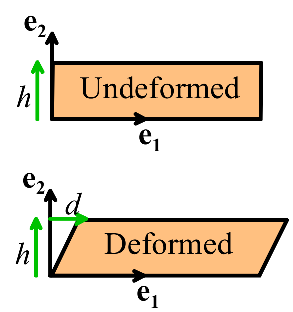
Chapter 2: Governing Equations

2.2 Mathematical Description of Shape Changes in Solids

1. A thin film of material is deformed in simple shear during a plate impact experiment, as shown in the figure.
   (a) Write down expressions for the displacement field in the film, in terms of x_1, x_2 d and h, expressing your answer as components in the basis shown.
   (b) Calculate the Lagrange strain tensor associated with the deformation, expressing your answer as components in the basis shown.
   (c) Calculate the infinitesimal strain tensor for the deformation, expressing your answer as components in the basis shown.
   (d) Find the principal values of the infinitesimal strain tensor, in terms of d and h



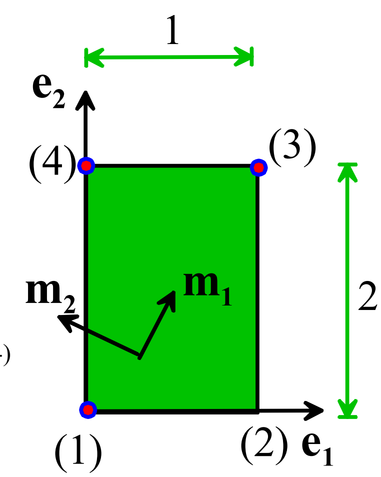
2. The figure shows a rectangular (2D) finite element. The displacement vector (as Cartesian components in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis, in arbitrary units) at each of its corners is

$$\mathbf{u}^{(1)} = [0,0]; \quad \mathbf{u}^{(1)} = \mathbf{u}^{(2)} = \mathbf{u}^{(4)} = [0,0]; \quad \mathbf{u}^{(3)} = [0.2, 0.4];$$

The displacement at an arbitrary point inside the element is computed using a linear interpolation between values at the four corners (this only works for the rectangle shown, the general formula for an arbitrary rectangular finite element is more complicated)

$$\mathbf{u} = \frac{1}{2}(2-x_2)(1-x_1)\mathbf{u}^{(1)} + \frac{1}{2}(2-x_2)x_1\mathbf{u}^{(2)} + \frac{1}{2}x_1x_2\mathbf{u}^{(3)} + \frac{1}{2}(1-x_1)x_2\mathbf{u}^{(4)}$$

where $\mathbf{x}$ is the position of a point in the element before deformation.



   (a) Find the components of $\nabla \mathbf{u}$ (give the answer as a 2x2 matrix, which will be a function of position) and hence calculate the infinitesimal strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}\left\{\nabla \mathbf{u} + \left(\nabla \mathbf{u}\right)^T\right\}$
   (b) Find the components of the deformation gradient (in 2D) $\mathbf{F}$ and hence deduce the components of the Lagrange strain tensor $\mathbf{E} = (\mathbf{F}^T\mathbf{F} - \mathbf{I})/2$

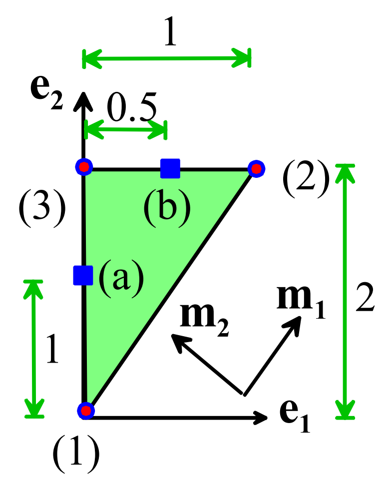
(c) Find

$$\int_A \nabla \cdot \mathbf{u} dA$$

where A denotes the area of the square. Check your answer using the divergence theorem; i.e. calculate $\int_C \mathbf{n} \cdot \mathbf{u} ds$ around the perimeter of the square

(d) Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be unit vectors parallel and perpendicular to a line from corner (1) to corner (2) of the element. Calculate the components of $\nabla \mathbf{u}$ in this basis. Check your answer by calculating $\nabla \cdot \mathbf{u}$ using the components in $\{\mathbf{m}_1, \mathbf{m}_2\}$.

3. The figure shows a triangular finite element. The displacement vector (as Cartesian components in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis, in arbitrary units) at each of its corners is $\mathbf{u}^{(1)} = [0,0]$; $\mathbf{u}^{(2)} = [0.1, 0.1]$; $\mathbf{u}^{(3)} = [0, 0.2]$; . The displacement at an arbitrary point inside the element is computed using a linear interpolation between values at the three corners (this only works for the triangle shown, it is not a general formula for all triangles)



$$\mathbf{u} = \frac{1}{2}(2 - x_2)\mathbf{u}^{(1)} + x_1 \mathbf{u}^{(2)} + \frac{1}{2}(x_2 - 2x_1)\mathbf{u}^{(3)}$$

where $\mathbf{x}$ is the position of a point in the element before deformation.

(a) Find the components of $\nabla \mathbf{u}$ (give the answer as a 2x2 matrix)

(b) Use the gradient to calculate $\mathbf{u}^a - \mathbf{u}^b$, where the two points a and b are shown in the figure.

(c) Calculate the divergence of $\mathbf{u}$ (i.e. $\nabla \cdot \mathbf{u}$)

(d) Find

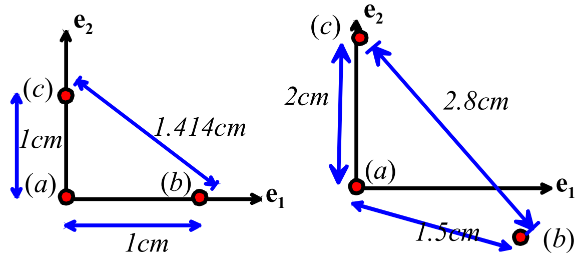
$$\int_A \nabla \cdot \mathbf{u} dA$$

where A denotes the area of the triangle. Check your answer using the divergence theorem; i.e. calculate $\int_C \mathbf{n} \cdot \mathbf{u} ds$ around the perimeter of the triangle

(e) Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be unit vectors parallel and perpendicular to the side (1-2) of the triangle. Calculate the components of $\nabla \mathbf{u}$ in this basis. Check your answer by calculating $\nabla \cdot \mathbf{u}$ using the components in $\{\mathbf{m}_1, \mathbf{m}_2\}$.

4. Find a displacement field that corresponds to a uniform Lagrange strain tensor E_{ij}. Is the displacement unique? Find a formula for the most general displacement field that generates a uniform Lagrange strain.

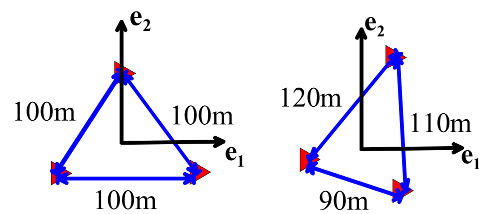
5. To measure the in-plane deformation of a sheet of metal during a forming process, three small hardness indentations are placed on the sheet. Using a travelling microscope, you determine that the initial lengths of the sides of the triangle formed by the three indents are 1cm, 1cm, 1.414cm, as shown in the picture below. After deformation, the sides have lengths 1.5cm, 2.0cm and 2.8cm.



(a) Calculate the components of the Lagrange strain tensor E_{11}, E_{22}, E_{12} in the basis shown.

(b) Calculate the components of the Eulerian strain tensor $E^*_{11}, E^*_{22}, E^*_{12}$ in the basis shown.

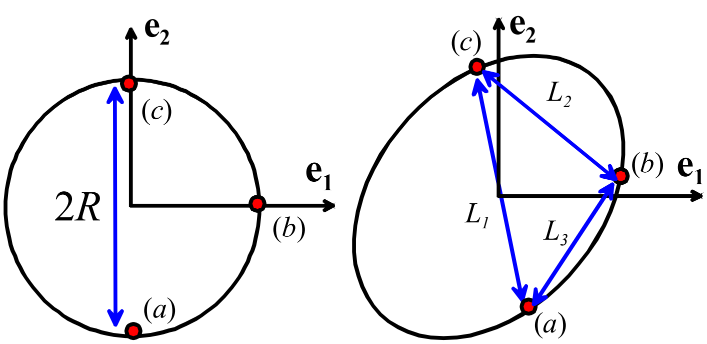
6. To track the deformation in a slowly moving glacier, three survey stations are installed in the shape of an equilateral triangle, spaced 100m apart, as shown in the picture. After a suitable period of time, the spacing between the three stations is measured again, and found to be 90m, 110m and 120m, as shown in the figure. Assuming that the deformation of the glacier is homogeneous over the region spanned by the survey stations, please compute:

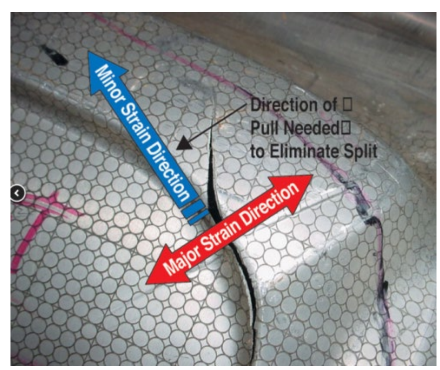


Before deformation

After deformation

Find the components of the Lagrange strain tensor associated with this deformation, expressing your answer as components in the basis shown.



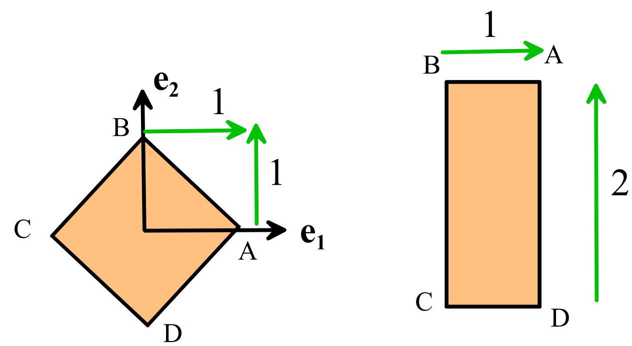


7. ‘Circle Grid Analysis’ is used to estimate strains induced in sheet metal parts during stamping or drawing. A grid of circles is etched on the sheet before it is deformed, and the strain at the center of each circle is then determined by measuring how the circles change their shape. The figure shows a typical example. Suppose that the three lengths L_1, L_2, L_3 on a circle with initial radius R are measured.

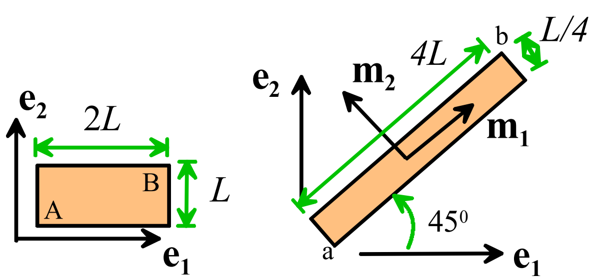
(a) Find a formula for the components of the 2D Lagrange strain tensor in terms of the lengths and R.

(b) Check that you get the correct solution for a rigid rotation (i.e. put in values for L_1, L_2, L_3 consistent with a rigid rotation)

8. The figure shows a material element on the surface of a specimen before and after deformation. Calculate the (2D) Lagrange strain tensor (as a 2x2 matrix)

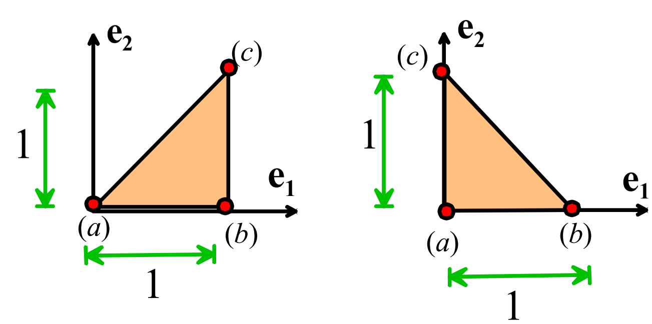


9. The figure shows the reference and deformed configurations for a solid. The out-of-plane dimensions are unchanged. Points a and b are the positions of points A and B after deformation. Determine



(a) The right stretch tensor $\mathbf{U}$, expressed as components in $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. (A 2x2 matrix is sufficient). There is no need for lengthy calculations – you may write down the result by inspection.
(b) The rotation tensor $\mathbf{R}$ in the polar decomposition of the deformation gradient $\mathbf{F}=\mathbf{RU}=\mathbf{VR}$
(c) The deformation gradient, expressed as components in $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$. Try to do this without using the basis-change formulas.

10. The figure shows a plane 2D triangular constant strain finite element before and after deformation. Calculate the Lagrange strain in the element.



11. Construct (i.e. find a displacement field) a homogeneous deformation that has the following properties:
    - The volume of the solid is doubled
    - A material fiber parallel to the $\mathbf{e}_1$ direction in the undeformed solid increases its length by a factor of $\sqrt{2}$ and is oriented parallel to the $\mathbf{e}_1+\mathbf{e}_2$ direction in the deformed solid
    - A material fiber parallel to the $\mathbf{e}_2$ direction in the undeformed is oriented parallel to the $-\mathbf{e}_1+\mathbf{e}_2$ direction in the deformed solid.
    - A material fiber parallel to the $\mathbf{e}_3$ direction in the undeformed solid preserves its length and orientation in the deformed solid

12. A rigid body motion is a nonzero displacement field that does not distort any infinitesimal volume element within a solid. Thus, a rigid body displacement induces no strain, and hence no stress, in the solid. The deformation corresponding to a 3D rigid rotation about an axis through the origin is

$$\mathbf{y}=\mathbf{R}\cdot\mathbf{x} \quad \text{or} \quad y_i = R_{ij}x_j$$

where $\mathbf{R}$ must satisfy $\mathbf{R}\cdot\mathbf{R}^T=\mathbf{R}^T\cdot\mathbf{R}=\mathbf{I}$, det($\mathbf{R}$)>0.
    (a) Show that the Lagrange strain associated with this deformation is zero.
    (b) As a specific example, consider the deformation

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This is the displacement field caused by rotating a solid through an angle θ about the $\mathbf{e}_3$ axis. Find the deformation gradient for this displacement field, and show that the deformation gradient tensor is orthogonal, as predicted above.
    (c) Show also that the infinitesimal strain tensor for this displacement field is not generally zero, but is of order θ^2 if θ is small.
    (d) If the displacements are small, we can find a simpler representation for a rigid body displacement. Consider a deformation of the form

$$y_i = x_i + \in_{ijk}\,\omega_j x_k$$

Here $\boldsymbol{\omega}$ is a vector with magnitude <<1, which represents an infinitesimal rotation about an axis parallel to $\boldsymbol{\omega}$. Show that the infinitesimal strain tensor associated with this displacement is always zero.
    (e) Show further that the Lagrange strain associated with this displacement field is

$$E_{ij}=\frac{1}{2}\left(\delta_{ij}\omega_k\omega_k-\omega_i\omega_j\right)$$

This is not, in general, zero. It is small if all $\omega_k \ll 1$.

13. A solid is subjected to a rigid rotation so that a unit vector **a** in the undeformed solid is rotated to a new orientation **b**. Find a rotation tensor **R** that is consistent with this deformation, in terms of the components of **a** and **b**. Is the rotation tensor unique? If not, find the most general formula for the rotation tensor.

14. The formula for the deformation due to a rotation through an angle θ about an axis parallel to a unit vector **n** that passes through the origin is

$$y_i = \left[\cos\theta\delta_{ij} + (1-\cos\theta)n_i n_j + \sin\theta \in_{ikj} n_k\right] x_j$$

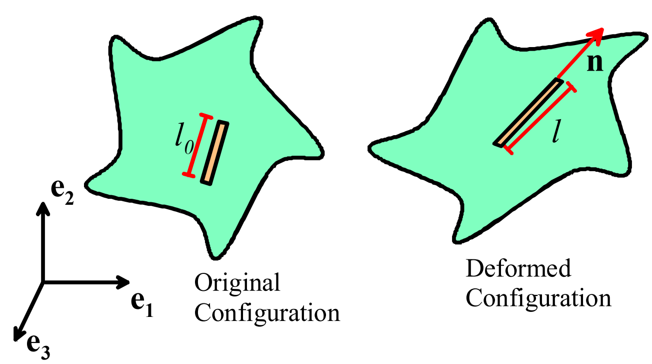
(a) Calculate the components of corresponding deformation gradient
(b) Verify that the deformation gradient satisfies $F_{ik}F_{jk} = F_{ki}F_{kj} = \delta_{ij}$
(c) Find the components of the *inverse* of the deformation gradient
(d) Verify that both the Lagrange strain tensor and the Eulerian strain tensor are zero for this deformation. What does this tell you about the distorsion of the material?
(e) Calculate the Jacobian of the deformation gradient. What does this tell you about volume changes associated with the deformation?

15. Show that the Lagrange strain **E**, the right Cauchy-Green deformation tensor **C** and the right stretch tensor **U** have the same principal directions (eigenvectors). Similarly, show that $\mathbf{E}^*, \mathbf{B}, \mathbf{V}$ have the same principal directions.

16. Show that the Eulerian strain tensor E_{ij}^* can be used to relate the length of a material fiber in a deformable solid before and after deformation, using the formula

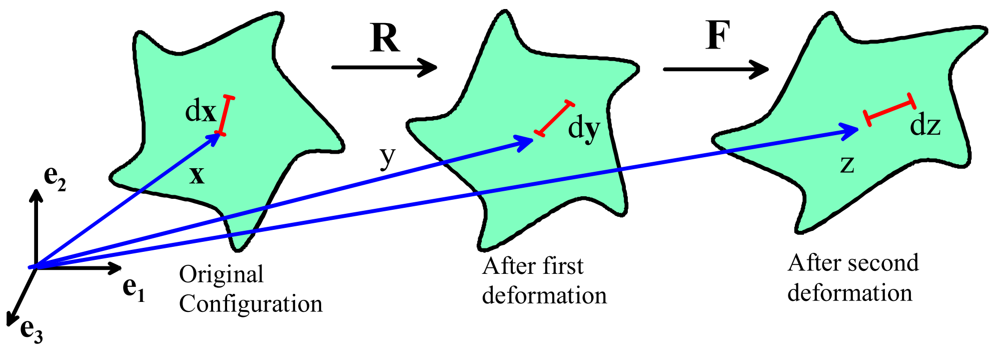
$$\frac{l^2 - l_0^2}{2l^2} = E_{ij}^* n_i n_j$$

where n_i are the components of a unit vector parallel to the material fiber after deformation.



17. The Lagrange strain tensor can be used to calculate the change in angle between any two material fibers in a solid as the solid is deformed. In this problem you will calculate the formula that can be used to do this. To this end, consider two infinitesimal material fibers in the undeformed solid, which are characterized by vectors with components $dx_i^{(1)} = l_1 m_i^{(1)}$ and $dx_i^{(2)} = l_2 m_i^{(2)}$, where $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$ are two unit vectors. Recall that the angle θ_0 between $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$ before deformation can be calculated from $\cos\theta_0 = m_i^{(1)} m_i^{(2)}$. Let $dy_i^{(1)}$ and $dy_i^{(2)}$ represent the two material fibers after deformation. Show that the angle between $dy_i^{(1)}$ and $dy_i^{(2)}$ can be calculated from the formula

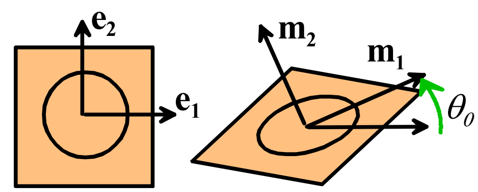
$$\cos\theta_1 = \frac{2E_{ij}m_i^{(1)}m_j^{(2)} + \cos\theta_0}{\sqrt{1+2E_{ij}m_i^{(1)}m_j^{(1)}}\sqrt{1+2E_{ij}m_i^{(2)}m_j^{(2)}}}$$



18. Suppose that a solid is subjected to a sequence of two homogeneous deformations (i) a rigid rotation **R**, followed by (ii) an arbitrary homogeneous deformation **F**. Taking the original configuration as reference, find formulas for the following deformation measures for the final configuration of the solid, in terms of **F** and **R**:
(a) The deformation gradient
(b) The Left and Right Cauchy-Green deformation tensors
(c) The Lagrange strain
(d) The Eulerian strain.

19. Repeat problem 18, but this time assume that the sequence of the two deformations is reversed, i.e. the solid is first subjected to an arbitrary homogeneous deformation **F,** and is subsequently subjected to a rigid rotation **R**.

20. A sheet of material is subjected to a two dimensional homogeneous deformation of the form
$$y_1 = A_{11}x_1 + A_{12}x_2 \qquad y_2 = A_{21}x_1 + A_{22}x_2$$
where A_{ij} are constants. Suppose that a circle of unit radius is drawn on the undeformed sheet. This circle is distorted to a smooth curve on the deformed sheet. Show that the distorted circle is an ellipse, with semi-axes that are parallel to the principal directions of the left stretch tensor **V,** and that the lengths of the semi-axes of the ellipse are equal to the principal stretches for the deformation.



21. The center of mass and the mass moment of inertia tensor in the reference and deformed configurations of a solid are (by definition)
$$r_i^{c0} = \frac{1}{M}\int_{V_0} x_i \rho_0 dV_0 \qquad I_{ij}^{c0} = \int_{V_0} \left(x_i - r_i^{c0}\right)\left(x_j - r_j^{c0}\right)\rho_0 dV_0$$
$$r_i^{c} = \frac{1}{M}\int_{V} y_i \rho dV \qquad I_{ij}^{c} = \int_{V} \left(y_i - r_i^{c}\right)\left(y_j - r_j^{c}\right)\rho dV$$
where ρ_0, ρ are the mass density of the solid in the reference and deformed configurations, $\mathbf{x}, \mathbf{y}$ are the positions of material particles in the reference and deformed configurations, and M is the total mass.

Suppose that a solid is subjected to a homogeneous deformation
$$y_i = A_{ik}x_k + c_i$$
where A_{ij} and c_i are constants.

(a) Find formulas for r_{ic}, I_{ij}^C in terms of r_i^{c0} I_{ij}^{c0}, A_{ij} and c_i.

(b) Suppose that A_{ij} is a rigid rotation (this means $A_{ik}A_{jk} = A_{ki}A_{kj} = \delta_{ij}$. Use the solution to (a) to show that the time derivative of I_{ij} can be expressed as

$$\frac{dI_{ij}^c}{dt} = W_{ik}I_{kj}^c - I_{ik}^c W_{kj}$$

where $W_{ik} = \dfrac{dA_{ik}}{dt}A_{jk}$ is the spin tensor.

(c) Suppose that a rigid body rotates with angular velocity ω_k and therefore has angular momentum

$$h_i = I_{ij}^c \omega_j$$

Use (b) to show that the time derivative of the angular momentum is

$$\frac{dh_i}{dt} = I_{ij}\frac{d\omega_j}{dt} + \in_{ijk} \omega_j I_{kl}\omega_l$$

22. Let **F** be a deformation gradient, **E** the Lagrange strain tensor and **D** the stretch rate tensor. Show that

$$\mathbf{D} = \mathbf{F}^{-T}\frac{d\mathbf{E}}{dt}\mathbf{F}^{-1}$$

23. Let **n** be a unit vector parallel to infinitesimal material fiber in a deforming solid., and let **D** and **W** denote the stretch rate and spin tensors. Show that

$$\frac{d\mathbf{n}}{dt} = \mathbf{Dn} + \mathbf{Wn} - (\mathbf{n}\cdot\mathbf{Dn})\mathbf{n}$$

24. The properties of many rubbers and foams are specified by functions of the following invariants of the left Cauchy-Green deformation tensor $B_{ij} = F_{ik}F_{jk}$.

$$I_1 = \mathrm{trace}(\mathbf{B}) = B_{kk}$$

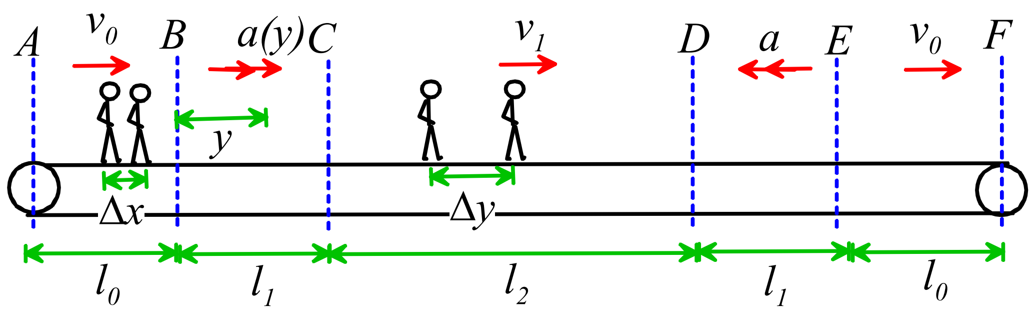
$$I_2 = \frac{1}{2}\left(I_1^2 - \mathbf{B}\cdot\cdot\mathbf{B}\right) = \frac{1}{2}\left(I_1^2 - B_{ik}B_{ki}\right)$$

$$I_3 = \det\mathbf{B} = J^2$$

(a) Verify that I_1, I_2, I_3 are invariants. The simplest way to do this is to show that I_1, I_2, I_3 are unchanged during a change of basis.

(b) In order to calculate stress-strain relations for these materials, it is necessary to evaluate derivatives of the invariants. Show that

$$\frac{\partial I_1}{\partial F_{ij}} = 2F_{ij}, \quad \frac{\partial I_2}{\partial F_{ij}} = 2\left(I_1F_{ij} - B_{ik}F_{kj}\right), \quad \frac{\partial I_3}{\partial F_{ij}} = 2I_3F_{ji}^{-1}$$



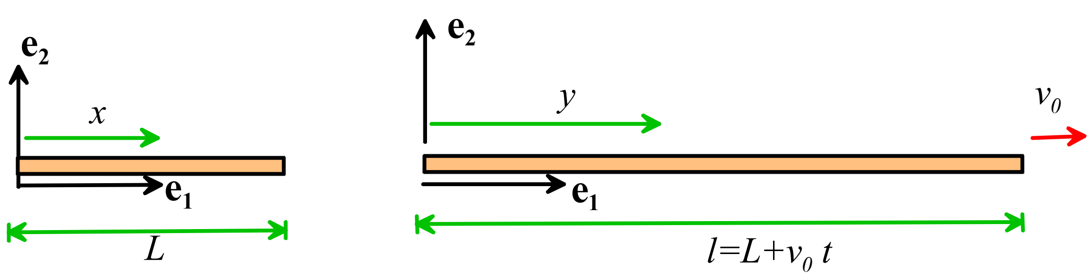
25. The figure shows a design for a high-speed moving walkway. A passenger standing on the walkway passes through five regions:
    (i) between A and B she moves at constant speed v_0;
    (ii) between B and C she accelerates (with an acceleration to be specified below);
    (iii) between C and D she moves with constant (high) speed v_1; and
    (iv) between D and E she decelerates
    (v) between E and F she travels at speed v_0 again.

    In this problem we will just focus on portion (ii) of the motion – i.e. between *B* and *C*.

    (a) Suppose that the walkway is designed so that the velocity varies linearly with distance between *B* and *C*. Assume that a person walks with speed *w* relative to the moving walkway. Determine her acceleration as a function of distance *y* from *B*, and also as a function of time after passing the point *B*. Find a formula for the maximum value of the acceleration, and identify the point where it occurs.

    (b) Suppose the walkway is designed instead so that a person standing on the track has constant acceleration *a*. Calculate the required velocity distribution *v(y)* as a function of distance *y* from B, and determine the acceleration of the person walking along the accelerating walkway as a function of *y* and also a function of *t*.

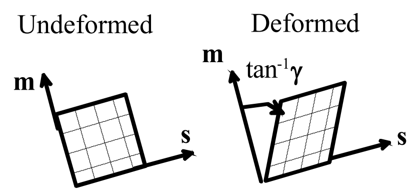
26. A rubber band has initial length *L*. One end of the band is held fixed. For time $t>0$ the other end is pulled at constant speed v_0. Following the usual convention, let *x* denote position in the reference configuration, and let *y* denote position in the deformed configuration. Assume one dimensional deformation.



    (a) Write down the position *y* of a material particle as a function of its initial position *x* and time *t*.
    (b) Hence, determine the velocity distribution as both a function of *x* and a function of *y*.
    (c) Find the deformation gradient (you only need to state the one nonzero component
    (d) Find the velocity gradient
    (e) Suppose that a fly walks along the rubber band with speed *w* relative to the band. Calculate the acceleration of the fly as a function of time and other relevant variables.

(f) Suppose that the fly is at $x=y=0$ at time $t=0$. Find how long it takes for the fly to walk to the other end of the rubber band, in terms of L, v_0 and w. It is easiest to do this by calculating dx/dt for the fly.

27. A single crystal deforms by shearing on a single active slip system as illustrated in the figure. The crystal is loaded so that the slip direction $\mathbf{s}$ and normal to the slip plane $\mathbf{m}$ maintain a constant direction during the deformation

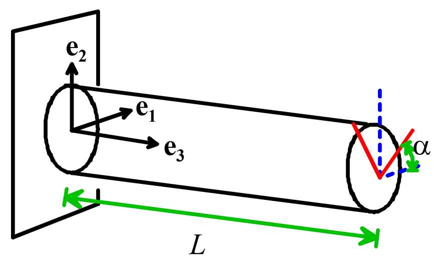


(a) Show that the deformation gradient can be expressed in terms of the components of the slip direction $\mathbf{s}$ and the normal to the slip plane $\mathbf{m}$ as $F_{ij} = \delta_{ij} + \gamma s_i m_j$ where γ denotes the shear, as illustrated in the figure.

(b) Suppose shearing proceeds at some rate $\dot{\gamma}$. At the instant when $\gamma = 0$, calculate (i) the velocity gradient tensor; (ii) the stretch rate tensor and (iii) the spin tensor associated with the deformation.

(c) Find an expression for the stretch rate and angular velocity of a material fiber parallel to a unit vector $\mathbf{n}$ in the deformed solid, in terms of $\dot{\gamma}, \mathbf{s}, \mathbf{m}$.

28. The displacement field in a homogeneous, isotropic circular shaft twisted through angle α at one end is given by



$$u_1 = x_1[\cos\left(\frac{\alpha x_3}{L}\right) - 1] - x_2 \sin\left(\frac{\alpha x_3}{L}\right)$$

$$u_2 = x_1 \sin\left(\frac{\alpha x_3}{L}\right) + x_2[\cos\left(\frac{\alpha x_3}{L}\right) - 1]$$

$$u_3 = 0$$

(a) Calculate the matrix of components of the deformation gradient tensor

(b) Calculate the matrix of components of the Lagrange strain tensor.

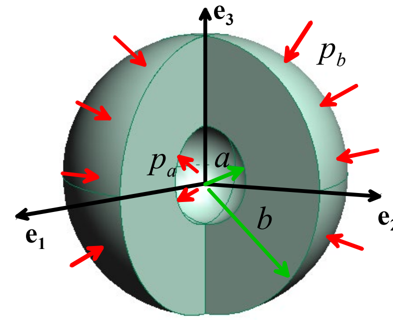
(c) At the point $\mathbf{x}=(a,0,0)$, find the stretch l/l_0 of material fibers that are parallel to the three basis vectors in the undeformed bar.

(d) At the point $\mathbf{x}=(a,0,0)$, find the stretch l/l_0 of material fibers that are parallel to unit vectors $\mathbf{m}_1 = (\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{2}$ and $\mathbf{m}_2 = (\mathbf{e}_2 - \mathbf{e}_3)/\sqrt{2}$ in the undeformed bar.

(e) Calculate the components of the infinitesimal strain tensor. Show that, for small values of α, the infinitesimal strain tensor is identical to the Lagrange strain tensor, but for finite rotations the two measures of deformation differ.

(f) At the point $\mathbf{x}=(a,0,0)$, use the infinitesimal strain tensor to obtain estimates for the stretch l/l_0 of material fibers that are parallel to unit vectors $\mathbf{m}_1 = (\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{2}$ and $\mathbf{m}_2 = (\mathbf{e}_2 - \mathbf{e}_3)/\sqrt{2}$ in the undeformed bar. How large can $\alpha a/L$ be before the error in this estimate reaches 1%?

29. A spherical shell (see the figure) is made from an incompressible material. In its undeformed state, the inner and outer radii of the shell are A, B. After deformation, the new values are a, b. The deformation in the shell can be described (in Cartesian components) by the equation

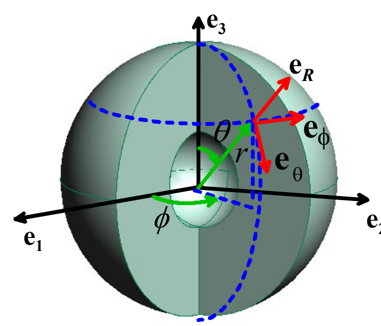


$$y_i = r\frac{x_i}{R} \qquad r = \left(R^3 + a^3 - A^3\right)^{1/3} \qquad R = \sqrt{x_k x_k}$$

(a) Calculate the components of the deformation gradient tensor
(b) Verify that the deformation is volume preserving
(c) Find the deformed length of an infinitesimal radial line that has initial length l_0, expressed as a function of R
(d) Find the deformed length of an infinitesimal circumferential line that has initial length l_0, expressed as a function of R
(e) Using the results of (c) and (d), write down the principal stretches for the deformation.
(f) Find the *inverse* of the deformation gradient, expressed as a function of y_i. You can do this by inspection, by inverting (a) (not recommended!), or by working out a formula that enables you to calculate x_i in terms of y_i and $r = \sqrt{y_i y_i}$ and differentiating the result. The first is quickest!

30. Suppose that the spherical shell described in Problem 3 is continuously expanding (visualize a balloon being inflated). The rate of expansion can be characterized by the velocity $v_a = da/dt$ of the surface that lies at R=A in the undeformed cylinder.
(a) Calculate the components of the deformation gradient tensor
(b) Calculate the velocity field $v_i = dy_i/dt$ in the sphere as a function of x_i
(c) Calculate the velocity field as a function of y_i
(d) Calculate the time derivative of the deformation gradient tensor calculated in problem 8
(e) Calculate the components of the velocity gradient $L_{ij} = \dfrac{\partial v_i}{\partial y_j}$ by differentiating the result of (b)
(f) Calculate the components of the velocity gradient using the results of (c) and 29(f)
(g) Calculate the stretch rate tensor D_{ij}. Verify that the result represents a volume preserving stretch rate field.

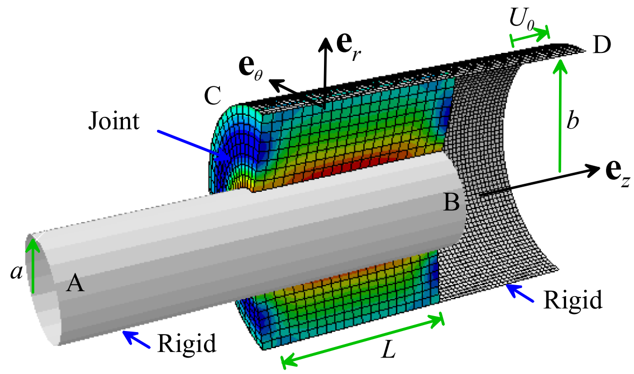
31. Repeat Problem 3(a), 3(f) and all of 4(b), 4(d), but this time solve the problem using spherical-polar coordinates, using the various formulas for vector and tensor operations given in the notes. In this case, you may assume that a point with position $\mathbf{x} = R\mathbf{e}_R$ in the undeformed solid has position vector



$$\mathbf{y} = \left(R^3 + a^3 - A^3\right)^{1/3} \mathbf{e}_R$$

after deformation.

32. The figure shows a cross-section through a joint connecting two hollow cylindrical shafts. The joint is a hollow cylinder with external radius b and internal radius a. It is bonded to the two rigid shafts AB and CD. Shaft AB is fixed (no translation or rotation), and an axial displacement $\mathbf{u} = U_0 \mathbf{e}_z$ is applied to the hollow cylinder CD. The displacement field in the joint can be shown to be approximately



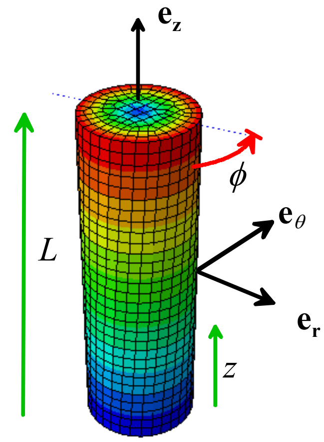
$$\mathbf{u} = \left[\frac{U_0}{\log(b/a)} \log(r/a) \right] \mathbf{e}_z$$

Find

(a) The displacement gradient;
(b) The deformation gradient
(c) The Lagrange strain tensor and
(d) The infinitesimal strain tensor field in the joint, expressing your answer as components in the cylindrical-polar basis shown in the figure

33. The twisted cylinder shown in the figure has a displacement field (in polar coordinates)

$$\mathbf{u} = \frac{\phi}{L} r z \mathbf{e}_\theta$$



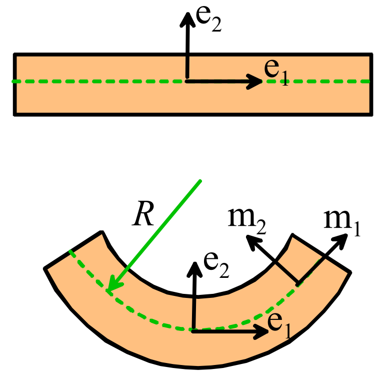
where ϕ is the angle of twist at the end of the cylinder. The contours show the magnitude of the displacement vector, to help visualize the deformation.

Calculate the length of a material fiber that lies on the outer surface of the cylinder (r=a) and is parallel to the $\mathbf{e}_z$ direction before the cylinder is twisted

34. An initially straight beam is bent into a circle with radius R as shown in the figure. Material fibers that are perpendicular to the axis of the undeformed beam are assumed to remain perpendicular to the axis after deformation, and the beam's thickness and the length of its axis are assumed to be unchanged. Under these conditions the deformation can be described as

$$y_1 = (R - x_2)\sin(x_1 / R) \qquad y_2 = R - (R - x_2)\cos(x_1 / R)$$

where, as usual $\mathbf{x}$ is the position of a material particle in the undeformed beam, and $\mathbf{y}$ is the position of the same particle after deformation.



(a) Calculate the deformation gradient field in the beam, expressing your answer as a function of x_1, x_2, and as components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ shown.

(b) Calculate the Lagrange strain field in the beam.

(c) Calculate the infinitesimal strain field in the beam.

(d) Compare the values of Lagrange strain and infinitesimal strain for two points that lie at $(x_1 = 0, x_2 = h)$ and $(x_1 = L, x_2 = 0)$. Explain briefly the physical origin of the difference between the two strain measures at each point. Recommend maximum allowable values of h/R and L/R for use of the infinitesimal strain measure in modeling beam deflections.

(e) Calculate the deformed length of an infinitesimal material fiber that has length l_0 and orientation $\mathbf{e}_1$ in the undeformed beam. Express your answer as a function of x_2.

(f) Calculate the change in length of an infinitesimal material fiber that has length l_0 and orientation $\mathbf{e}_2$ in the undeformed beam.

(g) Show that the two material fibers described in (3) and (f) remain mutually perpendicular after deformation. Is this true for *all* material fibers that are mutually perpendicular in the undeformed solid?

(h) Find the components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the Left and Right stretch tensors $\mathbf{U}$ and $\mathbf{V}$ as well as the rotation tensor $\mathbf{R}$ for this deformation. You should be able to write down $\mathbf{U}$ and $\mathbf{R}$ by inspection, without needing to wade through the laborious general process. The results can then be used to calculate $\mathbf{V}$.

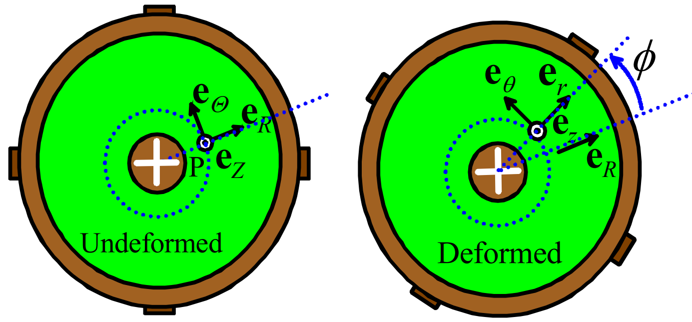
(i) Find the principal directions of $\mathbf{U}$ as well as the principal stretches. You should be able to write these down without doing any tedious calculations.

(j) Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be a basis in which $\mathbf{m}_1$ is parallel to the axis of the deformed beam, as shown in the figure. Write down the components of each of the unit vectors $\mathbf{m}_i$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Hence, compute the transformation matrix $Q_{ij} = \mathbf{m}_i \cdot \mathbf{e}_j$ that is used to transform tensor components from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{m}_1, \mathbf{m}_2\}$.

(k) Find the components of the deformation gradient tensor, Lagrange strain tensor, as well as $\mathbf{U}$ $\mathbf{V}$ and $\mathbf{R}$ in the basis $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. It is best to do these with a symbolic manipulation program.

(l) Find the principal directions of $\mathbf{V}$ expressed as components in the basis $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. Again, you should be able to simply write down this result.

35. The figure shows a test designed to measure the response of a polymer to large shear strains. The sample is a hollow cylinder with internal radius a_0 and external radius a_1. The inside diameter is bonded to a fixed rigid cylinder. The external diameter is bonded inside a rigid tube, which is rotated through an angle $\alpha(t)$. Assume that the specimen deforms as indicated in the figure, i.e. (a) cylindrical sections remain cylindrical; (b) no point in the specimen moves in the axial or radial directions; (c) that a cylindrical element of material at radius R rotates through angle $\phi(R,t)$ about the axis of the specimen. Take the undeformed configuration as reference. Let $(R,\boldsymbol{\Theta},Z)$ denote the cylindrical-polar coordinates of a material point in the reference configuration, and let $\{\mathbf{e}_R,\mathbf{e}_\Theta,\mathbf{e}_Z\}$ be cylindrical-polar basis vectors at $(R,\boldsymbol{\Theta},Z)$. Let (r,θ,z) denote the coordinates of this point in the deformed configuration, and let $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$ by cylindrical-polar basis vectors located at (r,θ,z).



(a) Write down expressions for (r,θ,z) in terms of $(R,\boldsymbol{\Theta},Z)$ (this constitutes the deformation mapping)

(b) Let P denote the material point at at $(R,\boldsymbol{\Theta},Z)$ in the reference configuration. Write down the reference position vector $\mathbf{X}$ of P, expressing your answer as components in the basis $\{\mathbf{e}_R,\mathbf{e}_\Theta,\mathbf{e}_Z\}$

(c) Write down the deformed position vector $\mathbf{x}$ of P, expressing your answer in terms of $(R,\boldsymbol{\Theta},Z)$ and basis vectors $\{\mathbf{e}_R,\mathbf{e}_\Theta,\mathbf{e}_Z\}$

(d) Find the components of the deformation gradient tensor $\mathbf{F}$ in $\{\mathbf{e}_R,\mathbf{e}_\Theta,\mathbf{e}_Z\}$. (Recall that the gradient operator in cylindrical-polar coordinates is $\nabla \equiv (\mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_\Theta \frac{1}{R}\frac{\partial}{\partial \boldsymbol{\Theta}} + \mathbf{e}_Z \frac{\partial}{\partial Z})$; recall also that $\frac{\partial \mathbf{e}_R}{\partial \Theta} = \mathbf{e}_\Theta$; $\frac{\partial \mathbf{e}_\Theta}{\partial \Theta} = -\mathbf{e}_R$)

(e) Show that the deformation gradient can be decomposed into a sequence $\mathbf{F} = \mathbf{R}\cdot\mathbf{S}$ of a simple shear $\mathbf{S}$ followed by a rigid rotation through angle ϕ about the $\mathbf{e}_Z$ direction $\mathbf{R}$. In this case the simple shear deformation will have the form

$$\mathbf{S} = \mathbf{e}_R \otimes \mathbf{e}_R + \mathbf{e}_\Theta \otimes \mathbf{e}_\Theta + \mathbf{e}_Z \otimes \mathbf{e}_Z + \alpha \mathbf{e}_\Theta \otimes \mathbf{e}_R$$

where α is to be determined.

(f) Find the components of $\mathbf{F}$ in $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$.

(g) Verify that the deformation is volume preserving

(h) Find the components of the right Cauchy-Green deformation tensors in $\{\mathbf{e}_R,\mathbf{e}_\Theta,\mathbf{e}_Z\}$

(i) Find the components of the left Cauchy-Green deformation tensor in $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$

(j) Find $\mathbf{F}^{-1}$ in $\{\mathbf{e}_R,\mathbf{e}_\Theta,\mathbf{e}_Z\}$.

(k) Write down the velocity field $\mathbf{v}$ in terms of (r,θ,z) in the basis $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$

(l) Calculate the spatial velocity gradient $\mathbf{L}$ in the basis $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$

36. Find a displacement field corresponding to a uniform infinitesimal strain field ε_{ij}. (Don't make this hard – think about what kind of function, when differentiated, gives a constant). Is the displacement unique?

37. Find a formula for the most general displacement field that generates *zero* infinitesimal strain.

38. The infinitesimal strain field in a long cylinder containing a hole at its center is given by

$$\varepsilon_{31} = -bx_2 / r^2 \qquad \varepsilon_{32} = bx_1 / r^2 \qquad r = \sqrt{x_1^2 + x_2^2}$$

(a) Show that the strain field satisfies the equations of compatibility.

(b) Show that the strain field is consistent with a displacement field of the form $u_3 = \theta$, where $\theta = 2b\tan^{-1} x_2 / x_1$. Note that although the strain field is compatible, the displacement field is *multiple valued* – i.e. the displacements are not equal at $\theta = 2\pi$ and $\theta = 0$, which supposedly represent the same point in the solid. Of course, displacement fields like this do exist in solids – they are caused by dislocations in a crystal.

39. Calculate the displacement field that generates the following 3D infinitesimal strain field

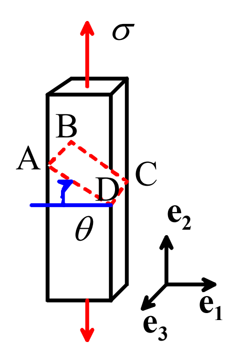
$$\varepsilon_{ij} = (1+\nu)(x_k x_k \delta_{ij} + 2x_i x_j) - (3-\nu)\delta_{ij}$$

40. Consider the 2D infinitesimal strain field

$$\varepsilon_{11} = \frac{1}{r^2} - \frac{2x_1^2}{r^4} \qquad \varepsilon_{22} = \frac{1}{r^2} - 2\frac{x_2^2}{r^4} \qquad \varepsilon_{12} = -2\frac{x_1 x_2}{r^4} \qquad r = \sqrt{x_1^2 + x_2^2}$$

Show that the strain field is compatible, and find the corresponding displacement field

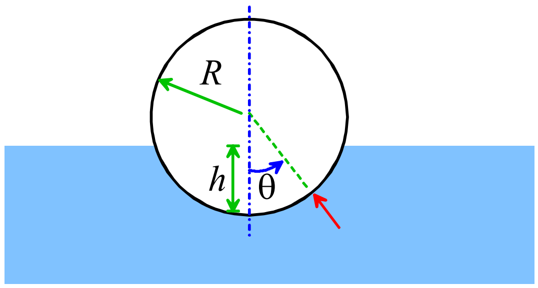
2.3 Mathematical Description of Internal Forces in Solids



1. A rectangular bar is loaded in a state of uniaxial tension with magnitude σ, as shown in the figure.
   (a) Write down the components of the stress tensor in the bar, using the basis vectors shown.
   (b) Calculate the components of the normal vector to the plane ABCD shown, and hence deduce the components of the traction vector acting on this plane, expressing your answer as components in the basis shown, in terms of q
   (c) Compute the normal and tangential tractions acting on the plane shown.

2. Consider a state of hydrostatic stress $s_{ij} = p d_{ij}$. Show that the traction vector acting on any internal plane in the solid (or, more likely, fluid!) has magnitude p and direction normal to the plane.

3. A cylinder of radius R is partially immersed in a static fluid.



(a) Recall that the pressure at a depth d in a fluid has magnitude rgd. Write down an expression for the horizontal and vertical components of traction acting on the surface of the cylinder in terms of q.
(b) Hence compute the resultant force (per unit out of plane distance) exerted by the fluid on the cylinder, in terms of ρ, g, h, R .

4. For the Cauchy stress tensor with components

$$\begin{bmatrix} 100 & 250 & 0 \\ 250 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}$$

(MPa) compute
(a) The traction vector acting on an internal material plane with normal $\mathbf{n} = (\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$
(b) The principal stresses
(c) The hydrostatic stress
(d) The deviatoric stress tensor
(e) The Von-Mises equivalent stress

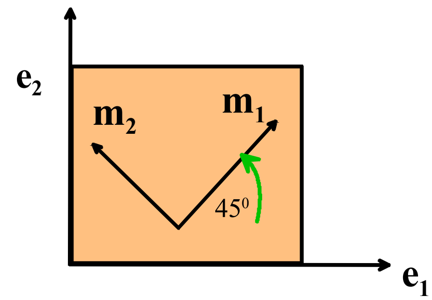
5. For the Cauchy stress tensor with components

$$\boldsymbol{\sigma} = \begin{bmatrix} 50 & 250 & 0 \\ 250 & 130 & -10 \\ 0 & -10 & 320 \end{bmatrix}$$

compute
(a) The traction vector acting on an internal material plane with normal $\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{e}_1 - \frac{1}{\sqrt{2}}\mathbf{e}_2$
(b) The principal stresses
(c) The hydrostatic stress
(d) The deviatoric stress tensor

(e) The Von-Mises equivalent stress (find the answer using the answers to both (b) and (d))

6. A sheet of material that lies in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ plane is subjected to a state of stress with the following properties:
   - The stress is in a state of *plane stress*
   - The principal stress directions are parallel to the $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ directions shown in the figure
   - The hydrostatic stress is zero
   - The Von-Mises stress has magnitude 250 MPa



(a) For the plane stress state, which components of stress are zero?

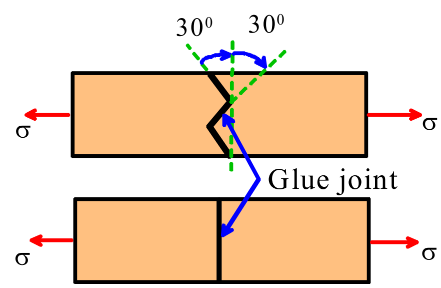
(b) Write down the formulas for hydrostatic stress and von-Mises stress in terms of the principal stresses. Hence, find the components of stress in the $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ basis (i.e. the principal basis). If you find more than one possible solution give them all…

(c) Hence, find the stress components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis

7. The figure shows two designs for a glue joint. The glue will fail if the stress acting normal to the joint exceeds 60 MPa, or if the shear stress acting parallel to the plane of the joint exceeds 300 Mpa.

(a) Calculate the normal and shear stress acting on each joint, in terms of the applied stress σ

(b) Hence, calculate the value of σ that will cause each joint to fail.

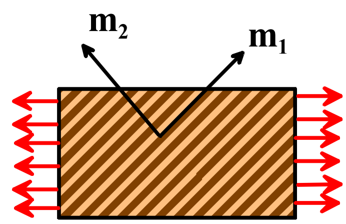


8. An internal surface plane that makes equal angles with each of the three principal stress directions is known as the *octahedral plane.* Show that the normal component of stress acting on this plane is $I_1/3$, and that the magnitude of the shear traction acting on the plane is

$$\frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2}=\sqrt{-2I_2'/3}$$

where $(\sigma_1, \sigma_2, \sigma_3)$ are the three principal stresses, and I_2' is the second invariant of the *deviatoric* stress tensor $S_{ij}=\sigma_{ij}-\sigma_{kk}\delta_{ij}/3$

9. The 'Tsai-Hill' criterion is used to predict the critical loads that cause failure in brittle laminated fiber-reinforced composites and wood. A specimen of laminated composite subjected to in-plane loading is sketched in the figure. The Tsai-Hill criterion assumes that a plane stress state exists in the solid. Let $\sigma_{11}, \sigma_{22}, \sigma_{12}$ denote the nonzero components of stress, with basis vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ oriented parallel and perpendicular to the fibers in the sheet, as shown. The Tsai-Hill failure criterion is



$$\left(\frac{\sigma_{11}}{\sigma_{TS1}}\right)^2 + \left(\frac{\sigma_{22}}{\sigma_{TS2}}\right)^2 - \frac{\sigma_{11}\sigma_{22}}{\sigma_{TS1}^2} + \frac{\sigma_{12}^2}{\sigma_{SS}^2} = 1$$

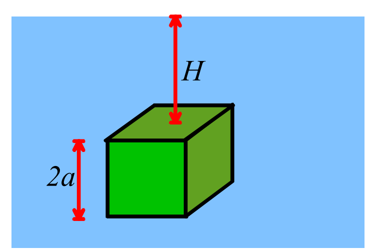
at failure, where σ_{TS1}, σ_{TS2} and σ_{SS} are material properties. (The material fails when the quantity on the left hand side of the equation is equal to 1) The table below gives data for the properties of a graphite-epoxy CFRP

Property	Value (MPa)
σ_{TS1}	2206
σ_{TS2}	56.5
σ_{SS}	110.3

Suppose that a specimen is loaded in uniaxial tension with the tensile axis at 45 degrees to the fiber direction, as shown in the figure. Calculate the maximum stress that the material can withstand (you will need to use the basis change formulas for a tensor).

10. Show that the hydrostatic stress σ_{kk} is invariant under a change of basis – i.e. if $\sigma_{ij}^{\mathbf{e}}$ and $\sigma_{ij}^{\mathbf{m}}$ denote the components of stress in bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$, respectively, show that $\sigma_{kk}^{\mathbf{e}} = \sigma_{kk}^{\mathbf{m}}$.

11. A rigid, cubic solid is immersed in a fluid with mass density ρ. Recall that a stationary fluid exerts a compressive pressure of magnitude $\rho g h$ at depth h.

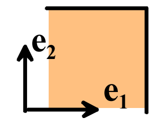


(a) Write down expressions for the traction vector exerted by the fluid on each face of the cube. You might find it convenient to take the origin for your coordinate system at the center of the cube, and take basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ perpendicular to the cube faces.

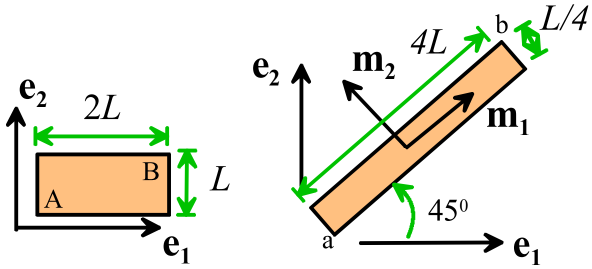
(b) Calculate the resultant force due to the tractions acting on the cube, and show that the vertical force is equal and opposite to the weight of fluid displaced by the cube.

(c) Show that the result of problem (b) applies to any arbitrarily shaped solid immersed below the surface of a fluid, i.e. prove that the resultant force acting on an immersed solid with volume V is $P_i = \rho g V \delta_{i3}$, where it is assumed that $\mathbf{e}_3$ is vertical.

12. A component contains a feature with a 90 degree corner as shown in the picture. The surfaces that meet at the corner are not subjected to any loading. List all the stress components that must be zero at the corner



13. The figure shows the reference and deformed configurations for a solid. The out-of-plane dimensions are unchanged. Points a and b are the positions of points A and B after deformation. The Cauchy stress in the solid is $\sigma \mathbf{m}_1 \otimes \mathbf{m}_1$



Determine:

(a) The components of Cauchy stress in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

(b) The components of Nominal stress **S** in both $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$.

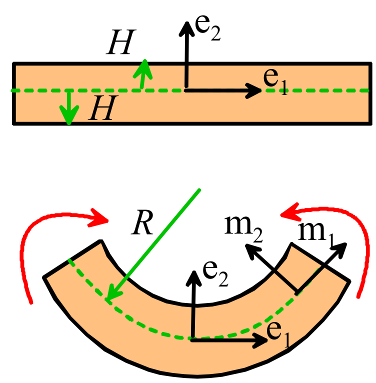
(c) The nominal stress in the mixed basis $S_{ij}\mathbf{e}_i \otimes \mathbf{m}_j$

(d) The components of material stress in both $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$

14. In this problem we consider further the beam bending calculation discussed in problem 2.2.34. Suppose that the beam is made from a material in which the Material Stress tensor is related to the Lagrange strain tensor by

$$\Sigma_{ij} = 2\mu E_{ij}$$

(this can be regarded as representing an elastic material with zero Poisson's ratio and shear modulus μ)



(a) Calculate the distribution of material stress in the bar, expressing your answer as components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis

(b) Calculate the distribution of nominal stress in the bar expressing your answer as components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis

(c) Calculate the distribution of Cauchy stress in the bar expressing your answer as components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis

(d) Repeat (a)-(c) but express the stresses as components in the $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ basis

(e) Calculate the distribution of traction on a surface in the beam that has normal $\mathbf{e}_1$ in the undeformed beam. Give expressions for the tractions in both $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$

(f) Show that the surfaces of the beam that have positions $x_2 = \pm h/2$ in the undeformed beam are traction free after deformation

(g) Calculate the resultant moment (per unit out of plane distance) acting on the ends of the beam.

15. A solid is subjected to some loading that induces a Cauchy stress $\sigma_{ij}^{(0)}$ at some point in the solid. The solid and the loading frame are then rotated together so that the entire solid (as well as the loading frame) is subjected to a rigid rotation R_{ij}. This causes the components of the Cauchy stress tensor to change to new values $\sigma_{ij}^{(1)}$. The goal of this problem is to calculate a formula relating $\sigma_{ij}^{(0)}$, $\sigma_{ij}^{(1)}$ and R_{ij}.
    (a) Let $n_i^{(0)}$ be a unit vector normal to an internal material plane in the solid before rotation. After rotation, this vector (which rotates with the solid) is $n_i^{(1)}$. Write down the formula relating $n_i^{(0)}$ and $n_i^{(1)}$
    (b) Let $T_i^{(0)}$ be the internal traction vector that acts on a material plane with normal $n_i^{(0)}$ in the solid before application of the rigid rotation. Let $T_i^{(1)}$ be the traction acting on the same material plane after rotation. Write down the formula relating $T_i^{(0)}$ and $T_i^{(1)}$
    (c) Finally, using the definition of Cauchy stress, find the relationship between $\sigma_{ij}^{(0)}$, $\sigma_{ij}^{(1)}$ and R_{ij}.

16. Repeat problem 5, but instead, calculate a relationship between the components of Nominal stress $S_{ij}^{(0)}$ and $S_{ij}^{(1)}$ before and after the rigid rotation.

17. Repeat problem 5, but instead, calculate a relationship between the components of material stress $\Sigma_{ij}^{(0)}$ and $\Sigma_{ij}^{(1)}$ before and after the rigid rotation.

18. Show that the von-Mises effective stress $\sigma_e = \sqrt{\frac{3}{2}\boldsymbol{\sigma}^{DEV}:\boldsymbol{\sigma}^{DEV}}$ where $\boldsymbol{\sigma}^{DEV} = \boldsymbol{\sigma} - trace(\boldsymbol{\sigma})\mathbf{I}/3$ is invariant under a change of basis.

19. One constitutive model for metallic glass (Anand and Su, J. Mech Phys. Solids **53** 1362 (2005)) assumes that plastic flow in the glass takes place by shearing on planes that are oriented at an angle θ to the principal stress directions, calculated as follows.
    - Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors parallel to the three principal stresses (with $\sigma_1 > \sigma_2 > \sigma_3$) and suppose that shearing takes place on a plane with normal $\mathbf{m}$, with shearing direction (tangent to the plane) $\mathbf{s}$.
    - Let $\tau = \mathbf{m}\cdot\boldsymbol{\sigma}\cdot\mathbf{s}$ and $p_n = -\mathbf{m}\cdot\boldsymbol{\sigma}\cdot\mathbf{m}$ denote the resolved shear stress and (compressive) normal stress acting on the shear plane.
    - The constitutive model assumes that shearing in the $\{\mathbf{e}_1, \mathbf{e}_3\}$ plane occurs on the plane for which
    $$f(\theta) = \tau(\theta) - \mu p(\theta)$$
    is a maximum with respect to θ. Here μ is a material property known as the 'internal friction coefficient.'
    Show that $f(\theta)$ is a maximum for
    $$\theta = \left\{\frac{\pi}{4} \pm \frac{\phi}{2}\right\} \qquad \phi = \tan^{-1}\mu$$

2.4 Equations of motion and equilibrium for deformable solids

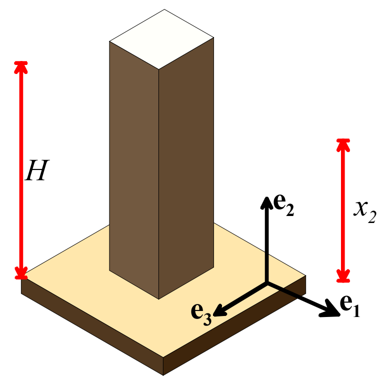
1. Show that the local mass balance equation

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} + \rho \frac{\partial v_i}{\partial y_i} = 0$$

can be re-written in spatial form as

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho v_i}{\partial y_i} = 0$$

2. A prismatic concrete column of mass density ρ supports its own weight, as shown below. (Assume that the solid is subjected to a uniform gravitational body force of magnitude g per unit mass).



(a) Show that the stress distribution

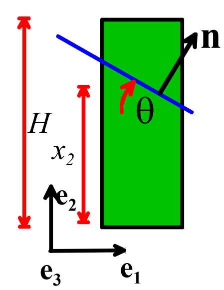
$$\sigma_{22} = -\rho g (H - x_2)$$

satisfies the equations of static equilibrium

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = 0$$

and also satisfies the boundary conditions $\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{0}$ on all free boundaries.

(b) Find a formula for the traction vector acting on a plane with normal $\mathbf{n} = \sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2$ at a height x_2



(c) Find the normal and tangential tractions acting on the plane with normal $\mathbf{n} = \sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2$ at a height x_2

(d) Suppose that the concrete contains a large number of randomly oriented microcracks. A crack which lies at an angle θ to the horizontal will propagate if

$$|\mathbf{T}_t| + \mu T_n > \tau$$

where μ is the friction coefficient between the faces of the crack and τ is a critical shear stress that is related to the size of the microcracks and the fracture toughness of the concrete.

Assume that $\mu = 1$. Find the orientation of the microcrack that is most likely to propagate. Hence, find an expression for the maximum possible height of the column.

3. Is the stress field given below in static equilibrium? If not, find the acceleration or body force density required to satisfy linear momentum balance

$$\sigma_{11} = C x_1 x_2 \qquad \sigma_{12} = \sigma_{21} = C(a^2 - x_2^2)$$
$$\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$$

4. Let f be a twice differentiable, scalar function of position. Derive a plane stress field from f by setting

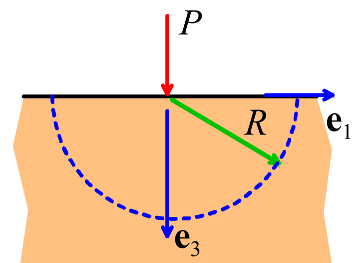
$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} \qquad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} \qquad \sigma_{12} = \sigma_{21} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$$

Show that this stress field satisfies the equations of stress equilibrium with zero body force.

5. The stress field

$$\sigma_{ij} = \frac{-3P_k x_k x_i x_j}{4\pi R^5} \qquad R = \sqrt{x_k x_k}$$

represents the stress in an infinite, incompressible elastic solid that is subjected to a point force with components P_k acting at the origin (you can visualize a point force as a very large body force which is concentrated in a very small region around the origin).



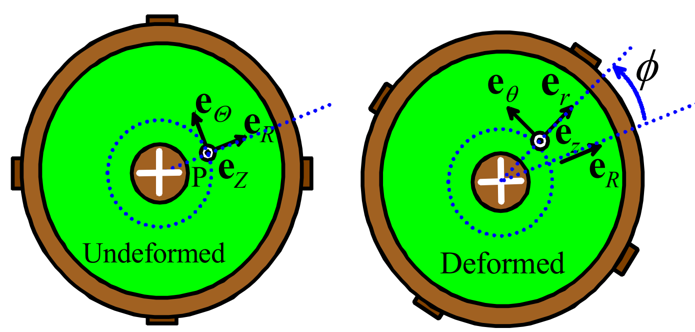
(a) Verify that the stress field is in static equilibrium

(b) Consider a spherical region of material centered at the origin. This region is subjected to (1) the body force acting at the origin; and (2) a force exerted by the stress field on the outer surface of the sphere. Calculate the resultant force exerted on the outer surface of the sphere by the stress, and show that it is equal in magnitude and opposite in direction to the body force.

6. In this problem, we consider the internal forces in the polymer specimen described in Problem 2.2.35 (you will need to solve 2.2.35 before you can attempt this one). Suppose that the specimen is homogeneous, has mass density ρ in the reference configuration, and may be idealized as a viscous fluid, in which the Kirchhoff stress is related to stretch rate by

$$\boldsymbol{\tau} = \mu \mathbf{D} + p\mathbf{I}$$

where p is an indeterminate hydrostatic pressure and μ is the viscosity.



(a) Find expressions for the Cauchy stress tensor as functions of R and t, expressing your answer as components in $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$

(b) Assuming quasi-static deformation (neglect accelerations), express the equations of equilibrium in terms of $\phi(R,t)$

(c) Solve the governing equation to calculate $\phi(r,t)$

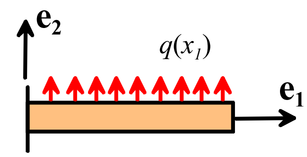
(d) Find the torque necessary to rotate the external cylinder

(e) Calculate the acceleration of a material particle in the fluid

(f) Estimate the rotation rate $\dot{\alpha}$ where inertia begins to play a significant role in determining the state of stress in the fluid

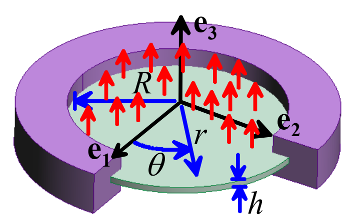
2.5 Work Done by Stresses; Principle of Virtual Work

1. The figure shows a cantilever beam that is subjected to surface loading $q(x_1)$ per unit length. The state of stress in the beam can be approximated by $\sigma_{11} = M(x_1)x_2 / I$, where $I = \int_A x_2^2 dA$ is the area moment of inertia of the beam's cross section and $M(x_1)$ is an arbitrary function (all other stress components are zero).

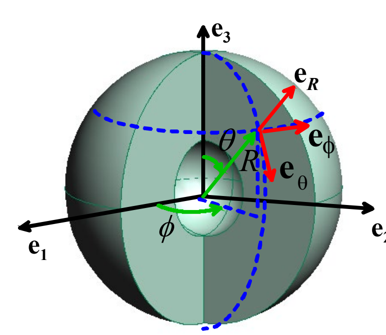


   (a) By considering a virtual velocity field of the form
$$\delta v_1 = -\frac{dw(x_1)}{dx_1}x_2 \qquad \delta v_2 = w(x_1)$$
   where $w(x_1)$ is an arbitrary function satisfying $w = 0$ at $x_1 = 0$, show that the beam is in static equilibrium if
$$\int_0^L M(x_1)\frac{d^2 w}{dx_1^2}dx_1 + \int_0^L q(x_1)w dx_1 = 0$$
   (b) By integrating the first integral expression by parts twice, show that the equilibrium equation and boundary conditions for $M(x_1)$ are
$$\frac{d^2 M}{dx_1^2} + q(x_1) = 0 \qquad M(x_1) = \frac{dM(x_1)}{dx_1} = 0 \qquad x_1 = L$$

2. The figure shows a plate with a clamped edge that is subjected to a pressure $p(x_1, x_2)$ on its surface. The state of stress in the plate can be approximated by
$$\sigma_{\alpha\beta} = M_{\alpha\beta}(x_1, x_2)x_3 / 3h^3 \qquad \sigma_{33} = \sigma_{3\alpha} = 0$$
where the subscripts α, β can have values 1 or 2, and $M_{\alpha\beta}(x_1, x_2)$ is a tensor valued function.



   (a) By considering a virtual velocity of the form
$$\delta v_\alpha = -\frac{\partial w}{\partial x_\alpha}x_3 \qquad \delta v_3 = w(x_1, x_2)$$
   where $w(x_1, x_2)$ is an arbitrary function satisfying $w = 0$ on the edge of the plate, show that the beam is in static equilibrium if
$$\int_A M_{\alpha\beta}(x_1)\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta}dA + \int_A p(x_1, x_2)w dA = 0$$
   (b) By applying the divergence theorem appropriately, show that the governing equation for $M_{\alpha\beta}(x_1, x_2)$ is
$$\frac{\partial^2 M_{\alpha\beta}}{\partial x_\alpha \partial x_\beta} + p = 0$$



3. The shell shown in the figure is subjected to a radial body force $\mathbf{b} = \rho b(R)\mathbf{e}_R$, and a radial pressure p_a, p_b acting on the surfaces at $R = a$ and $R = b$. The loading induces a spherically symmetric state of stress in the shell, which can be expressed in terms of its components in a spherical-polar coordinate system as $\sigma_{RR}\mathbf{e}_R \otimes \mathbf{e}_R + \sigma_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_{\phi\phi}\mathbf{e}_\phi \otimes \mathbf{e}_\phi$.

   (a) By considering a virtual velocity of the form $\delta\mathbf{v} = w(R)\mathbf{e}_R$, show that the stress state is in static equilibrium if

$$\int_a^b \left\{ \sigma_{RR}\frac{dw}{dR} + \left(\sigma_{\theta\theta} + \sigma_{\phi\phi}\right)\frac{w}{R} \right\} 4\pi R^2 dR - \int_a^b b(R)w(R)4\pi R^2 dR - 4\pi a^2 p_a w(a) + 4\pi b^2 p_b w(b) = 0$$

   for all *w(R)*.

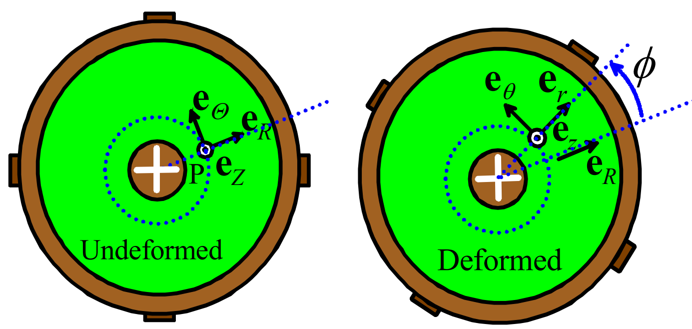
   (b) Hence, show that the stress state must satisfy

$$\frac{d\sigma_{RR}}{dR} + \frac{1}{R}\left(2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) + b = 0 \qquad \sigma_{RR} = -p_a \;\left(R = a\right) \qquad \sigma_{RR} = -p_b \;\left(R = b\right)$$

4. In this problem, we consider the internal dissipation in the polymer specimen described in Problem 2.2.35 and 2.4.6 (you will need to solve these two problems before you can attempt this one). Suppose that the specimen is homogeneous, has mass density ρ in the reference configuration, and may be idealized as a viscous fluid, in which the Kirchhoff stress is related to stretch rate by

$$\boldsymbol{\tau} = \mu\mathbf{D} + p\mathbf{I}$$

   where p is an indeterminate hydrostatic pressure and μ is the viscosity.



   (a) Calculate the rate of external work done by the torque acting on the rotating exterior cyclinder
   (b) Calculate the rate of internal dissipation in the solid as a function of *r*.
   (c) Show that the total internal dissipation is equal to the external work done on the specimen.

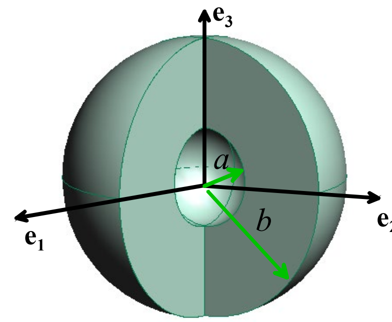
5. A solid with volume V is subjected to a distribution of traction t_i on its surface. Assume that the solid is in static equilibrium (this requires that t_i exerts no resultant force or moment on the boundary). By considering a virtual velocity of the form $\delta v_i = A_{ij}y_j$, where A_{ij} is a constant tensor, use the principle of virtual work to show that the average stress in a solid can be computed from the shape of the solid and the tractions acting on its surface using the expression

$$\frac{1}{V}\int_V \sigma_{ij}dV = \frac{1}{V}\int_S \frac{1}{2}\left(t_i y_j + t_j y_i\right)dA$$

6. A thick walled spherical shell is made from an incompressible linear viscous material, in which the Cauchy stress is related to the stretch rate **D** by

$$\boldsymbol{\sigma} = 2\mu\mathbf{D} + p\mathbf{I}$$

where p is a hydrostatic stress to be determined and μ is a material property (viscosity).



The solid is subjected to a radial gravitational body force

$$\rho\mathbf{b} = -\rho B_0 \frac{r-a}{b-a}\mathbf{e}_r$$

(a) Assume that the velocity field in the shell is radial $\mathbf{v} = v(r)\mathbf{e}_r$. Calculate the velocity gradient and stretch rate.

(b) Show that the incompressibility condition implies that

$$\frac{\partial v}{\partial r} + \frac{2}{r}v = 0$$

and hence find an expression for *v(r)* in terms of $v(a) = \dot{a}$.

(c) Hence, find an expression for the particle acceleration $\left.\frac{dv}{dt}\right|_{\mathbf{x}}$, in terms of r, $\dot{a}$ and $\ddot{a}$ Be careful with this – it is not just the partial time derivative of $v(r)$.

(d) Find an expression for the (total) rate of work done on the shell by gravity.

(e) Find an expression for the total kinetic energy of the shell, in terms of $\dot{a}$

(f) Calculate the time derivative of kinetic energy. Note that b is not constant.

(g) Find the total internal stress power, in terms of $\dot{a}$, μ

(h) Use the principle of virtual work to show that the stress state must satisfy

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) - \rho B_0 \frac{r-b}{a-b} = \rho \left.\frac{dv}{dt}\right|_{\mathbf{x}}$$

(i) Write down the boundary conditions for $p(r)$ at $r = a, b$.

(j) Hence, show that *a(t)* satisfies the differential equation

$$\frac{d^2a}{dt^2} = -B_0\frac{b}{2a} - \frac{da}{dt}\frac{4\mu}{\rho}\frac{a^2+b^2+ab}{a^2b^2} - \left(\frac{da}{dt}\right)^2 \frac{(b-a)(a^2+2ab+3b^2)}{2ab^3}$$

(k) Verify the result of (j) using energy methods

(l) Plot $a(t)$ with initial conditions $a = 1; b = 5; \dot{a} = 0$ for $B_0 = 1$ and $\mu/\rho = 0.1, 0.2, 0.4, 1.0, 2.0$ (you will need to solve the differential equation numerically).

2.6 The Laws of Thermodynamics for Deformable Solids

1 Define the *expended power* of external forces acting on a deformable solid (which could be a sub-volume within a larger body) by

$$W_{\exp} = \int_S \mathbf{t} \cdot \mathbf{v} + \int_R \rho \mathbf{b} \cdot \mathbf{v} - \frac{d}{dt} \int_R \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v}$$

Show that the expended power is zero for any rigid velocity field of the form

$$\mathbf{v}(\mathbf{y}, t) = \mathbf{v}_0(t) + \boldsymbol{\omega}(t) \times (\mathbf{y} - \mathbf{y}_0)$$

where $\mathbf{v}_0(t)$, $\boldsymbol{\omega}(t)$ are vector valued functions of time (but independent of position).

2. It is helpful to have versions of the first and second laws of thermodynamics in terms of quantities defined on the reference configuration. To this end, define :

- Deformation mapping $y_i(x_j)$
- Temperature θ
- Reference mass density ρ_0
- Specific internal energy ε
- Specific Helmholtz free ehergy ψ
- Specific entropy s
- External heat supply per unit deformed volume q
- Nominal stress and deformation gradient $\mathbf{S}$, $\mathbf{F}$
- Jacobian $J = \det(\mathbf{F})$
- Material stress and Lagrange strain rate $\boldsymbol{\Sigma}$, $\mathbf{E}$
- Referential heat flux $\mathbf{Q} = J\mathbf{F}^{-1}\mathbf{q}$

With these definitions, show the following identities:

$$\rho_0 \left. \frac{\partial \varepsilon}{\partial t} \right|_{\mathbf{x}=const} = S_{ij} \frac{dF_{ji}}{dt} - \frac{\partial Q_i}{\partial x_i} + Jq$$

$$\rho_0 \left. \frac{\partial \varepsilon}{\partial t} \right|_{\mathbf{x}=const} = \Sigma_{ij} \frac{dE_{ij}}{dt} - \frac{\partial Q_i}{\partial x_i} + Jq$$

$$S_{ij} \frac{dF_{ji}}{dt} - \frac{1}{\theta} Q_i \frac{\partial \theta}{\partial x_i} - \rho_0 \left(\frac{\partial \psi}{\partial t} + s \frac{\partial \theta}{\partial t} \right) \geq 0$$

$$\Sigma_{ij} \frac{dE_{ij}}{dt} - \frac{1}{\theta} Q_i \frac{\partial \theta}{\partial x_i} - \rho_0 \left(\frac{\partial \psi}{\partial t} + s \frac{\partial \theta}{\partial t} \right) \geq 0$$

2 Starting with the local form of the second law of thermodynamics and mass conservation

$$\rho \left. \frac{\partial s}{\partial t} \right|_{\mathbf{x}=const} + \frac{\partial (q_i / \theta)}{\partial y_i} - \frac{q}{\theta} \geq 0 \qquad \left. \frac{\partial \rho}{\partial t} \right|_{\mathbf{y}} + \frac{\partial \rho v_i}{\partial y_i} = 0$$

(the symbols have their usual meaning), derive the statement of the second law for a control volume

$$\frac{\partial}{\partial t} \int_R \rho s dV + \int_B \rho s (\mathbf{v} \cdot \mathbf{n}) dA + \int_B \frac{\mathbf{q} \cdot \mathbf{n}}{\theta} dA - \int_R \frac{q}{\theta} dV \geq 0$$

2.7 Transformation of kinematic and kinetic variables under changes of reference frame

1. Determine how the following quantities transform under a change of observer
   (a) The spatial heat flux vector $\mathbf{q}$ (recall that $\mathbf{q}\cdot\mathbf{n}dA$ gives the heat flux across a surface element with area dA and normal $\mathbf{n}$, and note that $\mathbf{n}$ is an objective vector, and that the all observers must see the same heat flux….)
   (b) The referential heat flux vector $\boldsymbol{\Theta} = J\mathbf{F}^{-1}\mathbf{q}$ (we normally use $\mathbf{Q}$ to denote referential heat flux, but in this section that symbol is already used to denote the rotation of the observer's frame).
   (c) The spatial gradient of a scalar function of position in a deformed solid $\mathbf{g} = \nabla_{\mathbf{y}}\phi(\mathbf{y})$
   (d) The material gradient of a function of particle position in a solid $\mathbf{G} = \nabla\phi(\mathbf{x})$

2. The expended power of external forces acting on a deformable solid was defined in Problem 2.6.1 as

$$W_{\exp}(\mathbf{t},\mathbf{b},\frac{d\mathbf{y}}{dt}) = \int_S \mathbf{t}\cdot\frac{d\mathbf{y}}{dt} + \int_R \rho\mathbf{b}\cdot\frac{d\mathbf{y}}{dt} - \frac{d}{dt}\int_R \frac{1}{2}\rho\frac{d\mathbf{y}}{dt}\cdot\frac{d\mathbf{y}}{dt}$$

Show that the external power is invariant to all changes of observer if and only if linear momentum and angular momentum are conserved

i.e.

$$W_{\exp}(\mathbf{t}^*,\mathbf{b}^*,\frac{d\mathbf{y}^*}{dt}) = W_{\exp}(\mathbf{t},\mathbf{b},\frac{d\mathbf{y}}{dt}) \qquad \forall \quad \mathbf{y}^* = \mathbf{y}_0^*(t) + \mathbf{Q}(t)(\mathbf{y}-\mathbf{y}_0)\ )$$

$$\Leftrightarrow \int_A \mathbf{t}dA + \int_R \rho\mathbf{b}dV = \frac{d}{dt}\int_R \rho\frac{d\mathbf{y}}{dt}dV \qquad \int_A \mathbf{y}\times\mathbf{t}dA + \int_R \mathbf{y}\times\rho\mathbf{b}dV = \frac{d}{dt}\int_R \mathbf{y}\times\rho\mathbf{v}dV$$

(Suggestion: It is best to re-write the kinetic energy term in terms of accelerations first. Then note that

$$\mathbf{Q}^T\frac{d\mathbf{y}_0^*}{dt} + \mathbf{Q}^T\frac{d\mathbf{Q}}{dt}(\mathbf{y}-\mathbf{y}_0)$$

Can be interpreted as the velocity field associated with a rigid body motion. You can use many of the steps from Problem 2.6.1)