.



6.3 Bounding Theorems in Plasticity and their Applications

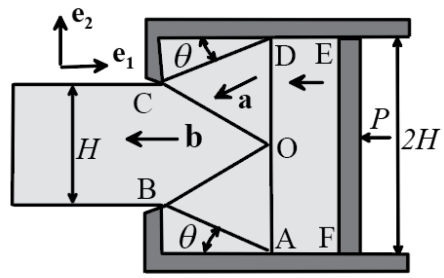
Problem 6.15 The figure shows a pressurized cylindrical cavity. The solid has yield stress in shear k. The objective of this problem is to calculate an upper bound to the pressure required to cause plastic collapse in the cylinder.



(a) Take a volume preserving radial distribution of velocity as the collapse mechanism. Calculate the strain rate associated with the collapse mechanism, in terms of the radial velocity da/dt at the inner wall of the cylinder.

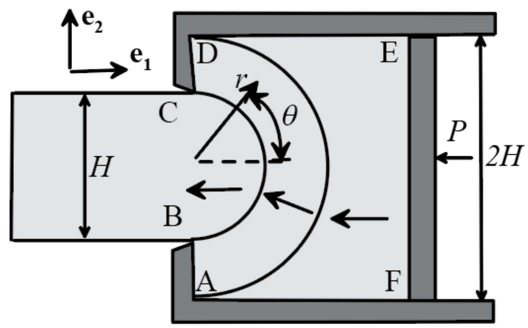
(b) Apply the upper bound theorem to estimate the internal pressure p at collapse. Compare the result with the exact solution

Problem 6.16 The figure shows a kinematically admissible velocity field for an extrusion process. The velocity of the solid is uniform in each sector, with velocity discontinuities across each line. The solid has uniaxial tensile yield stress Y.



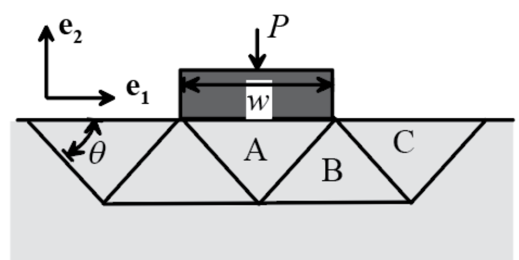
(a) .Assume the ram EF moves to the left at constant speed V. Calculate the velocity vectors **a** and **b** shown in the figure, and deduce the magnitude of the velocity discontinuity between neighboring regions.
(b) Hence, calculate the total plastic dissipation and obtain an upper bound to the extrusion force P per unit out-of-plane distance.
(c) Select the angle θ that gives the least upper bound.

Problem 6.17 The figure shows a kinematically admissible velocity field for an extrusion process. Material particles in the annular region ABCD move along radial lines. There are velocity discontinuities across the arcs BC and AD.



(a) Assume the ram EF moves to the left at constant speed V. Use flow continuity to write down the radial velocity of material particles inside the deforming region ABCD.
(b) Hence, calculate the strain rates in ABCD, as well as the velocity discontinuity across the surfaces BC and AD
(c) Calculate the plastic dissipation (neglect friction at the walls), and hence obtain an upper bound to the force P (you will need to evaluate an integral numerically).

Problem 6.18 The figure shows a proposed collapse mechanism for indentation of a rigid-plastic solid. Each triangle slides as a rigid block, with velocity discontinuities across the edges of the triangles.



(a) Assume that triangle A moves vertically downwards with speed V. Find the velocity of triangles B and C, and determine the velocity discontinuities between A/B and B/C
(b) Hence, obtain an upper bound to the force P
(c) Select the angle θ that minimizes the collapse load, and hence find the lowest upper bound.

Problem 6.19 The purpose of this problem is to extend the upper bound theorem to pressure-dependent (frictional) materials. Consider, in particular, a material with a yield criterion and plastic flow rule given by

$$f(\sigma_{ij}) = \sigma_e + \mu\sigma_{kk} - Y = 0$$

$$\dot{\varepsilon}_{ij}^p = \frac{\dot{\varepsilon}_e}{\sqrt{1+2\mu^2}}\frac{\partial f}{\partial \sigma_{ij}} = \frac{\dot{\varepsilon}_e}{\sqrt{1+2\mu^2}}\left(\frac{3S_{ij}}{2\sigma_e} + \mu\delta_{ij}\right)$$

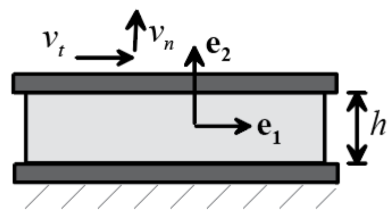
$$S_{ij} = \sigma_{ij} - (\sigma_{kk}\delta_{ij})/3 \qquad \sigma_e = \sqrt{3S_{ij}S_{ij}/2} \qquad \dot{\varepsilon}_e = \sqrt{2\dot{\varepsilon}_{ij}^p\dot{\varepsilon}_{ij}^p/3}$$

where Y, μ are material properties.

(a) Show that the rate of plastic work per unit volume $\dot{w}^p = \sigma_{ij}\dot{\varepsilon}_{ij}^p$ associated with a plastic strain rate $\dot{\varepsilon}_{ij}^p$ is related to the effective plastic strain rate by

$$\dot{w}^p = Y\dot{\varepsilon}_e / \sqrt{1+2\mu^2}$$

(b) We need to understand the nature of the plastic dissipation associated with velocity discontinuities in this material. We can develop the results for a velocity discontinuity by considering shearing (and associated dilatation) of a thin layer of material with uniform thickness h as indicated in the figure. Assume that the strain rate in the layer is homogeneous, and that the surface at $x_2 = h$ has a uniform tangential velocity v_t and normal velocity v_n. Show that:



(i) The rate of plastic work per unit area of the layer can be computed as

$$\sigma_{22}v_n + \sigma_{12}v_t = Y\sqrt{2v_n^2 + v_t^2} / \sqrt{3(1+2\mu^2)}$$

(ii) To satisfy the plastic flow rule the normal and tangential velocities must be related by

$$v_n = \mu\sqrt{3}v_t / \sqrt{1-4\mu^2}$$

Note that these results are independent of the layer thickness, and therefore (by letting $h \to 0$) also characterize the dissipation and kinematic constraint associated with a velocity discontinuity in the solid.

(c) To state the upper bound theorem for this material we introduce a kinematically admissible velocity field $\mathbf{v}$, which may have discontinuities across a set of surfaces $\hat{S}$ in the solid. Define the strain rate distribution associated with $\mathbf{v}$ as

$$\hat{\dot{\varepsilon}}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad \hat{\dot{\varepsilon}}_e = \sqrt{2\hat{\dot{\varepsilon}}_{ij}\hat{\dot{\varepsilon}}_{ij}/3}$$

The velocity field must satisfy $\hat{\dot{\varepsilon}}_{kk} = 3\mu\hat{\dot{\varepsilon}}_e / \sqrt{1+2\mu^2}$ in the interior of the solid, and must satisfy

$$[\![v_n]\!] = \mu[\![v_t]\!] \qquad [\![v_n]\!] = v_i m_i \qquad [\![v_t]\!] = \sqrt{(v_i - v_k m_k m_i)(v_i - v_k m_k m_i)} = \sqrt{v_i v_i - v_k m_k v_j m_j}$$

on $\hat{S}$, where m_i denotes a unit vector normal to $\hat{S}$. Define the plastic dissipation function as

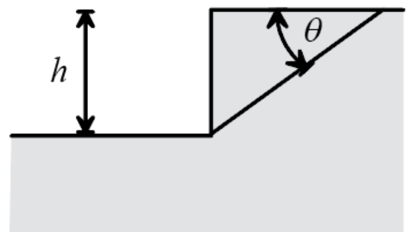
$$\mathbf{\Phi}(\mathbf{v}) = \int_R \frac{Y\hat{\dot{\varepsilon}}_e}{\sqrt{1+2\mu^2}}dV + \int_{\hat{S}} \frac{Y}{\sqrt{3(1-4\mu^2)}}[\![v_t]\!]dA - \int_R b_i v_i dV - \int_{\partial R} t_i^* v_i dA$$

Show that (i) $\mathbf{\Phi}(\dot{\mathbf{u}}) = 0$, where $\dot{\mathbf{u}}$ denotes the actual velocity field in the solid at collapse, and (ii) $\mathbf{\Phi}(\mathbf{v}) \geq 0$

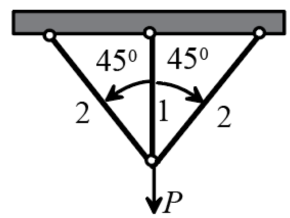
(d) Hence, show that an upper bound to the load factor at collapse can be calculated as

$$\beta_L \le \frac{\int_R \frac{Y\hat{\dot{\varepsilon}}_e}{\sqrt{1+2\mu^2}} dV + \int_{\hat{S}} \frac{Y}{\sqrt{3(1-4\mu^2)}} [\![v_t]\!] dA}{\int_R b_i v_i dV + \int_{\partial R} t_i^* v_i dA}$$

Problem 6.20 As an application of the results derived in the preceding problem, consider a soil embankment with vertical slope, as shown in the figure. The soil has mass density ρ and can be idealized as a frictional material with constitutive equation given in Problem 3. Using a collapse mechanism consisting of shearing and dilatation along the line AB shown in the figure (the angle θ for the optimal mechanism must be determined), calculate an upper bound to the admissible height h of the embankment.

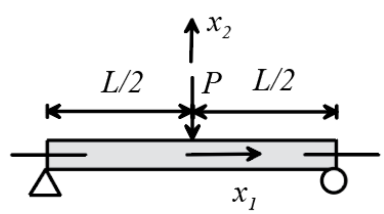


Problem 6.21 The figure shows a statically indeterminate structure. All bars have cross-sectional area A, Young's modulus E and uniaxial tensile yield stress Y. The solid is subjected to a cyclic load P with mean value $\bar{P}$ and amplitude ΔP as shown



(a) Calculate the load that will cause the structure to yield.
(b) Select an appropriate distribution of residual stress in the structure, and hence obtain a lower bound to the shakedown limit for the structure.
(c) Select possible cycles of plastic strain in the structure, and hence obtain an upper bound to the shakedown limit for the structure.
(d) Show the results of (a)-(c) on a graph of ΔP as a function of $\bar{P}$ as discussed in section 6.2.8 of Applied Mechanics of Solids.

Problem 6.22 The figure shows a beam with Young's modulus E, yield stress Y and area moment of inertia I that is subjected to cyclic three point bending. The applied load P varies cyclically with mean value $\bar{P}$ and amplitude ΔP. Assume that the beam has a rectangular cross-section, to keep the problem simple.

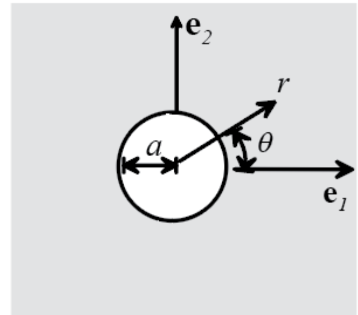


(a) Write down the stress field in the beam, as predicted by elementary beam theory
(b) Find a lower bound to the shakedown limit for the beam (finding a residual stress field that satisfies static equilibrium can be a chore – if you can't think of a better one you could try

$$\rho_{11} = \rho_0(1 - 3x_2 / h) \qquad x_2 > 0 \qquad \rho_{11} = -\rho_0(1 + 3x_2 / h) \quad x_2 < 0$$

where h is the thickness of the beam
(c) To find an upper bound, consider a mechanism which consists of a plastic strain increment $\Delta\varepsilon_{11}$ that occurs at the instant of maximum load in a vanishingly small volume of the beam located at $x_1 = h/2, x_2 = 0$, together with a strain increment $-\Delta\varepsilon_{11}$ at the instant of zero load.

Problem 6.23 The stress state induced by stretching a large plate containing a cylindrical hole of radius a at the origin parallel to the $\mathbf{e}_1$ direction is given by



$$\sigma_{11} = \sigma_0 \left(1 + \left(\frac{3a^4}{2r^4} - \frac{a^2}{r^2} \right) \cos 4\theta - \frac{3a^2}{2r^2} \cos 2\theta \right)$$
$$\sigma_{22} = \sigma_0 \left(\left(\frac{a^2}{r^2} - \frac{3a^4}{2r^4} \right) \cos 4\theta - \frac{a^2}{2r^2} \cos 2\theta \right)$$
$$\sigma_{12} = \sigma_0 \left(\left(\frac{3a^4}{2r^4} - \frac{a^2}{r^2} \right) \sin 4\theta - \frac{a^2}{2r^2} \sin 2\theta \right)$$
$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

where ν denotes Poisson's ratio. Suppose that the plate is subjected to a cyclic stress with $0 < \sigma_0 < \sigma_{\max}$ and has a yield stress Y. Find upper and lower bounds for the shakedown limit for the particular case $\nu = 0.3$.

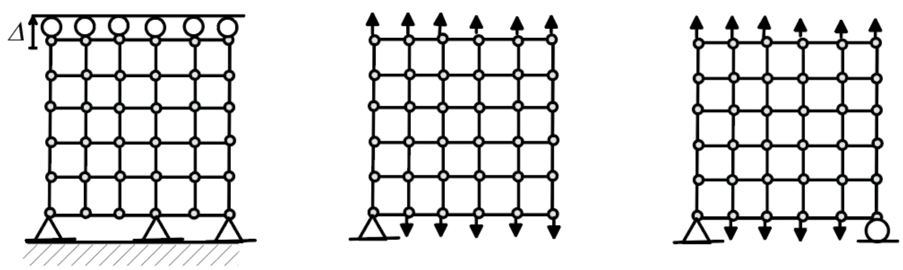
Chapter 7
Introduction to Finite Element Analysis

NOTE: The finite element analysis problems in this chapter have been designed to be solved using the 'leaning edition' of the commercial code ABAQUS/CAE (from Dassault Systemes Simulia). This code can be downloaded and used for non-commercial purposes without cost from http://www.simulia.com.

7.1 A Guide to Using Finite Element Software

Problem 7.1 Please answer the following general questions about finite element analysis

(a) What is the difference between a static and a dynamic FEA computation (please limit your answer to a sentence!)
(b) What is the difference between the displacement fields in 8 noded and 20 noded hexahedral elements?
(c) What is the key difference between the nodes on a beam element and the nodes on a 3D solid element?
(d) Which of the boundary conditions shown below properly constrain the solid for a plane strain static analysis?



(i) (ii) (iii)

(e) List three ways that loads can be applied to a finite element mesh
(f) In a quasi-static analysis of a ceramic cutting tool machining steel, which surface would you choose as the master surface, and which would you choose as the slave surface?
(g) You conduct an FEA computation to calculate the natural frequency of vibration of a beam that is pinned at both ends. You enter as parameters the Young's modulus of the beam E, its area moment of inertia I, its mass per unit length m and its length L. Work through the dimensional analysis to identify a dimensionless functional relationship between the natural frequency and other parameters. You can assume that only the product EI appears in the governing equations, (i.e. E and I are not separate parameters).

Problem 7.2 Please answer the following general questions about finite element analysis

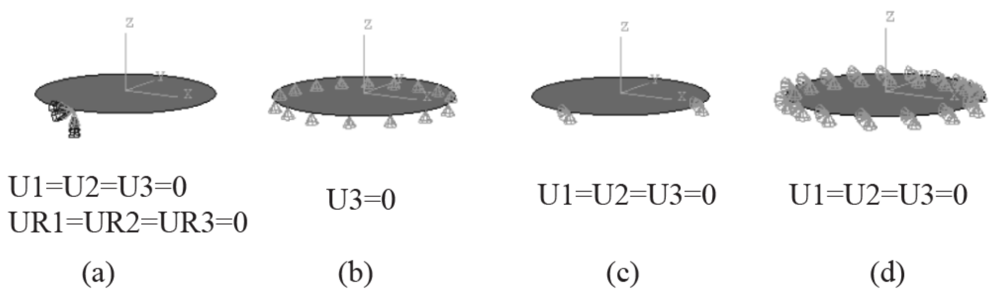
(a) Choose a suitable material model from the list below for each of the following applications:
(i) Calculating the natural frequencies of vibration of a wind-turbine blade
(ii) Calculating the energy absorbed by a crash-rail in a vehicle
(iii) Calculating the deformation of the eye induced by a noncontact 'air puff' tonometer
(iv) Calculating the contact pressure between an elastomeric aircraft door seal and the doorframe

Material models: (a) Linear elastic; (b) Hyperelastic; (c) Viscoelastic; (d) Elastic-plastic

(b) Two linear elasto-static analyses are conducted of the same part, with the same positions and numbers for the nodes, but in one simulation the displacement field is interpolated using linear elements; in the other, quadratic elements are used. Which analysis is likely to give the more accurate solution, and why?

(c) What is the 'default' boundary condition at the exterior of a finite element mesh (i.e. if no boundary conditions are entered for a node/element face, what loading or displacements will be applied in the simulation?)

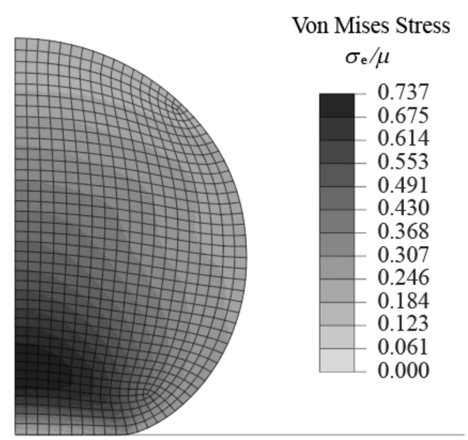
(d) The circular plate shown in the figure is meshed with plate elements for a static analysis (the plate will be loaded by pressure and in-plane traction applied to its surface). The conventions for labeling the constraints are that Un specifies that displacement parallel to basis vector $\mathbf{e}_n$ (where n=1,2,or 3) is constrained; while Urn indicates that rotation about the axis $\mathbf{e}_n$ is constrained at points marked by the arrows. Indicate which of the analyses will fail and which will succeed.



(e) If a dynamic simulation is conducted of the plates described in problem (d) which analyses will fail and which will succeed?

(f) Explain what is meant by (i) 'Newton iterations' in a static nonlinear finite element finite element simulation. List three reasons why Newton iterations may not converge.

(g) The figure shows a finite element simulation of an elastic cylinder with radius R, mass density ρ, Young's modulus E and Poisson's ratio ν colliding with a frictionless rigid surface. The goal of the simulation is to calculate the contact time T as a function of material properties and geometry. Re-write the relationship $T = f(E,\rho,\nu,R)$ in dimensionless form.



Problem 7.3 Please answer the following general questions about finite element analysis

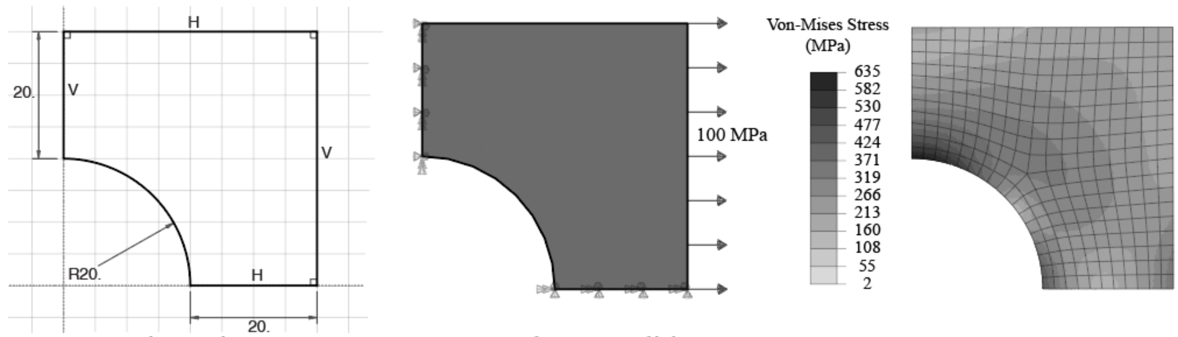
(a) What are the main differences between a small displacement/geometrically linear (NLGEOM OFF in ABAQUS) and a large displacement/geometrically nonlinear analysis (NLGEOM ON in ABAQUS)? (i.e. what approximations are made for NLGEOM OFF)

(b) Give three reasons why a nonlinear static simulation may not converge

(c) Suggest a suitable choice of material model for each of the following applications:
(i) Calculate stresses near the contact between two gear teeth
(ii) Model material removal in an orthogonal machining process

(d) Explain briefly what is meant by a finite element interpolation.

(e) Explain the difference between a static and an explicit dynamic FEA simulation



Dimensions Boundary Conditions Solution

Problem 7.4 The goal of this problem is to set up a basic 2D elasticity simulation in the ABAQUS 'learning edition' finite element software. The figure shows the problem to be solved: a thin plate containing a central hole is stretched horizontally by a horizontal stress applied to the right hand edge. Symmetry boundary conditions are imposed on the left and bottom boundaries, and the top surface is free of tractions. The plate is made from a linear elastic material with Youngs Modulus 210GPa and Poisson's ratio 0.3.

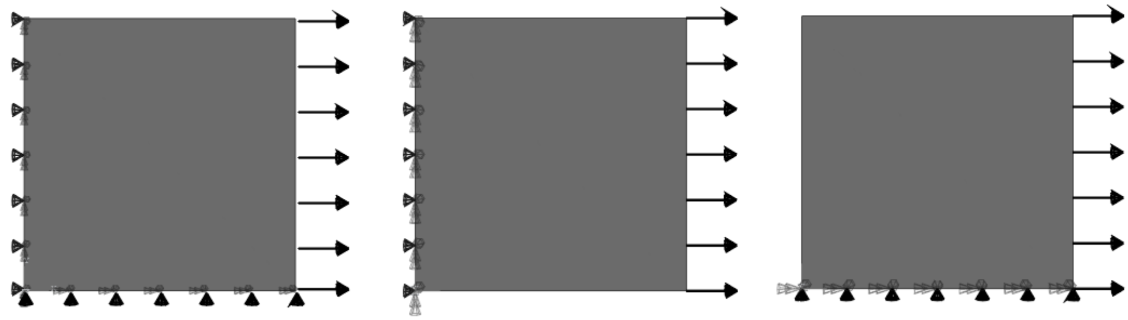
(a) Use the following procedure to set up the simulation:

- Open ABAQUS/CAE and use File> to create a new Model Database with a Standard/Explicit Model.
- The problem will be defined by working through a series of 'Modules' listed in the dropdown menu near the top of the window, starting with the 'Part' module. Use Part>Create on the top menubar to create a new part. Name it 'Plate.' In the options, select '2D Planar, Deformable, Shell, and enter an approximate size of 200 (the units in FEA are arbitrary. We will select mm as the length unit and Newtons as the force unit. The stresses are then in N / mm^2, which is equivalent to MPa).
- Use the sketch window to create a 2D part with geometry and dimensions shown in the figure.
- Save the model database to a file in case of unexpected crashes (repeat after each step!).
- In the dropdown 'Module' menu select 'Property.' Use Material>Create to create a material named 'Linear Elastic.' In the menu select Mechanical > Elasticity > Elastic. Enter 210000 for Young's modulus (the stress units are MPa for this simulation, so this is 210GPa), and 0.3 for Poisson's ratio. Select 'OK.' Next, use Section > Create to create a homogeneous solid section called Plate. (This step might seem a bit strange, but makes sense when you start using beam elements). Select 'Continue;' then select the Linear Elastic material in the dropdown menu and check the 'Plane stress thickness' box and enter 2mm. Select 'OK' Finally use Assign > Section; name the set 'Plate' select the solid and press 'Done.' Press OK in the next menu.
- Move on to the 'Assembly' module. Use Instance > Create; select Part-1; then press OK. Note that by default, you will need to create a mesh on the part, not the instance of the part in the assembly. You can change this if you want to create multiple instances of the part with different meshes.
- In the 'Step' module use Step > Create to create a Static, General load step called Step-1. Press 'Continue;' leave NLGEOM off (this will create a small deformation calculation); accept the defaults for the other options and press 'OK.' In later problems you will need to use the 'Output' menu to control what variables get saved to the output database in your simulation, but the defaults are fine for this problem.
- Skip the 'Interaction' module (this is used to define contacts) and move on to the 'Load' module. Use BC>Create to create a Mechanical symmetry constraint called 'Left Edge' in Step-1; then press 'continue.' Select the left edge of the plate; name the Set 'Left Edge' and press 'Done.' Select Xsymm from the menu, and press OK. Repeat this procedure to create a Ysymm constraint for

the bottom edge. Note that the window will show rotational constraints but these are ignored in the simulation. Next, use Load> Create to create a Pressure named 'Tensile applied stress' and press 'continue.' Select the right edge, name the surface 'Loaded Edge' and press 'done.' Enter a magnitude -100 (MPa) for the pressure and then press OK. At the end of this procedure the viewport should display the boundary conditions shown in the figure on the preceding page.

- Proceed to the Mesh module, and check the 'Part' radio button to specify the 'Object' to be meshed. Then select Mesh>Element type and select the part in the viewport. Select a Quadratic plane stress reduced integration element for both triangular and quadrilateral elements (You will need to scroll down the 'family' window to see the plane stress option. The remaining options can be left unchanged). Then use Mesh>Controls, and select a Structured meshing algorithm with Quad elements. Select Seed>Part and enter an element size of 2mm. Then use Mesh> Part to create the mesh.
- Skip to the Job module, and use Job>Create to create an analysis job named Problem4; press continue and then OK in the next menu. Use Job>Submit and select Problem4 to run the analysis.
- Use File>Set Work Directory to find the name of the directory that ABAQUS uses to store analysis results on your PC. Navigate to this directory in the file explorer of your computer – you will see a large number of files have been created: the .odb file is the 'output data base' with data in a format that can be read and processed by the Visualization module; the .inp file contains the analysis definition in ASCII format. If you open this file with a text editor you will see the raw data defining nodal coordinates, element connectivity; boundary conditions; properties and load definition. This file can be run outside of the ABAQUS GUI using the ABAQUS command prompt. The .sta file contains information about the analysis and will contain error messages if something went wrong. The .msg file lists internal parameters that were used in the analysis.
- To plot the results of your simulation, select the Visualization module in the viewport. Then, select File>Open and load the Problem4.odb file. If you select the icon with color contours next to the viewport you will see a contour plot and an image of the deformed mesh in the viewport. The Legend will tell you that Mises stress has been plotted. The deformation will appear unreasonably large: this is because by default the displacements are magnified to be visible. Select Options>Common and in the Basic tab select Uniform and enter a Value of 1 to see the actual displacements. The result should resemble the contours shown in the figure on the preceding page (but they will be in color on your PC). You can use Result>Field Output to bring up a window containing a list of other variables that can be plotted, or else use the drop-down menus near the top left of the viewport.

(b) Now that your simulation is set up, you can change the analysis by returning to the relevant section in the 'Module' dropdown menu. For example, to change the Young's Modulus, return to the Property module; select Material>Manager, and you will then be able to Edit the Linear Elastic material property that was defined in part (a). Or, if you prefer, you can right-click any of the icons in the model tree to the left of the viewport and edit the relevant information there. You might find it helpful to explore some of the options available in the menus on your own.



Case (a) Case (b) Case (c)

Problem 7.5 This problem uses ABAQUS to illustrate the effects of improper boundary constraints in a static finite element simulation.

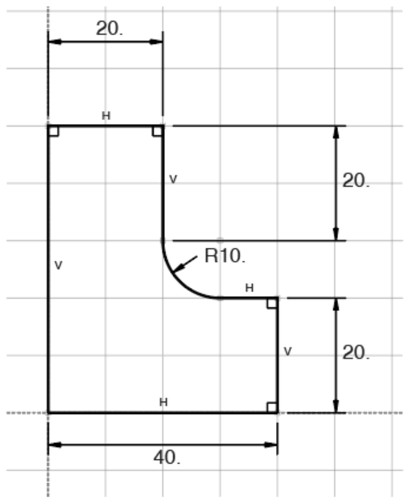
(a) Use the procedure described in problem 7.4 to create a new model database. Create a part that is a simple rectangular plate, as shown in the figure (the dimensions are not important). Assign the plate a homogeneous solid section with a linear elastic material property, with your favorite values for Young's modulus and Poisson's ratio. Create a static/general step with the default options. Mesh the plate with linear plane stress quadrilaterals (the mesh size is not important). For the first simulation, use sensible boundary conditions shown in case (a), with a symmetry boundary on the base and left hand sides of the plate, and a horizontal traction with a sensible magnitude on the right edge. Create a job and run the simulation. The analysis should run successfully. Check the output to make sure.

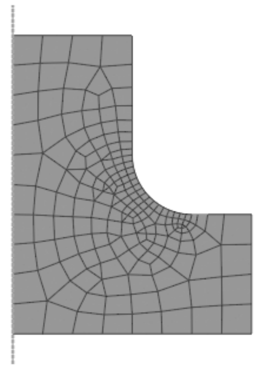
(b) Next, return to the Load module, use BC>Manager to see a list of constraints, and delete the constraint on the base of the plate, as shown by Case (b) of the figure. Run the analysis again. It should complete successfully. That is because although the plate is not fully constrained, an equilibrium solution does exist, and ABAQUS (somewhat fortuitously) has been able to find one. Of course, the solution is not unique, because the plate is free to move vertically.

(c) Repeat (b), but this time delete the constraint on the left edge of the plate, as shown by Case (c). This time the analysis will fail. Open the .msg file (use a text editor) in the ABAQUS working directory – you will see warnings that the equation solver has detected a singular stiffness matrix; and a message that an equilibrium solution has not been found. Open the .sta file to see a summary of the attempts to solve the equation system: you will see that ABAQUS has tried to reduce the time-step to improve the likelihood of finding a solution a few times, but without success.

(d) As a cautionary exercise, return to the 'step' module and use Step>manager to edit the step definition. In the 'Automatic Stabilization' dropdown menu select 'Specify damping factor' and use the default value for the damping. This adds some numerical damping to the equilibrium equations, which can sometimes resolve convergence problems in a sensible nonlinear static analysis. Run the simulation again. It will complete successfully. Check the solution – hopefully you will see immediately that the solution is complete nonsense! FEA analysis clearly needs to be used carefully – a single ill-advised mouse-click can produce misleading results.

(e) If you are curious, try experimenting with a few more boundary constraints – for example, try fixing displacements at some combination of corners of the plate. Try to come up with constraints that work; constraints that have nonunique solutions, and constraints that do not have equilibrium solutions.





Problem 7.6 The objective of this problem is to investigate the influence of element size on the FEA predictions of stresses near a stress concentration. The node limitation in the ABAQUS 'learning edition' restricts the study slightly (and in particular we can only test 2D elements) but the tests will give you a sense of the rules to follow when designing a mesh for a linear elastic analysis.

(a) Set up a finite element model of the axisymmetric stepped bar shown in the sketch.
  (i) Use mm and N for length and force units in your simulations.
  (ii) Use a linear elastic constitutive equation with $E = 210GNm^{-2}$, $\nu = 0.3$.
  (iii) For boundary conditions, use zero radial (horizontal) displacement on the axis of rotational symmetry (the vertical axis), zero vertical displacement on the base, and apply a uniform tensile axial traction with magnitude 100MPa on the top face of the bar. Run a quasi-static computation with NLGEOM off, and the default options for the step.

(b) By running simulations with a series of different finite element meshes with uniform size, and reading the predicted maximum value of the maximum principal stress from the contour plot in the viewport window, complete the table shown below. Note that you can display the location of the maximum stress using the Options>Contour menu in the Visualization module, and checking the box in the 'Limits' tab. FEA tends to underestimate the stress, so the meshes that give the highest stress are the best.

Seed size (mm)	Element type	Meshing algorithm	Max. principal stress (MPa)
4	Linear triangle	Structured tri	
4	Quadratic triangle	Structured tri	
4	Linear quad, reduced int	Structured quad	
4	Quadratic quad, reduced int	Structured quad	
4	Quadratic quad	Structured quad	
2	Linear triangle	Structured tri	
1.25	Linear quad, reduced int	Structured quad	
1.25	Linear quad	Structured quad	
2.5	Quadratic quad, reduced int	Structured quad	
2.5	Quadratic quad	Structured quad	
Custom	See part I	See part I	
Adaptive	See part (d)	See part (d)	

(c) In the Mesh module, delete the part seeds, and use Seed>Edges to design a finite element mesh with a finer mesh size around the fillet. An example mesh is shown in the figure on the preceding page. You should be able to seed the curved edge with a mesh size of order 1mm or less; and use biased seeding on the edges adjacent to the curve to transition to a mesh size of order 5mm on the remaining edges. Use the advancing front quad dominated meshing algorithm, with quadratic elements for both quads and triangles. Uncheck the 'modified formulation' and 'reduced integration' boxes for triangles and quads, respectively. Re-run the simulation and add the results to the table in part (b)

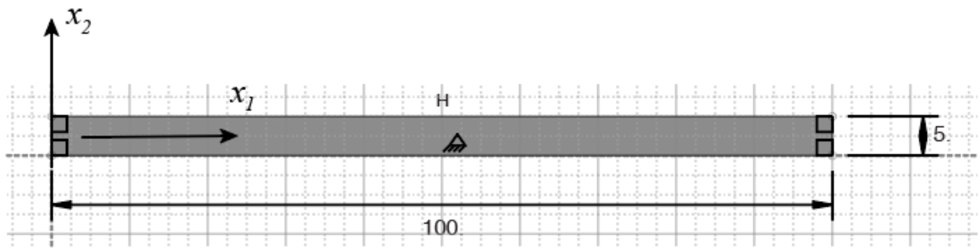
(d) Many FEA packages (including ABAQUS) have the ability to automatically optimize a finite element mesh in some way. It can be tricky to get this to work on a part with a very complex geometry and material model, but there is no problem with a simple part such as the one in this example. To try this feature,

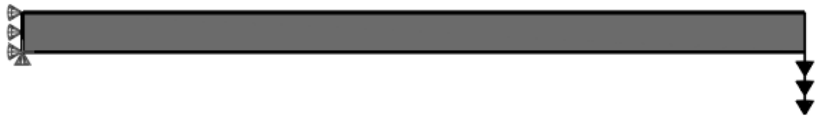
(i) Go to the Mesh module, and delete the mesh and edge seeds that were created in part (c). Then, set the mesh controls to the free advancing front algorithm, the element type to the settings in part (c), and create an initial mesh with coarse quadratic quad dominated elements (a mesh size of 6mm is fine).

(ii) Stay in the Mesh module, and use Adaptivity > Remeshing rule to create a remeshing rule for the simulation. In the 'Step and Indicator' tab select the Mises equivalent stress as the error indicator, and in the 'Constraints' tab check the 'Approx max number of elements box' and enter 200 for the max number of elements (you can use more if you have access to an unrestricted ABAQUS license).

(iii) In the Job module, select Adaptivity > create and set up an adaptivity job with the default options. Then select Adaptivity > submit to start the analysis. ABAQUS will run several times, and you should see the mesh changing in the viewport window.

(iv) To see the results, open the Adaptivity-1-iter3.odb database (or if your simulation took more than 3 iterations use the results of the last one). Find the maximum principal stress and add it to the table in part (a).





Problem 7.7 This problem will continue to explore the influence of mesh design on the accuracy of a finite element simulation, by examining the effects of using a mesh that contains distorted elements.

(a) To begin:

(i) Set up a finite element simulation of the slender beam with length $L=100$ and thickness $h=5$ shown in the figure, using mm and N for units (so stresses are in MPa).

(ii) Assign the beam a plane 2D section with thickness b=5mm (so the cross-section is square), with a linear elastic material with Young's modulus 200 GPa and Poisson's ratio 0.3.

(iii) Impose the boundary conditions shown in the figure: (i) a zero vertical displacement BC on the bottom left corner of the beam; a zero horizontal displacement on the left edge, and a uniform vertical traction of $p_2 = 1$ MPa acting vertically downwards on the right-hand face.

(iv) Mesh the beam with quadratic fully integrated quadrilateral elements (uncheck the 'reduced integration' box), with a mesh size of 1.5mm.

(v) Set up a static/general step with NLGEOM off, using the defaults for remaining options

(vi) For comparison, (approximate) Euler-Bernoulli beam theory predicts that the deflection of the beam's neutral section is

$$w = \frac{2p_2 x_1^2}{Eh^2}(3L - x_1) \qquad \sigma_{11} = \frac{12p_2}{h^2}(L - x_1)x_2$$

Run the analysis and verify that the finite element prediction for maximum stress and deflection of the beam agrees with Euler-Bernoulli theory.

(b) Euler-Bernoulli theory is accurate only if the beam length is significantly greater than its thickness. Repeat the simulations for a few beam lengths between $L=15mm$ and $L=100$mm. You can change the beam length by editing the part sketch. Use a mesh with the same dimensions as part (a).

(c) For a beam with length 100mm, run simulations with meshes listed in the table below (Delete the global seeds, and seed the horizontal and vertical edges of the beam with length ΔL and height Δh to obtain various element shapes). For each mesh, enter the predicted maximum stress and deflection in the table (you can do this quickly by plotting contours of S11 and U2).

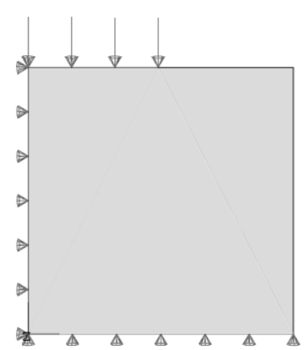
Seed sizes (mm)		Geometry	Order	Options	Max stress	Max deflection
Δh	ΔL				(MPa)	(mm)
1.5	1.5	Quad	Quadratic	-		
1.5	1.5	Quad	Linear			
1.5	1.5	Tri	Quadratic	Modified		
1.5	1.5	Tri	Linear	-		

Continued on next page						
Seed sizes (mm)		Geometry	Order	Options	Max stress (MPa)	Max deflection (mm)
Δh	ΔL					
5	5	Quad	Quadratic	Red. Int.		
5	5	Quad	Quadratic	-		
5	5	Tri	Quadratic	Modified		
5	5	Tri	Quadratic	-		
5	5	Quad	Linear	-		
5	5	Quad	Linear	Red. Int		
5	5	Tri	Linear			
0.5	5	Quad	Quadratic			
0.5	5	Quad	Linear			
0.5	5	Quad	Linear	Red. Int		
0.5	5	Quad	Linear	Inc. Modes		
0.5	5	Tri	Quadratic	Modified		
0.5	5	Tri	Linear			

Problem 7.8 This problem illustrates an issue with choosing elements for near-incompressible materials.

(a) Open (or if you skipped problem 7.4, create) the database for problem 7.4. Change the element type to *plane strain* linear quadrilateral elements with reduced integration off, and edit the material properties to change the Poisson's ratio to 0.4999. Submit the job. It will terminate with an error, because standard elements become very stiff for some modes of deformation if the material is incompressible – a phenomenon known as 'pressure locking.' ABAQUS uses 'B-Bar' elements (see chapter 8 of Applied Mechanics of Solids) designed to resist locking (and in fact will run successfully with this particular problem), but it is better to use hybrid elements for near incompressible materials, as indicated in the warning.

(b) Switch the element type to plane strain hybrid, and run the analysis. Compare the predicted maximum Von-Mises with the plane stress solution in problem 7.4.



Problem 7.9 This problem illustrates a phenomenon called 'hourglassing' that occurs in certain elements. Set up the analysis as follows

(i) Create a 50mmx50mm square block. After creating the part, use Tools>Partition to partition the top edge somewhere near its center, as shown in the figure. Assign the part a homogeneous solid section with a linear elastic material, using a Young's modulus of 100GPa and Poissons ratio of 0.25. Create an instance of the part, and create a static step with all the default options. Add the boundary conditions shown in the figure (use a pressure of 100MPa on the top edge).

(ii) Mesh the part with plane strain linear reduced integration quadrilaterals, but in the 'Element Control' menu check the radio button for 'Stiffness' next to the Hourglass control options, and then check the 'Specify' radio button next to the Hourglass Stiffness option and enter 1.e-05 for the stiffness value.

Run the analysis and plot the deformed mesh with the default option for the deformation scale factor. The deformation will be scary! Reduce the scale factor to 3 (use Tools>Common) to see the houglassing more clearly. Plot contours of Von-Mises stress – you will find they look reasonable.

Problem 7.10 This problem runs some simple uniaxial tensile tests on a 'Linear Elastic' material in ABAQUS, and illustrates the effect of assigning an 'orientation' to a part.

(a) To begin

(i) Create a 3D extruded tensile bar with a 1mmx1mm square cross section in the (x,z) plane, and a length of 2mm in the y direction. Create an instance of the part, and mesh it with a single 8 noded reduced integration element

(ii) Create a homogeneous solid section made from a linear elastic material with a Young's modulus of 100GPa and a Poisson's ratio of 0.25. (Work with units of N and mm, as usual). Assign the section to the part.

(iii) Create a general static step with time period of 1s with Automatic time increment of 1s, and initial and maximum increment sizes of 1s, and the NLGEOM parameter off. Create both a field and a history output request that will record the total strains E, elastic strains EE and the stresses S in the bar.

(iv) Apply boundary conditions to the specimen that will prevent vertical motion of its base, and which will prevent rigid body motion of the specimen, but which leave it free to strain in all direction. Apply a 100MPa tensile pressure to the top face.

(v) Run the analysis and (assuming the job completes successfully!) open the ODB in the visualization module.

(vi) Probe the stresses and strains in the element by using Result>Field output to select the variable you wish to see, then use Tools>Query and select the 'Probe values' option in the Query menu. Select Elements in the Probe Values menu, All Direct for the components, and Integration Point for the position. Then click on the element to see the values of your variables.

Find the stress and strain values predicted by ABAQUS, and compare them to the exact solution.

(b) Return to the 'step' module and edit the step to activate the NLGEOM parameter (this will do a finite strain calculation). Run the analysis again, and check the variables that are available in the Result>Field Output menu. You will find that the strain measure EE that is used in a geometrically linear simulation has been replaced with LE, the logarithmic strains. Find the predicted logarithmic strains.

(c) The stress components in part (a) are all displayed as components in the fixed, global coordinate system. This is not always the most convenient choice, and can be changed by assigning an 'orientation' to the part (or the instance of the part, if you prefer). Test this as follows:

(i) De-activate the NLGEOM flag in the 'step' module (and leave it inactive for the rest of this problem)

(ii) Return to the 'property' module and create a new set of basis vectors as follows: (i) select Tools>Datum…; (ii) check the CSYS radio button and select 3 points, (iv) enter a name for the coordinate system and select 'Rectangular' in the CSY creation menu, then enter (0,0,0); (1,1,0), (-1,1,0) for the 3 points. The new basis vectors will be displayed on the part.

(iii) Select Assign>Material Orientation, select the part, then select Datum CSYS List and select the new coordinate system you just defined. Press OK. A final menu will come up that allows you to apply an additional rotation to the coordinate system. Leave the 'None' radio button checked and press OK again.

(iv) Run the analysis again, and open the new ODB in the visualization module.

Check that the stress components (which are now in the local coordinate system) are predicted correctly by ABAQUS.

(d) Once a material orientation has been assigned to a part, vector and tensor quantities relevant to the part will all be displayed as components in that coordinate system. The coordinate system is attached to the material, and if a material point rotates, the coordinate system will rotate with it. To reset stress and strain components to the fixed global basis, you need to suppress or delete the coordinate system attached to the part (or its instance). To do this, go to the model tree to the left of the viewport, expand the 'part' and its 'Orientations', then right-click the coordinate system and select 'suppress.' Run the analysis again, and confirm that the stresses are computed in the global coordinate system.

(e) If you run simulations with anisotropic materials, you always need to specify the basis that is used to define vector or tensor valued material properties. This is done by assigning a material orientation to the part, using the same procedure outlined in part I. **This will also cause vector and tensor quantities to be displayed in the basis associated with the material**. Run a preliminary test of the anisotropic elastic constitutive law in ABAQUS as follows:

(i) 'Resume' the coordinate system that was suppressed in part (d)

(ii) Open the material manager, edit the material definition and change 'isotropic' to 'Engineering constants.' The variables in the table follow the definitions in Section 3.2.14 of Applied Mechanics of Solids. Enter values that will make the material isotropic, with a Young's modulus 100GPa and Poisson's ratio 0.25.

Run the analysis again, and check that ABAQUS reports the values you would expect for the strain components.

(f) Finally, run simulations that predict the strains in a gold specimen with crystallographic directions parallel to the coordinate system created in step (c) (elastic properties of gold are listed in Table 3.5 of Applied Mechanics of Solids – but note that E,ν are displayed to only 2 decimal places). Check the predictions using both the 'Engineering Constants' and the 'orthotropic' material options. Then (if you happen to have the resources to do so), check your prediction experimentally, and mail your used tensile specimen as a gift to the author.

Problem 7.11 This problem illustrates the 'hyperelastic' material in ABAQUS with a simple tensile test. Set up the analysis as follows:

(i) Create the same tensile specimen used in problem 7.10 (or you can copy and edit the model database – but if you do this be sure to suppress the 45 degree 'orientation' for the part).

(ii) Create (or edit) the material to assign an isotropic hyperelastic material behavior with a Neo-Hookean strain energy potential, and select 'coefficients' for the input source. Use a value of 0.5GPa for C10, and leave the data box for the coefficient D1 blank. The coefficient C10 is related to the shear modulus defined in Section 3.5.5 of Applied Mechanics of Solids by $\mu = 2C_{10}$, and if D1 is left blank, ABAQUS will use a value that makes the material approximately incompressible.

(iii) Assign the section to the part. If you created the material by editing a copy of the database from problem 7.10, you will need to use Section>manager to edit the section definition and select the new material; and use Section>Assignment manager to delete the old section assignment, and use Assign>Section to assign the modified section to the tensile bar.

(iv) Create a static step with time period 1, the default time incrementation method, and NLGEOM on.

(v) In the 'step' module use Set>Create to create a set the contains the one element in the mesh. Then, use 'Result>History Output' to create/edit a history output request for the total strains E; elastic strains EE; logarithmic strains LE and nominal strains NE, as well as stresses S for this set

(vi) Impose the same boundary conditions on the base of the tensile bar as problem 7.10, but instead of applying a pressure to the top face, apply a vertical displacement that will stretch the bar to a length of 6mm (a stretch of 3).

(vii) Mesh the part with a single 8 noded hybrid reduced integration element (incompressible hyperelastic materials can only be modeled with hybrid or plane stress elements).

(viii) Run the analysis and use Job>Monitor to see how the calculation has proceeded. You will see that the first attempt failed to converge, and ABAQUS automatically reduces the time increment to 0.25s. After that, several Newton-Raphson iterations are used to find the solution at the end of subsequent increments. This is because the equilibrium equation and the material constitutive law are both nonlinear.

(ix) Finally, return to the step module and change the time increment to a fixed time step of 0.1s.

Use your simulation to plot a graph of the Cauchy stress (the stress measure reported by ABAQUS) as a function of stretch (the principal nominal strains in ABAQUS are related to the principal stretches by $\varepsilon_i^{(nom)} = \lambda_i - 1$), and compare the prediction to the exact solution.

The easiest way to do this plot is as follows:

Open the output database in the visualization module. Then, use Result>History Output to plot the Cauchy stress S22 as a function of time. Next, use Tools>XYData>Edit and select the dataset listed in the tab. This will bring up two columns of data. You can click on the column you are interested in (the second column in this case) and cut and paste it into a spreadsheet or text file. You can repeat this process to extract the nominal strain NE22 (or maximum principal strain). Then you can use your favorite data analysis program to read the spreadsheet, manipulate the data, and plot the graph.

You can also (more painfully) do the plot in ABAQUS as follows:

After running the analysis again, use Tools>XYData>Manager and select 'Operate on XYData' to create, and then save Cauchy stress and nominal strains to datasets. Then use the 'operate on XYData' feature again to create a dataset with the principal stretch. Finally, use the combine("*x data*","*y data"*) operator in the top window to create a new dataset with principal stretch on the X axis and force on the Y axis. You can then plot the data. **Warning:** there is no way to permanently save the databases you create using these menus in ABAQUS – once you close the program they will disappear and you need to recreate them.

Problem 7.12 This problem illustrates the 'plastic' material in ABAQUS with a simple tensile test.

(a) Set up the analysis as follows:

(i) Create the same tensile specimen used in problem 7.10 (or you can copy and edit the model database – but if you do this, be sure to suppress the 45 degree 'orientation' for the part).

(ii) Create (or edit) the material to assign an elastic-isotropic hardening plastic material to the tensile bar (you need to create both 'elastic' and 'plastic' properties for the material) with Youngs modulus 100GPa, and stress/plastic strain points (500,0); (550,0.1); (600,0.2),(650,0.4)

(iii) Assign the section to the part. If you created the material by editing a copy of the database from problem 7.10, you will need to use Section>manager to edit the section definition and select the new material; and use Section>Assignment manager to delete the old section assignment, and use Assign>Section to assign the modified section to the tensile bar.

(iv) Create a static step with time period 4s, a fixed time increment of 0.05s, and NLGEOM on. Also, create/edit a history output request for the total strains E; elastic strains EE; plastic strains EP and the equivalent plastic strain PEEQ as well as stresses S.

(v) Impose the same boundary conditions on the base of the tensile bar as problem 9, but change the boundary condition applied to the top face to apply displacement U2 with magnitude 1, and create an 'amplitude' (click the icon to the right of the drop-down menu) with tabular data (0,0), (1,1.644), (3,-0.902), (4,0). This will subject the bar to a cycle of 60% true strain in tension and compression.

(vi) Mesh the part with a single 8 noded reduced integration element (make sure the 'hybrid' option is not selected).

(vii) Run the analysis and use Job>Monitor to see how the calculation has proceeded. You will see that Newton-Raphson iteration is used to solve the equilibrium equations, but because the time step is short, only a few iterations are needed at each increment.

Plot the true stress-v-true strain curve for the bar (the true strain is the 'logarithmic' strain in ABAQUS, and Cauchy – or true – stress is default stress measure). Explain the shape of the curve (you can change some of the input data to check that your explanation is correct).

(b) Return to the Property manager, and click the 'Suboptions' tab for the 'Plastic' material property. Select 'Rate Dependent' and use the 'Power Law' option. Enter 0.5 for the 'multiplier' $\dot{\varepsilon}_0$ and 10 for the 'exponent' n. The power-law rate dependence in ABAQUS relates the effective plastic strain rate to the effective stress by

$$\dot{\varepsilon}_e^p = \dot{\varepsilon}_0 \left\langle \frac{\sigma_e}{Y(\varepsilon_e^p)} - 1 \right\rangle^n$$

where Y is a function representing the tabular yield stress-v-strain curve and $<x>=x$ for $x>0$ and zero otherwise. Run the analysis with the rate dependent model and add a graph of stress-v-strain to your plot in part (a).

(c) Return to the Property manager, and create a new material with an elastic-Kinematic hardening plastic property, with Young's modulus 100GPa, Poisson's ratio 0.25, and yield stress-plastic strain data (0,550), (0.4,700). (you can only enter 2 pairs of points for kinematic hardening. The two points are used to define the hardening rate in a linear kinematic hardening law). Add the stress-strain curve for the new material to the graph in (a). Explain the ABAQUS prediction.

Problem 7.13 This problem illustrates the 'viscoelastic' material in ABAQUS with a simple tensile test.

(a) Set up the analysis as follows:

(i) Create the same tensile specimen used in problem 10, but edit the boundary conditions to apply a step compressive pressure of 100MPa to the end of the bar instead of the prescribed displacement in problem 10.

(ii) Create a viscoelastic material defined in the time domain with a Prony series. ABAQUS uses the following forms for the Prony series for shear and bulk modulus:

$$G(t)=\frac{E}{2(1+\nu)}\frac{1}{\beta_G}\left\{1-\sum_{i=1}^{N}g_i\left[1-\exp(-t/\tau_i)\right]\right\}$$

$$K(t)=\frac{E}{3(1-2\nu)}\frac{1}{\beta_K}\left\{1-\sum_{i=1}^{N}k_i\left[1-\exp(-t/\tau_i)\right]\right\}$$

where E and ν are the Young's modulus and Poisson's ratio entered via the 'elastic' properties of the material, $\beta_K=\beta_G=1$ if the 'Instantaneous' option is selected for the moduli time scale, and $\beta_G=1-\sum g_i$ and $\beta_K=1-\sum k_i$ if the 'Long-term' option is selected. In your simulation use 10GPa for Young's modulus, 0.25 for Poisson's ratio, select 'instantaneous' for the time-scale, and use $g_1=0.9,\ \tau_1=0.25s,\ k_1=0$ for the Prony series.

(iii) ABAQUS uses a time stepping option called 'Visco' instead of the Static/General step to model viscoelastic materials. If you set up your database by copying and editing one created in problems 8-10, use 'Replace' in the step module and select 'Visco' from the step menu (it is at the bottom of the list). If you created the database from scratch, just create a new 'Visco' time step. Use a time period of 1s with a fixed time increment of 0.05s.

(iv) Create a history output request that will save the total axial strain EE22 to the output database.

Plot a graph showing the ABAQUS prediction for the axial strain in the bar as a function of time.

(b) Calculate (by hand) the expected variation of strain with time for the problem analyzed with ABAQUS (you may find the solution to problem 3.33 helpful). Add the prediction to your graph in (a).

Problem 7.14 This problem illustrates the 'creep' material option in ABAQUS with a simple tensile test. ABAQUS offers too many different creep laws to review them all in detail here. The 'Power Law' model is closest to the creep law discussed in Section 3.8 of Applied Mechanics of Solids. ABAQUS models creep as a source of strain that acts in addition to elastic and plastic straining, so the total strain rate is decomposed as $\dot{\varepsilon} = \dot{\varepsilon}^{e} + \dot{\varepsilon}^{p} + \dot{\varepsilon}^{cr}$, where (for the power law model) the creep strain rate is related to the Von-Mises effective stress σ_e by

$$\dot{\varepsilon}_{ij}^{cr} = \dot{\varepsilon}_0 \left([m+1] \bar{\varepsilon}^{cr} \right)^{m/(m+1)} \left(\frac{\sigma_e}{q_0} \right)^{n/(m+1)} \frac{S_{ij}}{\sigma_e}$$

Here S_{ij} is the deviatoric stress; $\bar{\varepsilon}^{cr}$ is the accumulated creep strain, which has a rate $\dot{\varepsilon}_{cr} = \sqrt{2\dot{\varepsilon}_{ij}^{cr}\dot{\varepsilon}_{ij}^{cr}/3}$, and $(\dot{\varepsilon}_0, q_0, m, n)$ with $n > 1$, $-1 < m \le 0$ are material constants. A second version of the same creep law allows you to enter a parameter $A = 1/\left[\dot{\varepsilon}_0^{(m+1)} q_0^n \right]$ instead of the constants $(\dot{\varepsilon}_0, q_0)$ (this option is called the 'Strain Hardening' model).

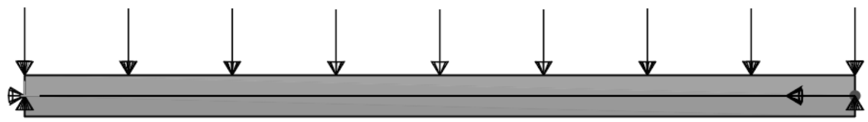
(a) Set up an analysis to test the 'Power' creep model as follows:
   (i) Create the same tensile specimen used in problems 7.10-7.13, but edit the boundary conditions to apply a velocity of 0.1mm/s to the end of the bar
   (ii) Assign a homogeneous solid section with an elastic-Power creep material model to the part, using a Youngs modulus of 100GPa, Poisson's ratio of 0.25, $\dot{\varepsilon}_0 = 0.0001\text{s}^{-1}$, $q_0 = 100$ MPa, *n=4*, *m=0*.
   (iii) ABAQUS uses a time stepping option called 'Visco' instead of the Static/General step to model creep (you can also use 'coupled temperature-displacement'). If you set up your database by copying and editing one created in problems 8-10, use 'Replace' in the step module and select 'Visco' from the step menu (it is at the bottom of the list). If you created the database from scratch, just create a new 'Visco' time step. Use a time period of 1s and initial increment size of 0.001s with Automatic time integration. The time stepping scheme used by ABAQUS may be unstable if the time increment is too large (the creep strain increment needs to be comparable to or less than the elastic strain increment). A 'strain error tolerance' must be entered to control the automatic time-step. For a first guess, you can try setting this parameter to $\sigma/(100E)$, where σ is the expected stress level and E is Young's modulus. If this gives strange results, try reducing it. For this problem we expect the stress to be of order 100MPa, so use 10^{-4} as an initial guess for this parameter.

   Run the analysis, and plot the variation of axial stress in the bar with time.

(b) Compare the FEA prediction in part (a) with the exact solution (see problem 3.56)

(c) Return to the 'Step' menu and change the tolerance value to 10^{-2}. Run the analysis again, and add the predicted stress to the plot in part (a).

Partitioned beam part



Boundary conditions

Problem 7.15 The goal of this problem is to show the effects of accounting for finite changes in the geometry of a component during a finite element analysis (this is controlled by the NLGEOM parameter in ABAQUS). As an example, you will calculate the deformed shape of a beam that is subjected to a small transverse load, together with a large axial load, with, and without, the NLGEOM parameter selected.

(a) Set up the analysis as follows:

(i) In a new model database, create a part that consists of the beam with geometry described in Problem 7.7. After exiting the sketch, use Tools>Partition in the Part module, select the 'face' radio button, and select the Sketch method. Then draw a rectangle that divides the beam into two parts down its mid-plane, as shown in the figure above (this is to make sure that the finite element mesh contains a node on the neutral plane at each end of the beam, so it can be loaded.

(ii) In the Property module assign the part a homogeneous solid section with out-of-plane thickness 5mm, and a linear elastic material property with Young's modulus 200GPa and Poisson's ratio 0.3.

(iii) Create an instance of the part in the Assembly module

(iv) In the Step module, create a General Static step named 'Preload' with the default options (this will have NLGEOM off). Then create a second step named 'Buckling' with a time period 1, and initial and maximum increment sizes of 0.1 (arbitrary units). To begin with, set NLGEOM on for the second step. Also, create a history output request that will save the displacement and concentrated force acting on the loaded reference point at the end of the bar for plotting later.

(v) In the 'Load' module, create boundary conditions that set U1=U2=0 on the neutral plane at the left hand end of the beam, and U2=0 on the neutral plane at the right hand end of the beam. Next, create a Load in the 'preload' step that applies a pressure with magnitude 1MPa on the top face of the beam (this will cause the beam to bend slightly before the axial load is applied). Make sure this load is propagated to the second step in the step manager menu. Then create a second load that applies a concentrated force with magnitude CF1=-20000N to the neutral plane at the right hand end of the beam. The axial load should be applied only in the 'Buckling' step. After this is completed, the loads should be displayed as shown the figure above.

(vi) In the Mesh module seed the part (or instance) with a mesh size 2.5mm, then mesh the part with quadratic plane stress reduced integration elements (the entire part can be meshed at once).

(vii) Create a Job with an inspirational name of your choice, and submit the analysis.

After the job has completed:

(viii) In the Job module use Job>Monitor to examine the time stepping history selected by ABAQUS to complete the analysis. You should see that at a time of about 1.5sec, the time step is cut back significantly. This is because the beam buckles, and the resulting rapid change in the geometry of the beam reduces the radius of convergence of the Newton iterations.

(ix) Plot the deformed shape of the beam at the end of step 2, and check that the beam has buckled.

(b) Plot the predicted load-v-displacement curve for the reference point at the end of the bar. Hence, compare the predicted buckling load with the Euler prediction (see Chapter 10 of applied mechanics of solids to review this topic)

(c) Next, return to the 'step' module and use Step>Manager to turn off NLGEOM for step 2 of the analysis. Examine the deformed shape of the beam, and repeat part (b) to plot the load-v-displacement curve.

Problem 7.16 This problem illustrates an explicit dynamic simulation in ABAQUS by analyzing plane wave propagation through a linear elastic solid.

(a) Set up the problem as follows:

(i) Create a 2D planar deformable part that has the shape of a rectangle with length 100 and height 1 (arbitrary units). Assign the part a homogeneous solid section with a Youngs modulus of 1000, Poisson's ratio of 0.25, and density 0.12 (arbitrary units). Create an instance of the part.

(ii) Create an Explicit Dynamic step with time period of 4 and defaults for all other options. Also, create a 'set' that contains the surface at the right hand end of the bar, and create a history output request that will save the horizontal velocity (U1) of the set at every time increment.

(iii) Set the vertical displacement on the top and bottom of the rectangle to zero, and apply an instantaneous pressure of 1 unit to the left hand end of the bar.

(iv) In the 'mesh controls' menu, make sure the 'explicit' radio button is selected, and select linear plane strain quadrilaterals as the element type (notice that the element library is much smaller for an explicit dynamic analysis). Seed the top and bottom edges of the bar with a 0.5 unit mesh size, the left and right edges with a 1 unit mesh size, and mesh the part (there should be one element through the thickness of the bar).

(v) Create a job with all the default options, and run it.

(vi) Use Job>Monitor to see how the analysis has proceeded. You will see that no Newton-Raphson iteration is used, and ABAQUS has taken a large number of very small time increments. If you animate the time history of stress, you will see a plane wave propagating down the bar and reflecting off the ends (only a small number of frames are written to the output database by default – you can edit the history request to request more frames to smooth out the animation. This will slow down the simulation, as saving data to permanent memory is slow).

(b) Calculate (by hand) the longitudinal wave speed in the bar. Calculate the ratio of the 'stable time step' in the job monitor to the time required for the wave to propagate through one element in the mesh.

(c) Plot a graph showing the variation of velocity at the end of the bar with time, and plot (on the same graph) the exact solution.

(d) As a test, return to the 'Step' menu and use Step>Manager to edit the step. Change the incrementation to 'Fixed' with a 'User defined time increment' of 0.0048 units. Also, create a set for the left end of the bar, and a history output request that will save the velocity at the left end of the bar at every increment to the output database. Plot the velocity of the left end of the bar as a function of time (you will see the instability caused by the time-step exceeding the limit for stability).

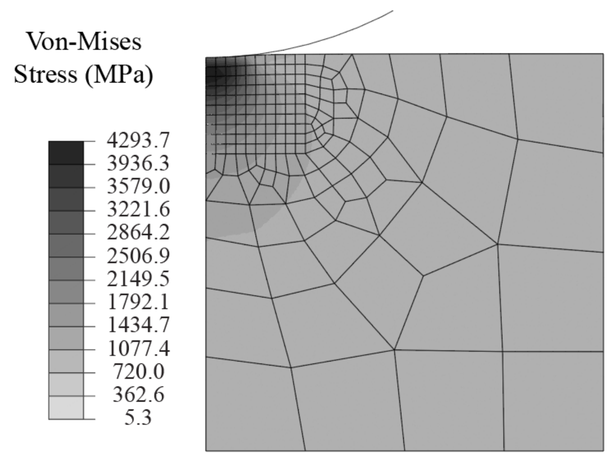
(e) Sometimes (although not in this example!) an explicit dynamic simulation can be speeded up by using 'mass scaling.' Since the stable time-step is determined by the time for a wave to propagate through the smallest element in the mesh, increasing the mass density will increase the stable time-step by reducing the wave speed. Mass scaling is a way to do this. Mass scaling is usually used in two situations: (i) when the mesh contains a few small elements in areas that are not of great interest (mass scaling is then applied only to the small elements); and (ii) to model quasi-static deformation in dissipative materials (e.g. plastic metal forming) where elastic wave propagation plays no role. Test mass scaling in ABAQUS as follows: return to the 'step' menu and in the Mass Scaling tab select 'Use scaling definitions below,' then use 'Create…' to open the mass scaling menu. Apply mass scaling to the whole model and at the beginning of the step. Select 'Scale by factor' and enter 4, then select OK. Change the step back to automatic time incrementation. Press OK and make sure the step menu has closed (otherwise the changes will not take effect) and run the analysis again. What would you expect the mass scaling to do to the wave speed and time step? Plot the velocity of the right hand end of the bar to confirm your prediction.

Problem 7.17 This problem demonstrates the 'implicit dynamic' time integration method in ABAQUS. Set up the problem defined in problem 7.16 (or copy the database).

(a) If you copied the database, go to the step module, and use the step manager to redefine the step to an implicit dynamic analysis; otherwise, create a new implicit dynamic step. Set the time period of the analysis to 4 units, and set the time increments to 'fixed' with a step size of 0.00421 units, and change the maximum number of time increments to 1000. Run the analysis (notice that it takes a very long time – this is because the implicit dynamic algorithm solves a large system of equations at each timestep) and plot the velocity of the end of the bar

(b) The implicit dynamic algorithm is clearly unsuitable for capturing deformation at the short time scales associated with elastic wave propagation. It is the only method capable of modeling dynamics at long time scales, however. Show this by editing the step to select Automatic time integration, and changing the analysis time to 1000units, and the initial increment size to 1unit. Monitor the job to watch the analysis proceed – you will see that the analysis takes large time steps, and uses Newton iteration to solve the equations of motion at each increment. Plot the velocity of the bar as a function of time, and compare it to the exact solution (from particle dynamics)

Problem 7.18 This problem illustrates how to use a rigid surface and surface interactions to analyze contact in ABAQUS. ABAQUS has a large library of numerical contact algorithms which can't be described in full here. This problem will test one of the options.



(a) Set up the problem as follows

(i) Create a part that is an axisymmetric analytical rigid surface, which has the shape of a quarter-circle with center at (0,10)mm and radius 10mm. After exiting the sketch use Tools>Reference point to create a reference point located at the origin (the point will move with the part even though it is not located on the surface).

(ii) Create a second part that is an axisymmetric deformable solid, which has the shape of a square with side length 10mm, with corners located at (0,0) and (10,-10)mm. After exiting the sketch use Tools>Partition>Face to create a square partition with side length 2.5mm with one corner coincident with the top left corner of the part.

(iii) Assign the second part a solid section with a linear elastic material property with Young's modulus 100GPa and Poissons ratio 0.25

(iv) Create instances of both parts in the assembly module.

(v) Create a Static/General step with NLGEOM on and a time period of 1s. Use automatic time incrementation with initial and maximum increment sizes 0.1s. Create a history output request that will record the vertical displacement and vertical force acting on the reference point. In addition, create a 'Set' that consists of the top surface of the partitioned region of the solid, and create a history output request for the contact stresses for this set., and the total vertical force due to contact pressure.

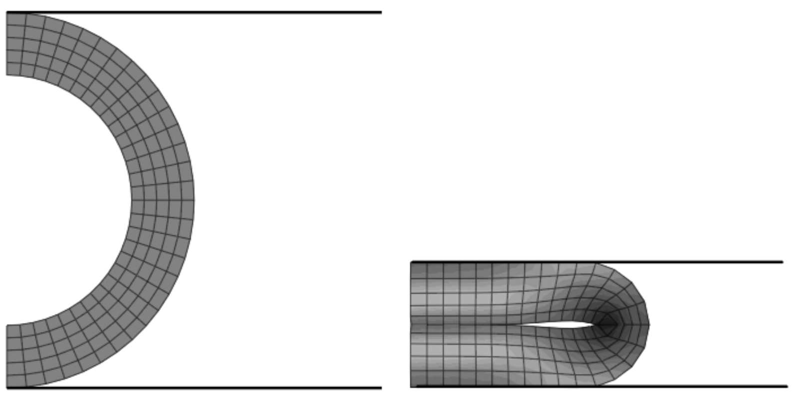
(vi) In the Interaction module use Interaction>Property to create a contact property called Frictionless_Hard_Contact with the default Hard contact model for normal contact in the 'Mechanical' menu and the frictionless option for tangential behavior. Then create an Interaction called 'Sphere_Flat' using the 'Surface-to-Surface' option (General Contact will work as well – the other option gives a slightly different set of menus, and has some capabilities for identifying contacting surfaces automatically). Select the rigid sphere as the 'main' (or 'master' in old terminology) with the arrow pointing to the outer surface of the sphere when you choose the direction. You can select either 'surface' or 'node region' for the secondary surface, and then select the entire top surface of the block in the viewport. Accept the defaults in the interaction menu.

(vii) Seed the edges of the square partitioned region with a 0.25mm mesh size, and seed the bottom and right edges of the full region with a 2.5mm mesh size. Finally, add biased seeds to the top and left edges of the outer region with 0.5mm size near the origin and 2.5mm size far from the origin. Create a mesh with quadratic quadrilateral reduced integration elements.

(viii) Apply boundary conditions to the deformable solid that prevent radial motion of the axis of symmetry and vertical motion of the base. Set the horizontal displacement and rotation (UR3) of the reference point on the rigid surface to zero, and apply a 15kN force vertically.

Plot contours of von-Mises stress in the solid and find the maximum value. Also, plot the force-v-displacement curve for the reference point on the sphere. Finally, find the maximum contact pressure at the end of the step (you can plot a line with color contours of pressure by selecting the CPRESS variable in the viewport; and the maximum value can be displayed in the viewport as with any contour plot).

(b) Compare the FEA predictions with the classical Hertz solution for a rigid sphere indenting a linear elastic half-space (see Chapter 5 of Applied Mechanics of Solids).



Problem 7.19 This example illustrates a more challenging contact problem by analyzing the compression of a hollow hyperelastic sphere between two rigid surfaces. The inner wall of the sphere contacts itself towards the end of the simulation. Set up the problem as follows:

(i) The sphere has inner radius 10mm and outer radius 15mm.

(ii) The material is a (near) incompressible neo-Hookean solid with shear modulus of 1GPa.

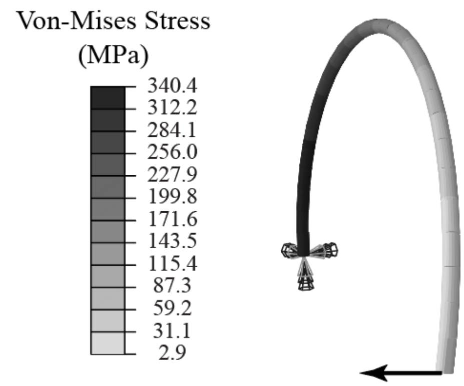
(iii) The sphere is meshed with linear reduced integration hybrid quadrilateral elements.

(iv) The sphere is compressed between two rigid surfaces; and the friction coefficient at all contacting surfaces is 0.25. (use the 'Penalty' option to specify the tangential interaction, and use the 'self-contact' option to account for the contact between the interior wall of the sphere.

(v) The bottom rigid surface is held fixed, and the top is subjected to a 20mm vertical displacement.

Plot a graph showing the force acting between the rigid surfaces as a function of their relative displacement.

Problem 7.20 This problem illustrates beam elements in ABAQUS.



Set up the problem as follows:

(i) Create a part that consists of a 3D wire feature that has the shape of a 180 degree semicircular arc with radius 20mm, with one end at the origin and its midpoint at (20,20) mm.

(ii) In the 'Property' module create a linear elastic material with Young's modulus 100GPa and Poisson's ratio 0.25. Use Profile>Create to create a circular cross section for the beam with radius 0.5mm. Create a 'Beam' section with the profile and material you just created.

(iii) Still in the 'Property' module, use Assign> Beam section orientation and when prompted enter (0,0,-1) for the direction of the n1 vector. This specifies the orientation of the beam's cross section (since the cross section is circular, this has no effect, but more generally the orientation will specify the directions of the principal axes of inertia). Finally, assign the beam section to the part.

(iv) Create an instance of the part, and a step with NLGEOM on, time period 1s, and initial increment size of 0.1s.

(v) Create a static step with NLGEOM on, a time period of 1s, and automatic incrementation with a 0.1s initial increment size.

(vi) For boundary conditions fix all the degrees of freedom (3 displacements and 3 rotations) at the left end of the beam, and apply a concentrated force with magnitude 1N in the out-of-plane direction (CF3) on the right end of the beam.

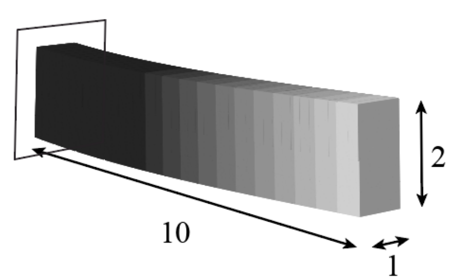
(vii) Seed the part with a 2mm element size, and select a linear Beam element with the default options for the element type. Mesh the part.

(viii) Create a job and run the analysis.

(ix) The deformation and stress in the beam is displayed more clearly if you select View>ODB display options and check the box to display the beam profile.

Calculate the displacement and rotation at the loaded end of the beam (you can write all the nodal displacements and rotations to a file by selecting Report>Field Output; selecting Unique Nodal for the position, and using the Setup tab to select the file name and format for the output. Then select 'Apply.' The file can be opened with a text editor).

Problem 7.21 This problem tests some of the options available in beam elements.



(a) Set up a model database that will predict the static deflection of the end of a cantilever beam with length 10 units and a rectangular profile with height 2 units and breadth 1 unit; clamped at one end, and subjected to a concentrated force of 1 unit on the other end. Create a step with NLGEOM off, time period of 1, automatic time integration with an initial step size of 1.

(i) When you create the section for the beam, select the 'Before analysis' radio button for section integration. This causes ABAQUS to calculate internal moments in the beam using stiffnesses calculated before the step. The option can only be used for linear elastic beams. The default 'During analysis' option can be used for any material.

(ii) The section definition menu now allows you to enter a value for Youngs modulus and shear modulus directly. Enter a value of 500 units for Young's modulus, and 200 for shear modulus (the shear modulus controls the shear stiffness of the beam, and is only used for certain choices of beam element, as the tests in this problem will demonstrate). You can enter 0.25 for Poisson's ratio, but the Poisson's ratio is used to account for changes in the shape of the cross-section in a geometrically nonlinear simulation and so will be ignored in this test.

(iii) Select a linear beam element and check the 'cubic formulation' for the beam type. This causes ABAQUS to use Euler-Bernoulli theory in its calculations. Shear modulus, and any additional options for specifying the shear stiffness of the cross-section are ignored.

(iv) Create a job and run the analysis.

Calculate the deflection of the end of the beam, and compare the FEA prediction with the analytical (Euler-Bernoulli) solution

(b) Return to the Mesh module, change the element type to 'shear flexible' linear elements (these use 'Timoshenko' theory for the beam, as discussed in Chapter 8 of Applied mechanics of solids), and re-calculate the tip deflection. Also, test the effect of reducing the shear modulus of the beam with the shear flexible elements. ABAQUS has a lot of options for controlling the shear stiffness of beams – too many to test here!

(c) Run the simulation again with 'quadratic' elements. Are these using Euler-Bernoulli theory?

(d) ABAQUS can model beams made from any material in the material library. Test this as follows:

(i) Create an elastic-perfectly plastic material with Youngs modulus 500 units, Poisson's ratio of 0.25, and a yield stress of 10 units.

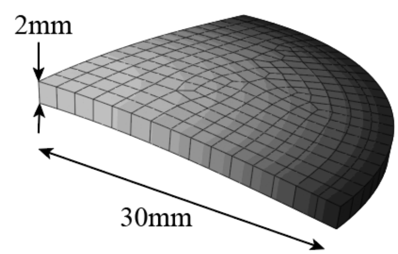
(ii) Change the section options to calculate the section integration during analysis.

(iii) Change the initial and maximum increment sizes in the step to 0.1 units. Create a history output request that will save the deflection and reaction force on the right hand end of the beam

(iv) Replace the concentrated force applied to the end of the beam with a prescribed vertical displacement of 1.5 units.

Plot the force-displacement curve for the load point on the beam, and compare the predicted collapse load with the analytical solution.

Problem 7.22 This problem runs a simple test of a 'shell' (or plate) element in ABAQUS.



(a) Set up the simulation as follows:

(i) Create a part that consists of a 3D deformable shell feature that has the shape of a ¼ circle with 30mm radius. The two straight edges should be horizontal and vertical in the sketch.

(ii) Create a linear elastic material with Young's modulus 75GPa and Poisson's ratio 0.25; then create a homogeneous shell section with this material, a shell thickness of 2mm, and the 'Bending only' option for the 'Idealization' (this will approximate classical plate theory)

(iii) Create an instance of the part, then create a static step with time period 1s and NLGEOM off.

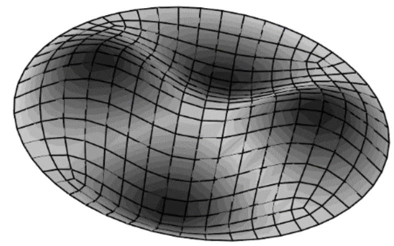
(iv) Create XSYMM and YSYMM boundary conditions on the horizontal and vertical edges; set U3=0 for the curved edge, and apply a pressure of $-5\mathrm{N/mm}^2$ to the plate surface

(v) Choose a Triangle meshing algorithm in the Mesh Controls; and for the element type use a linear triangular element with the 'small' membrane strain selected. Seed the part with a 2mm mesh size and mesh it.

(vi) Create a job and submit it.

Find the out-of-plane displacement at the center of the plate.

(b) Calculate (using Real Intelligence) the deflection of the center of the plate predicted by classical plate theory



Problem 7.23 The goal of this problem is to calculate the natural frequencies of vibration of a stretched membrane. It demonstrates several aspects of finite element analysis: (i) membrane elements; (ii) modeling thermal expansion; (iii) calculating natural frequencies of vibration, (iv) the consequences of an inappropriate choice of elements or procedures in a finite element calculation.

(a) Begin by setting up a stretched membrane, as follows:

(i) Create a 3D deformable planar shell part that is a circle with radius 25mm.Create a material with Young's modulus 7.5GPa, Poisson's ratio 0.25, mass density 0.001g/mm^3 (this is the mass unit with N, mm, and sec for force, length and time units) and a thermal expansion coefficient of 10^{-5} K^{-1}. Create a membrane shell section with thickness 1mm with this material, and assign it to the part. Create an instance of the part in the assembly.

(ii) Create a static step with NLGEOM on, with defaults for all remaining options.

(iii) Set displacements to zero in all 3 directions on the edge of the plate. Use Predefined Field>Create to set the temperature of the part to -100C (the part is stress free at 0C by default)

(iv) Use the 'medial axis' algorithm to mesh the part with linear quadrilateral fully integrated membrane elements (make sure the 'reduced integration' box is unchecked.

(v) Create a job and run the analysis.

Check the stress in the membrane, and verify with a hand calculation that the predicted stress is correct.

(b) Next, set up the vibration calculation as follows:

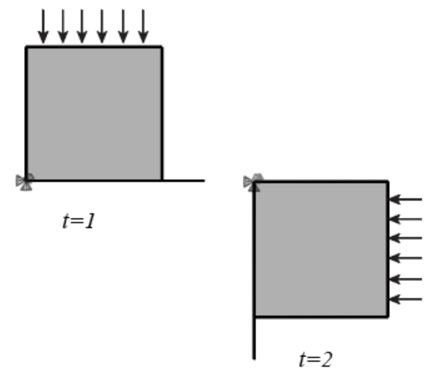
(i) Return to the 'Step' module and create a second step. In the step definition menu select 'Linear Perturbation' for the Procedure type, and select 'Frequency.' For the 'Number of eigenvalues requested' select 'Value' and enter 10 for the number of eigenvalues.

(ii) Run the analysis again. To view the predicted natural frequencies and mode shapes, use Result>Step/Frame in the visualization module and select the mode you would like to see. You can use the usual procedures to display the deformation associated with vibration mode in the viewport.

Compare the natural frequencies predicted by FEA for the first 3 vibration modes with the exact solution.

(c) Try running the simulation again with NLGEOM off for the two steps. You will get an error message and the simulation will fail. You can force ABAQUS to find some eigenvalues by switching to the 'Subspace' algorithm for the eigenvalue solver in the step definition menu. Try this, and plot the U1 component for the first (which has a repeated eigenvalue) and 3rd vibration modes. Check the out-of-plane displacements for all the vibration modes. Why are they all zero?

(d) Restore the 'step' choices to the original state (part c was clearly a bad idea!) Try changing the element type to a reduced integration linear quadrilateral (just check the 'Reduced Integration' box in the element options). Plot the shape of the first vibration mode. What has gone wrong with the simulation now?



Problem 7.24 This problem illustrates the rotation of an 'orientation' that is used to define tensor valued material properties and variables associated with a part.

(a) Set up the simulation as follows:

(i) Create a 1mmx1mm 2D deformable part consisting of a 1mmx1mm square, with one corner at the origin, and another at (1,1)mm. Assign the part a homogeneous solid section with a linear elastic material with Young's modulus 100 units and Poisson's ratio 0.25. Create a second part that consists of an analytical rigid surface, which consists of a line with one end at the origin and another at (1.5,0). Create a reference point for the rigid surface attached to the end at (0,0). Create instances of both parts.

(ii) Create a static step with time period 1s and NLGEOM on, then create a second step with period 1s, NLGEOM on, and initial and maximum increment sizes of 0.1s. Create a history output request that will report the stress components S11 and S22 during both steps.

(iii) Create a contact property with 'hard' normal contact and frictionless tangential contact. Take the rigid surface as the 'main' and the base of the square as the 'secondary' surfaces (you may need to suppress the rigid surface in the assembly viewport by right-clicking its icon in the model tree to be able to select the second surface).

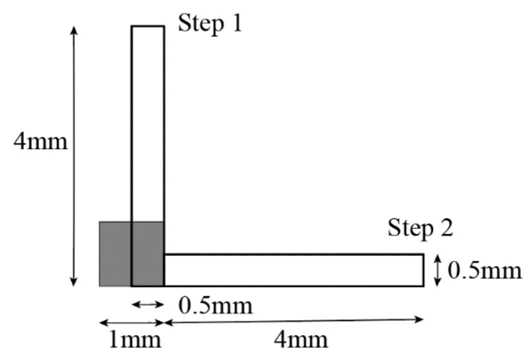
(iv) In the first step, create a boundary condition that will fix the bottom left corner of the square; fix all degrees of freedom for the rigid surface reference point; and create a 'load' that will apply a pressure of 1 unit to the top of the square. In the second step, edit the boundary condition for the reference point to apply a zero displacement and a rotation of -1.57rad.

(v) Mesh the part with a single plane stress reduced integration element.

(vi) Run the simulation and check that the part rotates correctly.

Plot a graph showing the variation of stress components S11 and S22 with time during the simulation.

(b) Return to the 'property' module and assign a material orientation to the square. Select 'Use Default' in the viewport when prompted, and use 'Global' to define the system (this just means that the initial basis used to define the orientation coincides with the global basis). Run the analysis and add the predicted variation of S11 and S22 to your plot from (a). Note that the stress components are now constant for time > 1 sec, because the basis rotates with the part.



Problem 7.25 This problem (i) illustrates the behavior of the 'Linear Elastic' material in ABAQUS® when it is subjected to large strains, (ii) tests some of the strain measures that are output by ABAQUS®; and (iii) gives some insight into the way ABAQUS® calculates deformation and stress measures. The problem is just a test, it is not an appropriate application for the 'Linear Elastic' material.

(a) To begin

(i) Create a 1mmx1mm 2D planar square part. Create an instance of the part, and mesh it with a single plane stress reduced integration linear quadrilateral element.

(ii) Create a homogeneous solid section with out-of-plane thickness of 1mm and made from a linear elastic material with a Young's modulus of 1GPa and a Poisson's ratio of 0.5 (this makes the material incompressible – it is not a good idea to run simulations with fully incompressible materials so ABAQUS® will adjust this to just below 0.5 and give a warning). Assign the material to the part.

(iii) Create two general static steps, each with time period of 1s, with Automatic time increment of 1s, and initial and maximum increment sizes of 1s, and the NLGEOM parameter on. Use Tools>Set to create a set containing the one element in the mesh, then create both a field output request and a history output request that will record the total strain, I; elastic strain (EE); Nominal strain (NE) and logarithmic strain (LE) in the element, as well as the stress (S).

(iv) Apply the following boundary conditions to the element: (i) during step 1, fix the right hand bottom corner of the square; apply a displacement of (0.5,0)mm to the bottom left corner, apply a displacement of (0,3)mm to the top right corner; and a displacement of (0.5,3) to the top left corner (this subjects the element to a volume preserving vertical stretch). (ii) Then in step 2, propagate the fixed boundary condition at the bottom right corner; apply a displacement (1,0.5)mm to the bottom left corner; a displacement of (4,-1)mm to the top right corner, and a displacement of (5,-0.5)mm to the top left corner (ABAQUS® measures displacements applied to a point from the original position of the node, so at the end of this step the stretched element will be rotated by 90 degrees in a clockwise sense)

(v) Run the analysis and check the deformed shape at the end of the two steps.

(b) Calculate (by hand) the following deformation measures, for comparison with the FEA predictions (only the in-plane components of the tensors are needed). See chapter 2 of Applied Mechanics of Solids for the definitions.

(i) The deformation gradient **F** at the end of steps 1 and 2

(ii) The Left Cauchy-Green deformation tensor **B** at the end of steps 1 and 2

(iii) The Left stretch tensor **V**, at the end of steps 1 and 2

(iv) The Eulerian nominal strain tensor $\mathbf{E}^{*}_{\text{nom}} = \mathbf{V} - \mathbf{I}$ at the end of steps 1 and 2.

(v) The Eulerian logarithmic strain at the end of step 1 and step 2

(vi) The stretch rate tensor **D** during step 1 (assume loading is at constant velocity – note that **D** will be a function of time)

(c) Find the 'elastic strain', Eulerian nominal strain, and Eulerian logarithmic strain computed by ABAQUS, and compare them with your hand calculations in part (b). You will find the nominal strain and logarithmic strain agree, but the 'elastic strain' does not correspond to any of the strain measures calculated in part (b)

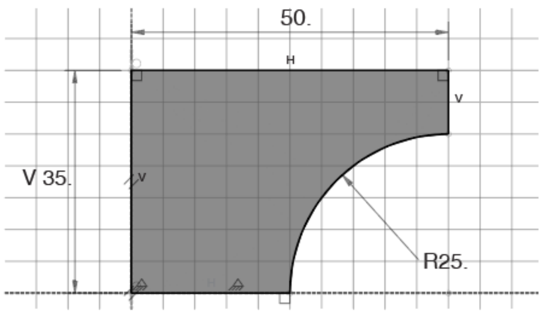
(d) Find the stresses predicted by ABAQUS at the ends of steps 1 and 2, and show that they correspond to stresses calculated using the linear elastic constitutive equations together with the 'elastic strain' measure computed by ABAQUS

(e) The 'elastic strain' and stresses observed in parts (c) and (d) are a bit mysterious. The following test gives a hint to what ABAQUS is doing. Return to the step module, and change the Incrementation for both steps 1 and 2 to a fixed time increment of 0.025s, and run the analysis again. Then, plot a graph showing the variation of the 'elastic strain' components during the two steps. Also, show that the 'elastic strains' at the end of the step are equal to the logarithmic strains.

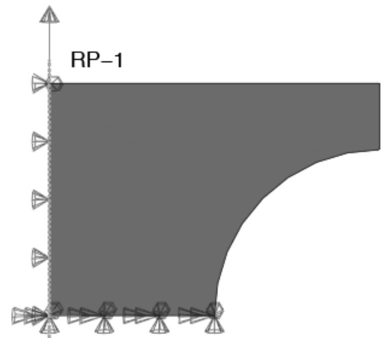
(f) The explanation for the unexpected results in I-I is that, during a static calculation for most materials, ABAQUS calculates the 'total' and 'elastic' strains from the stretch rate. The total strain is defined as

$$\boldsymbol{\varepsilon} = \int_0^t \mathbf{D}(t)\,dt$$

The elastic strain is calculated the same way, using the elastic part of the stretch rate (the distinction becomes relevant in an elastic-plastic material). The time integral is approximated by calculating the value of **D** at the mid-point of each increment, and multiplying by the time increment. Confirm this by calculating the value of **D** at time $t=0.5s$ using your answer to (b), and comparing the value to the result of (c). Also, show that at the end of step 1 your hand calculation predicts that the elastic strain $\boldsymbol{\varepsilon}^e$ is equal to the logarithmic strain.



Dimensions



Boundary Conditions

Problem 7.26 The goal of this problem is to demonstrate a finite element analysis of cyclic plastic deformation in a simple component.

(a) Set up the problem as follows

(i) Create a part representing a notched round (axisymmetric) bar with the geometry shown in the figure (use mm for dimensions and N for forces). Assign the part a homogeneous solid section with Young's modulus 200GPa and Poisson's ratio 0.3, and a kinematic hardening plastic response with yield stress 500 MPa at plastic strain of zero, and 600MPa at plastic strain of 0.5 (the linear kinematic hardening law requires two pairs of stress-plastic strain values to define the initial yield stress, and the subsequent constant hardening rate)

(ii) Create an instance of the part, and create a reference point named 'Load Point' located at the top left corner of the part.

(iii) Create a static step with initial increment and maximum increment size 0.1s, with the NLGEOM parameter active. In the step module create a History Output request that will save the vertical displacement (U2) and reaction force (RF2) at the reference point.

(iv) In the Interaction module, use Constraint>Create and select 'Coupling' from the menu. Select the reference point for the control point, and select the horizontal top surface of the bar for the surface. In the menu, check the box to constrain degree of freedom U2, and uncheck those for U1 and UR3. Use a kinematic coupling type (the default). Adding this constraint will allow you to load the top surface of the bar by applying displacements or forces to the reference point, and also to plot the force-v-displacement curve for the deforming bar.

(v) Create boundary conditions that will (i) apply symmetry conditions on the base of the bar; (ii) prevent horizontal displacement at the axis of rotational symmetry; and (iii) apply U1=UR3=0 and U2=0.1 to the reference point. At the end of this stage the boundary conditions should be displayed as shown in the figure.

(vi) Seed the part with a mesh size 1.25mm and mesh it with linear quadrilateral reduced integration axisymmetric elements, using the Structured meshing algorithm.

(vii) Return to the Step module, and create three additional steps after the first one. Each additional step should have initial and maximum increment sizes of 0.1s (the subsequent steps will apply unloading/reloading cycles).

(viii) Continuing in the step module, use Tools>Set to create a set called 'critical element,' check the 'element' radio button, and when prompted select the element on the lower left corner of the part (next to the axis of symmetry). Create a second history output request (make sure it is active in all the steps), which requests the plastic strain component PE22 for the critical element. This will enable you to determine the plastic strain amplitude.

(ix) Return to the Load module, and open the boundary condition manager. The BCs created for the first step should have propagated to the next three. Select the boundary condition for the reference point in step 2, and Edit it – change the U2 displacement from +1 to -1 (to unload the bar in step 2). Use a similar procedure to re-load the bar in step 3, and unload again in step 4.

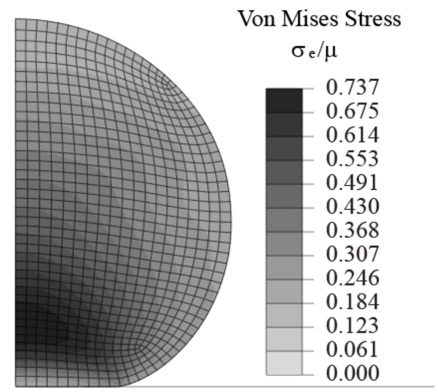
(x) Create a job, and run the analysis.

(xi) Check your analysis by animating the time history of equivalent plastic strain PEEQ in the bar. You should find that the maximum strain occurs in the critical element.

Use your results to plot the load-v-displacement curve for the specimen.

(b) Plot a graph showing the variation of axial plastic strain in the critical element as a function of time. Hence, estimate the plastic strain amplitude.

Problem 7.27 The figure shows an FEA simulation of the impact between a soft rubber sphere and a flat rigid frictionless surface. The goal of this problem is to run a finite element simulation of this problem.



- The sphere will be idealized as a near-incompressible neo-Hookean hyperelastic material with shear modulus μ and bulk modulus K and mass density ρ.
- The sphere has radius R and is launched with initial velocity V_0 towards the rigid surface
- Various quantities could be calculated in this analysis, and you should feel free to calculate anything that might be interesting. But to focus on something specific, we will calculate (i) The time that the sphere is in contact with the surface t_0; and (ii) the force of the impact $F(t)$.

Before doing any analysis,

(a) Make a list of the variables that appear in the problem definition. Find dimensionless expressions for the general functional dependence of the contact time t_0 and $F(t)$ on parameters in the problem. Note that since we are interested in an incompressible material we are interested only in the solution for the limit $K/\mu \to \infty$, which we will try to approximate with a suitably large value for K/μ

(b) Make a guess for how you would expect the contact time and maximum force to vary with relevant parameters (you don't need to be specific – just think about whether you would expect them to increase or decrease with impact velocity, for example. This will tell you how it varies with all the other parameters in the problem). You should be able to specify exactly how the force will scale with sphere radius.

(c) Would you expect the magnitude of the velocity of the center of mass of the sphere after rebound to be equal to, greater than, or less than its value before impact? Why?

(d) Set up a finite element simulation of the problem, as follows:

(i) In the 'part' module, create the sphere as a 2D axisymmetric solid. Use a radius *R=0.01* (arbitrary units).

(ii) Create a second part to serve as the rigid surface. This can be a 2D axisymmetric analytical or discrete rigid wireframe feature. You will need to add a reference point to the rigid object to be able to apply boundary conditions. In the Part module, use the Tools>Reference Point… menu to do this. It is best to put the reference point on the end of the surface at the axis of symmetry.

(iii) In the 'material' module, create a hyperelastic neo-hookean material with $C_{10} = 0.5$, and leave the *D* field blank (this makes ABAQUS use a default value to approximate incompressibility) and mass density $\rho = 1$ (again units are arbitrary since we are working in dimensionless form). Note that the parameter C_{10} in ABAQUS is related to the shear modulus by $\mu = 2C_{10}$ (the user manual gives more details of the options available in their hyperelatic material models). Create a homogeneous solid section with the hyperelastic material model, then assign it to the part.

(iv) In the 'Assembly' module, create instances of the sphere and the rigid plane. Place the sphere and the surface a small distance apart in the assembly. The exact value of the distance is not important.

(v) In the 'Step' module create an implicit dynamic step, with time period of order 0.06, initial and maximum increment sizes around 0.0005, and max increments of order 800. Make sure the NLGEOM parameter is set so you run a finite strain calculation. In the Output>Field Output Requests set the frequency of output to every 1 increment. Use the 'Tools' menu to create a Set that contains the reference point for the rigid surface (the reference point is a 'geometry'), and create a History Output that will ensure that reaction forces acting on this node are written to the output database. Add any other variables you might be interested in.

(vi) In the 'Interaction' module create an 'interaction property' that specifies a hard normal contact and frictionless tangential contact. Create an 'interaction' that defines the rigid surface as the 'main' (or 'master' in old terminology) surface, and the exterior of the sphere as the 'secondary' or 'slave' surface. Use 'Surface-to-Surface' contact, with finite sliding.

(vii) In the 'Load' module, constrain the radial displacement on the axis of symmetry of both solids. It is easier to move the rigid surface towards the sphere rather than to assign an initial velocity to the sphere and make the surface stationary, but you can do it either way. To constrain the rigid surface you need to apply boundary conditions to the reference point. Apply a vertical velocity of 0.15 (arbitrary units) and constrain the axial displacement and rotation of the reference point to zero. Add a zero axial displacement constraint to the symmetry axis of the sphere.

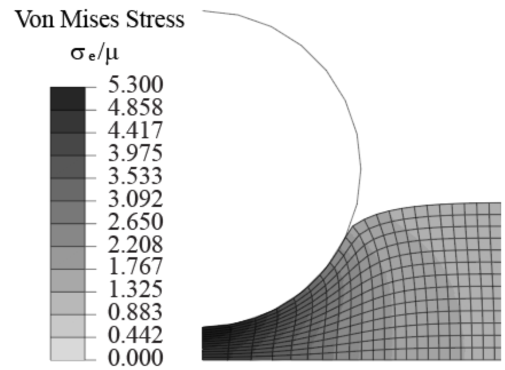
(viii) In the 'Mesh' module, mesh the sphere part (make sure the 'part' radio button is checked in to menu above the window) with hybrid reduced integration axisymmetric stress linear quadrilateral elements. Make sure you check the hybrid formulation box for both quad and triangular elements, because the mesh generator will probably use a combination of both (you can use the 'mesh controls' to specify this – the mesh in the figure was created using the medial axis algorithm with quad geometry enforced). Seeding the part with a mesh size of order 0.0005 should avoid the 1000 node limit in the 'learning edition' of the software. If you previously selected a 'discrete rigid' surface for the second part, you will also need to mesh the rigid surface as well as the deformable solid. An analytical rigid surface does not need to be meshed.

(ix) Use the default options for the job submission

When you have managed to get your simulation to run (watch an animation of the contours of Mises stress, to see what is happening), plot a graph showing the variation of the vertical reaction force acting on the rigid surface with time.

(e) Calculate the total impulse exerted on the sphere. Hence, calculate a value for the restitution coefficient of the impact (using rigid body collision formulas).

Problem 7.28 The figure shows an FEA simulation of a rigid sphere rebounding off a soft rubber thin film on a rigid substrate.



- The film is idealized as a near-incompressible neo-Hookean hyperelastic material with shear modulus μ, bulk modulus K and mass density ρ. We will focus on the incompressible limit where $K >> \mu$.
- The sphere has radius *R,* mass *m*, and is launched with initial velocity V_0 towards the film surface. During the impact the sphere has speed *V*(*t*)

The goal of this problem is to run a finite element simulation of the impact. Before doing any analysis:

(a) Write down the equations and boundary conditions that govern the solution. You will need equations for the film and for the rigid body motion of the sphere.

(b) Introduce the following normalized measures of stress, position before and after deformation, sphere speed and time

$$\hat{\sigma}_{ij} = \sigma_{ij} / \mu \qquad \hat{y}_i = y_i / R \qquad \hat{x}_i = x_i / R \qquad \hat{V} = V / V_0 \qquad \hat{t} = tV_0 / R$$

Re-write the governing equations in terms of these variables, and hence show that the solution is governed by three dimensionless groups:

$$\frac{\mu R^3}{mV_0^2}, \frac{\rho}{\mu} V_0^2, \frac{\mu}{K}$$

in addition to dimensionless geometric variables *H/R, L/R* . In the quasi-static limit for an incompressible material, only the first parameter is important.

(c) Now set up an ABAQUS simulation of the problem. Use the following procedure

(i) Make the sphere a 2D axisymmetric analytical rigid surface (if you can't get ABAQUS to accept a 180 degree arc try a discrete rigid surface instead). Use a radius *R=0.05* (arbitrary units). You will need to add a reference point to the rigid object to be able to apply boundary conditions. In the Part module, use the Feature>Reference Point… menu to do this. It is best to put the reference point on the axis of symmetry

(ii) Make the film a 3D axisymmetric deformable solid, with dimensions 0.2x0.05 units. Create a material with properties representing a hyperelastic neo-hookean material with $C_{10} = 0.5$ and mass density $\rho = 1$ (units are arbitrary since we are working in dimensionless form). Note that the parameter C_{10} in ABAQUS is related to the shear modulus by $\mu = 2C_{10}$ Leave the *D* field blank (this makes ABAQUS use a default value to approximate incompressibility). Create a homogeneous solid section with the hyperelastic material property and assign it to the part. Assign the sphere a mass of 0.2 units. (To do this use Special>Inertia menu in the property editor, and assign the mass to the Reference Point)

(iii) In the 'Assembly' module create instances of both parts and, place the sphere a small distance above the thin film surface. The distance is not important.

(iv) It is best to skip to the 'interaction' module before creating the step. The interaction between the sphere and the surface should be hard normal contact and frictionless tangential contact. The options for the contact (node region –v-surface) are not important – just use the surface to surface contact.

(v) Now go back to the 'Step' module and create an explicit dynamic step, with time period somewhere between 1 and 5, depending on how far above the film you start the sphere (the analysis should be long enough to simulate the whole contact event). You can accept defaults for all other parameters. Make sure the NLGEOM parameter is set so you run a finite strain calculation. In the Step module create a History output that will record the contact force during the collision – use Output > History Output Requests > Create, and select Interaction for the Domain; then you can select the surface interaction created while defining the contact. You can accept the default variables that will be saved, but you won't need most of them…. For the 'frequency' of history output select 'every n increments' and enter 1 for n. Also use the 'Field Output Requests' manager to edit the field output request (or create one if it did not appear by default). Make sure the box for stresses is checked, and for 'Frequency' select 'every x units of time' and enter 0.05 for x.

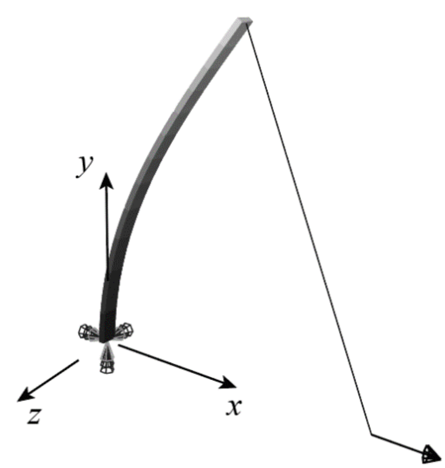
(vi) For boundary conditions, constrain the radial displacement on the axis of symmetry of both solids, and assign zero vertical displacement to the base of the film. Assign the sphere an initial downwards vertical velocity of 0.05 units (in the Load module use Predefined Field; select the Initial step in the window at the top, and check the Mechanical radio button – this will then allow you to select Velocity). Constrain the rotation of the reference point on the sphere to prevent rigid body rotation.

(vii) Mesh the solid with reduced integration axisymmetric stress linear quadrilateral elements (the default for quadrilateral elements in an explicit dynamic analysis). Choose a mesh size that you think is sensible. If you used a discrete rigid surface instead of an analytical rigid surface for the sphere you will need to mesh the rigid sphere as well as the block. To do this, seed the sphere in the usual way, then simply mesh the part (or instance if you made the mesh independent).

(viii) Use the default options for the job submission.

When you have managed to get your simulation to run (use the Animate > Time History menu to see an animation of the contours of Mises stress), plot a graph showing the variation of the vertical reaction force acting on the rigid surface with time. (The pressure can be found in the History output, and is named 'Total force due to contact pressure CFN2' in the list of variables available to plot)

Problem 7.29 This problem sets up up a simple finite element simulation to predict the static force-v-draw curve for a simple wooden longbow.



(a) Set up the simulation as follows:

(i) Because of symmetry, we only need to model half the bow. Create a 3D deformable part with wire base feature, with one end at the origin and the other at (0,830,0)mm (the undeformed bow is straight; it gets bent when the string is attached). Of course, this procedure will compute *half* the total draw force (there will be an additional force from the bottom half of the bow that has not been modeled).

(ii) Use a Young's modulus for Yew wood of 6.5 GPa (convert to N/mm^2 since lengths are in mm), a Poisson's ratio of 0.2 and make the cross-section a 40mmx12mm rectangle. Use Assign> Beam section orientation> to specify the orientation of the cross-section of the beam – make the normal to the flat side of the bow parallel to the global x direction.

(iii) Create a second part for the bow string. Place the top end of the string at (0,830,0)mm and the other at (0,790,0). Give the bow string a Young's modulus of 110 GPa; a Poisson's ratio of 0.25 and a circular cross section with radius 0.72mm.

(iv) Create two steps, both with 1 sec duration and NLGEOM on – these will be used to string the bow, and then draw it. Make sure NLGEOM is selected for both steps. Make the initial and maximum increment size 0.1s for the first step, and 0.025s for the second. Open the 'Field Output Request' manager and add requests to plot the section forces/moments. Also, create a Set containing the point at the bottom end of the bowstring, and use the History Output Request to save the displacement and reaction force for this set.

(v) To connect the bowstring to the bow, select Constraint > Create in the interaction module, select a tie constraint; in the menu below the viewport select Node Region, then select the end points of the string and bow as the master and slave (it doesn't matter which is which). If you need to, you can right click a part under the 'instance' branch of the model tree to hide it. In the 'edit constraint' menu uncheck the 'Tie rotational DOFs if applicable' option. This will allow the string to rotate freely relative to the end of the bow.

(vi) In the boundary condition module, create a BC for the 'initial' step that makes all the degrees of freedom (U and UR) to zero for the node on the bow that lies at the origin. This BC will be applied throughout the analysis by default.

(vii) We need to define boundary conditions for the two load steps. The first step will be used to string the bow. To do this we need to apply a force to bend the bow, and pull the string down so its bottom end lies at y=0. Apply a horizontal force (acting in the positive x direction) to the bow at its tip with magnitude 10N in the *x* direction. Create an 'amplitude' for this force; check the 'tabular' radio button, and enter a table that starts at 1 at time t=0 and ends at 0 at time t=1. Then select Amp-1 in the dropdown menu. This will bend the bow so it can be pulled down by the bowstring, but will remove the force by the end of the step. Next, create a BC that will set the vertical rotation UR2=0 for the bowstring, and in the same BC apply a vertical displacement to the free end of the bowstring that will pull it down to be level with the base of the bow (this will be U2=-40mm). A standard 'Ramp' amplitude will work. Finally open both the load and boundary condition managers and deactivate these this BC in step 2.

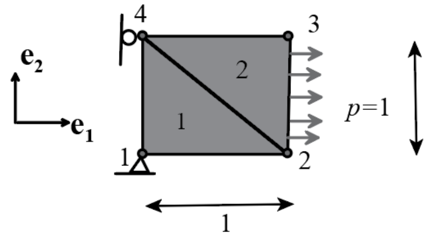
(viii) To draw the bow, (i) create a boundary condition that will fix the vertical displacement U2 at the bottom of the bowstring to remain constant in the second step. To do this open the BC manager, and make sure the BC you defined at the end of the string in Step 1 is deactivated in Step 2. Then create a new BC, select the bottom end of the bowstring, select 'Fixed at Current Position' in the Edit BC window, and check the box for U2. Next, create a second BC that will apply a constant horizontal velocity (V1) of 400 mm/sec to the bottom end of the bowstring. This will draw the bow by 400mm by the end of the step. Finally create a 3rd BC that will set the vertical component of rotation UR2 on the base of the bowstring to zero during the second step In the mesh module seed the bow and string with 10mm/20mm mesh size, respectively, and mesh both parts with the default beam element type.

(ix) Create a job and run the analysis.

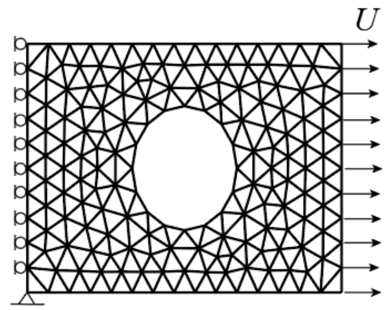
Plot a graph of the force-v-draw curve for the bow (don't forget to double the force, since only half the bow was modeled)

(b) As an experiment, remove the boundary condition that constrains the rotation of the bowstring about the *y* axis during the second step. Use the job monitor to see the progress of the analysis. You will see ABAQUS has trouble converging. There will also be some small errors in the force-v-time curve. Why does this occur?

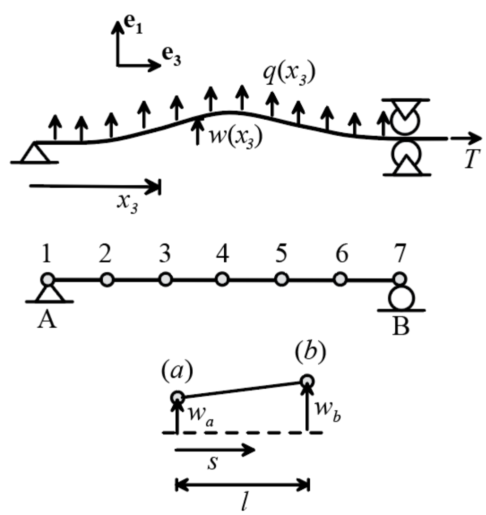
7.2 A Simple Finite Element Program



Problem 7.30 Modify the simple constant strain triangle FEA code with the name 'FEM_conststrain_triangles.m' posted at https://github.com/albower/Applied_Mechanics_of_Solids to solve problems involving plane stress deformation as well as plane strain. Test your code by solving the problem illustrated the figure. Use a value of 100 for Young's modulus and 0.25 for Poisson's ratio. Compare your FEA predictions to the exact solution, for both plane strain and plane stress elements.



Problem 7.31 Use your code from problem 7.30 together with the input file 'FEM_constrain_holeplate.txt' on the Github site to plot the stresses in a rectangular elastic solid that contains a hole at its center. The plate has a width of 6 units and height of 4 units, is fixed on its left edge, and has a prescribed horizontal displacement of 0.01 units on its right edge. Use a material with Youngs's modulus 100 units and Poisson's ratio 0.499. Find both a plane stress and a plane strain solution (the plane strain solution will demonstrate 'Volumetric locking')



Problem 7.32 The goal of this problem is to develop and apply a finite element method to calculate the shape of a tensioned, inextensible cable subjected to transverse loading (e.g. gravity or wind loading). The cable is pinned at A, and passes over a frictionless pulley at B. A tension T is applied to the end of the cable as shown. A (nonuniform) distributed load $q(x)$ causes the cable to deflect by a small distance $w(x)$ as shown. For w<<L, the potential energy of the system may be approximated as

$$\Pi = \int_0^L \frac{T}{2}\left(\frac{dw}{dx_3}\right)^2 dx_3 - \int_0^L q w dx_3$$

To develop a finite element scheme to calculate w, divide the cable into a series of 1-D finite elements as shown. Consider a generic element of length l with nodes a, b at its ends. Assume that the load q is uniform over the element, and assume that w varies linearly between values w_a, w_b at the two nodes.

(a) Write down an expression for w at an arbitrary distance s from node a, in terms of w_a, w_b, s and l. (assume a linear variation)

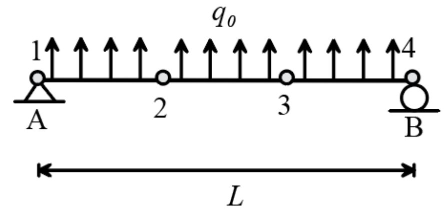
(b) Deduce an expression for dw/dx within the element, in terms of w_a, w_b and l

(c) Hence, calculate an expression for the contribution to the potential energy arising from the element shown, and show that element contribution to the potential energy may be expressed as

$$V^{\text{elem}} = \frac{1}{2}\left[w_a, w_b\right]\begin{bmatrix} T/l & -T/l \\ -T/l & T/l \end{bmatrix}\begin{bmatrix} w_a \\ w_b \end{bmatrix} - \left[w_a, w_b\right]\begin{bmatrix} ql/2 \\ ql/2 \end{bmatrix}$$

(d) Write down expressions for the element stiffness matrix and force vector.

(e) Consider the finite element mesh shown in the figure. The loading q_0 is uniform, and each element has the same length. The cable tension is *T*. Calculate the global stiffness matrix and residual vectors for the mesh, in terms of *T, L*, and q_0 (before any constraints are applied).

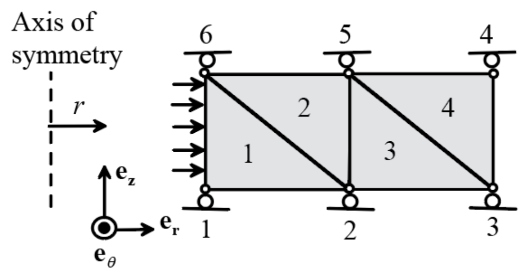


(f) Show how the global stiffness matrix and residual vectors must be modified to enforce the constraints $w_1 = w_4 = 0$

(g) Hence, calculate values of *w* at the two intermediate nodes.

(h) Write a simple MATLAB script to solve the problem for an arbitrary number of elements with equal spacing between nodes (you can just use the pattern you see in (e); you don't need to write a fancy code). Compare the FEA predictions with the exact solution.

Problem 7.33 The goal of this problem is to modify the simple plane stress/plane strain FEA code used in problem 7.30 so that it will solve problems involving *axially symmetric* solids. The figure shows a representative problem to be solved. The mesh represents a slice through an axially symmetric cylinder, which is prevented from stretching vertically, and pressurized on its interior surface. The solid is meshed using axisymmetric triangular elements (so the elements extend all the way around the circumference of the cylinder; but the out-of-plane parts of the elements are all hard coded and are not displayed in the mesh). The displacements are interpolated in each element as



$$u_r(r,z) = u_r^{(a)} N_a(r,z) + u_r^{(b)} N_b(r,z) + u_r^{(c)} N_c(r,z)$$
$$u_z(r,z) = u_z^{(a)} N_a(r,z) + u_z^{(b)} N_b(r,z) + u_z^{(c)} N_c(r,z)$$

where

$$N_a(x_1,x_2) = \frac{\left(x_2 - x_2^{(b)}\right)\left(x_1^{(c)} - x_1^{(b)}\right) - \left(x_1 - x_1^{(b)}\right)\left(x_2^{(c)} - x_2^{(b)}\right)}{\left(x_2^{(a)} - x_2^{(b)}\right)\left(x_1^{(c)} - x_1^{(b)}\right) - \left(x_1^{(a)} - x_1^{(b)}\right)\left(x_2^{(c)} - x_2^{(b)}\right)}$$

$$N_b(x_1,x_2) = \frac{\left(x_2 - x_2^{(c)}\right)\left(x_1^{(a)} - x_1^{(c)}\right) - \left(x_1 - x_1^{(c)}\right)\left(x_2^{(a)} - x_2^{(c)}\right)}{\left(x_2^{(b)} - x_2^{(c)}\right)\left(x_1^{(a)} - x_1^{(c)}\right) - \left(x_1^{(b)} - x_1^{(c)}\right)\left(x_2^{(a)} - x_2^{(c)}\right)}$$

$$N_c(x_1,x_2) = \frac{\left(x_2 - x_2^{(a)}\right)\left(x_1^{(b)} - x_1^{(a)}\right) - \left(x_1 - x_1^{(a)}\right)\left(x_2^{(b)} - x_2^{(a)}\right)}{\left(x_2^{(c)} - x_2^{(a)}\right)\left(x_1^{(b)} - x_1^{(a)}\right) - \left(x_1^{(c)} - x_1^{(a)}\right)\left(x_2^{(b)} - x_2^{(a)}\right)}$$

are the same interpolation functions used to analyze plane deformations, and

$$(u_r^{(a)}, u_z^{(a)}),\ (u_r^{(b)}, u_z^{(b)}),\ (u_r^{(c)}, u_z^{(c)})$$

are the 2D displacement vectors at the three corners of the elements.

(a) The nonzero strain components in the element (in cylindrical-polar cords) can be expressed as

$$[\varepsilon] = [B][u]$$

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{\theta\theta} \\ 2\varepsilon_{rz} \end{bmatrix} \qquad [u] = \begin{bmatrix} u_r^{(a)} & u_z^{(a)} & u_r^{(b)} & u_z^{(b)} & u_r^{(c)} & u_z^{(c)} & u_r^{(d)} & u_z^{(d)} \end{bmatrix}^T$$

where [B] is a 4x4 matrix containing derivatives of the interpolation functions, and (for axisymmetric elements) some additional terms that are functions of the interpolation functions (not their derivatives). Find a formula for [B]. You can assume that, since the deformation is axisymmetric, the nonzero strain components are

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \quad 2\varepsilon_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}$$

[B] will be very similar to the 3x4 matrix used for plane strain and stress problems, but needs an extra row to calculate $\varepsilon_{\theta\theta}$.

(b) Let $\left[\sigma\right] = \left[\sigma_{rr} \quad \sigma_{zz} \quad \sigma_{\theta\theta} \quad \sigma_{rz}\right]$ denote the stress in the element. Find a matrix $\left[D\right]$ that satisfies $\left[\sigma\right] = \left[D\right]\left[\varepsilon\right]$

(c) Write down an expression for the strain energy density U^{el} within the element, in terms of the vectors and matrices $[u],[B],[D]$

(d) The total strain energy of each element must be computed. Note that each element represents a cylindrical region of material around the axis of symmetry. The total strain energy in this material is

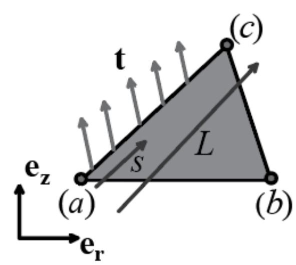
$$W^{el} = \int_{A_{el}} 2\pi r U^{el} drdz$$

The energy can be computed with sufficient accuracy by evaluating the integrand at the centroid of the element, and multiplying by the area of the element, with the result

$$W^{el} = 2\pi A_{el} \bar{r}\, \bar{U}^{el}$$

where $\bar{r}$ denotes the radial position of the element centroid, and $\bar{U}^{el}$ is the strain energy density at the element centroid. Use this result to deduce an expression for the element stiffness.

(e) The contribution to the potential energy from the pressure acting on element faces must also be computed. Following the procedure used for plane problems, the potential energy is



$$P = -\int_0^L 2\pi r\, t_i u_i ds$$

where

$$u_i = u_i^{(a)} \frac{s}{L} + u_i^{(c)}\left(1 - \frac{s}{L}\right) \qquad r = r^{(a)} \frac{s}{L} + r^{(c)}\left(1 - \frac{s}{L}\right)$$

and $u_i^{(a)}, u_i^{(c)}$ denote the displacements at the ends of the element face, and $r^{(a)}, r^{(c)}$ denote the radial position of the ends of the element face. Calculate an expression for P of the form

$$P^{\text{element}} = -\left[t_1 A \quad t_2 A \quad t_1 B \quad t_2 B\right] \cdot \left[u_1^{(a)} \quad u_2^{(a)} \quad u_1^{(c)} \quad u_2^{(c)}\right]$$

where A and B are constants (which depend on r^a, r^c) that you must determine. The element force vector for axisymmetric elements follows as

$$[r] = \left[t_1 A \quad t_2 A \quad t_1 B \quad t_2 B\right]$$

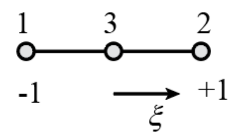
(f) Modify the simple constant strain triangle FEA code with the name 'FEM_conststrain_triangles.m' posted at https://github.com/albower/Applied_Mechanics_of_Solids to calculate the stress in a pressurized cylinder, which has inner radius 1, exterior radius 3, and is subjected to pressure *p=1* on its internal bore (all in arbitrary units), and deforms under plane strain conditions (axially). The figure on the preceding page shows an example mesh with 6 nodes, with the appropriate boundary conditions. The length in the *z* direction is not important – choose anything sensible. You can run simulations with Young's modulus *E=100* and Poissons ratio 0.3. Compare the FEA solution for displacements and stresses with the exact solution.

Chapter 8
Theory and Implementation of the Finite Element Method

Note: The problems in this chapter make use of demonstration finite element codes provided with the text 'Applied Mechanics of Solids' (2nd edition), Taylor and Francis, (2026). The codes can be found on the Github site https://github.com/albower/Applied_Mechanics_of_Solids.

8.1 Generalized Finite Element Method for Static Linear Elasticity

Problem 8.1 Consider a one-dimensional isoparametric quadratic element, illustrated in the figure, and described in more detail in Section 8.1.5 of Applied Mechanics of Solids.



(a) Suppose that the nodes have coordinates $x_1^1 = 0$, $x_1^3 = 1$, $x_1^2 = 2$. Using a parametric plot, construct graphs showing the spatial variation of displacement in the element, assuming that the nodal displacements are given by
   (i) $u_1^1 = 1,\ u_1^2 = 0,\ u_1^3 = 0$
   (ii) $u_1^1 = 0,\ u_1^2 = 1,\ u_1^3 = 0$
   (iii) $u_1^1 = 0,\ u_1^2 = 0,\ u_1^3 = 1$
   (iv) $u_1^1 = 0,\ u_1^2 = 0.5,\ u_1^3 = 1$

(b) Suppose that the nodes have coordinates $x_1^1 = 0$, $x_1^3 = 1.75$, $x_1^2 = 2$. Plot graphs showing the spatial variation of displacement in the element for each of the four sets of nodal displacements given in (a). What does the result tell you about the placement of the middle node in a quadratic element?

Problem 8.2 Consider the finite element scheme to calculate displacements in an axially loaded 1D bar, described in Section 8.1.5 of Applied Mechanics of Solids.

(a) Calculate an exact analytical expression for the 3x3 stiffness matrix

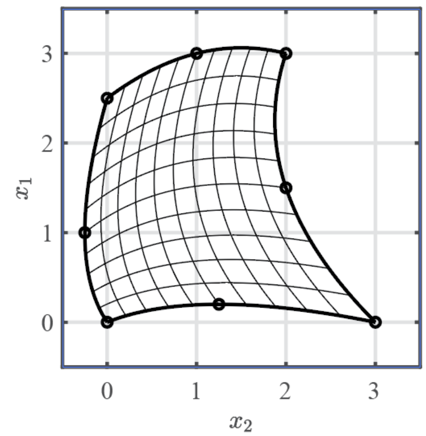
$$k_{ab} = \frac{2\mu(1-\nu)}{1-2\nu} h^2 \int_{x^1}^{x^2} \frac{\partial N^a(x)}{\partial x} \frac{\partial N^b(x)}{\partial x} dx$$

for a quadratic isoparametric 1-D element illustrated in the figure shown with the preceding problem (assume the nodes are located at $x^1 = -2,\ x^2 = 2 \quad x^3 = 0$).

(b) Write a simple code to integrate the stiffness matrix numerically, using the procedure described in 8.1.5 and compare the result with the exact solution (for this test, choose material and geometric parameters that give $2\mu(1-\nu)h^2/(1-2\nu) = 1$) Try integration schemes with 1, 2 and 3 integration points.

(c) Repeat the calculations in (a) and (b) for an element with nodes at $x^1 = -2,\ x^2 = 2 \quad x^3 = 0.5$

Problem 8.3 Consider the mapped, 8 noded isoparametric element illustrated in the figure. Write a simple program to plot a grid showing lines of $\xi_1 = \text{constant}$ and $\xi_2 = \text{constant}$ in the mapped element, as shown. Plot the grid with the following sets of nodal coordinates



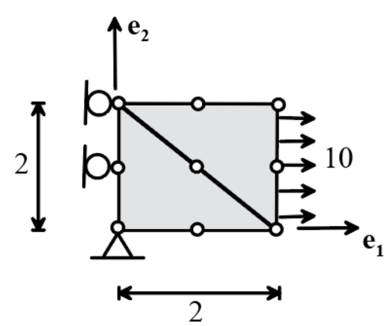
(a) $x_1^{(1)} = 0,\ x_2^{(1)} = 0,\ x_1^{(2)} = 2.0,\ x_2^{(2)} = 0.0,$
$x_1^{(3)} = 2.0,\ x_2^{(3)} = 2.0,\ x_1^{(4)} = 0.0,\ x_2^{(4)} = 2.0,$
$x_1^{(5)} = 1.0,\ x_2^{(5)} = 0.0,\ x_1^{(6)} = 2.0,\ x_2^{(6)} = 1.0,$
$x_1^{(7)} = 1.0,\ x_2^{(7)} = 2.0,\ x_1^{(8)} = 0.0,\ x_2^{(8)} = 1.0$

(b) $x_1^{(1)} = 0,\ x_2^{(1)} = 0,\ x_1^{(2)} = 2.0,\ x_2^{(2)} = 2.0,$
$x_1^{(3)} = 2.0,\ x_2^{(3)} = 6.0,\ x_1^{(4)} = 0.0,\ x_2^{(4)} = 4.0,$
$x_1^{(5)} = 1.0,\ x_2^{(5)} = 1.0,\ x_1^{(6)} = 2.0,\ x_2^{(6)} = 4.0,$
$x_1^{(7)} = 1.0,\ x_2^{(7)} = 5.0,\ x_1^{(8)} = 0.0,\ x_2^{(8)} = 2.0$

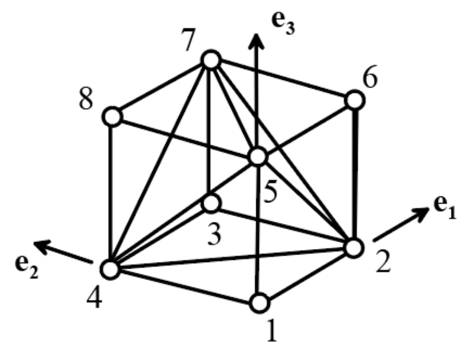
(c) $x_1^{(1)} = 0,\ x_2^{(1)} = 0,\ x_1^{(2)} = 2.0,\ x_2^{(2)} = 0.,\ x_1^{(3)} = 2.0,\ x_2^{(3)} = 2.0,\ x_1^{(4)} = 0.0,\ x_2^{(4)} = 2.0$
$x_1^{(5)} = 1.5,\ x_2^{(5)} = 1.0,\ x_1^{(6)} = 2.0,\ x_2^{(6)} = 1.0,\ x_1^{(7)} = 1.0,\ x_2^{(7)} = 2.0,\ x_1^{(8)} = 0.0,\ x_2^{(8)} = 1.0$

Note that for the latter case, there is a region in the element where $\det\left(\partial x_i / \partial \xi_j\right) < 0$. This is unphysical. Consequently, if elements with curved sides are used in a mesh, they must be designed carefully to avoid this behavior. In addition, quadratic elements can perform poorly in large displacement analyses.

Problem 8.4 Set up an input file for the general 2D/3D linear elastic finite element code provided on the Github site for Applied Mechanics of Solids to test the 6 noded plane triangular elements. Run the test shown in the figure (dimensions and loading are in arbitrary units), and use a Young's modulus and Poissons ratio $E = 1000, \nu = 0.3$. Compare the FEA solution to the exact solution (try both plane stress and plane strain).



Problem 8.5 Set up an input file for the general 2D/3D linear elastic finite element code provided on the Github site for Applied Mechanics of Solids to test the 4 noded tetrahedral elements. Run the test shown in the figure. Take the sides of the cube to have length 2 (arbitrary units), and take $E=10, \nu=0.3$. Run the following boundary conditions:

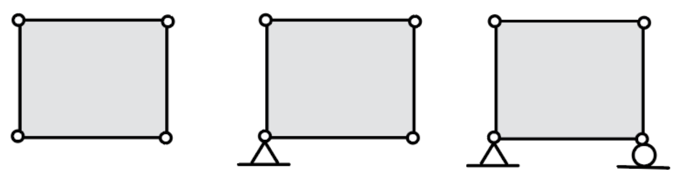


(a) $u_1=u_2=u_3=0$ at node 1, $u_2=u_3=0$ at node 2, and $u_3=0$ at nodes 3 and 4. The faces at $x_3=2$ subjected to uniform traction $t_3=2$ (arbitrary units)

(b) $u_1=u_2=u_3=0$ at node 1, $u_2=u_1=0$ at node 5, and $u_2=0$ at nodes 2 and 6. The face at $x_2=2$ subjected to uniform traction $t_2=2$ (arbitrary units)

In each case compare the finite element solution with the exact solution (they should be equal)

Problem 8.6 Add lines to the 2D/3D linear elastic finite element code provided on the Github site for Applied Mechanics of Solids to compute the determinant and the eigenvalues of the global stiffness matrix.

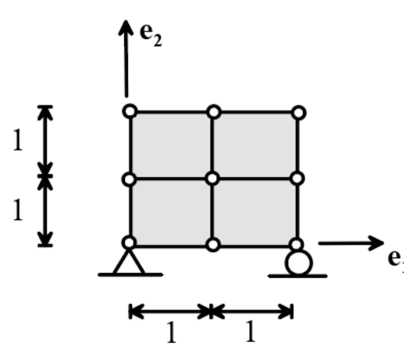


(a) Calculate the determinant and eigenvalues for a mesh containing a single 4 noded quadrilateral element, with each of the boundary conditions shown in the figure. (use sensible values for dimensions, Young's modulus and Poisson's ratio).

(b) For each case that has zero eigenvalues, choose one of the zeros, and find the eigenvector corresponding to the zero eigenvalue. Set the displacements equal to this eigenvector, and plot the shape of the deformed element (the code will do this automatically once the displacement has been defined). Also, check the stresses associated with this displacement field (also automatic – check the results file for the data).

(c) Briefly discuss the implications of the results on the nature of solutions to the finite element equations.

Problem 8.7 Set up an input file for the general 2D/3D linear elastic finite element code provided on the Github site for Applied Mechanics of Solids to test the mesh shown in the figure. Use material properties $E=100, \nu=0.3$ and assume plane strain deformation. Run the following tests



(a) Calculate the determinant of the global stiffness matrix

(b) Change the code so that the element stiffness matrix is computed using only a single integration point. Calculate the determinant and eigenvalues of the global stiffness matrix. Also, find an eigenvector corresponding to one of the zero eigenvector, and set the displacements equal to this eigenvector. Plot the deformed mesh with these displacements (you may need to reduce the scale factor for the displacements for this case to see the deformation clearly). Check the stresses associated with the deformation.

Problem 8.8 Extend the 2D/3D linear elastic finite element code provided on the Github site for Applied Mechanics of Solids to solve 3D problems involving anistropic elastic solids with cubic symmetry. This will require the following steps:

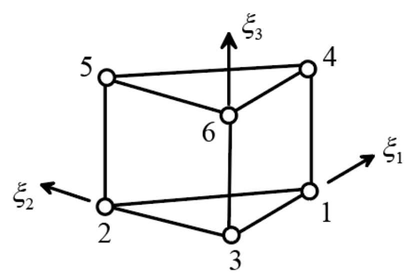
- The elastic constants E,ν,μ for the cubic crystal must be read from the input file
- The orientation of the crystal must be read from the input file. The orientation of the crystal can be specified by the components of vectors parallel to the [100] and [010] crystallographic directions.
- The parts of the code that compute the element stiffness matrix and (during post-processing) stress will need to be modified to use elastic constants for a cubic crystal. The calculation is complicated by the fact that the components of must be expressed in the *global* coordinate system, instead of a coordinate system aligned with the crystallographic directions.

Test your code by using it to compute the stresses and strains in a uniaxial tensile specimen made from a cubic crystal, in which the unit vectors parallel to [100] and [010] directions have components $\mathbf{n}_{[100]} = \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_3 \quad \mathbf{n}_{[010]} = -\sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_3$. Mesh the specimen with a single cubic 8 noded brick element. Apply boundary conditions to the element that will induce a state of uniaxial tension of 100MPa parallel to the $\mathbf{e}_3$ axis. Run the following tests:

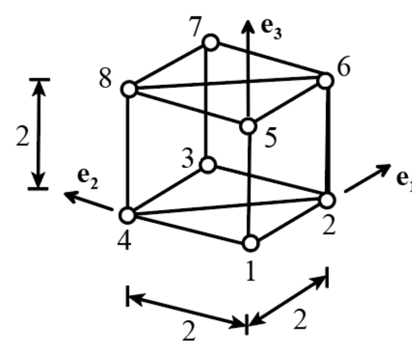
(a) Verify that if E,ν,μ are given values that represent an isotropic material, the stresses and strains the element are independent of θ.

(b) Use values for E,ν,μ representing gold, with $\theta = 45^0$. Compare your predictions with the simulation done in ABAQUS in problem 7.10(f)

Problem 8.9 Implement the linear 3D wedge-shaped element shown in the figure the 2D/3D linear elastic finite element code provided on the Github site for Applied Mechanics of Solids. To construct the shape functions for the element, use the shape functions for a linear triangle to write down the variation with (ξ_1,ξ_2), and use a linear variation with ξ_3. Assume that $0 \le \xi_i \le 1$. You will need to add the shape functions and their derivatives to the appropriate functions in the code, as well as the integration points and weights for this element. In addition, you will need to modify the functions that compute the traction vectors associated with pressures acting on the element faces.



Test your code by meshing a cube with two wedge-shaped elements as shown in the figure. Take the sides of the cube to have length 2 (arbitrary units), and take $E=10, \nu=0.3$. Run the following boundary conditions:

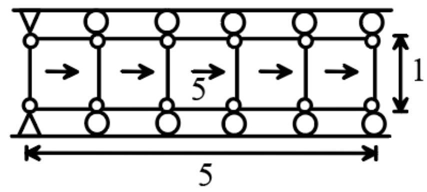


(a) $u_1 = u_2 = u_3 = 0$ at node 1, $u_2 = u_3 = 0$ at node 2, and $u_3 = 0$ at nodes 3 and 4. The faces at $x_3 = 2$ subjected to uniform traction $t_3 = 2$ (arbitrary units)

(b) $u_1 = u_2 = u_3 = 0$ at node 1, $u_2 = u_1 = 0$ at node 5, and $u_2 = 0$ at nodes 2 and 6. The face at $x_2 = 2$ subjected to uniform traction $t_2 = 2$ (arbitrary units)

In each case compare the finite element solution with the exact solution (they should be equal)

Problem 8.10 Extend the 2D/3D linear elastic finite element code provided on the Github site for Applied Mechanics of Solids to solve problems that include body forces. This will require the following steps



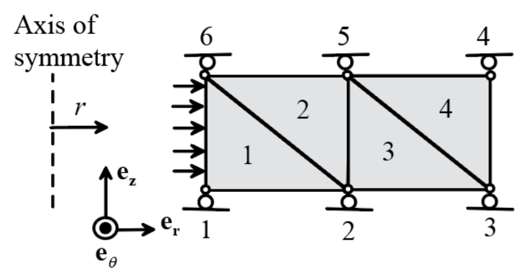
- You will need to read a list of elements subjected to body forces, and the body force vector for each element. The list can be added to the end of the input file.
- You will need to write a function to calculate the contribution from an individual element to the global system of finite element equations. This means evaluating the integral

$$\int_{V_e^{(l)}} b_i N^a(\mathbf{x}) dV$$

  over the volume of the element. The integral should be evaluated using numerical quadrature – the procedure (other than the integrand) is essentially identical to computing the stiffness matrix.
- You will need to add the contribution from each element to the global force vector. It is simplest to do this by modifying the procedure called globaltraction().

Test your code by using it to calculate the stress distribution in a 1D bar which is constrained as shown in the figure, and subjected to a uniform body force with magnitude 1 unit. Use 4 noded plane strain quadrilateral elements, and set $E = 10, \nu = 0.25$ (arbitrary units) and take the body force to have magnitude 5. Compare the displacements and stresses predicted by the finite element computation with the exact solution.

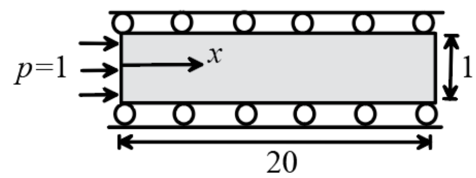
Problem 8.11 Extend the general 2D/3D linear elastic code provided on the Github site for Applied Mechanics of Solids to solve axisymmetric boundary value problems for linear elastic solids. You can follow the approach outlined (for constant strain triangles) in Problem 7.33. Modifications will include:



- In the functions that calculate the element stiffness matrix and stress and strain for output, change the matrix [B] that maps displacements to strains to include the hoop strains, and modify the matrix [D] of material properties. You can use the 'elident' flag to identify axisymmetric elements. In addition, change the integration scheme to account for the hoop component of the volume integral.
- Modify the functions that calculate the external force vectors for the element to account for the hoop component of the area (and volume, if you included body forces) integral.

Test your code by calculating the stress and displacement in an internally pressurized cylinder that deforms in plane strain and comparing the prediction to the analytical solution. You can mesh the cylinder with a single row of elements (you could try quadrilateral elements instead of triangles)

8.2 The Finite Element Method for Dynamic Linear Elasticity



Problem 8.12 Set up the general purpose 2D/3D dynamic finite element code provided on the Github site for Applied Mechanics of Solids to solve the 1D wave propagation problem shown in the figure. Use material properties $E=10, \nu=0.25, \rho=1$ (arbitrary units). Mesh the bar with 20 square 4 noded quadrilateral plane strain elements. Assume that the left hand end of the bar is subjected to a constant traction with magnitude *p=1* at time *t=0*.

(a) Calculate the maximum time step for a stable explicit computation.

(b) Run an explicit dynamics computation (Newmark parameters $\beta_1 = 1/2, \beta_2 = 0$, and a lumped mass matrix) with a time step of 90% of the stable limit. Use the simulation to plot the time variation of velocity at *x=0*. Use 150 time steps. Compare the numerical solution with the exact solution (the two will not agree exactly – the coarse mesh cannot resolve the sharp wave-front which causes some ringing and dispersion in the numerical solution).

(c) Repeat (b) with a time step equal to 10% greater than the critical timestep for instability.

(d) Repeat (c) with Newmark parameters $\beta_1 = \beta_2 = 1/2$.

(e) Repeat (a) and (b) with Newmark parameters $\beta_1 = \beta_2 = 1$. Try a time step equal to twice the theoretical stable limit.

Problem 8.13 The 2D/3D dynamic finite element code provided on the Github site for Applied Mechanics of Solids uses the row-sum method to compute lumped mass matrices.

(a) Use the row-sum method to compute lumped mass matrices for linear and quadratic triangular elements with sides of unit length and mass density $\rho = 1$ (you can simply print out the matrix from the code)

(b) Use the row-sum method to compute lumped mass matrices for linear and quadratic quadrilateral elements with sides of unit length and mass density $\rho = 1$

(c) Modify the finite element code to compute the lumped mass matrix using the scaled diagonal method instead of the row-sum method (you can write your code to use the 'lumpedmass' variable to switch from one method to another so both are available to your customers). Repeat (a) and (b) for the quadratic elements.

(d) Repeat problem 8.12(b) (using Newmark parameters $\beta_1 = 1/2, \beta_2 = 0$) using the scaled diagonal version of the lumped mass matrix, using quadratic quadrilateral elements to mesh the solid. Use a time-step of 20% of the critical value found in 12(a) (quadratic elements become unstable at lower time-steps because of the nonuniform mass distribution between the nodes) and run for 1000 steps. Compare the results with the predictions using a full mass matrix. (If you are curious you could try the row-sum lumped mass matrix as well, but the results will always be unstable!)

Problem 8.14 Write a script that uses the Newmark algorithm to integrate the equation of motion

$$\frac{d^2u}{dt^2}+u=0$$

(the harmonic oscillator, with time normalized to give a normalized period of oscillation 2π). Use initial conditions $u=1,\ du/dt=0$. In addition to the *u,v*, have your code calculate the normalized energy $E=(u^2+v^2)/2$ as a function of time.

(a) Test your code by plotting the variation of *u* with time using Newmark parameters $\beta_1=1/2, \beta_2=0$ and a time step $\Delta t=0.025$. Run for 1000 time-steps.

(b) Plot the variation of total energy with time using the time-step in (a), 300 time-steps, and $\beta_1=1/2, \beta_2=0$; $\beta_1=1/2, \beta_2=1/2$, $\beta_1=1, \beta_2=1/2$.

(c) Use numerical experiments to find the critical timestep for instability with $\beta_1=1/2, \beta_2=0$.

Problem 8.15 Consider the equation of motion for the harmonic oscillator

$$\frac{d^2u}{dt^2}+u=0$$

(a) Show that the Newmark algorithm for this ODE can be re-written as a recursion relation $\mathbf{P}\mathbf{r}_{n+1}=\mathbf{Q}\mathbf{r}_n$, where

$$\mathbf{P}=\begin{bmatrix}1+\beta_2\Delta t^2/2 & 0\\ \Delta t\beta_1 & 1\end{bmatrix}\qquad \mathbf{Q}=\begin{bmatrix}1-(1-\beta_2)\Delta t^2/2 & \Delta t\\ \left(\beta_1-1\right)\Delta t & 1\end{bmatrix}\qquad \mathbf{r}=\begin{bmatrix}u\\ v\end{bmatrix}$$

(b) Show that recursion relation in part (a) has the solution

$$\mathbf{r}_n=A\mathbf{q}_1\lambda_1^n+B\mathbf{q}_2\lambda_2^n$$

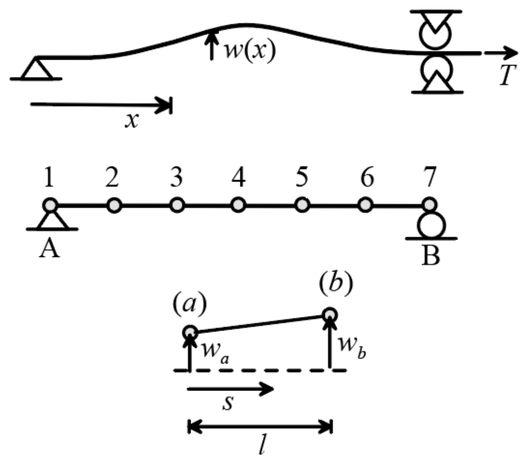
where (λ_1,λ_2), $(\mathbf{q}_1,\mathbf{q}_2)$ are the eigenvalues and eigenvectors of the amplification matrix $\mathbf{A}=\mathbf{P}^{-1}\mathbf{Q}$.

(c) The solution is stable as long as $|\lambda_i|\le 1$ (note that λ_i is complex for small time-steps). Find the eigenvalues for the explicit Newmark algorithm $\beta_1=1/2, \beta_2=0$, and hence calculate the stable time-step.

(d) Find $|\lambda_i|$ for the mid-point version of the Newmark algorithm $\beta_1=1/2, \beta_2=1/2$

(e) Find $|\lambda_i|$ for the fully implicit version of the Newmark algorithm $\beta_1=1, \beta_2=1$ (assume $\Delta t<4$)

Problem 8.16 In this problem you will implement a simple 1D finite element method to calculate the motion of a stretched vibrating string. The string has mass per unit length *m*, and is stretched by applying a tension *T* at one end. Assume small deflections. The equation of motion for the string (see Section 10.3.1 of Applied Mechanics of Solids) is



$$T\frac{d^2w}{dx^2}=m\frac{d^2w}{dt^2}$$

and the transverse deflection must satisfy $w=0$ at $x=0, x=L$.

(a) **Weak form of the equation of motion**. Let δw be a kinematically admissible variation of the deflection, satisfying $\delta w=0$ at $x=0, x=L$. Show that if

$$\int_0^L m\frac{d^2w}{dt^2}\delta w dx+\int_0^L T\frac{dw}{dx}\frac{d\delta w}{dx}dx=0$$

for all admissible δw, then *w* satisfies the equation of motion.

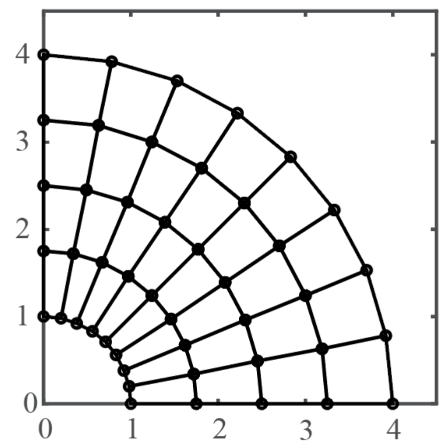
(b) Finite **element equations.** Introduce a linear finite element interpolation as illustrated in the figure. Calculate expressions for the full and lumped element mass matrices as well as th stiffness matrix (you can calculate analytical expressions for the matrices, or evaluate the integrals numerically. You may already have found the stiffness matrix if you solved problem 7.32)

(c) **Implementation:** Write a simple code to compute and integrate the equations of motion using the Newmark time integration procedure. Test your code by computing the motion of the string with different initial conditions. For example, try $w(x,t=0)=(L/10)\sin(n\pi x/L)$, *n=1,2….* These are the mode shapes for string, so the vibration should be harmonic. Use geometric and material parameters *L=10*, *m=1*, *T=1* (arbitrary units). Choose sensible values for the mesh size, Newmark parameters and time-step. Plot the variation of displacement at the center of the string for *n*=1 and *n*=3, and compare the prediction with the exact solution.

Problem 8.17 Modify the 1D code described in the preceding problem to calculate the natural frequencies and mode shapes for the stretched string. Calculate the lowest 3 nonzero natural frequencies and plot the corresponding mode shapes, using the same parameters as problem 8.16. Compare the numerical results with the exact solution.

8.3 Finite Element Method for Hypoelastic Materials

Problem 8.18 Set up an input file for the example hypoelastic finite element code on the Github site for Applied Mechanics of Solids to calculate the deformation and stress in a hypoelastic pressurized cylinder deforming under plane strain conditions. Use the mesh shown in the figure, with appropriate symmetry boundary conditions on $x_1 = 0$ and $x_2 = 0$. Apply a pressure of 50 (arbitrary units) to the internal bore of the cylinder and leave the exterior surface free of traction. Use the following material properties: $\sigma_0 = 10, \varepsilon_0 = 0.001, n = 4, \nu = 0.3$. Plot a graph showing the variation of the radial displacement of the inner bore of the cylinder as a function of the internal pressure. Note that the example code uses fully integrated elements and so seriously underestimates the displacement because of volumetric locking – see the problems 8.30-8.32 for a fix for this problem.



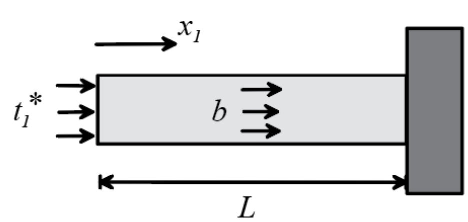
Problem 8.19 Consider solving the following coupled nonlinear equations for x and y using Newton-Raphson iteration

$$x\left(x^2 + y^2 + 1\right)^{1/3} - 5 = 0 \qquad\qquad y\left(x^2 + y^2 + 1\right)^{1/3} + 3 = 0$$

(a) The correction $d\mathbf{w}$ to a current approximation $\mathbf{w} = [x, y]$ for the solution is found by solving a pair of linear equations of the form $\mathbf{K}d\mathbf{w} = -\mathbf{r}$. Find formulas for **K** and **r**.

(b) Write a script to solve the equations.

Problem 8.20 Consider the simple 1D bar shown in the figure. Assume that the bar deforms in a state of uniaxial stress, so that σ_{11} is the only nonzero stress component. In addition, suppose that the bar has a uniaxial stress-strain curve



$$\sigma(\varepsilon) = \begin{cases} \text{sign}(\varepsilon)\sigma_0 \left[\sqrt{\dfrac{1+n^2}{(n-1)^2} - \left(\dfrac{n}{n-1} - \dfrac{|\varepsilon|}{\varepsilon_0} \right)^2} - \dfrac{1}{n-1} \right] & |\varepsilon| \le \varepsilon_0 \\ \text{sign}(\varepsilon)\sigma_0 \left(\dfrac{|\varepsilon|}{\varepsilon_0} \right)^{1/n} & |\varepsilon| \ge \varepsilon_0 \end{cases}$$

(a) Show that the virtual work principle can be reduced to

$$\int_0^L \sigma_{11} \frac{\partial \delta v_1}{\partial x_1} dx_1 - \int_0^L b \delta v_1 dx_1 - t_1^* \delta v_1(0) = 0$$

where δv_i is a kinematically admissible variation of the horizontal displacement u_1.

(b) Write down a 1D piecewise-linear finite element interpolation for u_1 and δv_1, and hence use the virtual work equation to obtain a nonlinear system of equations for nodal values of u_1^a.

(c) The nonlinear equations can be solved using a Newton-Raphson procedure, by repeatedly solving a system of linear equations

$$K_{ab} dw_1^b + R^a - F^a = 0$$

where dw_1^b denotes a correction to the current approximation to u_1^a, and K_{ab} and R^a are assembled from element stiffness matrices and residual force vectors. Give expressions for the contribution to K_{ab} and R^a from one element, in terms of interpolation functions N^a and the material tangent stiffness $D = \partial\sigma / \partial\varepsilon$.

(d) Implement the procedure outlined in part (c) as a simple 1D finite element code to find the displacement and stress in the bar. Test your code by (i) calculating a numerical solution with material properties $\sigma_0 = 5$, $n=4$, $\varepsilon_0 = 0.01$ and loading $b=0$, $t_1^* = 10$ (compare the FEA prediction for the strain with the exact solution); and (ii) calculating a numerical solution for a bar with length $L=10$ with material properties $\sigma_0 = 5$, $n=5$, $\varepsilon_0 = 0.01$ and loading $b=1$, $t_1^* = 0$ (plot a graph of the variation of displacement along the length of the bar)

Problem 8.21 Extend the example hypoelastic finite element code on the Github site for Applied Mechanics of Solids to solve problems using plane stress elements. This will require the following steps:

- When calculating the stress, the out-of-plane strain ε_{33} must be calculated, by solving the equation $\sigma_{33}(\varepsilon_{33}) = 0$ using Newton-Raphson iteration. You can save some busy-work by noting that the derivative $d\sigma_{33} / d\varepsilon_{33}$ required in this calculation is one of the components of the material stiffness for a general 3D stress state.
- The 2D material tangent stiffness matrix must account for the plane stress condition. This sounds painful, but the plane stress stiffness can always be easily extracted from the general 3D case. Recall that (for a 3D problem) stress and strain are stored in 6 dimensional vectors σ_i, ε_i. The tangent stiffness is stored in a 6x6 matrix D_{ij}, which relates an infinitesimal stress increment to a strain increment by

$$d\sigma_i = D_{ij} d\varepsilon_j \qquad i, j = 1 \ldots 6$$

For plane stress deformation, the out-of-plane strain increment $d\varepsilon_3$ is determined from the condition that the out-of-plane stress $d\sigma_3$ must vanish. This shows that

$$d\varepsilon_3 = -\sum_{k \neq 3} D_{3k} d\varepsilon_k \Big/ D_{33}$$

The components of the 3x3 plane stress tangent matrix d_{ij} follow by substituting this result into the 3D formula, so that

$$\begin{aligned}
d_{11} &= D_{11} - D_{13} D_{31} / D_{33} \\
d_{22} &= D_{22} - D_{23} D_{32} / D_{33} \\
d_{33} &= D_{44} - D_{43} D_{34} / D_{33} \\
d_{12} &= d_{21} = D_{12} - D_{13} D_{32} / D_{33} \\
d_{13} &= d_{13} = D_{14} - D_{34} D_{13} / D_{33} \\
d_{23} &= d_{32} = D_{24} - D_{34} D_{23} / D_{33}
\end{aligned}$$

Implement these steps in the sample code, and test your solution by plotting the variation of stress with strain in a quadrilateral 4 noded element with sides of unit length that is loaded in uniaxial tension (choose sensible values for the material properties and applied stress, and compare the prediction to the exact solution).

8.4 Finite Element Method for Large Deformations: Hyperelastic Materials

Problem 8.22 Write an input file for the demonstration hyperelastic finite element code on the Github site for Applied Mechanics of Solids to calculate the stress in a Neo-Hookean tensile specimen with shear modulus $\mu = 1$ and bulk modulus $K = 1000$ subjected to uniaxial tensile stress. It is sufficient to model the specimen using a single 8 noded brick element. Use the code to plot a graph showing the uniaxial Cauchy stress as a function of the stretch ratio $\lambda = l / l_0$. Compare the finite element solution with the exact solution for an incompressible Neo-Hookean solid.

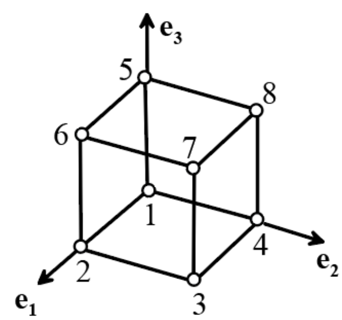
Problem 8.23 Extend the hyperelastic finite element code to solve problems involving a Mooney-Rivlin material. This will require the following steps:

(a) Calculate the tangent stiffness C^e_{ijkl} for the Mooney-Rivlin material

(b) Modify the hyperelastic FEA code to model the Mooney-Rivlin material. Test your code by plotting the variation of Cauchy stress with stretch for a material that is subjected to biaxial tension. Use material properties $\mu_1 = 1GPa,\ \mu_2 = 0.1GPa,\ K = 10000GPa$, and compare the prediction with the exact solution (for an incompressible material).

8.5 Finite Element Method for Inelastic Materials

Problem 8.24 Set up the demonstration viscoplastic finite element code provided on the Github site for Applied Mechanics of Solids to calculate the stress-strain relation for a viscoplastic material under uniaxial tension. The code uses a stress-v-strain rate relation given by



$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij}$$

$$\dot{\varepsilon}^e_{ij} = \frac{1+\nu}{E}\left(\dot{\sigma}_{ij} - \frac{\nu}{1+\nu}\dot{\sigma}_{kk}\delta_{ij}\right)$$

$$\dot{\varepsilon}^p_{ij} = \dot{\varepsilon}_0\left(\frac{\sigma_e}{\sigma_0}\right)^m \frac{3}{2}\frac{S_{ij}}{\sigma_e}$$

$$\sigma_0 = Y\left(1 + \frac{\varepsilon_e}{\varepsilon_0}\right)^{1/n} \qquad S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3 \qquad \sigma_e = \sqrt{\frac{3}{2}S_{ij}S_{ij}} \qquad \dot{\varepsilon}_e = \sqrt{\frac{2}{3}\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij}}$$

where $Y\, n,\ m,\ \dot{\varepsilon}_0$, and ε_0 are material properties. Mesh the specimen with a single 8 noded brick element with sides of unit length, using the mesh shown in the figure. Apply the following boundary constraints to the specimen: $u_1 = u_2 = u_3 = 0$ at node 1; $u_2 = u_3 = 0$ at node 2, $u_3 = 0$ at nodes 3 and 4. Apply a uniform normal traction whose magnitude increases from 0 to 20 (arbitrary units) in time of 2 units on the top face of the element. Run a simulation with the following material parameters:

$$E = 1000,\quad \nu = 0.3,\quad Y = 15,\ \varepsilon_0 = 0.5,\ n = 10,\quad \dot{\varepsilon}_0 = 0.1,\quad m = 10$$

(a) Plot a graph showing the variation of stress with strain for the material (use the traction and displacement applied to the element to compute stress and strain).

(b) Modify the boundary conditions so that only the constraint $u_3 = 0$ is enforced at node 2. Run the code to attempt to find a solution (you will have to abort the calculation). Explain why the Newton iterations do not converge.

(c) Repeat (a) with m=500 (you will have to abort the calculation). The code fails because rounding errors make it impossible to calculate the vanishingly small plastic strain rate at low stress levels. For such large m values it is better to use a rate independent material model.

Problem 8.25 The sample FEA code for viscoplasticity described in the preceding problem implements plane strain and 3D elements. Extend the code to solve problems with plane stress 2D elements. This will require the following steps:

(a) The procedure that is used to calculate the stress $\sigma_{ij}^{(n+1)}$ at the end of a time increment must be modified to enforce the plane stress condition $\sigma_{33}^{(n+1)}=0$. For a given applied strain increment $(\Delta\varepsilon_{11},\Delta\varepsilon_{22},\Delta\varepsilon_{12})$, this condition is a nonlinear function of the unknown out-of-plane strain increment $\Delta\varepsilon_{33}$ and the effective plastic strain increment $\Delta\varepsilon_e$. Modify the procedure described in Section 8.5.4 of Applied Mechanics of Solids to solve concurrently for $\Delta\varepsilon_{33}$ and $\Delta\varepsilon_e$.

(b) Devise a way to modify the 6x6 material tangent stiffness matrix **D** to extract the 3x3 plane stress version of the matrix.

(c) Implement parts (a) and (b) in the sample code, and test your element by plotting the uniaxial stress-strain curve for the viscoplastic solid, using a single 4 noded plane stress quadrilateral element (you can check your prediction by comparing it to the solution using 3D elements).

Problem 8.26 Modify the viscoplastic finite element code provided on the Github site for Applied Mechanics of Solids to solve problems involving a rate independent, power-law isotropic hardening elastic-plastic solid, with incremental stress-strain relations

$$\Delta\varepsilon_{ij}=\Delta\varepsilon_{ij}^{e}+\Delta\varepsilon_{ij}^{p}$$

$$\Delta\varepsilon_{ij}^{e}=\frac{1+\nu}{E}\left(\Delta\sigma_{ij}-\frac{\nu}{1+\nu}\Delta\sigma_{kk}\delta_{ij}\right)\qquad \Delta\varepsilon_{ij}^{p}=\Delta\varepsilon_{e}\frac{3}{2}\frac{S_{ij}}{\sigma_{e}}$$

$$S_{ij}=\sigma_{ij}-\sigma_{kk}\delta_{ij}/3\qquad \sigma_e=\sqrt{\frac{3}{2}S_{ij}S_{ij}}\qquad \Delta\varepsilon_e=\sqrt{\frac{2}{3}\Delta\varepsilon_{ij}^{p}\Delta\varepsilon_{ij}^{p}}$$

and a yield criterion

$$\sigma_e-Y_0\left(1+\frac{\varepsilon_e}{\varepsilon_0}\right)^{1/n}=0$$

Your solution should include the following steps:

(a) Devise a method for calculating the stress $\sigma_{ij}^{(n+1)}$ at the end of a load increment. Use a fully implicit computation, in which the yield criterion is exactly satisfied at the end of the load increment. Your derivation should follow closely the procedure described in Section 8.5.4, except that the relationship between $\sigma_e^{(n+1)}$ and $\Delta\varepsilon_e$ must be calculated using the yield criterion, and you need to add a step to check for elastic unloading.

(b) Calculate the tangent stiffness $\partial\sigma_{ij}^{(n+1)}/\partial\Delta\varepsilon_{kl}$ for the rate independent solid, by differentiating the result of (a).

(c) Implement the results of (a) and (b) in a finite element code (it is easiest to add a rate independent option to the viscoplatic code). Test your code by using it to calculate the stress-strain relation for the rate independent material under uniaxial tension, and compare the prediction to the exact solution. Use the mesh, loading and boundary conditions described in Problem 8.24, and use material properties $E=1000,\ \ \nu=0.3\ \ Y=15,\ \varepsilon_0=0.5\ \ n=10$, with a traction of 16 units applied to the top face of the element.

Problem 8.27 Modify the viscoplastic finite element code described in Section 8.5.7 of Applied Mechanics of Solids to solve problems involving a rate independent, linear kinematic hardening elastic-plastic solid, with incremental stress-strain relations

$$\Delta\dot{\varepsilon}_{ij} = \Delta\dot{\varepsilon}_{ij}^{e} + \Delta\dot{\varepsilon}_{ij}^{p}$$

$$\Delta\varepsilon_{ij}^{e} = \frac{1+\nu}{E}\left(\Delta\sigma_{ij} - \frac{\nu}{1+\nu}\Delta\sigma_{kk}\delta_{ij}\right) \qquad \Delta\varepsilon_{ij}^{p} = \Delta\varepsilon_{e}\frac{3}{2}\frac{S_{ij}-\alpha_{ij}}{Y}$$

$$S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3 \quad \sigma_{e} = \sqrt{\frac{3}{2}(S_{ij}-\alpha_{ij})(S_{ij}-\alpha_{ij})} \quad \Delta\varepsilon_{e} = \sqrt{\frac{2}{3}\Delta\varepsilon_{ij}^{p}\Delta\varepsilon_{ij}^{p}}$$

and a yield criterion and hardening law

$$\sigma_{e} - Y = 0 \qquad \Delta\alpha_{ij} = c\Delta\varepsilon_{e}\frac{\left(S_{ij}-\alpha_{ij}\right)}{Y}$$

Your solution should include the following steps:

(a) Devise a method for calculating the stress $\sigma_{ij}^{(n+1)}$ at the end of a load increment. Your derivation should follow closely the procedure described in Section 8.5.4 of Applied Mechanics of Solids, except that
(i) you will need to update (and store) the current position of the center of the yield surface $\alpha_{ij}^{(n+1)}$;
(ii) the relationship between $\sigma_{e}^{(n+1)}$, $\Delta\varepsilon_{e}$ and $\alpha_{ij}^{(n)}$ must be calculated using the yield criterion, and
(iii) you need to add a step to check for elastic unloading.
Use a fully implicit update for both the stress and the center of the yield locus (so the change in their values is calculated using the hardening rule and flow rule applied at the end of the time-step).

(b) Calculate the tangent stiffness $C_{ijkl}^{ep} = \partial\sigma_{ij}^{(n+1)} / \partial\Delta\varepsilon_{kl}$ for the kinematically hardening material

(c) Implement the kinematic hardening law in a copy of the viscoplastic finite element code posted on the Github site for Applied Mechanics of Solids. Test your code by using it to calculate the stress-strain relation for the kinematically hardening material under uniaxial tension, and compare the prediction to the exact solution. Use the mesh, loading and boundary conditions described in Problem 24, and use material properties $E = 1000$, $\nu = 0.3$ $Y = 15$, $c = 10$. Apply a cycle of traction that varies between ± 18 units to the top face of the element. Compare your FEA predictions to the exact solution.

Problem 8.28 The goal of this problem is to develop a finite element code to solve *dynamic* problems involving viscoplastic materials. Dynamic problems for nonlinear materials are usually solved using explicit Newmark time integration, which is very straightforward to implement. As usual, the starting point for the finite element method is the virtual work principle

$$\int_{R}\rho\frac{\partial^{2}u_{i}}{\partial t^{2}}\delta v_{i}dV + \int_{R}\sigma_{ij}[\varepsilon_{kl}]\frac{\partial\delta v_{i}}{\partial x_{j}}dV - \int_{R}b_{i}\delta v_{i}dV - \int_{\partial_{2}R}t_{i}^{*}\delta v_{i}dA = 0$$

$$u_{i} = u_{i}^{*} \qquad \text{on } \partial_{1}R$$

(a) By introducing a finite element interpolation, show that the virtual work principle can be reduced to a system of equations of the form

$$\left(M_{ab}\ddot{u}_{i}^{b} + R_{i}^{a} - F_{i}^{a}\right) = 0$$

and give expressions for $M_{ab}, R_{i}^{a}, F_{i}^{a}$.

(b) To implement the finite element method, it is necessary to calculate the stress σ_{ij} in the solid. In this problem, the solid will be idealized as a viscoplastic material with constitutive equations described in problem 24. Since very small time-steps must be used in an explicit dynamic calculation, it is sufficient to integrate the constitutive equations with respect to time using an explicit method, in which the plastic strain rate is computed based on the stress at the start of a time increment. Show that the stress $\sigma_{ij}^{(n+1)}$ at time $t+\Delta t$ can be expressed in terms of the stress $\sigma_{ij}^{(n)}$ at time *t*, the increment in total strain $\Delta\varepsilon_{ij}$ during the time interval Δt and material properties as

$$\sigma_{ij}^{(n+1)} = \sigma_{ij}^{(n)} + \frac{E}{1+\nu}\left(\Delta e_{ij} - \Delta t\dot{\varepsilon}_0\left(\frac{\sigma_e^{(n)}}{\sigma_0^{(n)}}\right)^m \frac{3}{2}\frac{S_{ij}^{(n)}}{\sigma_e^{(n)}}\right) + \frac{E}{3(1-2\nu)}\Delta\varepsilon_{kk}\delta_{ij}$$

where

$$\Delta e_{ij} = \Delta\varepsilon_{ij} - \Delta\varepsilon_{kk}\delta_{ij}/3 \qquad \sigma_0 = Y\left(1+\frac{\varepsilon_e^{(n)}}{\varepsilon_0}\right)^{1/n}$$

(c) The equations of motion can be integrated using an explicit Newmark method using the following expressions for the acceleration, velocity and displacement at the end of a generic time-step

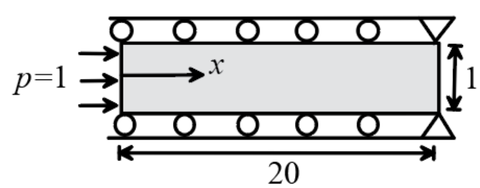
$$u_i^a(t+\Delta t) \approx u_i^a(t) + \Delta t\dot{u}_i^a(t) + \frac{\Delta t^2}{2}\ddot{u}_i^a(t)$$

$$\ddot{u}_i^a(t+\Delta t) = M_{ab}^{-1}\left[-R_i^b[u_i^a(t+\Delta t)] + F_i^b(t)\right]$$

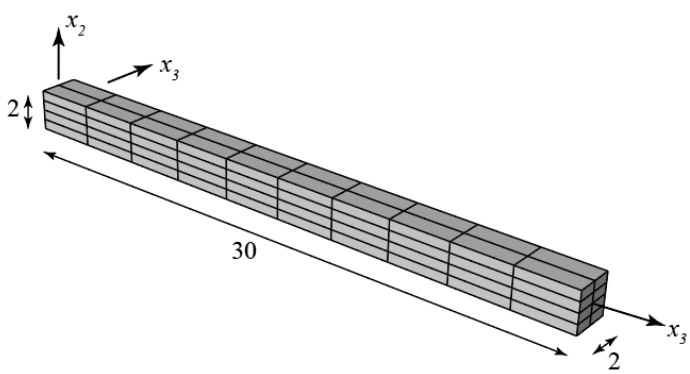
$$\dot{u}_i^a(t+\Delta t) \approx \dot{u}_i^a(t) + \Delta t\left[(1-\beta_1)\ddot{u}_i^a(t) + \beta_1\ddot{u}_i^a(t+\Delta t)\right]$$

A lumped mass matrix should be used to speed up computations. Note that the residual force vector R_i^a is a function of the displacement field in the solid. It therefore varies with time, and must be re-computed at each time step. This also means that you must apply appropriate constraints to nodes with prescribed accelerations at each step. Modify the elastodynamic FEA code provided on the Github site for Applied Mechanics of solids to implement this procedure (you can restrict your code to plane strain or 3D).

(d) Test your code by solving the 1D wave propagation problem shown in the figure. Use material properties $E=1000, \nu=0.25, \rho=1, Y=1, \varepsilon_0=0.1, n=5, \dot{\varepsilon}_0=1, m=4$ (arbitrary units). Mesh the bar with 20 square 4 noded quadrilateral plane strain elements. Assume that the left hand end of the bar is subjected to a constant pressure with magnitude 1. Take $\beta_1 = 0.5$ in the Newmark integration, and use 1000 time steps with step size 75% of the maximum stable time-step (for an elastic bar). Plot a graph showing the displacement of the bar at the left end of the bar as a function of time. Compare the solution to the displacement of an elastic bar with the same density and elastic properties (you can just set $\dot{\varepsilon}_0 = 0$ in your input file to make the material elastic).



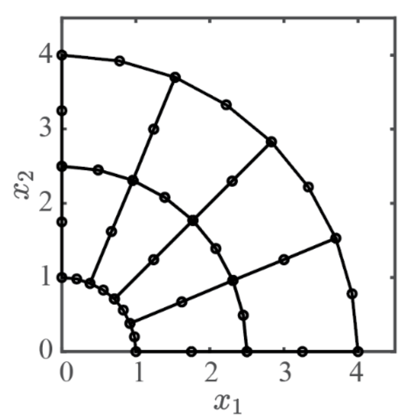
8.6 Advanced Element Formulations: Incompatible Modes; Reduced Integration; and Hybrid Elements



Problem 8.29 A FE code demonstrating 'Incompatible mode' elements is posted on the Github site for Applied Mechanics of Solids.

(a) Modify the code so that you can switch between standard isoparametric elements and 'Incompatible mode' elements using the element identifier. (Note that the stiffness for standard elements is computed as part of the stiffness for Incompatible mode elements).

(b) Test the 3D 8 noded 'Incompatible mode' element in the code by using it to predict the deflection of a cantilever beam subjected to a transverse force. Use the mesh shown in the figure. Fix all three components of displacement at the nodes on the left hand end of the beam, and apply a traction (0,-2,0) (arbitrary units) on the faces of the elements on the right hand end of the beam. If you want to avoid the tedious work of writing a script or typing the nodal coordinates and element connectivity into the intput file, you could use the learning edition of ABAQUS® to generate the mesh. To do this, create a part, boundary conditions, step, mesh, and job in the usual way, then in the job module select Job>Write input. This will create a file with a .inp extension in the ABAQUS® working directory. If you open this file with a text editor, you will see the nodal coordinates, element connectivity, and lists of nodes and elements with boundary conditions in the file. You can use these (with some minor editing) in the input file for the demonstration code. Compare the displacements predicted by the finite element code with the analytical solution (with both standard and Incompatible mode elements)



Problem 8.30 Volumetric locking can be a serious problem in computations involving nonlinear materials. In this problem, you will demonstrate, and correct, locking in a finite element simulation of a pressurized hypoelastic cylinder.

(a) Set up an input file for the example hypoelastic finite element code provided on the Github site for Applied Mechanics of Solids to calculate the deformation and stress in a hypoelastic pressurized cylinder deforming under plane strain conditions. Use the mesh shown in the figure, with appropriate symmetry boundary conditions on $x_1 = 0$ and $x_2 = 0$. Apply a pressure of 40 (arbitrary units) to the internal bore of the cylinder and leave the exterior surface free of traction. Use the following material properties: $\sigma_0 = 10,\ \varepsilon_0 = 0.001,\ n = 4,\ \nu = 0.3$. Plot a graph of the variation of the radial displacement of the inner bore of the cylinder as a function of the applied pressure.

(b) Edit the code to reduce the number of integration points used to compute the element stiffness matrix from 9 to 4. (You could make your code switch to reduced integration automatically by assigning a different element identifier to the full and reduced integration elements). Repeat the calculation in (a). Note the substantial discrepancy between the results of (a) and (b) – this is caused by locking. The solution in (b), which uses reduced integration, is the more accurate of the two. Note also that using reduced integration improves the rate of convergence of the Newton-Raphson iterations.

Problem 8.31 Reduced integration (without modification) will only work for a few elements. To show this,

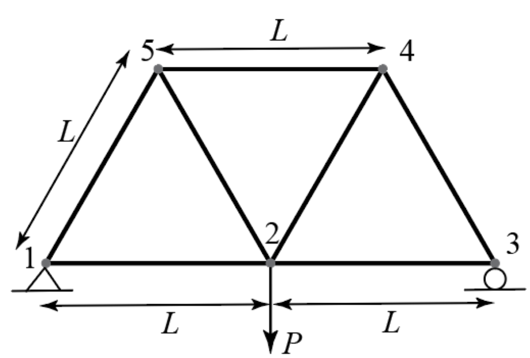
(a) Edit the example finite element code for hypoelastic materials provided on the Github site for Applied Mechanics of Solids to allow the code to use 4 noded linear reduced integration quadrilateral elements. Set up an input file to calculate the deformation and stress in a hypoelastic pressurized cylinder deforming under plane strain conditions, using the mesh of four noded quadrilaterals shown in the figure, with appropriate symmetry boundary conditions on $x_1 = 0$ and $x_2 = 0$. Apply a pressure of 10 (arbitrary units) to the internal bore of the cylinder and leave the exterior surface free of traction. Use the following values for the properties of the hypoelastic material: $\sigma_0 = 10, \varepsilon_0 = 0.001, n = 4, \nu = 0.3$. Plot the deformed mesh (you will see severe hourglassing)

(b) Modify the code to add hourglass control to the element. Repeat the simulation in part (a) with the pressure increased to a magnitude of 40 units. Plot the deformed mesh to confirm that hourglassing has been eliminated, and plot a graph that shows the variation of the radial displacement of the inner bore of the cylinder as a function of the applied pressure, for reduced integration elements with hourglass control as well as standard fully integrated elements.

Problem 8.32 B-bar elements are frequently used to avoid volumetric locking in finite element simulations of nonlinear materials. Test the B-bar method for hypoelastic materials by extending the sample finite element code for hypoelastic materials provided on the Github site for Applied Mechanics of Solids to use B-bar elements (you can consult the sample code that uses B-bar elements for linear elastic solids to guide your coding). Test your code using the input file created in Problem 8.31. Plot a graph that shows the variation of the radial displacement of the inner bore of the cylinder as a function of the applied pressure, for Bbar elements as well as standard fully integrated elements.

8.7 Structural Elements: Two-force Members, Beams, and Plates

Problem 8.33 A sample code to analyze small deformation of a linear elastic truss is provided on the Github site for Applied Mechanics of Solids. Set up an input file for the code that will calculate the deformation of the Warren truss shown in the figure. Set the length, Youngs modulus and cross-sectional area of the members to 4m, 210GPa and 0.0004m^2, and use a load of $P=100$kN. Compare the member forces predicted by FEA with the exact solution.



Problem 8.34 The goal of this problem is to modify the sample code described in problem 8.33 to predict the deformation and internal forces induced in a structure by a change in length of one of its members by thermal expansion.

(a) Write down the principle of virtual work that includes the additional term associated with the thermal expansion.

(b) Hence, find the system of linear equations that determines the displacements of the joints in the truss.

(c) Modify the sample code to include the additional term that accounts for thermal expansion. Test your code by using it to calculate the joint deflections in the structure analyzed in Problem 8.33, caused by a 50 $^{\circ}$C increase in temperature of member 4-5. Take the thermal expansion coefficient to be $10^{-5}\ ^{\circ}C^{-1}$. Compare the FEA prediction with the exact solution.

Problem 8.35 Extend the sample code to analyze small deformation of a linear elastic truss provided on the Github site for Applied Mechanics of Solids to analyze dynamic motion of the truss. To this end:

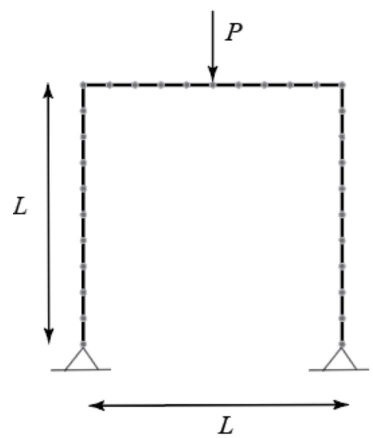
(a) Assume that each member in the truss has a uniform cross section and mass density, and that the velocity of its axis varies linearly between the ends. Show that the total kinetic energy of a member bounded by nodes a and b is related to the nodal velocities by

$$T = \frac{1}{2}\begin{bmatrix} \mathbf{v}^a & \mathbf{v}^b \end{bmatrix} \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \mathbf{v}^a \\ \mathbf{v}^b \end{bmatrix}$$

Hence, find a formula for the (full) mass matrix of the element.

(b) Modify the sample FEA code to calculate the natural frequencies of vibration and mode shapes for a truss. Test your code by predicting the natural frequencies and mode shapes for the structure shown in problem 8.33. Take the density of steel to be 8000kgm^{-3}. List the predicted (nonzero) frequencies, and plot the mode shapes for the two lowest frequency modes.

Problem 8.36 Sample codes to calculate the deflections of linear elastic beam structures (with both Euler-Bernoulli theory and Timoshenko theory) are provided on the Github site for Applied Mechanics of Solids.



(a) Set up an input file to calculate the deflection of the simple arch structure shown in the figure, using Euler-Bernoulli theory. The structure can be modeled as a single beam. Use $E = 210$ $\mu = 80$ GPa for Young's and shear modulus, and use section properties $I_{11} = 1.86 \times 10^{-4}$, $I_{12} = 0$, $I_{22} = 6 \times 10^{-6}$ $I_{33} = 1.92 \times 10^{-4}$ m^4 and a cross-sectional area of 0.0081m^2 (these represent an ASTM A6 S15x42.9 section). Assume that the $\mathbf{e}_1$ direction for the cross section is out of the plane of the figure. Take $L=10$m and P=100kN. Note that the code simulates 3D motion, so be careful to apply boundary conditions that will prevent any out-of-plane motion of the structure. Find the deflection of the load point (you may find it helpful to plot the shape of the deformed structure. The magnitude of the displacements will need to be scaled to see the deformation clearly)

(b) Calculate the deflection of the load point using Timoshenko theory. Use a shear coupling coefficient of 0.8333.

(c) Repeat the tests in (a) and (b) with a much smaller arch with length $L=2m$. Explain the behavior that you see (in a sentence or two!)

Problem 8.37 A sample code to calculate the deflections of linear elastic beam structures using Timoshenko beam theory is provided on the Github site for Applied Mechanics of Solids. Extend the code to calculate the bending moment and shear force at the integration points in the beam elements (report the results as components in the local basis aligned with the beam axis and cross-section). Use your code to plot the variation of bending moment and shear force in an end loaded cantilever beam (plot your results in a normalized form that is independent of the values used for parameters). Compare the prediction with the exact solution.

Problem 8.38 A sample code to calculate the deflections of linear elastic beam structures using Euler-Bernoulli beam theory is provided on the Github site for Applied Mechanics of Solids. Extend the code to calculate the bending moment at the integration points in the beam elements (report the results as components in the local basis aligned with the beam axis and cross-section). Devise a way to determine the shear force at the center of each beam element. Use your code to plot the variation of bending moment and shear force in an end loaded cantilever beam (plot your results in a normalized form that is independent of the values used for parameters). Compare the prediction with the exact solution.

Problem 8.39 The goal of this problem is to extend the finite element method to analyze dynamic motion of a beam.

(a) Find an expression for the 3D velocity field in a beam, in terms of the velocity $d\mathbf{u}/dt$ of its neutral line, and the angular velocity $d\boldsymbol{\theta}/dt$ of the plane normal to the centerline of the undeformed beam, in terms of the position (x_1, x_2) relative to the neutral line. Express your answer as components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ oriented with the beam.

(b) Hence, show that the kinetic energy of the beam can be expressed as

$$T = \frac{1}{2}\int_0^L \mu dx_3$$

$$\mu = \frac{d}{dt}\left[u_1, u_2, u_3, \theta_1, \theta_2, \theta_3\right]\begin{bmatrix} \rho A & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho A & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho A & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho I_{11} & -\rho I_{12} & 0 \\ 0 & 0 & 0 & -\rho I_{12} & \rho I_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho I_{33} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

where ρ is the mass density of the beam, A is its cross-sectional area, and $I_{\alpha\beta}$ are the area moments of inertia of the cross-section.

(c) Use the result of (b) to find a formula for the mass/inertia matrix of a beam element.

(d) Implement the results of part (c) in the demonstration codes for Euler and Timoshenko beams posted on the Github site for Applied Mechanics of Solids. Test your code by using it to calculate the natural frequencies of vibration for a cantilever beam with length L, Young's modulus E, shear modulus μ, mass density ρ, cross-sectional area A and moments of inertial I. The beam will have transverse, axial and torsional modes of vibration: report the 3 lowest frequencies for each mode in a form that is independent of the values used for material properties and geometry of the beam. For the Timoshenko beam, transverse frequencies will be a function of the normalized bending to shear stiffness $EI/(\beta\mu AL^2)$. Take $EI/(\beta\mu AL^2) < 0.02$ to approximate a long beam (you can explore the effects of this parameter if you are curious). In the limit of $EI/(\beta\mu AL^2) < 0.02$, does the Euler beam or Timoshenko beam give more accurate results?

Problem 8.40 Example FEA codes to calculate the deflection of a pressurized circular plate with a simply supported edge can be found on the Github site for Applied Mechanics of Solids. Modify the code that implements Kirchhoff plate theory to predict the deflection of a plate with a clamped boundary. Plot the variation of deflection and slope of the plate as a function of radial position, and compare the result with the exact solution.

Problem 8.41 Extend the example FEA code on the Github site for Applied Mechanics of Solids that analyzes plate bending using Kirchhoff theory to calculate the internal bending moments at the element integration points. Test your code by plotting the variation of the radial and hoop components of moment for a plate with a clamped edge, and compare the prediction with the analytical solution. It is simplest to plot the numerical data as a scatter plot for a suitable subset of the elements – e.g. all the elements with two corners along the line $\theta = 0$. The data will be very noisy. You can get better results by plotting the average moment at the three integration points of the element as a function of the position of the centroid.

Problem 8.42 Extend the example FEA code on the Github site for Applied Mechanics of Solids that analyzes plate bending using Reissner-Mindlin theory to calculate the internal bending moments and shear forces at the element integration points.

(a) Test your code by plotting the variation of the radial and hoop components of moment for a plate with a clamped edge, and compare the prediction with the analytical solution. Use material and geometric parameters such that $Eh^2 / (\beta \mu R^2) < 10^{-3}$ to approximate a thin plate.

(b) Plot a graph comparing the magnitude of the shear forces predicted by FEA with the exact solution. You will find that the FEA greatly overestimates the shear forces, particularly in thin plates. This is because of shear locking. The 'Min-3' element uses an artificial correction to the stiffness matrix that mitigates the effects of locking on the predicted deflections and moments, but this results in a spuriously large shear deformation. You will find the problem is less severe for larger values of $Eh^2 / (\beta \mu R^2)$, but the shear forces induced by bending are never predicted accurately with this element. Fortunately in most practical applications the bending moments are of more interest than the shear forces.

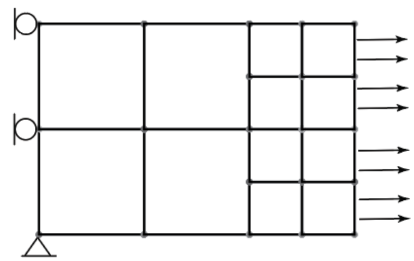
Problem 8.43 Use example FEA code that analyzes plate bending using Kirchhoff theory (posted on https://github.com/albower/Applied_Mechanics_of_Solids/tree/main) to test the effects of an in-plane biaxial force per unit length T_0 on the deformation of the plate. Use a pinned boundary condition on the edge of the plate Plot a graph that compares the deflection and slope of the plate with the exact solution for a pressurized stretched membrane, using parameters that makes the ratio of bending to membrane stiffness $Eh^3 / [(1-\nu^2) T_0 R^2] \le 0.1$ (this ensures that the membrane forces dominate over bending resistance).

Problem 8.44 Use example FEA code that analyzes plate bending using Reissner-Mindlin theory to test the effects of an in-plane biaxial force per unit length T_0 on the deformation of the plate. Use a pinned boundary condition on the edge of the plate.

(a) Plot a graph that compares the deflection and slope of the plate with the exact solution for a pressurized stretched membrane, using parameters that makes the ratio of bending to membrane stiffness $Eh^3 / [(1-\nu^2) T_0 R^2] \le 0.1$ (this ensures that the membrane forces dominate over bending resistance).

(b) Edit your code to set the parameter β_* that scales the shear contribution to the element stiffness matrix to $\beta_* = 1$, and repeat the test in part (a).

8.8 Constraints, Interfaces and Contact



Problem 8.45 The goal of this problem is to develop a 2D penalty constraint that can be used to connect two Incompatible (2D) finite element meshes. Let *a* and *b* denote two adjacent nodes on the face of a 2D element in one mesh, with coordinates $\mathbf{x}^a, \mathbf{x}^b$. Let *c* denote a node with coordinates $\mathbf{x}^c$ that is connected to an element in the second mesh. Assume that, before deformation, *c* is co-linear with *a* and *b*. Nodes *a,b,c* must be constrained so that the two meshes have equal displacements at position $\mathbf{x}^c$. Assume small deformations, to keep the calculations simple.

(a) Assume that the displacements vary linearly between *a* and *b*. Find a constraint equation relating the displacement vectors $\mathbf{u}^a, \mathbf{u}^b$ and $\mathbf{u}^c$, in terms of their coordinates.

(b) Assume that the constraint element will have degrees of freedom $[u_1^a, u_2^a, u_1^b, u_2^b, u_1^b, u_2^b]$. Find formulas for the element force vector and element stiffness matrix associated with the constraint, in terms of the penalty stiffness *G* defined in Section 8.8.1 of Applied Mechanics of Solids and other relevant variables.

(c) A demonstration FEA code that implements a simple tie constraint between two nodes is posted on the Github site for Applied Mechanics of Solids. Modify this code to impose the constraint described in this problem. Test your code by using it to bond the two Incompatible meshes shown in the figure. Only two constraint elements are needed for this example. Apply boundary conditions that will induce a state of uniaxial horizontal stress (you could use either plane strain or plane stress elements). Plot the deformed mesh with and without the constraints. As a further study, you could explore the influence of choosing a range of values for the penalty stiffness.

Problem 8.46 Implement the constraint described in problem 8.45 as a Lagrange multiplier. To this end:

(a) Assume that the constraint element will have degrees of freedom $[u_1^a, u_2^a, u_1^b, u_2^b, u_1^b, u_2^b, \lambda_1, \lambda_2]$, where λ_i are the two Lagrange multipliers for the constraint. Find expressions for the element force vector and stiffness matrix.

(b) A demonstration FEA code that implements a simple tie constraint between two nodes is posted on the Github site for Applied Mechanics of Solids. Modify this code to impose the constraint described in this problem. Test your code using the procedure described in problem 8.45(c).

Problem 8.47 An irreversible cohesive interface law relates the normal and tangential tractions (T_n, T_1, T_2) acting on the two solids adjacent to the interface to the relative displacement $(\Delta_n, \Delta_1, \Delta_2)$ of two initially coincident points on the two solids by

$$T_n = \begin{cases} k_0(1-D)\Delta_n + \eta d\Delta_n / dt & \Delta_n \ge 0 \\ k_1 \Delta_n & \Delta_n \le 0 \end{cases}$$

$$T_1 = k_0(1-D)\Delta_1 + \eta d\Delta_1 / dt \qquad T_2 = k_0(1-D)\Delta_2 + \eta d\Delta_2 / dt$$

where k_0, k_1 are the elastic stiffness of the interface in tension and compression (k_1 is a penalty stiffness that prevents the two solids from overlapping, so $k_1 >> k_0$), η is a small viscosity, and *D* is a parameter that quantifies the damage inflicted on the interface. The damage parameter evolves with the separation of the interface according to

$$\frac{dD}{dt}=\begin{cases} 0 & \tau < \dfrac{2(1-D)\sigma_{max}}{2-D} \quad \text{or} \quad d\lambda/dt<0 \quad \text{or} \quad D=1 \\ (2-D)\dfrac{1}{\lambda}\dfrac{d\lambda}{dt} & \text{Otherwise} \end{cases}$$

where σ_{max} is the maximum stress that the interface can withstand, and

$$\tau=\begin{cases}\sqrt{T_n^2+T_1^2+T_2^2} & \Delta_n \ge 0 \\ \sqrt{T_1^2+T_2^2} & \Delta_n \le 0\end{cases} \qquad \lambda=\begin{cases}\sqrt{\Delta_n^2+\Delta_1^2+\Delta_2^2} & \Delta_n \ge 0 \\ \sqrt{\Delta_1^2+\Delta_2^2} & \Delta_n \le 0\end{cases}$$

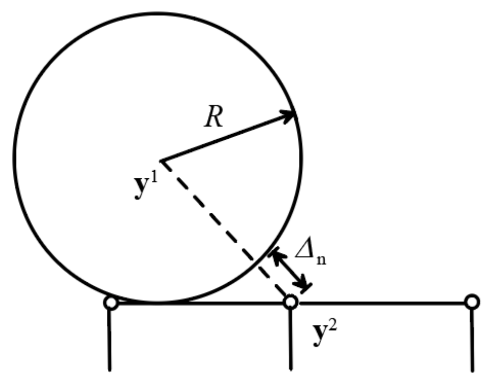
The goal of this problem is to implement this law in an FEA code.

(a) Note that the solution is time and history dependent, so the simulation will need to load the solid in a series of time increments. Let $(\Delta_n^N, \Delta_1^N, \Delta_2^N)$ denote the relative displacement at time t_N, and $(\Delta_n^{N+1}, \Delta_1^{N+1}, \Delta_2^{N+1})$ denote the separation at time t_{N+1}. Find formulas for (T_n, T_1, T_2) at time t_{N+1}, using an implicit integration scheme for D (i.e. find the change in D during the time-step using values for relevant variables at the end of the time-step).

(b) Hence, design a procedure for calculating the (discontinuous) tangent matrix for the cohesive zone law.

(c) A sample finite element code that implements a reversible cohesive zone law is posted on the Github site for Applied Mechanics of Solids. Modify the code to implement the irreversible law. Design some appropriate tests to check your code.

Problem 8.48 The goal of this problem is to develop a finite element code that will simulate contact between a frictionless cylindrical (or spherical, in 3D) analytical rigid surface and a deformable solid with a hypoelastic stress-strain law. A rigid surface always acts as the 'master' in a contact simulation, and some suitable subset of nodes on the deformable solid may act as 'slaves' if they contact the rigid surface. The figure shows a representative slave node near the master surface. The center of the sphere is identified by a reference node that is located at (deformed) position $\mathbf{y}^1$, while the node is at $\mathbf{y}^2$. The cylinder has radius R. The reference node has displacement degrees of freedom



(a) Write down an expression for the gap Δ_n between the node and the rigid surface.

(b) Assume that the constraint $\Delta_n = 0$ will be imposed by a contact element that has degrees of freedom stored as $\left[u_1^1, u_2^1, u_1^2, u_2^2, \lambda\right]$ (with a similar extension to 3D) Find an expression for the residual force vector and stiffness for a contact element that will enforce the constraint $\Delta_n = 0$ using a Lagrange Multiplier.

(c) A finite element code that models contact between two deformable hypoelastic solids is posted on the Github site for Applied Mechanics of Solids. Modify this code to solve this problem. You may find it helpful to modify the stiffness matrix for the deformable hypoelastic material to use the B-bar method (see problem 8.32) to prevent volumetric locking, but the contact algorithm will work for fully integrated elements. Test your code by simulating contact between a rigid cylinder and a hypoelastic block that is supported on a rigid flat surface. Choose appropriate values for dimensions and material properties.

Chapter 9
Modeling Material Failure

9.1 Summary of Mechanisms of Fracture and Fatigue under Static and Cyclic Loading

Problem 9.1 Summarize the main differences between a ductile and a brittle failure of a material. List a few examples of each.

Problem 9.2 What is the difference between a static fatigue failure and a cyclic fatigue failure?

Problem 9.3 Explain what is meant by a plastic instability, and explain the role of plastic instability in causing failure.

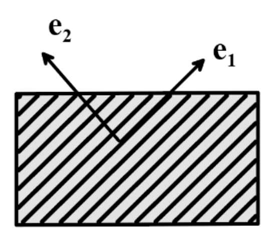
Problem 9.4 Explain the difference between `High cycle fatigue' and `Low cycle fatigue'

Problem 9.5 Summarize the main features and mechanisms of material failure under cyclic loading. List variables that may influence fatigue life.

9.2 Stress and Strain Based Fracture and Fatigue Criteria

Problem 9.6 A flat specimen of glass with fracture strength σ_{TS}, Young's modulus E and Poisson's ratio ν is indented by a hard metal sphere with radius R, Young's modulus E_2 and Poisson's ratio ν_2. Using solutions for contact stress fields given in Chapter 5.4.10 of Applied Mechanics of Solids, calculate a formula for the load P that will cause the glass to fracture, in terms of geometric and material parameters. You can assume that the critical stress occurs on the surface of the glass, and neglect friction.

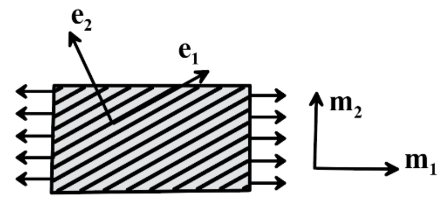
Problem 9.7 The figure shows a fiber reinforced composite laminate.



- When loaded in uniaxial tension parallel to the fibers, it fails at a stress of 500MPa.
- When loaded in uniaxial tension transverse to the fibers, it fails at a stress of 250 MPa.
- When loaded at 45 degrees to the fibers, it fails at a stress of 223.6 MPa
- Failure in the laminate is to be predicted using the Tsai-Hill criterion

$$\left(\frac{\sigma_{11}}{\sigma_{TS1}}\right)^2 + \left(\frac{\sigma_{22}}{\sigma_{TS2}}\right)^2 - \frac{\sigma_{11}\sigma_{22}}{\sigma_{TS1}^2} + \frac{\sigma_{12}^2}{\sigma_{SS}^2} = 1$$

(a) Use the measurements to calculate values for the parameters $\sigma_{TS1}, \sigma_{TS2}, \sigma_{SS}$.



(b) The laminate is then loaded in uniaxial tension at 30 degrees to the fibers. Calculate the expected failure stress under this loading, assuming that the material can be characterized using the Tsai-Hill failure criterion.

Problem 9.8 A number of cylindrical specimens of a brittle material with a 1cm radius and length 4cm are tested in uniaxial tension. It is found 60% of the specimens withstand a 150MPa stress without failure; while 30% withstand a 170 MPa stress without failure.

(a) Calculate values for the Weibull parameters σ_0 and m for the specimens

(b) Suppose that a second set of specimens is made from the same material, with length 8cm and radius 1cm. Calculate the stress level that will cause 50% of these specimens to fail.

Problem 9.9 A beam with length L, and rectangular cross-section $b \times h$ is made from a brittle material with Young's modulus E, Poisson's ratio ν, and the failure probability distribution of a volume V_0 is characterized by Weibull parameters σ_0 and m.

(a) Suppose that the beam is loaded in uniaxial tension parallel to its length. Calculate the stress level σ_T corresponding to 63% probability of failure, in terms of geometric and material parameters.

(b) Suppose that the beam is loaded in 3 point bending. Let $\sigma_R = 3PL/(bh^2)$ denote the maximum value of stress in the beam (predicted by beam theory). Find an expression for the stress distribution in the beam in terms of σ_R (use the standard Euler beam theory approximation)

(c) Hence, find an expression for the value of σ_R that corresponds to 63% probability of failure in the beam. Calculate the ratio σ_R / σ_T.

Problem 9.10 A glass shelf with length L and rectangular cross-section $b \times h$ (with h the shelf thickness) is used to display rich, heavy, chocolate cakes in a store. As a result, it subjected to a daily cycle of load (which may be approximated as a uniform pressure acting on its surface) of the form $p(t) = p_0(1 - t/T)$ where $0 < t < T$ is the time the store has been open, and T is the total time the store is open each day. As received, the shelf has a tensile strength σ_{TS0}, and the glass can be characterized by static fatigue parameters α and m. Find an expression for the life of the shelf (in number of days), in terms of relevant parameters. For extra credit, find an expression for the life of the people that eat the cakes.

Problem 9.11 A cylindrical concrete column with cross-sectional area A and length L is subjected to a monotonically increasing compressive axial load P. Assume that the material can be idealized using the constitutive law given in Section 9.2.4 of Applied Mechanics of Solids, with the compressive yield stress-v-plastic strain of the form

$$Y(\bar{\varepsilon}^p) = \sigma_0 \left(\frac{\bar{\varepsilon}^p}{\varepsilon_0} \right)^{1/m}$$

where σ_0, ε_0 and m are material properties. Assume small strains, and a homogeneous state of stress and strain in the column. Neglect elastic deformation and assume small strains, for simplicity.

(a) Calculate the relationship between the axial stress P/A and strain δ/L, in terms of the plastic properties c, σ_0, ε_0 and m

(b) Calculate the volume change of the column, in terms of P/A, c, σ_0, ε_0 and m

(c) Suppose that the sides of the column are subjected to a uniform radial pressure q. Repeat the calculations in parts (a) and (b).

Problem 9.12 Suppose that the column described in problem 9.11 is encased in a steel tube, with (small) wall thickness t. The steel can be idealized as a rigid perfectly plastic material with yield stress Y. Calculate the relationship between the axial stress P/A and strain δ/L, in terms of geometric and material properties (assume the structure is stress free before it the force P is applied).

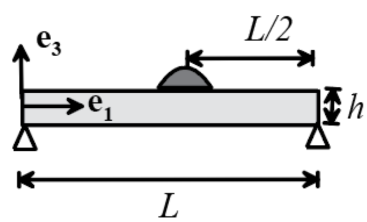
Problem 9.13 A specimen of steel has a yield stress of 500MPa. Under fully reversed cyclic loading at a stress amplitude of 200 MPa it is found to fail after 10^4 cycles, while at a stress amplitude of 100 MPa it fails after 10^5 cycles. This material is to be used to fabricate a plate, with thickness h, containing circular holes with radius a. The plate will be subjected to constant amplitude fully reversed cyclic uniaxial stress far from the holes, and must have a life of at least 10^5 cycles. What is the maximum stress amplitude (far from the hole) that the plate can withstand?

Problem 9.14 A spherical pressure vessel with internal radius a and external radius b=1.5a is repeatedly pressurized from zero internal pressure to a maximum value p. The sphere has yield stress Y, ultimate tensile strength σ_{UTS} and its fatigue behavior (under fully reversed uniaxial tension) can be characterized by Basquin's law $\sigma_a N^b = C$ You can assume that the elastic stresses in the vessel are given by

$$\sigma_{rr} = -p\frac{a^3(b^3 - r^3)}{r^3(b^3 - a^3)} \qquad \sigma_{\theta\theta} = p\frac{a^3(b^3 + 2r^3)}{2r^3(b^3 - a^3)}$$

Find an expression for the fatigue life of the vessel in terms of p, and relevant geometric and material properties. Assume that the effects of mean stress can be approximated using Goodman's rule. Assume that $p/Y < 2(1 - a^3/b^3)/3$, so the stress remains below yield.

Problem 9.15 The figure shows a solder joint on a printed circuit board. The printed circuit board can be idealized as a pinned-pinned beam with thickness h, length L, Young's modulus E and mass density ρ. The board vibrates in its fundamental mode with angular frequency $\omega = (\pi^2 h/L^2)\sqrt{E/(12\rho)}$ and mode shape $u_3 = A\sin(\pi x_1/L)$. The yield stress of solder is so low it can be neglected, and it is firmly bonded to the printed circuit board. As a result, it is subjected to a cyclic plastic strain equal to the strain at the surface of the beam. The fatigue life of solder can be characterized by a Coffin-Manson law $\Delta\varepsilon^p N^b = C$. Find an expression for the time to failure of the solder joint, in terms of relevant geometric and material parameters.



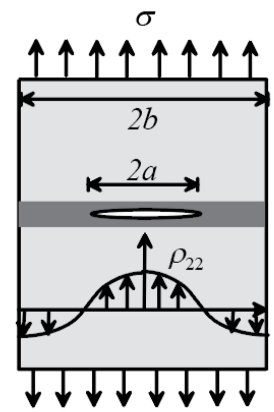
Problem 9.16 A specimen of steel is tested under cyclic loading. It is found to have a fatigue threshold $\sigma_0 = 75MPa$, and fails after 10^5 cycles when tested at a stress amplitude $\sigma_a = 1.5\sigma_0$ and 10^3 cycles when tested at a stress amplitude of $2\sigma_0$ Suppose that, in service, the material spends 80% of its life subjected to stress amplitudes $\sigma_a < \sigma_0$, 10% of its life at $\sigma_a = 1.1\sigma_0$, and the remainder at $\sigma_a = 1.2\sigma_0$. Calculate the life of the component during service (assume that the mean stress $\sigma_m = 0$ during both testing and service).

9.3 Modeling Failure by Crack Growth: Linear Elastic Fracture Mechanics

Problem 9.17 Using the equations for the crack tip fields, calculate the von-Mises equivalent stress distribution ahead of a mode I crack tip (assume plane strain deformation, with a representative value for Poisson's ratio). Hence, plot approximate contours of successive yield zones around the crack tip.

Problem 9.18 Briefly describe the way in which the concept of stress intensity factor can be used as a fracture criterion.

Problem 9.19 A welded plate with fracture toughness K_{IC} contains a residual stress distribution



$$\rho_{22} = \sigma_R \left[\frac{a}{\sqrt{x_1^2 + a^2}} - \frac{a}{b} \operatorname{arcsinh}\left(\frac{b}{a}\right) \right]$$

along the line of the weld. A crack with length $2a$ lies on the weld line. The solid is subjected to a uniaxial tensile stress $\sigma_{22} = \sigma_0$. Find an expression for the critical value of σ_0 that will cause the weld to fracture, in terms of K_{IC}, σ_R and a. The following integral will be helpful

$$\int_{-1}^{1} \frac{1}{\sqrt{x_1^2 + 1^2}} \sqrt{\frac{1+x_1}{1-x_1}} dx_1 \approx 2.622$$

Problem 9.20 Hard, polycrystalline materials such as ceramics often contain a distribution of inter-granular residual stress. The objective of this problem is to estimate the influence of this stress distribution on crack propagation through the material. Assume that

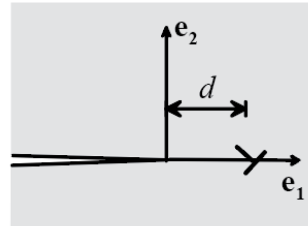
- The solid has mode I fracture toughness K_{IC}
- As a rough estimate, the residual stress distribution can be idealized as $\sigma_{22} = \sigma_R \sin(\pi x_1 / L)$, where L is of the order of the grain size of the solid and σ_R is the magnitude of the stress.
- A long (semi-infinite) crack propagates through the solid – at some time *t,* the crack tip is located at $x_1 = c$
- The solid is subjected to a remote stress, which induces a mode I stress intensity factor K_I^∞ at the crack tip

(a) If the solid is free of residual stress, what value of K_I^∞ that causes fracture.

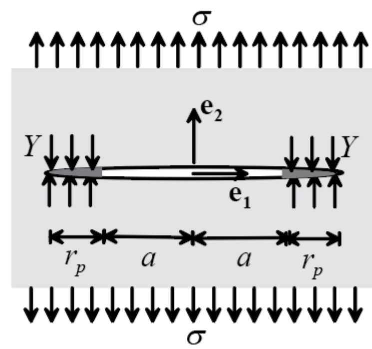
(b) Calculate the stress intensity factor induced by the residual stress distribution, as a function of *c*.

(c) What value of K_I^∞ is necessary to cause crack propagation through the residual stress field? What is the maximum value of K_I^∞, and for what crack tip position *c* does it occur?

Problem 9.21 A dislocation, with burgers vector $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$ and line direction $\mathbf{e}_3$ lies a distance *d* ahead of a semi-infinite crack. The solid is otherwise stress free. Calculate the crack tip stress intensity factors.



Problem 9.22 The figure shows a simple model that is used to estimate the size of the plastic zone at a crack tip. The crack, with length $2a$, together with the plastic zones with length r_p, are considered together to be a crack with length $2(a+r_p)$. The solid is loaded by uniform stress σ at infinity. The region with length r_p near each crack tip is subjected to traction Y acting to close the crack. Using the solutions in Section 9.3.3 of Applied Mechanics of Solids, calculate an expression for the Mode I crack tip stress intensity factor. Show that $K_I = 0$ if

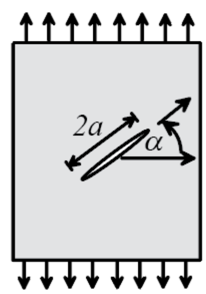


$$\frac{r_p}{a} = \sec\left(\frac{\pi\sigma}{2Y}\right) - 1$$

Problem 9.23 Suppose that an ASTM compact tension specimen is used to measure the fracture toughness of a steel. The specimen has dimensions $W = 40mm$ and $B = 20$ mm. The crack length was 18.5 mm, and the fracture load was 15kN.

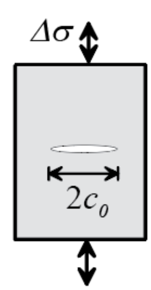
(a) Calculate the fracture toughness of the steel.
(b) If the steel has yield stress 800MPa, was this a valid measurement?

Problem 9.24 Find expressions for the Mode I and II stress intensity factors for the angled crack shown, in terms of the vertical stress magnitude σ. If $\alpha = 45^0$, what is the initial direction of crack propagation? You could try to check your prediction by tearing a center-cracked specimen of paper.



Problem 9.25 A large solid contains a crack with initial length $2c_0$. The solid has plane-strain fracture toughness K_{IC}, and under cyclic loading the crack growth rate obeys Paris law

$$\frac{da}{dN} = C\left(\Delta K_I\right)^n$$



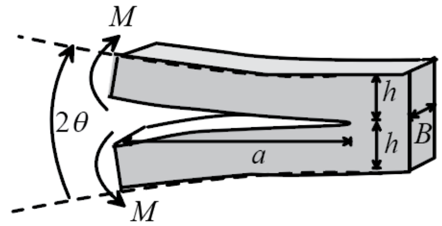
(a) Suppose that the material is subjected to a cyclic uniaxial stress with amplitude $\Delta\sigma$ and mean stress $\Delta\sigma$ (so the stress varies between 0 and $2\Delta\sigma$). Calculate the critical crack length that will cause fracture, in terms of K_{IC} and $\Delta\sigma$
(b) Calculate an expression for the number of cycles of loading that are necessary to cause a crack to grow from an initial length $2c_0$ to fracture under the loading described in (a), in terms of the stress amplitude
(c) Show that, if $2\Delta\sigma\sqrt{\pi c_0}/K_{IC} << 1$, the number of cycles to failure can be expressed in the form of Basquin's law as $N\left(\Delta\sigma\right)^b = D$, where *b* and *D* are constants. Give expressions for *b* and *D* in terms of the initial crack length and the material properties in Paris law.

9.4 Energy Methods in Fracture Mechanics

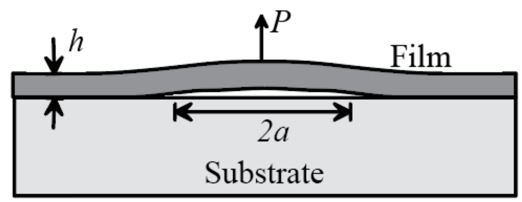
Problem 9.26 The figure shows a double-cantilever beam fracture specimen that is loaded by applying moments to the ends of the beams. Define the compliance of the solid as $C = 2\theta / M$

(a) Derive an expression relating the crack tip energy release rate to compliance and dimensions of the specimen.

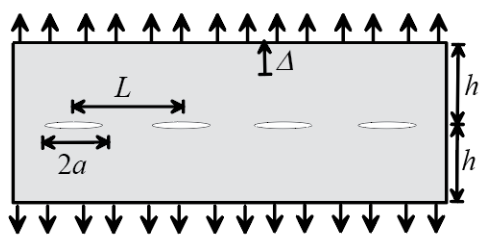
(b) Use the result of (a) to calculate the crack tip stress intensity factors for the specimen in terms of M and relevant geometric and material properties



Problem 9.27 The figure shows a thin film with thickness h, Young's modulus E and Poisson's ratio ν. To test the interface between film and substrate, a delaminated region of the film with width $2a$ is subjected to a vertical force (per unit out of plane distance) P. By idealizing the delaminated region as a beam, and using the method of compliance, estimate the crack tip energy release rate in terms of P and relevant material and geometric properties.



Problem 9.28 The figure shows a thin rectangular strip of material with height $2h$ and out-of-plane thickness B. The material can be idealized as a linear elastic solid with Young's modulus E and Poisson's ratio ν. The strip is damaged by an array of widely-spaced cracks with length $2a$ and spacing $L >> a$. It is loaded by a uniform tensile traction t acting on the top and bottom surface, which induces a displacement Δ of the top surface relative to the bottom. The sides are free of traction.

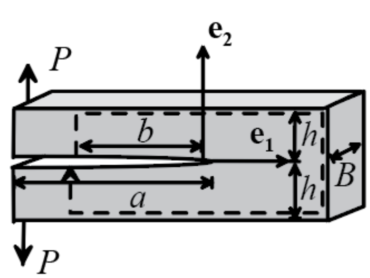


(a) Write down an expression for the displacement Δ of the top of undamaged strip (i.e. with no cracks), in terms of t and relevant material constants and geometric lengths. Assume plane strain deformation.

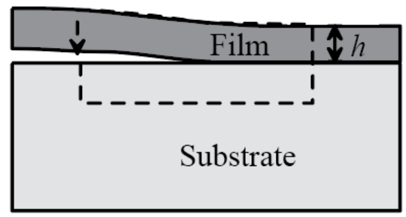
(b) Write down a relationship between the compliance of the cracked strip and the crack tip energy release rate.

(c) Estimate the crack tip energy release rate using the energy release rate for an isolated crack in an infinite solid (this is valid for $a/L<<1$). Hence, find an expression for the compliance $\Delta / (BLt)$ of the cracked strip, in terms of relevant geometric and material parameters.

Problem 9.29 The figure shows a double cantilever beam specimen that is loaded by forces applied to the ends of the beams. Evaluate the J integral around the path shown to calculate the crack tip energy release rate. You can use elementary beam theory to estimate the strain energy density, stress, and displacement in the two cantilevers. The solution must, of course, be independent of b.

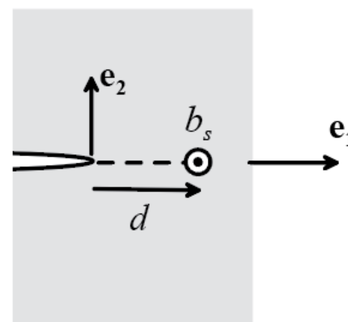


Problem 9.30 The figure shows a thin film with thickness h, thermal expansion coefficient α_f, Young's modulus E and Poisson's ratio ν on a large substrate with thermal expansion coefficient α_s. The film is initially perfectly bonded to the substrate and stress free. The system is then heated to raise its temperature by ΔT, inducing a thermal stress in the film. As a result, the film delaminates from the substrate, as shown in the figure.



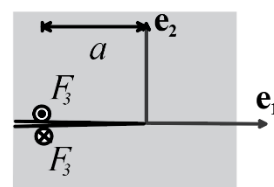
(a) Calculate the state of stress a distance $d >> h$ ahead of the advancing crack tip
(b) Assume that the film is stress free a distance $d >> h$ behind the crack tip. By directly calculating the change in energy of the system as the crack advances, find an expression for the crack tip energy release rate
(c) Check your answer to (b) by evaluating the J integral around the path indicated in the figure.

Problem 9.31 The figure shows a screw dislocation with burgers vector b_s a distance d ahead of a semi-infinite crack in an infinite, isotropic, linear elastic solid with shear modulus μ. The solid is stress free at infinity. The goal of this problem is to compute the crack tip stress intensity factors and the driving force on the dislocation. You will find the following integral useful in your calculations



$$\int_{-\infty}^{0} \frac{1}{(x-\xi)\sqrt{-\xi}} d\xi = \frac{\pi}{\sqrt{x}}$$

(a) Write down the stress state induced by an isolated screw dislocation at the origin of an infinite solid, expressing your answer as components in the basis shown.
(b) The stress intensity factor due to a pair of equal and opposite point forces acting on a semi-infinite crack (see the figure) is



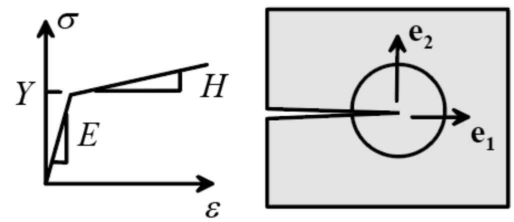
$$K_{III} = \frac{2F_3}{\sqrt{2\pi a}}$$

Use this result to calculate the stress intensity factor induced at the crack tip by the dislocation.
(c) Hence calculate the crack tip energy release rate, and deduce the configurational force acting on the dislocation. Is the dislocation attracted to, or repelled from the crack tip?

9.5 Plastic Fracture Mechanics

Problem 9.32 Explain briefly the main concepts underlying the use of the *J* integral as a fracture criterion in components experiencing large-scale plastic deformation.

Problem 9.33 The figure shows the tip of a semi-infinite crack in an elastic-plastic material with a bi-linear uniaxial stress-strain curve, as indicated in the figure. To provide some insight into the nature of the crack tip fields, the constitutive behavior can be approximated as an incompressible isotropic hypoelastic material, characterized by a strain energy density W such that $\sigma_{ij} = \partial W / \partial \varepsilon_{ij} + p\delta_{ij}$, which has a bi-linear uniaxial stress-strain curve with a yield stress *Y*, as shown in the figure.



Suppose that the solid is subjected to remote mode I loading (so that the shear stresses $\sigma_{12} = \sigma_{13} = 0$ on $x_2 = 0$).

(a) Construct the strain energy density and give the full stress-strain equations for the hypoelastic material, using the approach described in Section 3.3 of Applied Mechanics of Solids.

(b) Consider a material point that is very far from the crack tip, and so is subjected to a very low stress. Write down the asymptotic stress field in this region, in terms of an arbitrary constant K_I^∞ that characterizes the magnitude of the remote mode I loading (don't try to derive the fields: use your physical intuition and known solutions for cracks in materials with a linear stress-v-strain relation).

(c) Consider a material point that is very close to the crack tip, and so is subjected to a very large stress. Write down the asymptotic stress field in this region, in terms of an arbitrary constant K_I^{tip} that characterizes the magnitude of the near tip stresses.

(d) Using the path independence of the *J* integral, find a relationship between K_I^∞, K_I^{tip}, and the slopes E, H of the uniaxial stress-strain curve.

(e) Suppose that the material fractures when the stress at a small distance d ahead of the crack tip reaches a critical magnitude σ_0. Assume that the critical distance is much smaller than the region of high stress considered in (c). Calculate the critical value of K_I^∞ that will cause the crack to grow, in terms of *H,E* σ_0, and *d.*

(f) Consider a finite sized crack with length 2*a* in the hypoelastic material. Assume that the solid is subjected to a remote uniaxial stress σ acting perpendicular to the crack plane far from the crack. Discuss qualitatively how the stress field around the crack evolves as the remote stress is increased. Discuss the implications of this behavior on the validity of the fracture criterion derived in (e).

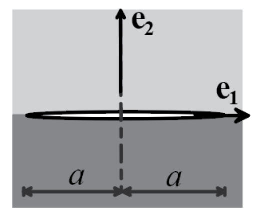
9.6 Linear Elastic Fracture Mechanics of Interfaces

Problem 9.34 Calculate values for the elastic constants α, β and the crack tip singularity parameter ε for the following bi-material interfaces:

(a) Aluminum on glass
(b) A glass fiber in a PVC matrix
(c) Nickel on titanium carbide
(d) Copper on Silicon

Problem 9.35 A center-cracked bi-material specimen is made by bonding Al to Silicon Nitride. It contains a crack with length 5mm, and is loaded to failure in uniaxial tension. It is found to fail at a stress level $\sigma_{22} = 400\,\text{MPa}$.

(a) Calculate the fracture toughness and the corresponding phase angle of loading, using a characteristic length $L = 100\mu\text{m}$
(b) Calculate the fracture phase angle if $L = 1\text{mm}$ is chosen as the characteristic length.



Problem 9.36 A bi-material interface is made by bonding two materials together. The material above the interface has shear modulus and Poisson's ratio μ_1, ν_1; the material below the crack has shear modulus and Poisson's ratio μ_2, ν_2 Due to roughness, a residual stress distribution

$$\sigma_{22}(x_1) = \sigma_0 \sin(2\pi x_1 / L)$$

acts on the bi-material interface. Suppose that the interface contains a long (semi-infinite) crack, with crack tip located at $x_1 = c$. Calculate the crack tip stress intensity factors as a function of the elastic mismatch parameter ε and other relevant parameters.

Chapter 10
Approximate Theories for Solids with Special Shapes: Rods, Beams, Membranes, Plates and Shells

10.1 Preliminaries: Dyadic Notation for Vectors and Tensors

Problem 10.1 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis. Express the identity tensor as a dyadic product of the basis vectors

Problem 10.2 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ be two Cartesian bases. Show that the tensor $\mathbf{R} = \mathbf{m}_i \otimes \mathbf{e}_i$ (with a sum on © can be visualized as a rigid rotation (you can show that **R** is an orthogonal tensor, for example, or calculate the change in length of a vector that is multiplied by **R**).

Problem 10.3 Let **a** and **b** be linearly independent vectors (satisfying $\mathbf{a}\cdot\mathbf{b} \neq 0$). Let $\mathbf{S} = \mathbf{a} \otimes \mathbf{b}$. Find an expression for all the vectors **u** that satisfy $\mathbf{Su} = 0$

Problem 10.4 Find the eigenvalues and eigenvectors of the tensor $\mathbf{S} = \mathbf{a} \otimes \mathbf{b}$ in terms of **a**, **b**, and their magnitudes (Don't forget to find *three* independent eigenvectors).

Problem 10.5 Let $\mathbf{a}_i$ be three linearly independent vectors. Define $\mathbf{a}^i$ to be three vectors that satisfy

$$\mathbf{a}^i \cdot \mathbf{a}_i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and let $g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j$, $g^i_j = \mathbf{a}^i \cdot \mathbf{a}_j$ and $g^{ij} = \mathbf{a}^i \cdot \mathbf{a}^j$ denote the dot products of these vectors.

(a) Find expressions for $\mathbf{a}^i$ in terms of vector and scalar products of $\mathbf{a}_i$

(b) Let $\mathbf{S} = S_{ij}\mathbf{a}^i \otimes \mathbf{a}^j$ be a general second order tensor. Find expressions for S^{ij}, $S^i_{\cdot j}$, $S_j^{\cdot i}$ satisfying

$$\mathbf{S} = S^{ij}\mathbf{a}_i \otimes \mathbf{a}_j = S^i_{\cdot j}\mathbf{a}_i \otimes \mathbf{a}^j = S_j^{\cdot i}\mathbf{a}^j \otimes \mathbf{a}_i$$

in terms of S_{ij} and g_{ij}, g^i_j and g^{ij}

(c) Calculate $\left(\mathbf{a}_i \otimes \mathbf{a}^i\right)\cdot\left(\mathbf{a}_j \otimes \mathbf{a}^j\right)$. What does the tensor $\left(\mathbf{a}_i \otimes \mathbf{a}^i\right)$ represent?

(d) Express $\left(\mathbf{a}_i \otimes \mathbf{a}^i\right)$ in terms of $\left(\mathbf{a}_i \otimes \mathbf{a}_j\right)$ and appropriate combinations of g_{ij}, g^i_j and g^{ij}

(e) Express $\left(\mathbf{a}_i \otimes \mathbf{a}^i\right)$ in terms of $\left(\mathbf{a}^i \otimes \mathbf{a}^j\right)$ and appropriate combinations of g_{ij}, g^i_j and g^{ij}

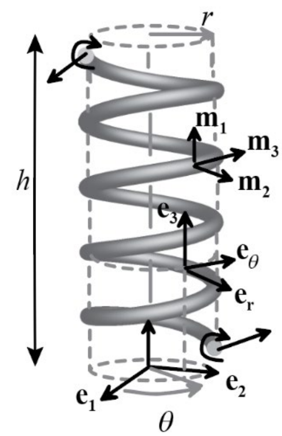
(f) Let **F** denote a homogeneous deformation gradient, satisfying $\mathbf{b}_i = \mathbf{F}\mathbf{a}_i$. Express **F** in terms of dyadic products of $\mathbf{a}^i$ and $\mathbf{b}_i$.

(g) Find an expression for $\mathbf{F}^T\mathbf{F}$ in terms of scalar products of $\mathbf{b}_i$ and dyadic products of $\mathbf{a}^i$, i.e. find components U_{ij} satisfying $\mathbf{F}^T\mathbf{F} = U_{ij}\mathbf{a}^i \otimes \mathbf{a}^j$

(h) Find an expression for the Lagrange strain tensor $\mathbf{E}=(\mathbf{F}^T\mathbf{F}-\mathbf{I})/2$ in terms of dyadic products of $\mathbf{a}^i$, i.e. find E_{ij} satisfying $(\mathbf{F}^T\mathbf{F}-\mathbf{I})/2=E_{ij}\mathbf{a}^i\otimes\mathbf{a}^j$, in terms of scalar products of $\mathbf{b}_i$ and appropriate combinations of g_{ij}, g^i_j and g^{ij}

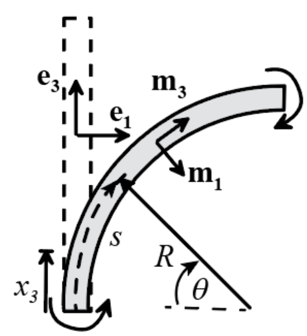
10.2 Motion and Deformation of Slender Rods

Problem 10.6 The figure shows an inextensible rod that is bent into a helical shape. The shape of the helix can be characterized by the radius *r* of the generating cylinder, and the number of turns *n* in the helix per unit axial length. Consider a point on the axis of the rod specified by the polar coordinates (r,θ,z).



(a) Write down an expression for z in terms of *r, n* and θ.
(b) Write down the position vector of the point as components in the $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ basis.
(c) Calculate an expression for the unit vector $\mathbf{m}_3$ that is tangent to the rod, in terms of the basis vectors $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and appropriate coordinates. You can simplify your answer by writing it in terms of the helix angle α, defined as $\tan\alpha=1/(2\pi rn)$
(d) Assume that $\mathbf{m}_2$ is perpendicular to the axis of the cylinder. Use this and the solution to (c) to find expressions for the basis vectors $\mathbf{m}_1,\mathbf{m}_2$ in terms of $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$.
(e) Calculate the normal and binormal vectors to the curve and hence deduce an expression for the torsion of the curve.
(f) Deduce an expression for the curvature vector of the rod.

Problem 10.7 An initially straight, inextensible slender bar with length *L* and circular cross-section with radius *a* is bent into a circle with radius *R* by terminal couples, as shown in the figure. Assume that cross-sections of the rod remain circles with radius *a* and remain transverse to the axis of the rod after deformation, and that the mid-section of the rod does not change its length.



(a) Write down an expression for the position vector $\mathbf{r}(x_3)$ of a point on the axis of the deformed rod, expressing your answer as components in the $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ basis (the origin should be at the base of the undeformed rod)
(b) Find an expression for the basis vectors $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$ as a function of arc-length *s*, expressing each unit vector as components in $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Hence find an expression for the orthogonal tensor **R** that maps $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ onto $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$.
(c) Write down the curvature vector for the deformed rod, and verify that

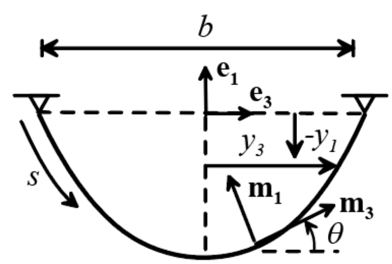
$$\frac{d\mathbf{m}_1}{ds}=-\kappa_2\mathbf{m}_3+\kappa_3\mathbf{m}_2 \quad \frac{d\mathbf{m}_2}{ds}=\kappa_1\mathbf{m}_3-\kappa_3\mathbf{m}_1 \quad \frac{d\mathbf{m}_3}{ds}=-\kappa_1\mathbf{m}_2+\kappa_2\mathbf{m}_1$$

(d) Write down expressions for the deformation gradient in the rod in the rod in terms of (R,θ,x_1), where x_1 is the distance of a material particle from the neutral section, expressing your answer as both components in the mixed basis, i.e. $\mathbf{F}=F_{ij}\mathbf{m}_i\otimes\mathbf{e}_j$, as well as $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$.

(e) Find an expression for the Lagrange strain tensor in the rod, expressing your answer as components in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Neglect second-order terms.
(f) Assume that the rod is made of a material that can be idealized using the generalized Hooke's law described in Section 3.4 of Applied Mechanics of Solids. Deduce expressions for the Material stress (in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ components) and Cauchy stress (in $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$). Assume – as is usual in beam theory – that only the axial stress is nonzero, and neglect second order terms in y_1 / R).
(g) Calculate the resultant internal moment and force acting on a generic internal cross-section of the rod.
(h) Show that the internal moment satisfies the equations of equilibrium.

10.3 Simplified Versions of the General Theory of Deformable Rods

Problem 10.8 Consider a flexible, inextensible cable subjected to transverse loading $\mathbf{p} = -p\mathbf{e}_1$ (e.g. due to gravity) as illustrated in the figure.

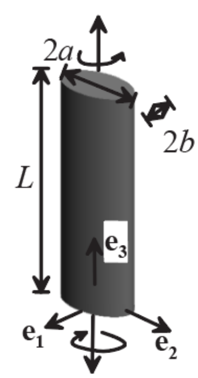


(a) Express the basis vector $\mathbf{e}_1$ in terms of $\mathbf{m}_1, \mathbf{m}_3$ and (derivatives of) y_1, y_3.
(b) Find an expression for the curvature vector for the cable in terms of (derivatives of) y_1, y_3
(c) Find the two equilibrium equations relating the axial tension T_3 to the external loading and geometry of the cable, by substituting $M_1 = M_2 = M_3 = T_1 = T_2 = 0$ into the general equations of motion for a rod. Show that the equations can be rearranged into the form

$$\frac{d}{dy_3}\left(T_3 \frac{1}{\sqrt{1+\left(dy_1 / dy_3\right)^2}} \right) = 0 \qquad\qquad \frac{C}{\sqrt{1+\left(dy_1 / dy_3\right)^2}} \frac{d^2 y_1}{dy_3^2} = p$$

where C is a constant of integration

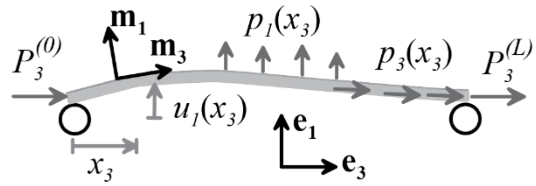
Problem 10.9 Consider a long, straight rod, with axis parallel to $\mathbf{e}_3$, which is subjected to pure twisting moments $\mathbf{Q} = Q\mathbf{e}_3$ acting at its ends. The rod may be idealized as a linear elastic solid with shear modulus μ. The deformation of the rod can be characterized by the twist $\psi(x_3)$ and the transverse displacement of the cross-section $u_3(x_\alpha)$. Assume that the only nonzero internal moment component is M_3, and the nonzero internal stress components are $\sigma_{3\alpha}$. Simplify the general governing equations for a deformable rod to obtain:



(a) A simplified expression for the curvature tensor $\boldsymbol{\kappa}$ for the deformed rod, in terms of $\psi(x_3)$
(b) Equations of equilibrium and boundary conditions for M_3 and $\sigma_{3\alpha}$
(c) Expressions relating $\sigma_{3\alpha}$ to $\psi(x_3)$ and the warping function *w*. Show that the equilibrium equation for $\sigma_{3\alpha}$ reduces to the governing equation for the warping function given in Section 10.2.10 of Applied Mechanics of Solids

(d) Expressions relating M_3 to $\psi(x_3)$.

Problem 10.10 An initially straight beam, with axis parallel to the $\mathbf{e}_3$ direction and principal axes of inertia parallel to $\mathbf{e}_1, \mathbf{e}_2$ is subjected to a force per unit length $\mathbf{p} = p_1^{(\mathbf{e})}\mathbf{e}_1 + p_3^{(\mathbf{e})}\mathbf{e}_3$. The beam has Young's modulus E and mass density ρ, and its cross-section has area A and principal moments of inertia I_1, I_2, I_3. Assume that a large axial internal force $N\mathbf{m}_3$ is developed in the beam, either by a horizontal force per unit length acting on the beam, or by horizontal forces acting at the ends of the beam. Suppose that the beam experiences a *finite* transverse displacement $\mathbf{u} = u_1\mathbf{e}_1 + u_3\mathbf{e}_3$, so that (expanding to second order in du_1/dx_3) the stretch of the beam and its curvature must be approximated by



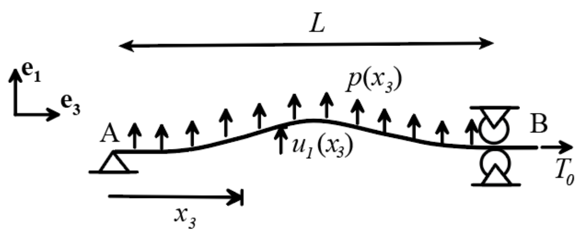
$$\frac{ds}{dx_3} \approx \left(1 + \frac{\partial u_3}{\partial x_3}\right)\sqrt{1 + \left(\frac{\partial u_1}{\partial x_3}\right)^2} \approx 1 + \frac{\partial u_3}{\partial x_3} + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}\right)^2 \qquad \boldsymbol{\kappa} \approx \frac{d^2 u_1}{dx_3^2}\mathbf{e}_2$$

Show that the static equilibrium equations for the displacement components can be reduced to

$$EI_2 \frac{\partial^4 u_1}{\partial x_3^4} - EA\left(\frac{\partial u_3}{dx_3} + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}\right)^2\right)\frac{\partial^2 u_1}{\partial x_3^2} + p_3^{(\mathbf{e})}\frac{du_1}{dx_3} - p_1^{(\mathbf{e})}\left(1 - \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}\right)^2\right) = 0$$

$$EI_2\left(\frac{\partial^2 u_1}{\partial x_3^2}\right)^2 + EA\frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{dx_3} + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}\right)^2\right) + p_3^{(\mathbf{e})}\left(1 - \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}\right)^2\right) + \frac{\partial u_1}{\partial x_3}p_1^{(\mathbf{e})} = 0$$

Problem 10.11 The goal of this problem is to derive the equation of motion for an inextensible stretched string subjected to small displacements by a direct application of the principle of virtual work. Assume that at some instant the string has transverse deflection $u_1(x_3)$ and velocity $v_1(x_3)$ as indicated in the figure.



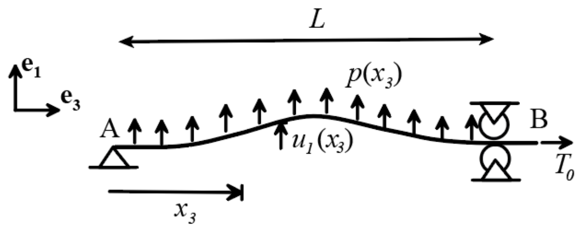
(a) Write down an expression for the relative velocity of the end B of the string relative to A, in terms of $u_1(x_3)$ and $v_1(x_3)$. Expand the expression, accurate to second order in $u_1(x_3)$

(b) Write down the rate of virtual work done by the transverse forces $p_1(x_3)$ in terms of a virtual velocity $\delta v_1(x_3)$

(c) Write down the rate of virtual done by the applied tension T_0.

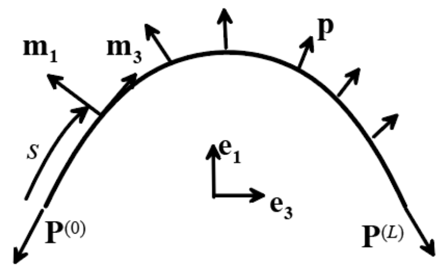
(d) Hence use the principle of virtual work to derive the equation of motion for the string.

Problem 10.12 The figure shows an inextensible cable supported at its ends and stretched by a force T_0 acting on its right end. It is subjected to a transverse force $p_1(x_3)$ per unit length that causes it to deflect upwards by a small distance $u_1(x_3)$. Show that the potential energy of the system can be approximated by



$$V \approx \frac{T_0}{2}\int_0^L \left(\frac{du_1}{dx_3}\right)^2 dx_3 - \int_0^L p_1 u_1 dx_3$$

Problem 10.13 The figure shows an inextensible, flexible cable, subjected to a transverse force per unit length $\mathbf{p} = p_i \mathbf{m}_i$ and forces $\mathbf{P}^{(0)} = P_3^{(0)}\mathbf{m}_3$ and $\mathbf{P}^{(L)} = P_3^{(L)}\mathbf{m}_3$ acting at its ends. In a flexible cable, the area moments of inertia can be neglected, so that the internal moments $M_i \approx 0$. In addition, the axial tension



$$T_3 = \int_A \sigma_{33} dA$$

is the only nonzero internal force.

(a) Show that under these conditions the general virtual work equation given in Sect 10.2.9 of Applied Mechanics of Solids reduces to

$$\int_0^{L_0} \frac{d\delta\dot{s}}{dx_3} T_3 dx_3 + \int_0^L \rho A a_i \delta v_i ds - \int_0^L \left(p_i \delta v_i\right) ds - \left[P_3^{(0)}\delta v_3\right]_{x_3=0} - \left[P_3^{(L)}\delta v_3\right]_{x_3=L} = 0$$

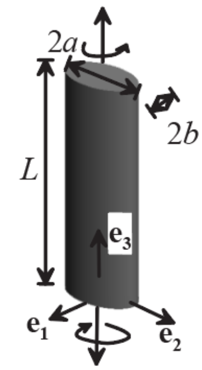
where δv_i is the virtual velocity of the cable, and $\delta\dot{s}$ is the corresponding rate of change of arc-length along the cable.

(b) Show that if the virtual work equation is satisfied for all δv_i and compatible $\delta\dot{s}$, the internal force and curvature κ_α of the cable must satisfy

$$T_3\kappa_2 + p_1 = \rho A a_1 \qquad -T_3\kappa_1 + p_2 = \rho A a_2 \qquad \frac{dT_3}{ds} + p_3 = \rho A a_3$$

$$T_3 = -P_3^{(0)} \qquad x_3 = 0 \qquad T_3 = P_3^{(L)} \qquad x_3 = L$$

10.4 Exact Solutions to Problems Involving Slender Rods



Problem 10.14 A slender, linear elastic rod has shear modulus μ and an elliptical cross-section with semi-axes *a,b*, as illustrated in the figure. It is subjected to equal and opposite axial couples with magnitude *Q* on its ends. Using the general theory of slender rods, and assuming that the rod remains straight:

(a) Write down the curvature vector in the rod, in terms of the rotation ψ of its cross section about its axis.

(b) Write down the internal force and moment distribution in the rod, in terms of *Q*.

(c) Show that the function

$$w = -\frac{x_1 x_2 (a^2 - b^2)}{\left(a^2 + b^2\right)}$$

satisfies the governing equation and boundary conditions for the warping function for an elliptical cross-section.

(d) Show that the modified polar moment of area for the cross-section is

$$J_3 = \frac{\pi a^3 b^3}{\left(a^2 + b^2\right)}$$

(e) Use the warping function and modified polar moment of area together with the force-deformation relations in Sect 10.2.10 of Applied Mechanics of Solids to find an expression relating *Q* to ψ , and determine the stresses in the rod in terms of *Q* and the geometry and material properties of the rod.

(f) Recall that (see Sect 10.2.4 of Applied Mechanics of Solids) a point that has initial position $\mathbf{x} = x_i \mathbf{e}_i$ in the rod displaces to

$$\mathbf{y}(x_k) = \mathbf{r}(x_3) + \eta_i(x_k)\mathbf{m}_i(x_3)$$

after deformation, where $\mathbf{m}_i$ are basis vectors that rotate with the rod's cross-section, and

$$\eta_\alpha = x_\alpha + f_{\alpha\beta}(x_3)x_\beta \qquad \eta_3 = u_3(x_\beta, x_3)$$

with

$$u_3 = \kappa_3 w(x_1, x_2)$$

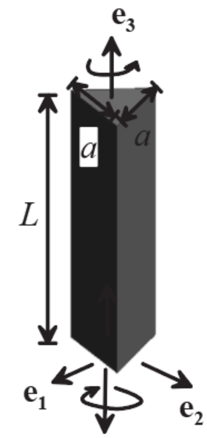
For a rod with traction free sides (see Sect 10.2.10 of Applied Mechanics of Solids)

$$f_{11} = f_{22} = -\nu\left((ds / d\overline{s})^2 - 1\right)/2\ , \ f_{12} = 0$$

where ν is Poisson's ratio. Use these results and those of (a)-(i) to find a formula for the displacement vector in the fixed basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. You can leave $\psi(x_3)$ in your formula to avoid making it too long.

(g) Find an expression for the critical moment *Q* that will cause the shaft to yield, in terms of the geometry of the shaft and the yield stress *Y*.

Problem 10.15 A slender, linear elastic rod has shear modulus μ and an equilateral triangular cross-section with side length a, as illustrated in the figure. It is subjected to equal and opposite axial couples with magnitude Q on its ends. Using the general theory of slender rods, and assuming that the rod remains straight:

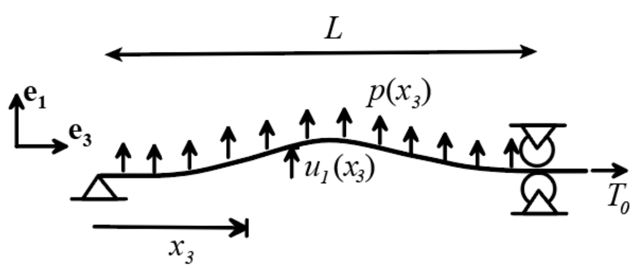


(a) Write down the curvature vector in the rod, in terms of the rotation ψ of its cross section about its axis.
(b) Write down the internal force and moment distribution in the rod, in terms of Q.
(c) The warping function and modified polar moment of area for a triangular cross-section are

$$w = \frac{x_2(x_2^2 - 3x_1^2)}{a\sqrt{3}} \qquad J_3 = \frac{a^4\sqrt{3}}{80}$$

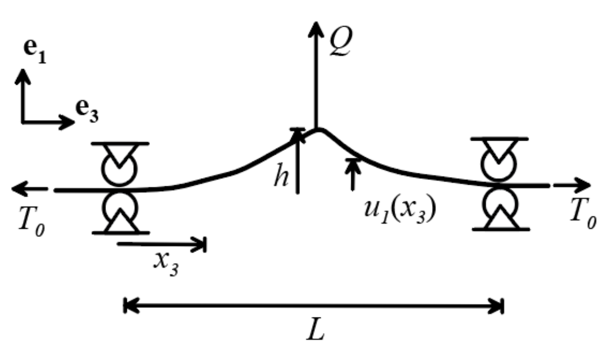
Use these together with the force-deformation relations in Sect 10.2.10 of Applied Mechanics of Solids to find an expression relating Q to ψ , and determine the stresses in the rod in terms of Q and the geometry and material properties of the rod.
(d) Find a formula for the rotation of the shaft's cross section
(e) Find an expression for the critical couple Q that will cause the shaft to yield

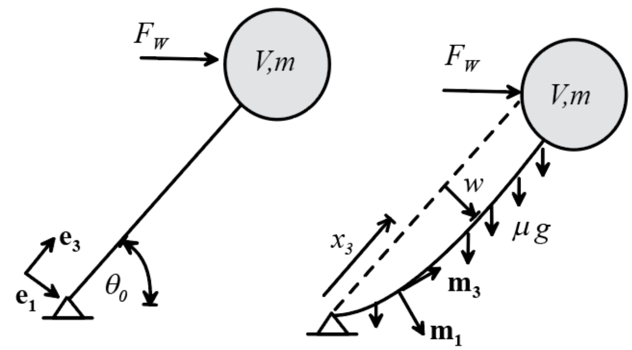
Problem 10.16 The figure shows a flexible string, which is supported at both ends and subjected to a tensile force T_0. The string is subjected to a uniform transverse for p per unit length.



(a) Calculate the deflection $u_1(x_3)$ of the string, assuming small deflections.
(b) An overhead transmission cable with mass density ρ and tensile strength σ_{UTS} has length L and a uniform cross-sectional area A. IEEE standard 524 recommends that the tension in the cable should not exceed 10% of the strength of the cable. Find a formula for the sag at mid-span of a cable that is subjected to the maximum allowable tension (you can assume small deflections).
(c) Find the maximum length of a high conductivity copper cable with mass density 9 g/cm^3 and tensile strength 170 MPa if h/L is required to be less than 0.1

Problem 10.17 A cable with weight per unit length p is stretched by tension T_0 and subjected to a transverse force Q at mid-span. Calculate the deflection h of the cable at mid-span. What value of Q (in term of p,L) will make the displacement at mid-span zero? You can assume small deflections.





Problem 10.18 The figure shows a spherical balloon with volume V and mass m tethered by a cable with length L and mass μ per unit length. The balloon is subjected to a lateral wind load F_W

(a) For the limiting case of a massless cable $(\mu = 0)$) , the cable remains straight and has a constant internal tension T. Find formulas for the angle θ_0 and T in terms of V,m, F_W and air density ρ .

(b) If the cable mass is significant the cable will deflect as shown in the figure. Show that, for a small deflection, the force per unit length acting on the cable can be approximated as

$$\mathbf{p} \approx \mu g\left(\cos\theta_0 + \sin\theta_0 \frac{dw}{dx_3}\right)\mathbf{m}_1 - \mu g\left(\sin\theta_0 - \cos\theta_0 \frac{dw}{dx_3}\right)\mathbf{m}_2$$

where g is the gravitational acceleration.

(c) Write down the curvature vector for the cable, in terms of w (assume small deflections)

(d) Assume (using the usual approximation for a flexible, inextensible cable) that the only nonzero internal force and moment in the cable is $\mathbf{T} = T_3\mathbf{m}_3$. Show that the equilibrium equations for the cable in the $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ basis (see Sect 10.2.9 of Applied Mechanics of Solids) can be approximated to first order in w and zeroth order in T_3 as

$$T_3 \frac{d^2 w}{dx_3^2} + \mu g \sin\theta_0 \frac{dw}{dx_3} + \mu g \cos\theta_0 \approx 0 \qquad \frac{dT_3}{dx_3} - \mu g \sin\theta_0 \approx 0$$

and show that they can be combined to simplify the first equation to

$$\frac{d}{dx_3}\left(T_3 \frac{dw}{dx_3}\right) + \mu g \cos\theta_0 \approx 0 \qquad \frac{dT_3}{dx_3} - \mu g \sin\theta_0 \approx 0$$

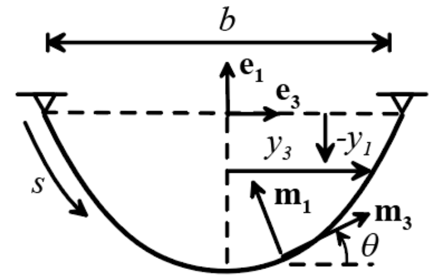
(e) Write down the boundary condition for the tension at $x_3 = L$, as well as the boundary conditions for w at $x_3 = 0$ and $x_3 = L$

(f) Hence, solve the equation system in part (d) to show that the cable deflection w is

$$w = \frac{b}{c}\left\{-x_3 + \frac{T_0}{c}\log\frac{T_0 - c(L - x_3)}{T_0 - cL}\right\}$$

$$c = \mu g \sin\theta_0 \qquad b = \mu g \cos\theta_0 \qquad T_0 = \sqrt{F_W^2 + (\rho g V - mg)^2}$$

Problem 10.19 The figure shows a flexible cable with length L and weight m per unit length hanging between two supports under uniform vertical gravitational loading. In a flexible cable, the area moments of inertia can be neglected, so the internal moments $M_i \approx 0$



(a) Write down the curvature vector of the cable in terms of the angle $\theta(s)$ shown in the figure

(b) Hence, show that the equations of equilibrium for the cable reduce to

$$\frac{dT_3}{ds} - m\sin\theta = 0 \qquad T_3\frac{d\theta}{ds} - m\cos\theta = 0$$

(c) Hence, show that $T_3\cos\theta = H$, where H is a constant (hint: eliminate m from the equilibrium equations). Interpret the equation physically.

(d) Use (b) to show that $d(T_3\sin\theta)/ds = m$. Interpret this equation physically.

(e) Deduce that

$$w(s) = \tan\theta = \frac{m(s-L/2)}{H} \qquad \Rightarrow \qquad \frac{dw}{dy_3} = \frac{m\sqrt{1+w^2}}{H}$$

(here w is introduced simply as a variable to denote $\tan\theta$)

(f) Hence, deduce that $w(y_3) = dy_1/dy_3 = \sinh(my_3/H)$ and calculate y_1 as a function of y_3, in terms of H, b and m

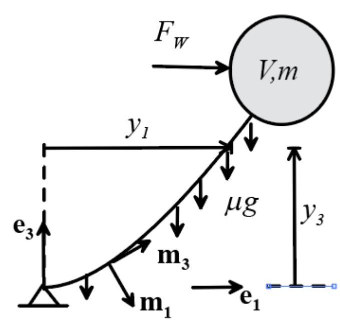
(g) Use the solution to (f) to find a formula for the length of the cable in terms of m,b,H, and hence show that the unknown constant H can be found by solving the equation

$$\frac{Lm}{2H} = \sinh\left(\frac{mb}{2H}\right)$$

(h) Finally, find a formula for the tension in the cable, in terms of H, L, m, b and s.

(i) Show that, as $mb/H \to 0$ the full solution approaches the small deflection solution calculated in Problem 10.15 (with an appropriate change of sign!).

Problem 10.20 The figure shows a spherical balloon with volume V and mass m tethered by a cable with length L and mass μ per unit length. The balloon is subjected to a lateral wind load F_W.



(a) Use the method of problem 10.19 to show that the deflected shape of the cable is given by the parametric equation

$$\eta_3 = \frac{\alpha}{\beta}\log\left\{\frac{\beta\xi+1-\beta+\sqrt{(\beta\xi+1-\beta)^2+\alpha^2}}{1-\beta+\sqrt{(1-\beta)^2+\alpha^2}}\right\}$$

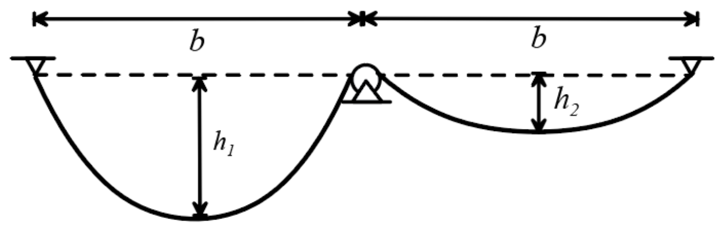
$$\eta_1 = \frac{1}{\beta}\left\{\sqrt{(\beta\xi+1-\beta)^2+\alpha^2} - \sqrt{(1-\beta)^2+\alpha^2}\right\}$$

where

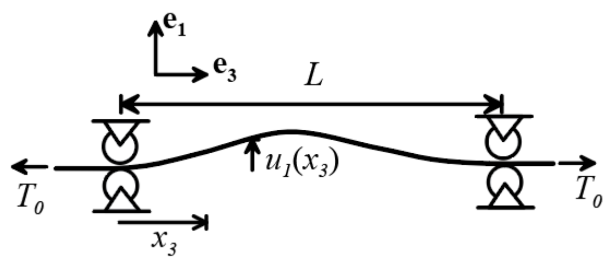
$$\alpha = \frac{F_W}{\rho g V - mg} \qquad \beta = \frac{\mu g L}{\rho g V - mg} \qquad \eta_i = \frac{y_i}{L} \qquad \xi = \frac{s}{L}$$

(b) Plot the deflected shape of the cable for some representative values of α, β (if you are interested you could also plot the approximate solution obtained in problem (5) for comparison. Note that the approximate solution predicts unphysical behavior if the cable weight is too large)

Problem 10.21 The figure shows an inextensible cable with weight per unit length m, and length L^* that is pinned at both ends. The cable is supported by a frictionless pulley midway between the two ends. Find all the possible equilibrium values of the sags h_1, h_2 of the cable. Display your results by plotting a graph showing the equilibrium values of h_1 / b as a function of $L^* / (2b)$. You will need to solve problem 10.19 before attempting this one.

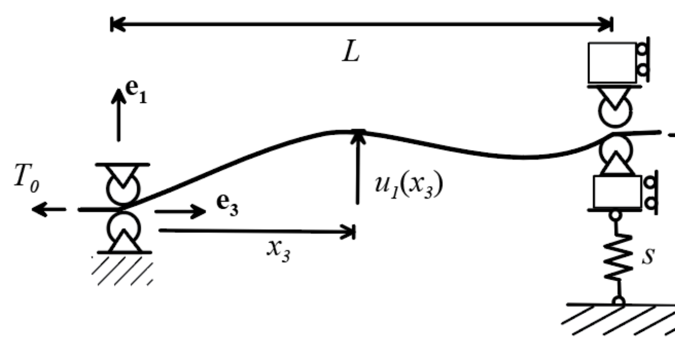


Problem 10.22 The figure shows a flexible string, which is supported at both ends and subjected to a tensile force T_0. The string has mass per unit length m and can be approximated as inextensible. Calculate the natural frequencies of vibration and the corresponding mode shapes, assuming small transverse deflections.



Problem 10.23 Estimate the fundamental frequency of vibration for the stretched string described in the preceding problem using the Rayleigh-Ritz method (problem 10.11 gives the formula for the potential energy of a stretched string, if you need it). Use the approximation $u_1(x_3) = Ax_3(L - x_3)$ for the mode shape. Compare the estimate with the exact solution derived in the preceding problem.

Problem 10.24 A cable with mass m per unit length is stretched by a tension T_0. The end at $x_3 = 0$ is pinned, while the support at $x_3 = L$ can move vertically, and is held in place by a spring with stiffness s The goal of this problem is to calculate the natural frequencies of vibration.



(a) Write down the boundary condition for the transverse displacement u_1 of the cable at $x_3 = 0$.
(b) Show that the boundary condition at $x_3 = L$ is

$$T_0 \frac{du_1}{dx_3} + su_1 = 0$$

(assume small deflections, and that $u_1(L) = 0$ when the system is in its static equilibrium configuration).

(c) State the equation of motion for the cable, and show that the standing wave solution $u_1 = \sin\omega t(A\sin kx_3 + B\cos kx_3)$ satisfies the equation. Give the relation between wave number *k*, natural frequency ω and wave speed *c*.

(d) Show that the natural frequencies are given by

$$\omega_n = \frac{\beta_n}{L}\sqrt{\frac{T_0}{m}}$$

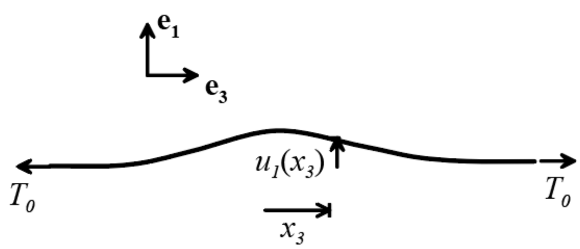
where β_n are the roots of the equation

$$\beta_n \cos\beta_n + \frac{Ls}{T_0}\sin\beta_n = 0$$

(e) Find formulas (in terms of T_0, L, m) for the lowest natural frequency of the system for

$$\frac{Ls}{T_0} = 0\,,\quad \frac{Ls}{T_0} \to \infty \text{ , and } \frac{Ls}{T_0} = 1$$

Problem 10.25 The figure shows a small transverse deflection in part of an infinitely long cable with mass per unit length *m* that is stretched by an axial tension T_0.



(a) Show that the travelling wave solution

$$u_1 = f(x_3 - ct) + g(x_3 + ct)$$

satisfies the equation of motion, and give an expression for the wave speed *c*.

(b) Suppose that at time *t=0* the cable has a deflection and velocity

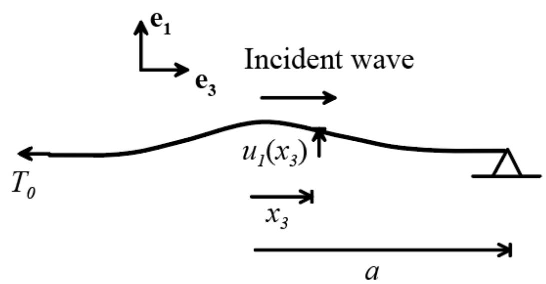
$$u_1(x_3, t=0) = u^{(0)}(x_3) \qquad \frac{\partial u_1}{\partial t} = v^{(0)}(x_3)$$

Show that the initial conditions are satisfied by *f* and *g* given by

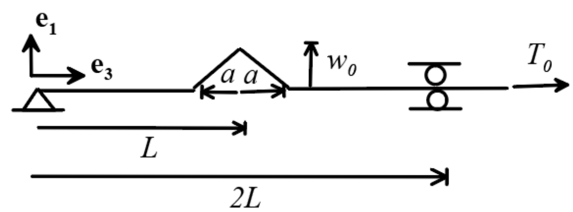
$$f(\lambda) = [u^{(0)}(\lambda) - \frac{1}{c}\int_{-\infty}^{\lambda} v^{(0)}(\xi)d\xi - A]/2 \qquad g(\lambda) = [u^{(0)}(\lambda) + \frac{1}{c}\int_{-\infty}^{\lambda} v^{(0)}(\xi)d\xi + A]/2$$

where *A* is an arbitrary constant.

Problem 10.26 The figure shows a semi-infinite cable with mass per unit length *m* that is stretched by an axial tension T_0 at $x_3 = -\infty$ and is pinned at $x_3 = a$. A wave with transverse displacement $u_1 = f(t - x_3/c)$ travels along the string towards the fixed end. Use the method applied to analyze reflection of plane waves in Section 4.3 of Applied Mechanics of Solids to find an expression for the displacement in the string, accounting for the zero displacement boundary condition at $x_3 = a$



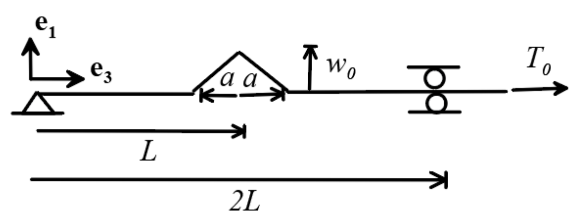
Problem 10.27 The figure shows a stretched string with an initial triangular displacement near its center. Assume the string has a wave speed c. Sketch the shape of the string at the following times (you don't need to solve all the equation from scratch, just use your physical understanding of wave propagation and reflection to work out what you expect to see)



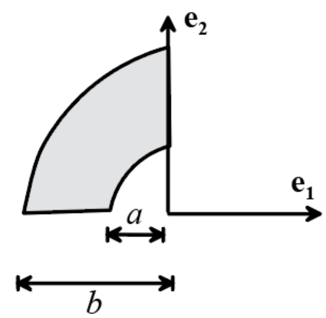
(a) $t = L/(2c)$

(b) $t = 3L/(2c)$

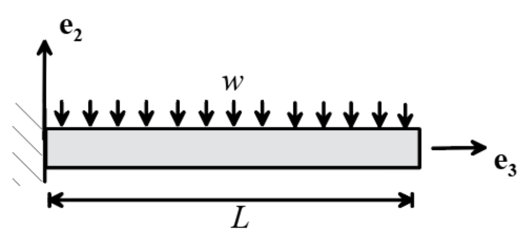
Problem 10.28 The string shown in the figure has mass per unit length m and is stretched by an axial tension T_0 . It is prevented from moving in a vertical direction at its ends $x_3 = 0, x_3 = L$. At time *t=0* it is released from rest with the displacement distribution shown in the figure. Draw the shape of the string at time $t = L\sqrt{m/T_0}$



Problem 10.29 The figure shows the cross-section of a beam. Find the position of the centroid, and calculate the area moments of inertia of the cross section. Simplify your answer by assuming that *b=a+t*, with *t<<a* (i.e. take the Taylor expansion to first order in *t*).



Problem 10.30 Suppose that a cantilever beam with length L and cross-section analyzed in problem 3 is subjected to a uniform load w acting in the (negative) $\mathbf{e}_2$ direction.



(a) By considering equilibrium of a section of the right hand end of the beam, write down the bending moment components M_1, M_2 in the beam (be careful with the sign convention. Note that $\mathbf{e}_1$ points into the plane of the figure.

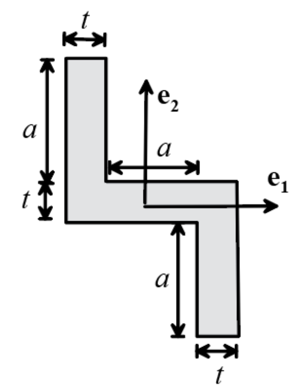
(b) Write down the curvature vector in terms of derivatives of u_1, u_2

(c) Use (a), (b) and the moment-curvature relations to show that

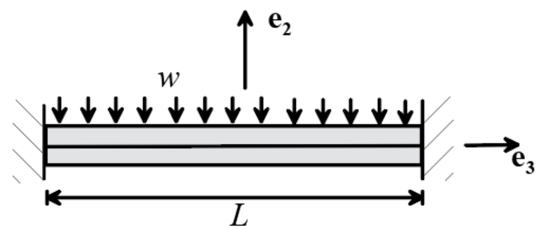
$$\frac{d^2}{dx_3^2}\begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} = \frac{1}{E(I_{11}I_{22} - I_{12}^2)}\begin{bmatrix} I_{22} & I_{12} \\ I_{12} & I_{11} \end{bmatrix}\begin{bmatrix} w(L^2 - x_3^2)/2 \\ 0 \end{bmatrix}$$

(d) Hence, calculate the deflection of the end of the beam

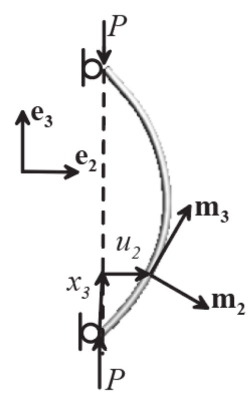
Problem 10.31 The figure shows the cross-section of a beam. Calculate the area moment of inertia tensor for the beam section shown (simplify the solution by assuming $t<<a$)



Problem 10.32 Suppose that a beam with length L and cross-section analyzed in problem 17 is subjected to a uniform load w acting in the (negative) $\mathbf{e}_2$ direction. Calculate the deflection of the beam at mid-span.



Problem 10.33 The figure shows an initially straight, inextensible elastic rod, with Young's modulus E, length L and principal in-plane moments of area $I_1 = I_2 = I$, which is subjected to end thrust. The ends of the rod are constrained to travel along a line that is parallel to the undeformed rod, but the ends are free to rotate.



(a) Show that for the problem of interest the general equations for the deflection of a straight beam subjected to significant axial force given in Section 10.3.3 of Applied Mechanics of Solids can be simplified to

$$EI\frac{d^4u_2}{dx_3^4} + P\frac{d^2u_2}{dx_3^2} = 0$$

while the boundary conditions are

$$u_2 = 0 \qquad \frac{d^2u_2}{dx_3^2} = 0 \qquad (x_3 = 0,\ x_3 = L)$$

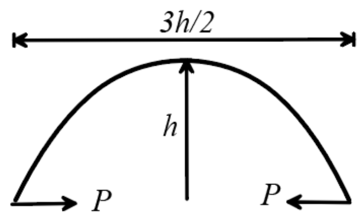
(b) Show that the differential equation and boundary conditions in part (a) have solution

$$u_2 = A\sin\lambda L = 0$$

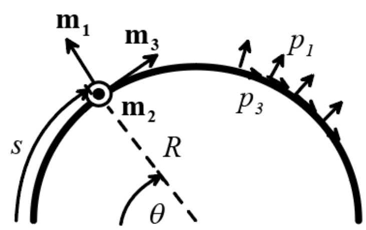
with $\lambda L = n\pi$ where $n \ge 1$ is an integer, and A an arbitrary constant. Hence, find a formula for the smallest value of the (normalized) axial thrust $PL^2/(EI)$ that is necessary to produce the lateral deflection.

(c) Show that the critical force found in part (b) agrees with the value found in Section 10.4.3 of Applied Mechanics of Solids. Also, show that for small deflections the shape of the buckled rod predicted by the exact post-buckling solution agrees with the approximate buckling mode found in part (b).

Problem 10.34 An initially straight tent-pole with Young's modulus E and hollow circular cross-section with external radius a, moment of inertia I is to be bent into an arc with height h and base $3h/2$ as shown in the figure. Calculate expressions for the force P required to bend the pole into shape; the total length of the pole (in terms of h) and the maximum stress in the pole



Problem 10.35 The figure shows a rod which is a circular arc with radius R in its stress free configuration, and is subjected to load per unit length $\mathbf{p}=p_1\mathbf{m}_1+p_3\mathbf{m}_3$ and forces $\mathbf{P}^{(0)}$, $\mathbf{P}^{(L)}$ on its ends. The rod has equal principal moments of area I. The loads cause a small change in its shape. In this problem, we shall neglect out-of-plane deformation and twisting of the rod, for simplicity. Let $\bar{s}=R\theta$ denote the arc length measured along the undeformed rod, and let $\mathbf{u}=u_1(\theta)\mathbf{m}_1+u_3(\theta)\mathbf{m}_3$ the displacement of the rod's centerline.



(a) Note that approximate expressions for the resulting (small) change in arc length and curvature of the rod can be calculated using the time derivatives given in Section 10.2.3 of Applied Mechanics of Solids. Hence, show that

- The derivative of the change in arc-length of the deformed rod is

$$\frac{d\delta s}{d\bar{s}}=\frac{1}{R}\left(\frac{du_3}{d\theta}+u_1(\theta)\right)$$

- The change in curvature vector is

$$\delta\boldsymbol{\kappa}=\frac{1}{R^2}\left(\frac{d^2u_1}{d\theta^2}-\frac{du_3}{d\theta}\right)\mathbf{m}_2$$

(b) The geometric terms in the equilibrium equations listed in Section 10.2.9 can be approximated using the geometry of the undeformed rod. Show that internal forces $\mathbf{T}=T_1\mathbf{m}_1+T_3\mathbf{m}_3$ and internal moment $\mathbf{M}=M_2\mathbf{m}_2$ must satisfy the following static equilibrium equations

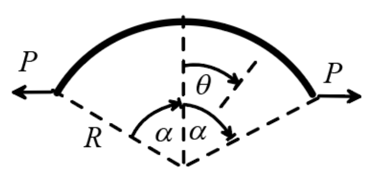
$$\frac{dT_1}{d\theta}-T_3+Rp_1=0 \qquad \frac{dT_3}{d\theta}+T_1+Rp_3=0 \qquad \frac{dM_2}{d\theta}+RT_1=0$$

(c) Assume that the rod is elastic, with Young's modulus E and area moment of inertia $I_1=I_2=I$, and can be idealized as inextensible. Show that under these conditions the transverse and axial displacements u_1,u_3 must satisfy

$$\frac{EI}{R^4}\left(\frac{d^6u_3}{d\theta^6}+2\frac{d^4u_3}{d\theta^4}+\frac{d^2u_3}{d\theta^2}\right)+\frac{dp_1}{d\theta}+p_3=0 \qquad\qquad u_1=-\frac{du_3}{d\theta}$$

and write down expressions for the boundary conditions at the ends of the rod.

(d) As a particular example, consider a rod which is a semicircular arc between $\theta=\pm\alpha$, subjected to equal and opposite forces acting on its ends, as shown in the figure. Assume that the displacement and rotation of the rod vanish at $\theta=0$. Calculate $u_1(\theta)$ and $u_3(\theta)$ for the rod.

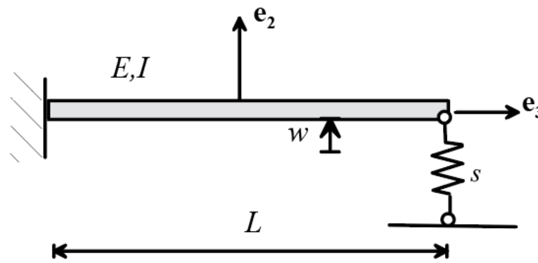


Problem 10.36 An initially straight, elastic rod with Young's modulus E, shear modulus μ, length L, area moments of inertia $I_1 = I_2 = I$ and axial effective inertia J_3 is subjected to equal and opposite axial couples $\mathbf{Q} = \pm Q\mathbf{e}_3$ on its ends, which remain fixed in direction as the rod deforms.

(a) Show that the straight rod, with an appropriate twist is a possible equilibrium configuration for all values of Q, and calculate the value of twist

(b) Show that, for a critical value of Q, the rod may adopt a helical shape, with one complete turn and arbitrary height h and radius r. Calculate the critical value of Q

(c) What can you infer about the stability of a straight rod subjected to end couples?



Problem 10.37 The figure shows an idealization that is sometimes used to analyze the vibration of the cantilever in an atomic force microscope. The goal of this problem is to repeat some of their calculations. The figure shows the problem to be solved: the cantilever is idealized as a beam with modulus E and area moment of inertia I, mass density ρ and cross sectional area A. The interaction of the microscope tip with the surface of the specimen is approximated by a spring with stiffness s (we use s instead of the usual k because k is used for the wave number). Assume that the spring is free of force when the displacement of the cantilever is zero.

(a) Write down the differential equation governing flexural vibration of the cantilever, and by considering solutions of the form $w = \cos(\omega t + \phi) f(x_3)$ show that the equation is satisfied by a solution of the form

$$f(x_3) = A\sin kx_3 + B\cos kx_3 + C\sinh kx_3 + D\cosh kx_3$$

Find the relationship between $k, \omega,$ and $\beta = \sqrt{EI/(\rho A)}$

(b) Write down the boundary conditions for the transverse displacement w, and show that they can be arranged into the following form

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -S & -C & \Sigma & \chi \\ -(kL)^3 C - \mu S & (kL)^3 S - \mu C & (kL)^3 \chi - \mu \Sigma & (kL)^3 \Sigma - \mu \chi \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mu = \frac{sL^3}{EI} \quad S = \sin(kL) \quad C = \cos(kL) \quad \Sigma = \sinh(kL) \quad \chi = \cosh(kL)$$

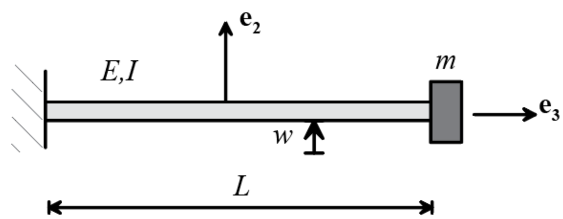
(c) Hence, show that the wave numbers for the vibration modes satisfy

$$(kL)^3 \cos(Lk)\cosh(Lk) + \mu\cosh(Lk)\sin(Lk) - \mu\cos(Lk)\sinh(Lk) + (kL)^3 = 0$$

(d) Calculate the lowest natural frequency of the beam (in terms of β and L) without the spring on its end (i.e. $\mu = 0$). You will need to solve the equation $\cos(Lk)\cosh(Lk) + 1 = 0$ numerically.

(e) Plot a graph of $\omega(\mu)/\omega(0)$, where $\omega(0)$ is the frequency of the cantilever without the spring on its end (i.e. the solution to (d)) as a function of μ in the range $0 < \mu < 150$.

Problem 10.38 Functionalized cantilevers are sometimes used as chemical or biochemical mass sensors. The basic principle is to detect the change in resonant frequency of the cantilever when a small mass is adsorbed on its tip. The goal of this problem is to provide the necessary relationship between mass and natural frequency. Assume that the beam has modulus E and area moment of inertia I, mass density ρ and cross sectional area A, and has a small mass (with negligible mass moment of inertia) attached to its tip.



(a) Show that the transverse force acting on the right end of the cantilever is related to the displacement at its tip by

$$T_2 = -m\frac{d^2 w}{dt^2}$$

(b) Write down the differential equation governing flexural vibration of the cantilever, and by considering solutions of the form $w = \cos(\omega t + \phi) f(x_3)$ show that the equation is satisfied by a solution of the form

$$f(x_3) = A\sin kx_3 + B\cos kx_3 + C\sinh kx_3 + D\cosh kx_3$$

Write down the relationship between k, ω, and $\beta = \sqrt{EI / \rho A}$

(c) Write down the boundary conditions for the transverse displacement w, and show that they can be arranged into the following form

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -S & -C & \Sigma & \xi \\ -C+\mu kLS & S+\mu kLC & \chi+\mu kL\Sigma & \Sigma+\mu kL\xi \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where

$$\mu = \frac{m}{A\rho L} \qquad S = \sin(kL) \quad C = \cos(kL) \qquad \Sigma = \sinh(kL) \qquad \chi = \cosh(kL)$$

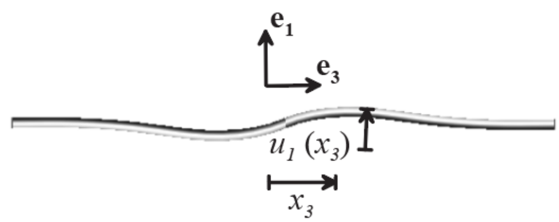
(d) Hence, show that the wave numbers for the vibration modes satisfy

$$\cos(Lk)\cosh(Lk) - \mu kL\cosh(Lk)\sin(Lk) + \mu kL\cos(Lk)\sinh(Lk) + 1 = 0$$

(e) Find a formula for the lowest natural frequency of the beam (in terms of β and L) without the mass on its end (i.e. $\mu = 0$). You will need to solve the equation $\cos(Lk)\cosh(Lk) + 1 = 0$ numerically.

(f) What would you expect the lowest natural frequency to be in the limit of very large μ? (give a formula in terms of μ, β, L).

(g) Plot a graph of $\omega(\mu) / \omega(0)$, where $\omega(0)$ is the frequency of the cantilever without a tip mass (i.e. the solution to 3.5) as a function of μ in the range $0 < \mu < 0.2$. Suppose that it is possible to detect a 2% change in frequency, and a typical cantilever has a mass of about 30 ng (nanograms) – your graph should show that the measurement would be able to measure a mass of 200 pg (picograms) or so.



Problem 10.39 The figure shows an infinitely long beam with Youngs modulus *E*, mass density ρ, cross sectional area *A* and inertia components $I_{11} = I_{22} = I$ and $I_{12} = 0$. The beam is at rest and has a transverse displacement

$$u_1(x_3) = u^{(0)}(x_3)$$

at time *t=0*. The goal of this problem is to find the subsequent motion of the beam.

(a) Write down the equation governing transverse motion of the beam, assuming small deflections with no axial force in the beam.

(b) Show that, although the governing equation resembles the wave equation in a flexible cable with wave speed $\beta = \sqrt{EI / \rho A}$, the propagating wave solution $u_1 = f(x_3 - \beta t) + g(x_3 + \beta t)$ does not, in general satisfy the equation.

(c) Propagating wave solutions do exist, for special initial displacements. Show that

$$u_1(t, x_3) = U_0 \left[\cos\left\{ k(x_3 - \beta k t) \right\} + \cos\left\{ k(x_3 + \beta k t) \right\} \right] / 2$$

is a possible solution, and determine the speed of the wave (note that the wave speed depends on its wavelength. Waves of this kind are said to be 'dispersive')

(d) More generally, show that a solution of the form

$$u_1(t, x_3) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} U_0(k) \left[\exp\left\{ ik(x_3 - \beta k t) \right\} + \exp\left\{ ik(x_3 + \beta k t) \right\} \right] dk$$

satisfies the governing equation.

(e) The solution in (d) is an inverse Fourier transform. Therefore, selecting

$$U_0(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^{(0)}(x_3) \exp(ikx_3) dx_3$$

(the forward Fourier transform) will satisfy the initial condition. As a specific example, show that for an initial displacement given by

$$u^{(0)} = \exp(-x_3^2)$$

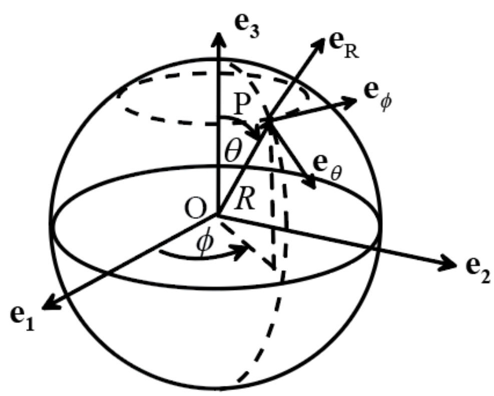
the solution is

$$u_1(t, x_3) = \frac{1}{4\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\frac{-k^2}{4} \exp(ikx_3)(\exp(ik^2 \beta t) + \exp(-ik^2 \beta t)) dk$$

Plot the solution for $-20 < x_3 < 20$ at times $\beta t = 0, \beta t = \pi / 2, \beta t = \pi$. You should see the dispersion clearly.

10.5 Motion and Deformation of Thin Shells

Problem 10.40 A spherical-polar coordinate system is to be used to describe the deformation of a spherical shell with radius R. The two angles (θ,ϕ) illustrated in the figure are to be used as the coordinate system (ξ_1,ξ_2) for this geometry.



(a) Write down the position vector $\overline{\mathbf{r}}$ in terms of (θ,ϕ), expressing your answer as components in the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ shown in the figure.

(b) Calculate the covariant basis vectors $\left(\overline{\mathbf{m}}_1,\overline{\mathbf{m}}_2,\overline{\mathbf{m}}_3\right)$ in terms of $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and (R,θ,ϕ). Choose the order for $\left(\overline{\mathbf{m}}_1,\overline{\mathbf{m}}_2,\overline{\mathbf{m}}_3\right)$ so that $\overline{\mathbf{m}}_3$ points out of the sphere.

(c) Calculate the contravariant basis vectors $\left(\overline{\mathbf{m}}^1,\overline{\mathbf{m}}^2,\overline{\mathbf{m}}^3\right)$ in terms of $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and (R,ϕ,θ)

(d) Calculate the covariant, contravariant and mixed components of the curvature tensor $\overline{\boldsymbol{\kappa}}$ for the shell

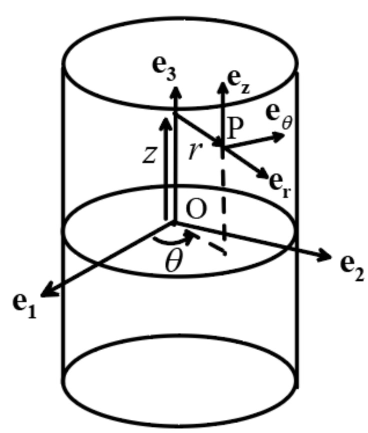
(e) Find the components of the Christoffel symbol $\Gamma^i_{\alpha\beta}$ for the coordinate system

(f) Suppose that under loading the shell simply expands radially to a new radius r. Find the components of the mid-plane Lagrange strain tensor $\boldsymbol{\gamma} = \gamma_{\alpha\beta}\,\overline{\mathbf{m}}^\alpha \otimes \overline{\mathbf{m}}^\beta = \dfrac{1}{2}\left(g_{\alpha\beta} - \overline{g}_{\alpha\beta}\right)\overline{\mathbf{m}}^\alpha \otimes \overline{\mathbf{m}}^\beta$ and the components of the curvature change tensor $\Delta\boldsymbol{\kappa} = (\kappa^\alpha_\beta - \overline{\kappa}^\alpha_\beta)\overline{g}_{\lambda\alpha}\overline{\mathbf{m}}^\lambda \otimes \overline{\mathbf{m}}^\beta$

(g) Suppose that the shell is elastic, with Young's modulus E and Poisson's ratio ν. Calculate the contravariant components of the internal force $T^{\alpha\beta}$ and internal moment $M^{\alpha\beta}$ induced by the radial expansion described in part (f) (you can use the approximate expressions relating $T^{\alpha\beta}$ and $M^{\alpha\beta}$ to $\gamma_{\alpha\beta},\Delta\kappa_{\alpha\beta}$)

(h) Find the physical components of the internal force and moment, expressing your answer as components in the spherical-polar basis of unit vectors $\{\mathbf{e}_R,\mathbf{e}_\theta,\mathbf{e}_\phi\}$

Problem 10.41 A cylindrical-polar coordinate system is to be used to describe the deformation of a cylindrical shell with radius r. The angles and axial distance (θ,z) illustrated in the figure are to be used as the coordinate system (ξ_1,ξ_2) for this geometry.



(a) Write down the position vector $\overline{\mathbf{r}}$ in terms of (θ,z), expressing your answer as components in the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ shown in the figure.

(b) Calculate the covariant basis vectors $\left(\overline{\mathbf{m}}_1,\overline{\mathbf{m}}_2,\overline{\mathbf{m}}_3\right)$ in terms of $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and (r,θ,z). Choose the order for $\left(\overline{\mathbf{m}}_1,\overline{\mathbf{m}}_2,\overline{\mathbf{m}}_3\right)$ so that $\overline{\mathbf{m}}_3$ points out of the cylinder.

(c) Calculate the contravariant basis vectors $\left(\overline{\mathbf{m}}^1,\overline{\mathbf{m}}^2,\overline{\mathbf{m}}^3\right)$ in terms of $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and (θ,z)

(d) Calculate the covariant, contravariant and mixed components of the metric tensor $\bar{\mathbf{g}}$

(e) Calculate the covariant, contravariant and mixed components of the curvature tensor $\bar{\boldsymbol{\kappa}}$ for the shell

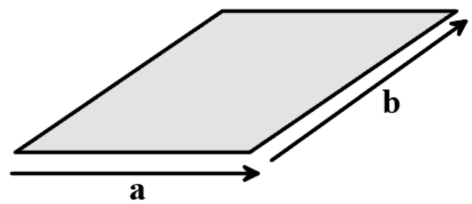
(f) Find the components of the Christoffel symbol $\bar{\Gamma}^{i}_{\alpha\beta}$ for the undeformed shell

(g) Suppose that under loading the shell simply expands radially to a new radius ρ, without axial stretch. Find the covariant components of the mid-plane Lagrange strain tensor $\boldsymbol{\gamma}$ and the covariant components of the curvature change tensor $\Delta\boldsymbol{\kappa}$

(h) Suppose that the shell is elastic, with Young's modulus E and Poisson's ratio ν. Calculate the contravariant components of the internal force $T^{\alpha\beta}$ and internal moment $M^{\alpha\beta}$.

(i) Find the physical components of the internal force and moment, expressing your answer as components in the spherical-polar basis of unit vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$

Problem 10.42 The figure illustrates a flat plate with area A, whose geometry can be described by two vectors $\mathbf{a}$ and $\mathbf{b}$ parallel to two sides of the triangle. The position vector of a point in the plate is to be characterized using a coordinate system (ξ_1, ξ_2) by setting $\bar{\mathbf{r}} = \xi_1 \mathbf{a} + \xi_2 \mathbf{b}$ where $0 \le \xi_1 \le 1$, $0 \le \xi_2 \le 1$.



(a) Calculate the covariant basis vectors $(\bar{\mathbf{m}}_1, \bar{\mathbf{m}}_2, \bar{\mathbf{m}}_3)$ in terms of $\mathbf{a}$ and $\mathbf{b}$

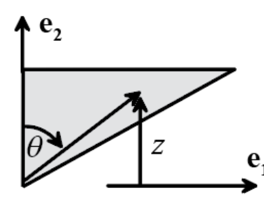
(b) Calculate the contravariant basis vectors $(\bar{\mathbf{m}}^1, \bar{\mathbf{m}}^2, \bar{\mathbf{m}}^3)$ in terms of $\mathbf{a}$ and $\mathbf{b}$

(c) Calculate the covariant, contravariant and mixed components of the metric tensor $\bar{\mathbf{g}}$

(d) Suppose that the plate is subjected to a homogeneous deformation, so that after deformation its sides are $\lambda\mathbf{a}, \lambda\mathbf{b}$ Find the mid-plane Lagrange strain tensor.

(e) Suppose that the plate is elastic, with Young's modulus E and Poisson's ratio ν. Calculate the contravariant components of the internal force $T^{\alpha\beta}$

Problem 10.43 The figure illustrates a triangular plate. The position points in the plate is to be characterized using the height z and angle θ as the coordinate system (ξ_1, ξ_2).



(a) Calculate the covariant basis vectors $(\bar{\mathbf{m}}_1, \bar{\mathbf{m}}_2, \bar{\mathbf{m}}_3)$ expressing your answer as components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ shown in the figure.

(b) Calculate the contravariant basis vectors $(\bar{\mathbf{m}}^1, \bar{\mathbf{m}}^2, \bar{\mathbf{m}}^3)$ as components in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

(c) Calculate the covariant, contravariant and mixed components of the metric tensor $\bar{\mathbf{g}}$

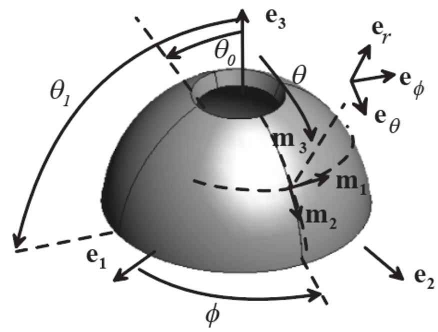
(d) Find the components of the Christoffel symbol $\bar{\Gamma}^{i}_{\alpha\beta}$ for the undeformed plate

(e) Suppose that the plate is subjected to a homogeneous deformation such that the position vector of a point that lies at (z, θ) in the undeformed shell has position vector $(\lambda z, \theta)$ after deformation Find the mid-plane Lagrange strain tensor $\boldsymbol{\gamma} = \gamma_{\alpha\beta} \bar{\mathbf{m}}^{\alpha} \otimes \bar{\mathbf{m}}^{\beta}$, in terms of z, θ, λ

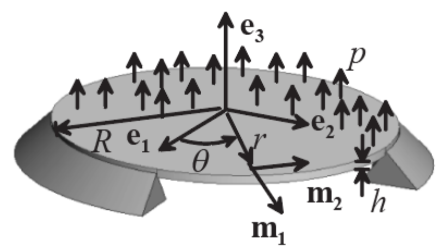
(f) Suppose that the plate is elastic, with Young's modulus E and Poisson's ratio ν. Calculate the contravariant components of the internal force $T^{\alpha\beta}$

10.6 Simplified Versions of General Shell Theory: Flat Plates and Membranes

Problem 10.44 Consider a shell that is so thin that the internal moments $M_{\alpha\beta}$, as well as the rotational inertia ρh^3 and external couple q^{β} all vanish. Find the simplified equations of motion for the internal forces $T^{\alpha\beta}$ and the transverse force V^{α} in terms of relevant geometric parameters.



Problem 10.45 The figure shows a thin circular plate with thickness *h,* mass density ρ, Young's modulus *E* and Poisson's ratio ν that is simply supported at its edge and is subjected to a pressure distribution acting perpendicular to its surface. The goal of this problem is to derive the equations governing the transverse deflection $u_3(r,\theta)$ of the plate in terms of the cylindrical-polar coordinate (r,θ) system shown in the figure. The algebra in this problem is tedious, and is best done using a symbolic manipulation program.



(a) Write down the position vector of a point on the mid-plane of the undeformed plate in terms of (r,θ), expressing your answer as components in the $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ basis.

(b) Calculate the basis vectors $\left(\bar{\mathbf{m}}_1,\bar{\mathbf{m}}_2,\bar{\mathbf{m}}_3\right)$ and $\left(\bar{\mathbf{m}}^1,\bar{\mathbf{m}}^2,\bar{\mathbf{m}}^3\right)$, expressing your answer as components in the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ shown in the figure.

(c) Calculate the covariant and contravariant components of the metric tensor $\bar{\mathbf{g}}$

(d) Assume that the displacement field in the plate is $\mathbf{u}=u_3(r,\theta)\mathbf{e}_3$. Find the basis vectors $\left(\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\right)$ for the deformed plate, neglecting terms of order $\left(\partial u_3/\partial r\right)^2, \left(\partial u_3/\partial\theta\right)^2$ and higher.

(e) Show that, to within the same order of accuracy as part (c), the curvature tensor has components

$$\kappa_{11}=-\frac{\partial^2 u_3}{\partial r^2} \qquad \kappa_{12}=\kappa_{21}=\frac{1}{r}\frac{\partial u_3}{\partial\theta}-\frac{\partial^2 u_3}{\partial r\partial\theta} \qquad \kappa_{22}=-r\frac{\partial u_3}{\partial r}-\frac{\partial^2 u_3}{\partial\theta^2}$$

(f) Express the internal moments $M^{\alpha\beta}$ in the plate in terms of E,ν and u_3 and its derivatives. Use the approximate expression relating curvature to moment. Hence, show that the components of internal moment $M_{rr},M_{\theta\theta},M_{r\theta}$ in polar coordinates are

$$M_{rr}=-\frac{Eh^3}{12(1-\nu^2)}\left\{\frac{\partial^2 u_3}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial u_3}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_3}{\partial\theta^2}\right)\right\}$$

$$M_{\theta\theta}=-\frac{Eh^3}{12(1-\nu^2)}\left\{\left(\frac{1}{r}\frac{\partial u_3}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_3}{\partial\theta^2}\right)+\nu\frac{\partial^2 u_3}{\partial r^2}\right\}$$

$$M_{r\theta}=M_{\theta r}=\frac{Eh^3}{12(1+\nu)}\left(\frac{1}{r}\frac{\partial u_3}{\partial\theta}-\frac{\partial^2 u_3}{\partial r\partial\theta}\right)$$

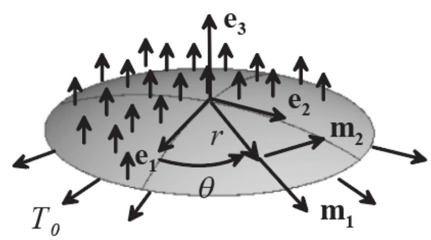
(g) Write down the equations of motion for the plate in terms of $M^{\alpha\beta}$ and V^{α} (note that $T^{\alpha\beta}=0$, and neglect second order terms, as well as the rotational inertia of the plate) and show that they can be reduced to

$$\frac{\partial M^{11}}{\partial r}+\frac{\partial M^{21}}{\partial \theta}+\frac{1}{r}M^{11}-rM^{22}-V^{1}=0$$
$$\frac{\partial M^{12}}{\partial r}+\frac{\partial M^{22}}{\partial \theta}+\frac{3}{r}M^{12}-V^{2}=0$$
$$\frac{\partial V^{1}}{\partial r}+\frac{\partial V^{2}}{\partial \theta}+\frac{1}{r}V^{1}+p=\rho h\frac{\partial^2 u_3}{\partial t^2}$$

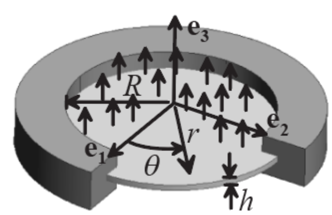
(h) Hence, show that the transverse displacement must satisfy the following governing equation

$$\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 u_3}{\partial r^2}+\frac{1}{r}\frac{\partial u_3}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_3}{\partial \theta^2}\right)+\rho h\frac{\partial^2 u_3}{\partial t^2}=p_3$$

10.7 Solutions to Problems Involving Membranes, Plates and Shells

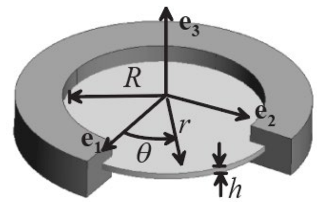


Problem 10.46 A thin circular membrane with radius R is subjected to in-plane boundary loading that induces a uniform biaxial membrane tension with magnitude T_0. Vertical displacement of the membrane is prevented at $r=R$. The membrane is subjected to a uniform out-of-plane pressure with magnitude p on its surface. Calculate the displacement field in the membrane, assuming small deflections. This problem can be solved quite easily using either Cartesian or polar coordinates.



Problem 10.47 The figure shows a thin circular plate with thickness h, Young's modulus E and Poisson's ratio ν. The edge of the plate is clamped, and its surface is subjected to a uniform out-of-plane pressure with magnitude p. Calculate the displacement field and internal moment and shear force in the plate, assuming small deflections. This problem can easily be solved using Cartesian or polar coordinates (see problem 10.44 for the governing equations in polar coordinates).

Problem 10.48 The figure shows a circular elastic plate with Young's modulus E, Poissions ratio ν. The plate has thickness h and radius R, and is clamped at its edge. The goal of this problem is to calculate the mode shapes and natural frequencies of vibration of the plate. The solution to Problem 10.44 may be helpful.



(a) Show that the governing equations for a vibrating plate are satisfied by the general solution

$$u_3(r,\theta)=\Big(AJ_n(k_{(m,n)}r)\sin(n\theta+\theta_0)+BY_n(k_{(m,n)}r)\sin(n\theta+\theta_1)$$
$$+CI_n(k_{(m,n)}r)\sin(n\theta+\theta_2)+DK_n(k_{(m,n)}r)\sin(n\theta+\theta_3)\Big)\cos(\omega_{(m,n)}t+\phi)$$

where A,B,C,D, ϕ $\theta_0 ... \theta_3$ are arbitrary constants, J_n, Y_n are Bessel functions of the first and second kinds, and I_n, K_n are modified Bessel functions of the first and second kinds, with order n, while $k_{(m,n)}$ and $\omega_{(m,n)}$ are a wave number and vibration frequency that are related by

$$k_{(m,n)}^2=\omega_{(m,n)}\sqrt{12(1-\nu^2)\rho/\left(Eh^2\right)}$$

(b) Show that most general solution with bounded displacements and curvature at $r=0$ has the form

$$u_3(r,\theta)=\Big(\left[A_1J_n(k_{(m,n)}r)+B_1I_n(k_{(m,n)}r)\right]\sin(n\theta)$$
$$+\left[A_2J_n(k_{(m,n)}r)+B_2I_n(k_{(m,n)}r)\right]\cos(n\theta)\Big)\cos(\omega_{(m,n)}t+\phi)$$

(c) Write down the boundary conditions for u_3 at $r=R$, and hence show that the wave numbers $k_{(m,n)}$ are roots of the equation

$$J_n(k_{(m,n)}R)I_{n+1}(k_{(m,n)}R)+I_n(k_{(m,n)}R)J_{n+1}(k_{(m,n)}R)=0$$

(d) Show that the corresponding mode shapes are given by

$$U_{(m,n)}(r,\theta)=A\left[I_n(k_{(m,n)}R)J_n(k_{(m,n)}r)-J_n(k_{(m,n)}R)I_n(k_{(m,n)}r)\right]\sin(n\theta+\theta_1)$$

where A and θ_1 are arbitrary constants.

(e) Calculate the normalized natural frequencies of vibration

$$\omega_{(m,n)}\sqrt{\frac{12(1-\nu^2)\rho R^4}{Eh^2}}=\left(Rk_{(m,n)}\right)^2$$

for $0<n<3$, $1<m<3$ and tabulate your results.

(f) Plot the mode shapes for the (distinct) modes with the lowest four frequencies.

Problem 10.49 Repeat the preceding problem for a plate with a simply supported edge. You should find that the wave numbers $k_{(m,n)}$ are the roots of the equation

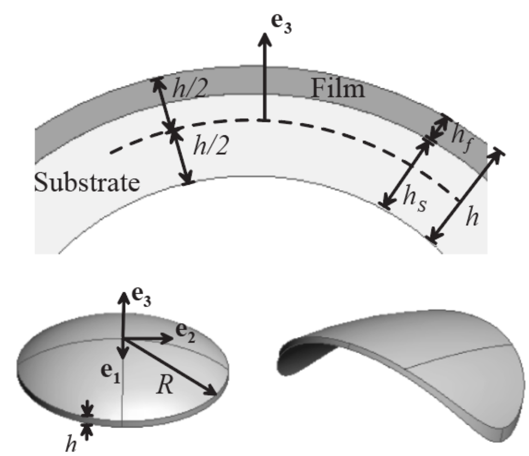
$$(1-\nu)\left[J_n(k_{(m,n)}R)I_{n+1}(k_{(m,n)}R)+I_n(k_{(m,n)}R)J_{n+1}(k_{(m,n)}R)\right]-2k_{(m,n)}RI_n(k_{(m,n)}R)J_n(k_{(m,n)}R)=0$$

Tabulate the normalized natural frequencies for $\nu=0.3$.

Problem 10.50 Use Rayleigh's method, taking the static deflection under uniform pressure as the guess for the mode shape, to estimate the fundamental frequency of vibration of the clamped circular plate described in Problem 10.48. Compare the approximate solution with the exact solution.

Problem 10.51 Use Rayleigh's method, with a suitable guess for the mode shape, to estimate the fundamental frequency of vibration of the circular plate described in Problem 10.49. Compare the approximate solution with the exact solution.

Problem 10.52 A thin film with thickness h_f is deposited on the surface of a circular wafer with radius R and thickness h_s. Both film and substrate have Young's modulus E, and Poisson's ratio ν. An inelastic strain $\varepsilon_{11}^p = \varepsilon_{22}^p = \varepsilon_0$ is introduced into the film by some external process (e.g. thermal expansion), which generates stresses in the film, and also causes the substrate to bend. In Section 10.7.3 of Applied Mechanics of Solids, expressions were derived relating the substrate curvature to the mismatch strain in the film. These expressions are only valid if the substrate curvature is small. For mismatch strains, the wafer buckles, as shown in the figure. The goal of this problem is to estimate the critical value of mismatch strain that will cause the wafer to buckle.



Deformed shape for low mismatch

Deformed shape for high mismatch

(a) Assume that the displacement of the midplane of the wafer-film system is

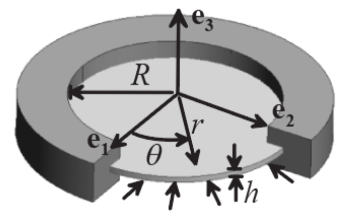
$$u_3 = \kappa_1 x_1^2 / 2 + \kappa_2 x_2^2 / 2 \qquad u_1 = A_1 x_1 + A_2 x_1^3 + A_3 x_1 x_2^2 \qquad u_2 = B_1 x_1 + B_2 x_2^3 + B_3 x_2 x_1^2$$

Using Von-Karman plate theory, calculate the distribution of strain in the film and substrate, and hence deduce the total strain energy density of the system. Calculate the values of A_i, B_i that minimize the potential energy of the plate, and hence show that the two curvatures satisfy

$$\kappa_1 \kappa_2^2 (1-\nu^2) R^4 + 16h^2 \kappa_1 + 16h^2 \nu \kappa_2 - C\varepsilon_0 = 0$$
$$\kappa_2 \kappa_1^2 (1-\nu^2) R^4 + 16h^2 \kappa_2 + 16h^2 \nu \kappa_1 - C\varepsilon_0 = 0$$

(where $h = h_f + h_s$) and find a formula for the constant C. It is best to do this calculation with a symbolic manipulation program.

(b) Hence, plot a graph showing the equilibrium values of the normalized curvature $\kappa_1 R^2 \sqrt{1+\nu} / (4h)$ as a function of a suitably normalized measure of mismatch strain, for a Poisson's ratio $\nu = 0.3$.



Problem 10.53 The figure shows an elastic plate with Young's modulus E, Poisson's ratio ν and thermal expansion coefficient α. The plate is circular, with radius R and its edge is clamped. The plate is initially stress free and is then heated to raise its temperature by ΔT, inducing a uniform internal force $T_{\alpha\beta} = -Eh\alpha\Delta T\delta_{\alpha\beta}/(1-\nu)$ in the plate. The goal of this problem is to calculate the critical temperature that will cause the plate to buckle, using the governing equations listed in Section 10.6.2 of Applied Mechanics of Solids.

(a) Assume an axially symmetric buckling mode, so that $u_3 = w(r)$ with $r = \sqrt{x_\alpha x_\alpha}$. Show that w satisfies the governing equation

$$\frac{Eh^3}{12(1-\nu^2)}\left(\frac{d^4w}{dr^4} + \frac{2}{r}\frac{d^3w}{dr^3} - \frac{1}{r^2}\frac{d^2w}{dr^2} + \frac{1}{r^3}\frac{dw}{dr}\right) + \frac{Eh\alpha\Delta T}{(1-\nu)}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) = 0$$

(b) Show that the general solution to this equation is

$$w(r) = A + B\log(r) + CJ_0(k_n r) + DY_0(k_n r)$$

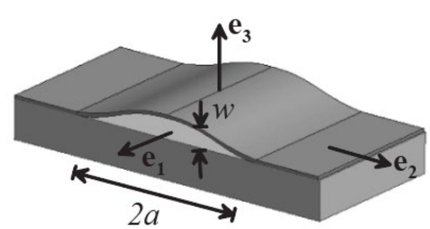
where A,B,C,D, are arbitrary constants, J_0, Y_0 are Bessel functions of the first and second kind of order zero, and k_n is the wave number for the nth buckling mode. Find an expression for k_n in terms of ΔT and relevant geometric and material properties.

(c) Show that the wave number k_n satisfies $J_1(k_n R) = 0$ and deduce an expression for the critical temperature change at which the plate can first buckle.

(d) Give an expression for the buckling mode associated with this temperature

Problem 10.54 Repeat the preceding problem for a plate with a simply supported edge (take $\nu = 0.3$ to calculate the expression for the critical temperature). You should find that the wave numbers k_n are the roots of the equation

$$(1-\nu)J_1(Rk_n) - Rk_n J_0(Rk_n) = 0$$



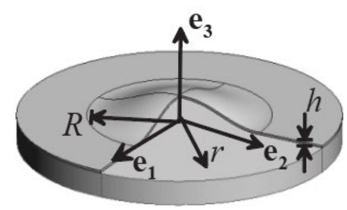
Problem 10.55 The interface between a thin film with thickness h and its rigid substrate contains a tunnel crack with length $2a$. The film is initially stress free, then heated to raise its temperature by ΔT, inducing a uniform biaxial stress field $\sigma_{\alpha\beta} = -E\alpha\Delta T\delta_{\alpha\beta}/(1-\nu)$ in the film. At a critical temperature, the film buckles as shown in the figure. For $a >> h$, the buckled film can be modeled as a plate with clamped edge. The goal of this problem is to calculate the critical temperature required to cause the film to buckle, and to calculate the crack tip energy release rate as the buckled film delaminates from the substrate. Assume that the displacement of the mid-plane of the film (taking the strained flat film as the reference configuration) has the form $\mathbf{u} = u(x_2)\mathbf{e}_2 + w(x_2)\mathbf{e}_3$, and use the Von-Karman plate bending theory of Section 10.6.3 of Applied Mechanics of Solids.

(a) Write down expressions for the mid-plane strain $\gamma_{\alpha\beta}$ and the curvature change tensor $\Delta\kappa_{\alpha\beta}$ for the film in terms of u and w, and hence find a formula for the internal force $T_{\alpha\beta}$ and moment $M_{\alpha\beta}$ in terms of u, w, ΔT and material properties.
(b) Hence, show that the equations of equilibrium in terms of u and w reduce to

$$\frac{dT_{22}}{dx_2}=0 \qquad \frac{d^4w}{dx_2^4}+\frac{12(1-\nu^2)}{Eh^3}T_{22}\frac{d^2w}{dx_2^2}=0 \qquad T_{22}=\frac{Eh}{(1-\nu^2)}\left[-(1+\nu)\alpha\Delta T+\frac{du}{dx_2}+\frac{1}{2}\left(\frac{dw}{dx_2}\right)^2\right]$$

(c) Assume that $w=dw/dx_2=0$ at $x_2=\mp a$. Hence, find a formula for the smallest value of T_{22} for which a solution with nonzero w exists.
(d) Deduce an expression for the critical temperature required to cause the film to buckle
(e) Assume that ΔT exceeds the critical value. Find formulas for u and w.
(f) Hence calculate the decrease in energy of the film caused by the buckling.
(g) Use the preceding result to deduce the crack tip energy release rate.

Problem 10.56 The interface between a thin film and its substrate contains a circular crack with radius R. The film has Young's modulus E, Poisson's ratio ν and thermal expansion coefficient α. The substrate can be idealized as rigid. The film is initially stress free, then heated to raise its temperature by ΔT, inducing a uniform biaxial stress field $\sigma_{\alpha\beta}=-E\alpha\Delta T\delta_{\alpha\beta}/(1-\nu)$ in the film. At a critical temperature, the film buckles as shown in the figure. When the film buckles, some of the strain energy in the film is relaxed. This relaxation in energy can cause the film to delaminate from the substrate. The goal of this problem is to find an approximate solution to the post-buckled shape of the film, and hence to estimate the crack tip energy release rate. The simplest approach is to assume a displacement field and minimize the potential energy of the system.



(a) Assume that the deformation of the plate is axially symmetric, so that the displacement field can be expressed as $\mathbf{u}=u(r)\mathbf{e}_r+w(r)\mathbf{e}_z$. Show that the Von-Karman approximations for the mid-plane strain and curvature reduce (in polar coordinates) to

$$\gamma_{rr}=\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2 \qquad \gamma_{\theta\theta}=\frac{u}{r} \qquad \Delta\kappa_{rr}=-\frac{d^2w}{dr^2} \qquad \Delta\kappa_{\theta\theta}=-\frac{1}{r}\frac{dw}{dr}$$

(b) As a suitable kinematically admissible displacement field, assume that

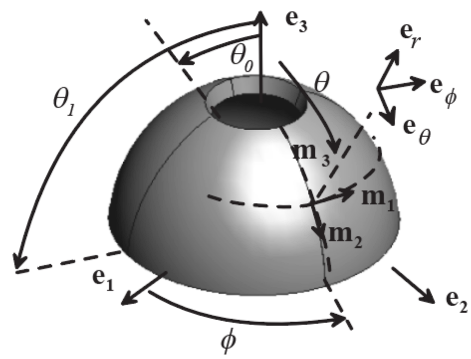
$$w=A\left(R^2-r^2\right)^2 \qquad u=Br(R-r)$$

where A and B are constants to be determined. Using a symbolic manipulation program, find the strain energy released by the delamination, and maximize it to eliminate A and B. Take Poisson's ratio $\nu=1/3$ for simplicity.
(c) Compare the critical buckling temperature predicted by the approximate solution with the exact value calculated in problem 10.52
(d) Find a formula for the energy release rate (per unit length of crack front)

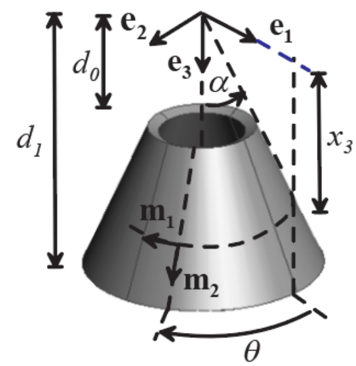
Problem 10.57 The figure shows a thin-walled, spherical dome with radius *R,* thickness *h* and mass density ρ. The dome is open at its top, so that the shell is bounded by spherical polar angles $\theta_0 < \theta < \theta_1$.

(a) Calculate the internal forces induced by gravitational loading of the structure, using the membrane theory of shells in Section 10.7.7 of Applied Mechanics of Solids.

(b) Suppose that a dome is to be designed with $\theta_1 = 5\pi/12$. What is the minimum allowable value for θ_0 to ensure that the membrane forces are compressive everywhere?



Problem 10.58 The figure shows a thin-walled conical shell with thickness *h* and mass density ρ. Calculate the internal forces induced by gravitational loading of the structure, using the membrane theory of shells in Section 10.7.7 of Applied Mechanics of Solids. Use the cylindrical-polar coordinates $(\xi_1 = \theta, \xi_2 = x_3)$ as the coordinate system for the calculation. Find the physical components of the membrane force in a suitable basis of unit vectors that lies in the plane of the cone.



Appendices

A1: Vectors and Matrices

Problem A.1 Calculate the magnitudes of each of the vectors listed below

(a) $\mathbf{r} = 3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$

(b) $\mathbf{r} = 16\mathbf{i} + 6\mathbf{j}$

(c) $\mathbf{r} = -9.6\mathbf{j} + 2.4\mathbf{i} - 4.6\mathbf{k}$

Problem A.2 A vector has magnitude 3, and **i** and **j** components of 1 and 2, respectively. Calculate its **k** component.

Problem A.3 Find the dot products of the pairs of vectors listed below

(a) $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

(b) $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} + 6\mathbf{k}$

(c) $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{b} = \xi\mathbf{i} - \eta\mathbf{j} + \varsigma\mathbf{k}$

Problem A.4 Calculate the angle between the following pairs of vectors i.e. find the angle $\theta(\mathbf{a},\mathbf{b})$ between **a** and **b** in each case

(a) $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

(b) $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} - 6\mathbf{k}$

Problem A.5 Two vectors **a** and **b** have magnitudes $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and the angle between them is $\theta(\mathbf{a},\mathbf{b}) = 30^0$. Calculate the magnitude of $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

Problem A.6 The 'direction cosines' of a vector **a** in a Cartesian basis $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ are defined as $(\cos[\theta(\mathbf{a},\mathbf{i})],\cos[\theta(\mathbf{a},\mathbf{j})],\cos[\theta(\mathbf{a},\mathbf{k})])$. Find the direction cosines of the following vectors

(a) $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

(b) $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$

Problem A.7 Let **{i,j,k}** be a Cartesian basis. A vector **a** has magnitude 4 and subtends angles of 30 degrees and 100 degrees to the **i** and **k** directions, respectively. Calculate the components of **a** in the basis **{i,j,k}**

Problem A.8 Find the cross products of the vectors listed below

(a) $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

(b) $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} + 6\mathbf{k}$

(c) $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{b} = \xi\mathbf{i} - \eta\mathbf{j} + \varsigma\mathbf{k}$

Problem A.9 Two vectors **a** and **b** have magnitudes $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$, $|\mathbf{a}+\mathbf{b}| = 5$. Calculate $|\mathbf{a}\times\mathbf{b}|$

Problem A.10 Let $\mathbf{a}=3\mathbf{i}-4\mathbf{j}$, $\mathbf{b}=\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ be two vectors, and let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ be a third vector with unknown components

(a) Solve the equation $\mathbf{a}+\mathbf{r}=\mathbf{b}$

(b) Solve the equations

$$\mathbf{r}\cdot\mathbf{a}=0 \qquad \mathbf{r}\cdot\mathbf{b}=0 \qquad \mathbf{r}\cdot\mathbf{r}=1$$

Problem A.11 Let $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ be a Cartesian basis, and let $\mathbf{a}=5\mathbf{i}+6\mathbf{j}-3\mathbf{k}$, $\mathbf{b}=3\mathbf{j}+6\mathbf{k}$, $\mathbf{c}=45\mathbf{i}-30\mathbf{j}+15\mathbf{k}$ be three vectors.

(a) Verify that **a**, **b** and **c** are mutually perpendicular and that $\mathbf{c}=\mathbf{a}\times\mathbf{b}$

(b) In view of (a), three unit vectors $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ parallel to **a b** and **c** can form a second Cartesian basis. Calculate the components of $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ in the $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ basis.

(c) Let **r**=4**i**+6**k**. Calculate the components of **r** in $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. (Use your answer to (b) to calculate the required dot products in the formula)

Problem A.12 Let **a** and **b** be two unit vectors (which need not be orthogonal). Let α and β be two scalars, and let **c** be a vector such that

$$\mathbf{c}=\alpha\mathbf{a}+\beta\mathbf{b}$$

(a) Prove that

$$\alpha=\frac{(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b})-(\mathbf{a}\cdot\mathbf{c})}{1-(\mathbf{a}\cdot\mathbf{b})^2} \qquad \beta=\frac{(\mathbf{a}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b})-(\mathbf{b}\cdot\mathbf{c})}{1-(\mathbf{a}\cdot\mathbf{b})^2}$$

(b) Let $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ be a Cartesian basis. Consider the three vectors $\mathbf{a}=(3/5)\mathbf{i}+(4/5)\mathbf{j}$, $\mathbf{b}=\mathbf{k}$, $\mathbf{c}=\mathbf{i}+\mathbf{k}$. For this set of vectors, calculate values for

$$\alpha=\frac{(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b})-(\mathbf{a}\cdot\mathbf{c})}{1-(\mathbf{a}\cdot\mathbf{b})^2} \qquad \beta=\frac{(\mathbf{a}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b})-(\mathbf{b}\cdot\mathbf{c})}{1-(\mathbf{a}\cdot\mathbf{b})^2}$$

(c) By substituting values, show that if **a, b, c**, α and β have the values given in (b), then $\mathbf{c}\neq\alpha\mathbf{a}+\beta\mathbf{b}$

(d) Explain briefly why the vectors used in (b,c) cannot satisfy $\mathbf{c}=\alpha\mathbf{a}+\beta\mathbf{b}$ for any values of α and β.

Problem A.13 Here is a nice matrix.

$$[A]=\begin{bmatrix}1 & 3 & 0\\ 3 & 2 & 0\\ 0 & 0 & 5\end{bmatrix}$$

(a) Find $\det([A])$

(b) Find $[A]^{-1}$ (Don't try to use the general expression for the inverse of a matrix – this matrix can be inverted trivially)

(c) Find the eigenvalues and eigenvectors of $[A]$. (You can write down one of the eigenvalues and eigenvectors by inspection. The other two can be found using the formulae for a 2×2 matrix)

Problem A.14 Consider a square, symmetric matrix

$$[A]=\begin{bmatrix} 31 & -5\sqrt{3} \\ -5\sqrt{3} & 21 \end{bmatrix}$$

(a) Find the spectral decomposition of $[A]$, i.e. find a diagonal matrix $[\Lambda]$ and an orthogonal matrix $[Q]$ such that $[A]=[Q][\Lambda][Q]^T$

(b) Hence, calculate $[A]^{1/2}$

Problem A.15 The exponential of a matrix is defined as

$$\exp([A])=\sum_{k=1}^{\infty}\frac{[A]^k}{k!}$$

(a) Show that the exponential of a diagonal matrix $[\Lambda]=\mathrm{diag}(\lambda_1,\lambda_2...\lambda_n)$ is given by

$$\exp([\Lambda])=\mathrm{diag}\left[\exp(\lambda_1),\exp(\lambda_2)...\exp(\lambda_n)\right]$$

(b) Let $[A]$ be a square, symmetric $n\times n$ matrix, and let $[\Lambda]=\mathrm{diag}(\lambda_1,\lambda_2...\lambda_n)$ and $[Q]$ be a diagonal and orthogonal matrix, respectively, such that $[A]=[Q][\Lambda][Q]^T$. Find an expression for $\exp([A])$ in terms of $[Q]$ and $[\Lambda]$

(c) Calculate the exponential of the matrix given in problem A.14.

A2 Introduction to Tensors

Problem A.16 Let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be a Cartesian basis. Vector **u** has components $[1,2,0]$ in this basis, while tensors **S** and **T** have components

$$\mathbf{T}\equiv\begin{bmatrix} 1 & \sqrt{6} & 0 \\ \sqrt{6} & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad \mathbf{S}\equiv\begin{bmatrix} 1 & -1 & 3 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$$

(a) Calculate the components of the following vectors and tensors

(i) $\mathbf{v}=\mathbf{Tu}$ (ii) $\mathbf{v}=\mathbf{u}\cdot\mathbf{T}$ (iii) $\mathbf{V}=\mathbf{S}+\mathbf{T}$ (iv) $\mathbf{V}=\mathbf{ST}$ (v) $\mathbf{V}=\mathbf{S}^T$

(b) Find the eigenvalues and the components of the eigenvectors of **T**.

(c) Denote the three (unit) eigenvectors of **T** by $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$ (It doesn't matter which eigenvector is which, but be sure to state your choice clearly). Let $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$ be a new Cartesian basis. Write down the components of **T** in $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$.

(d) Calculate the components of **S** in the basis $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$.

Problem A.17 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis, and let $\mathbf{m}_1 = (5\mathbf{e}_1 + 6\mathbf{e}_2 - 3\mathbf{e}_3)$, $\mathbf{m}_2 = (3\mathbf{e}_2 + 6\mathbf{e}_3)$, $\mathbf{m}_3 = (3\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3)$ be three vectors.

(a) Calculate the components in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the tensor **T** that satisfies $\mathbf{m}_i = \mathbf{T}\mathbf{e}_i$.

(b) Calculate the components of **T** in a basis of unit vectors parallel to $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$.

Problem A.18 Let **S** be a tensor, and let

$$[S^{(\mathbf{e})}] = \begin{bmatrix} S_{11}^{(\mathbf{e})} & S_{12}^{(\mathbf{e})} & S_{13}^{(\mathbf{e})} \\ S_{21}^{(\mathbf{e})} & S_{22}^{(\mathbf{e})} & S_{23}^{(\mathbf{e})} \\ S_{31}^{(\mathbf{e})} & S_{32}^{(\mathbf{e})} & S_{33}^{(\mathbf{e})} \end{bmatrix} \qquad [S^{(\mathbf{m})}] = \begin{bmatrix} S_{11}^{(\mathbf{m})} & S_{12}^{(\mathbf{m})} & S_{13}^{(\mathbf{m})} \\ S_{21}^{(\mathbf{m})} & S_{22}^{(\mathbf{m})} & S_{23}^{(\mathbf{m})} \\ S_{31}^{(\mathbf{m})} & S_{32}^{(\mathbf{m})} & S_{33}^{(\mathbf{m})} \end{bmatrix}$$

denote the components of **S** in Cartesian bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$, respectively. Show that the trace of **S** is invariant to a change of basis, i.e. show that $\text{trace}\left\{[S^{(\mathbf{e})}]\right\} = \text{trace}\left\{[S^{(\mathbf{m})}]\right\}$ (it is easiest to solve this problem using index notation, discussed in Appendix C of Applied Mechanics of Solids).

Problem A.19 Show that the inner product of two tensors is invariant to a change of basis (it is easiest to solve this problem using index notation, discussed in Appendix C).

Problem A.20 Show that the eigenvalues of a tensor are invariant to a change of basis. Are the eigenvectors similarly invariant?

Problem A.21 Let **S** be a real symmetric tensor with three distinct eigenvalues λ_i and corresponding eigenvectors $\mathbf{m}^{(i)}$. Show that $\mathbf{m}^{(i)} \cdot \mathbf{m}^{(j)} = 0$ for $i \neq j$.

Problem A.22 Let **S** be a real symmetric tensor with three distinct eigenvalues λ_i and corresponding *normalized* eigenvectors $\mathbf{m}^{(i)}$ satisfying $\mathbf{m}^{(i)} \cdot \mathbf{m}^{(i)} = 1$. Use the results of A.21 to show that

$$\mathbf{S}\mathbf{b} = \sum_{i=1}^{3} \lambda_i (\mathbf{m}^{(i)} \cdot \mathbf{b})\, \mathbf{m}^{(i)}$$

for any arbitrary vector **b**, and hence deduce that

$$\mathbf{S} = \sum_{i=1}^{3} \lambda_i\, \mathbf{m}^{(i)} \otimes \mathbf{m}^{(i)}$$

Problem A.23 Use the results of problem A.22 to find a way to calculate the square root of a real, symmetric tensor.

Problem A.24 Let **n** be the dual vector of a skew tensor **W**. What is $\mathbf{W}\mathbf{n}$?

Problem A.25 Let **S** and **W** denote a symmetric and a skew tensor, respectively. Calculate $\mathbf{S} : \mathbf{W}$

Problem A.26 Let

$$\mathbf{W} \equiv \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$$

Find expressions for the eigenvalues and eigenvectors of $\mathbf{T}$ in terms of its components a_{ij}. Note that the eigenvalues and eigenvectors may be complex.

Problem A.27 Let $\mathbf{W}$ be a skew tensor and $\mathbf{I}$ the identity tensor.
(a) Show that $\mathbf{I}+\mathbf{W}$ is nonsingular
(b) Show that $\mathbf{R}=(\mathbf{I}-\mathbf{W})(\mathbf{I}+\mathbf{W})^{-1}$ is orthogonal.

Problem A.28 Let $\mathbf{a},\mathbf{b}$ be two (not necessarily orthogonal) unit vectors. Find formulas for the eigenvalues and eigenvectors of $\mathbf{S}=\mathbf{a}\otimes\mathbf{a}+\mathbf{b}\otimes\mathbf{b}$, in terms of $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$. (Don't use the standard formulas to do this. You can write down one eigenvalue and eigenvector by inspection. This and the symmetry of **S** then tells you something about the direction of the other two eigenvectors. You can use that insight to find the remaining eigenvectors, and finally deduce the eigenvalues).

Problem A.29 Let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be a Cartesian basis. Let **R** be a proper orthogonal tensor, and let $\mathbf{m}_i=\mathbf{R}\mathbf{e}_i$
(a) Show that $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$ is also a Cartesian basis (i.e. show that $\mathbf{m}_i$ are orthogonal unit vectors
(b) Let $R_{ij}^{(\mathbf{e})},R_{ij}^{(\mathbf{m})}$ denote the components of **R** in $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$, respectively. Show that $R_{ij}^{(\mathbf{e})}=R_{ij}^{(\mathbf{m})}$.

Problem A.30 Let $\mathbf{R}$ be a proper orthogonal tensor. Let I_1,I_2,I_3 be the three invariants of **R** defined as $I_1=\text{trace}(\mathbf{R})$, $I_2=(I_1^2-\mathbf{R}\cdot\cdot\mathbf{R})/2$ $I_3=\det(\mathbf{R})$. Show that $I_1=I_2$ (but do this without any index notation manipulations. Start by writing down the characteristic equation for **R**).

Problem A.31 Let $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$ be a (not necessarily Cartesian) basis, and let

$$g_{ij}=\mathbf{m}_i\cdot\mathbf{m}_j \qquad \mathbf{T}=T_{ij}\mathbf{m}_i\otimes\mathbf{m}_j \qquad \mathbf{S}=S_{ij}\mathbf{m}_i\otimes\mathbf{m}_j \qquad \mathbf{U}=\mathbf{ST}$$

Find an expression for the components of $\mathbf{U}$ in $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$.

Problem A.32 Let **S** be a non-singular second order tensor with invariants I_1,I_2,I_3. Show that

$$\mathbf{S}^{-1}=(\mathbf{S}^2-I_1\mathbf{S}+I_2\mathbf{I})/I_3$$

A3 Index Notation

Problem A.33 Which of the following equations are valid expressions using index notation? If you decide an expression is invalid, state which rule is violated.

(a) $\sigma_{ij} = C_{klij}\varepsilon_{kl}$

(b) $\in_{kkk} = 0$

(c) $\dfrac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \dfrac{\partial^2 u_i}{\partial t^2}$

(d) $\in_{ijk}\in_{ijk} = X$

Problem A.34 Which of the following equations are valid expressions using index notation? If you decide an expression is invalid, state which rule is violated.

(a) $S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}$

(b) $\in_{ijk}\in_{kkj} = 0$

(c) $\rho \dfrac{\partial u_i}{\partial t} = \dfrac{\partial}{\partial x_j} C_{ijkl} \dfrac{\partial u_k}{\partial x_l}$

Problem A.35 Match the meaning of each index notation expression shown below with an option from the list

(a) $\lambda = T_{ij}S_{ij}$ (b) $E_{ij} = T_{ik}S_{kj}$ (c) $E_{ij} = S_{ki}T_{kj}$ (d) $a_i = \in_{kij} b_j c_k$ (e) $\lambda = a_i b_i$

(f) δ_{ij} (g) $T_{ij}m_j = \lambda m_i$ (h) $a_i = S_{ij}b_j$ (i) $A_{ki}A_{kj} = \delta_{ij}$ (j) $A_{ij} = A_{ji}$

(1) Product of two tensors
(2) Product of the transpose of a tensor with another tensor
(3) Cross product of two vectors
(4) Product of a vector and a tensor
(5) Components of the identity tensor
(6) Equation for the eigenvalues and eigenvectors of a tensor
(7) Contraction of a tensor
(8) Dot product of two vectors
(9) The definition of an orthogonal tensor
(10) Definition of a symmetric tensor

Problem A.36 What is the value of $\in_{ijk}\in_{ijk}$?

Problem A.37 Let $S_{ij} = P_{ij} - P_{kk}\delta_{ij}/3$. Calculate S_{kk} (a tensor with this property is called *deviatoric*, and **S** is called the *deviatoric part* of **P**)

Problem A.38 Write out in full the three equations expressed by

$$\frac{1}{1-2\nu}\frac{\partial^2 u_k}{\partial x_k \partial x_i}+\frac{\partial^2 u_i}{\partial x_k \partial x_k}+\frac{b_i}{\mu}=\frac{\rho}{\mu}\frac{\partial^2 u_i}{\partial t^2}$$

Problem A.39 Let **a, b, c** be three vectors. Use index notation to show that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

Problem A.40 Let **A** and **B** be tensors with components A_{ij} and B_{ij}. Use index notation to show that

$$(\mathbf{AB})^T=\mathbf{B}^T\mathbf{A}^T$$

Problem A.41 Let **A**, **B** and **C** be tensors with components A_{ij}, B_{ij} and C_{ij}. Use index notation to show that

$$(\mathbf{AB}):\mathbf{C}=\left(\mathbf{A}^T\mathbf{C}\right):\mathbf{B}=\left(\mathbf{C}\mathbf{B}^T\right):\mathbf{A}$$

Problem A.42 Use index notation rules to show that $\nabla\times\nabla\times\mathbf{u}=\nabla\nabla\cdot\mathbf{u}-\nabla^2\mathbf{u}$

Problem A.43 Let **a**, **b** be unit vectors, let **I** denote the identity tensor, and define a tensor **S** as

$$\mathbf{S}=(\mathbf{a}\times\mathbf{b})\otimes(\mathbf{a}\times\mathbf{b})$$

Show that

$$\mathbf{S}=\mathbf{I}\left[1-(\mathbf{a}\cdot\mathbf{b})^2\right]-\mathbf{a}\otimes\mathbf{a}-\mathbf{b}\otimes\mathbf{b}+(\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a})$$

Problem A.44 Let

$$C_{ijkl}=\frac{E}{2(1+\nu)}\left(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}\right)+\frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl}$$

$$S_{ijkl}=\frac{1+\nu}{2E}\left(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}\right)-\frac{\nu}{E}\delta_{ij}\delta_{kl}$$

be two tensors. Calculate $T_{ijkl}=S_{ijpq}C_{pqkl}$

Problem A.45 Verify that

$$S^{-1}_{ji}=\frac{1}{2\det(\mathbf{S})}\in_{ipq}\in_{jkl}S_{pk}S_{ql}$$

(i.e. use the index notation rules to show that $S^{-1}_{ji}S_{im}=\delta_{jm}$)

Problem A.46 Let $J=\det(\mathbf{S})$. Show that

$$\frac{\partial J}{\partial S_{mn}}=JS^{-1}_{nm}.$$

Problem A.47 Let $R=\sqrt{x_k x_k}$. Calculate

$$\frac{\partial \log(R)}{\partial x_i} \text{ and } \frac{\partial^2 \log(R)}{\partial x_i \partial x_i}$$

Problem A.48 The stress-strain relations for an isotropic, linear elastic material are

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}$$

Calculate the inverse relation giving stresses in terms of strains.

Problem A.49 Let σ_{ij} denote a symmetric second order tensor, and let

$$\sigma_e = \sqrt{S_{ij}S_{ij}} \qquad S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$$

Show that

$$\frac{\partial\sigma_e}{\partial\sigma_{ij}} = \frac{S_{ij}}{\sigma_e}$$

Problem A.50 The stress in a hypoelastic material is related to its strain energy density by

$$\sigma_{ij} = \frac{\partial U}{\partial\varepsilon_{ij}}$$

Find the stress for a material with strain energy density given by

$$U = \frac{1}{2}KI_1^2 + \frac{2n\sigma_0\varepsilon_0}{n+1}\left(\frac{I_2}{\varepsilon_0^2}\right)^{(n+1)/2n}$$

where

$$I_1 = \varepsilon_{kk} \qquad I_2 = \frac{1}{2}\left(\varepsilon_{ij}\varepsilon_{ij} - \varepsilon_{kk}\varepsilon_{pp}/3\right)$$

Problem A.51 Show that

$$\det(\mathbf{S}) = \frac{1}{6}\in_{ijk}\in_{lmn} S_{il}S_{jm}S_{kn} = \frac{1}{6}S_{ii}\left(S_{jj}S_{kk} - 3S_{kj}S_{jk}\right) + \frac{1}{3}S_{ji}S_{kj}S_{ik}$$

Problem A.52 Let $R_{ij} = \cos\theta\delta_{ij} + n_i n_j(1-\cos\theta) - \sin\theta\in_{ijk} n_k$ where n_k are the components of a unit vector. Calculate $R_{ik}R_{jk}$. (You should know what the answer is, of course, but see if you can verify the result using index notation manipulations).

Problem A.53 The strain energy density of a hyperelastic material with a compressible Neo-Hookean constitutive relation is given by

$$\bar{U} = \frac{\mu_1}{2}(\bar{I}_1 - 3) + \frac{K_1}{2}(J-1)^2$$

where

$$\bar{I}_1 = \frac{F_{ki}F_{ki}}{J^{2/3}} \qquad J = \det(\mathbf{F})$$

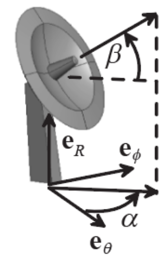
and **F** is the deformation gradient. The Cauchy stress is related to the deformation gradient by

$$\sigma_{ij} = \frac{1}{J}F_{ik}\frac{\partial\bar{U}}{\partial F_{jk}}$$

Evaluate the derivative to find an explicit formula for σ_{ij} in terms of F_{ij}. The result of problem A.46 is useful when differentiating *J*.

A4 Polar Coordinates

Problem A.54 A geo-stationary satellite orbits the earth at radius 41000 km in the equatorial plane, and is positioned at 0º longitude. A satellite dish located in Providence, Rhode Island (Longitude $41^0 40.3'\mathrm{N}$, Latitude $71^0 34.6'\mathrm{W}$) is to be pointed at the satellite. In this problem, you will calculate the angles α and β to position the satellite. Let $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ be a Cartesian basis with origin at the center of the earth, $\mathbf{k}$ pointing to the North Pole and $\mathbf{i}$ pointing towards the intersection of the equator (0º latitude) and the Greenwich meridian (0º longitude). Define a spherical-polar coordinate system (R,ϕ,θ) with basis vectors $\{\mathbf{e}_R,\mathbf{e}_\phi,\mathbf{e}_\theta\}$ in the usual way. Take the earth's radius as 6000 km.



(Due South)

(a) Write down the values of (R,ϕ,θ) for Providence, Rhode Island

(b) Write down the position vector of the satellite in the Cartesian $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ coordinate system

(c) Hence, find the position vector of the satellite relative to the center of the earth in the $\{\mathbf{e}_R,\mathbf{e}_\phi,\mathbf{e}_\theta\}$ basis located at Providence, Rhode Island.

(d) Find the position vector $\overrightarrow{PS}$ of the satellite relative to Providence, Rhode Island, in terms of basis vectors $\{\mathbf{e}_R,\mathbf{e}_\phi,\mathbf{e}_\theta\}$ located at Providence, Rhode Island.

(e) Find the components of a unit vector parallel to $\overrightarrow{PS}$, in terms of basis vectors $\{\mathbf{e}_R,\mathbf{e}_\phi,\mathbf{e}_\theta\}$ located at Providence, Rhode Island.

(f) Hence, calculate the angles α and β

Problem A.55 Calculate the gradient $\mathbf{v}\nabla$ and divergence $\nabla\cdot\mathbf{v}$ of the following vector fields

(a) $\mathbf{v}=R^2\mathbf{e}_R$

(b) $\mathbf{v}=R\sin\theta\mathbf{e}_\phi$

(c) $\mathbf{v}=\mathbf{e}_R/R^2$

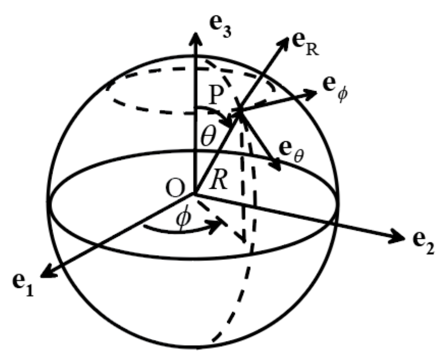
Problem A.56 Show that the components of the gradient of a vector field in spherical-polar coordinates is

$$\mathbf{v}\nabla\equiv\begin{bmatrix}\dfrac{\partial v_R}{\partial R} & \dfrac{1}{R}\dfrac{\partial v_R}{\partial\theta}-\dfrac{v_\theta}{R} & \dfrac{1}{R\sin\theta}\dfrac{\partial v_R}{\partial\phi}-\dfrac{v_\phi}{R}\\ \dfrac{\partial v_\theta}{\partial R} & \dfrac{1}{R}\dfrac{\partial v_\theta}{\partial\theta}+\dfrac{v_R}{R} & \dfrac{1}{R\sin\theta}\dfrac{\partial v_\theta}{\partial\phi}-\cot\theta\dfrac{v_\phi}{R}\\ \dfrac{\partial v_\phi}{\partial R} & \dfrac{1}{R}\dfrac{\partial v_\phi}{\partial\theta} & \dfrac{1}{R\sin\theta}\dfrac{\partial v_\phi}{\partial\phi}+\cot\theta\dfrac{v_\theta}{R}+\dfrac{v_R}{R}\end{bmatrix}$$

Problem A.57 Show that the components of the gradient of a vector field in cylindrical-polar coordinates are

$$\mathbf{v}\nabla \equiv \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_r}{\partial \theta} - \dfrac{v_\theta}{r} & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial \theta} + \dfrac{v_r}{r} & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix}$$

Problem A.58 In this problem you will derive the expression given in Appendix D of Applied Mechanics of Solids for the gradient operator associated with polar coordinates.



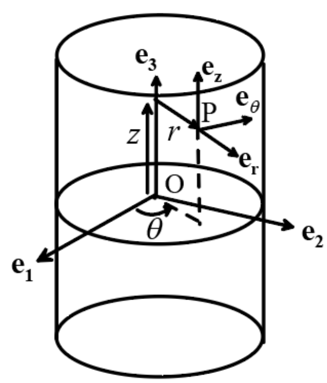
(a) Consider a scalar field $f(R,\theta,\phi)$. Write down an expression for the change df in f due to an infinitesimal change in the three coordinates (R,θ,ϕ) to first order in $(dR,d\theta,d\phi)$.

(b) Write down an expression for the change in position vector $d\mathbf{r}$ due to an infinitesimal change in the three coordinates (R,θ,ϕ), to first order in $(dR,d\theta,d\phi)$, expressing your answer as components in the $\{\mathbf{e}_R,\mathbf{e}_\theta,\mathbf{e}_\phi\}$ basis.

(c) Hence, find expressions for $(dR,d\theta,d\phi)$ in terms of $(\mathbf{e}_R\cdot d\mathbf{r},\mathbf{e}_\theta\cdot d\mathbf{r},\mathbf{e}_\phi\cdot d\mathbf{r})$

(d) Finally, substitute the result of (c) into the result of (a) to obtain an expression relating df to $d\mathbf{r}$. Rearrange the result into the form $df = [\nabla f]\cdot d\mathbf{r}$, and hence deduce the expression for the gradient operator.

Problem A.59 In this problem you will derive the expression given in Appendix D of Applied Mechanics of Solids for the gradient operator associated with polar coordinates.



(a) Consider a scalar field $f(r,\theta,z)$. Write down an expression for the change df in f due to an infinitesimal change in the three coordinates (r,θ,z), to first order in $(dr,d\theta,dz)$.

(b) Write down an expression for the change in position vector $d\mathbf{r}$ due to an infinitesimal change in the three coordinates (r,θ,z), to first order in $(dr,d\theta,dz)$, expressing your answer as components in the $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$ basis.

(c) Hence, find expressions for $(dr,d\theta,dz)$ in terms of $(\mathbf{e}_r\cdot d\mathbf{r},\mathbf{e}_\theta\cdot d\mathbf{r},\mathbf{e}_z\cdot d\mathbf{r})$

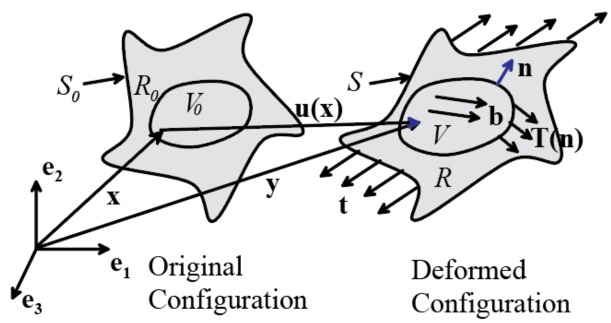
(d) Finally, substitute the result of (c) into the result of (a) to obtain an expression relating df to $d\mathbf{r}$. Rearrange the result into the form $df = [\nabla f]\cdot d\mathbf{r}$, and hence deduce the expression for the gradient operator.

A5 Miscellaneous Derivations

Problem A.60 Let A_{ij} be a time varying tensor. Show that

$$\frac{dA_{pq}^{-1}}{dt} = -A_{pi}^{-1}\frac{dA_{ij}}{dt}A_{jq}^{-1}$$

Problem A.61 Consider a deformable solid. Let V_0 denote a closed region within the undeformed solid, and let V be the same material region of the solid in the deformed configuration. Let A_0 denote the area of the material surface ∂V_0 surrounding V_0, and let A denote the area of the surface ∂V after deformation. Let $\mathbf{x}$ denote the position of a material particle in the solid before deformation and let $\mathbf{y}$ be the position of the same point after deformation. Define



$$F_{ij} = \frac{\partial y_i}{\partial x_j} \qquad J = \det(\mathbf{F}) \qquad L_{ij} = \frac{dF_{ik}}{dt}F_{kj}^{-1}$$

Show that

$$\frac{dA}{dt} = \frac{d}{dt}\int_{\partial V} dA = \int_{\partial V}\left(L_{kk}\delta_{ij} - L_{ij}\right)n_i n_j dA$$

where $\mathbf{n}$ is a unit vector normal to the deformed surface.