Course:	QUANTUM COMPUTING		
Professor:	P. Cremonesi M. Ferrari Dacrema	**Date:**	05/09/2024
Last Name:		**Codice Persona:**	
First Name:		**Grade:**	

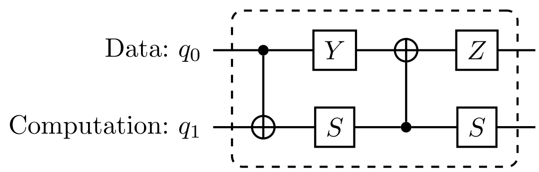
Exam Rules:

- Exam duration: 2 h
- Students must use a pen to write answers. Please do not use a pencil or write answers in red.
- Students are NOT allowed to use notes, books or any other materials as well as electronic devices.
- Students are NOT allowed to copy someone else's answers or let others copy from them, as well as use any electronic devices. Those in violation of this rule will receive a zero grade for the exam and will be excluded from the next exam call.
- Write your answers using these sheets.

1 Deutsch Algorithm (11 points)

1. Briefly state the definition of the problem that is solved by the Deutsch algorithm, and then by the Deutsch-Jozsa algorithm. Draw the quantum circuits used by both algorithms and state their speedup compared to a classical algorithm.

2. Consider a function implemented with the following quantum circuit. Apply the Deutsch algorithm to this function, write the full quantum circuit, show its outcome and state, based on that, whether the function is constant or balanced.



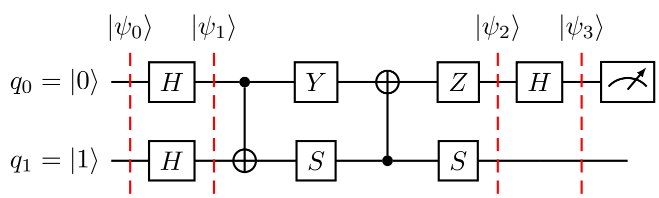
The S gate is as follows:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Important: Motivate your answer by showing all stages of the computation.

Solution

The full circuit for the Deutsch algorithm is as follows:



The outcome of the circuit is computed as follows:

$$|\psi_0\rangle = |0, 1\rangle$$
$$|\psi_1\rangle = (H \otimes H)\,|0, 1\rangle = |+, -\rangle$$

$$\begin{aligned}
|\psi_2\rangle &= (Z \otimes S)\, C_{X,q_1 q_0}\, (Y \otimes S)\, C_{X,q_0 q_1}\, |+, -\rangle \\
&= (Z \otimes S)\, C_{X,q_1 q_0}\, (Y \otimes S)\, C_{X,q_0 q_1}\, \frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right) \\
&= (Z \otimes S)\, C_{X,q_1 q_0}\, (Y \otimes S)\, \frac{1}{2}\left(|00\rangle - |01\rangle + |11\rangle - |10\rangle\right) \\
&= (Z \otimes S)\, C_{X,q_1 q_0}\, \frac{1}{2}\left(i\,|00\rangle + |01\rangle + |11\rangle + i\,|10\rangle\right) \\
&= (Z \otimes S)\, \frac{1}{2}\left(i\,|00\rangle + |11\rangle + |01\rangle + i\,|10\rangle\right) \\
&= \frac{1}{2}\left(i\,|00\rangle - i\,|11\rangle + i\,|01\rangle - i\,|10\rangle\right)
\end{aligned}$$

There is a global phase of i that can be removed

$$
\begin{aligned}
&= \frac{1}{2}\left(|00\rangle - |11\rangle + |01\rangle - |10\rangle\right) \\
&= |-,+\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_3\rangle &= (H \otimes I)\,|-,+\rangle \\
&= H\,|-\rangle \otimes |+\rangle \\
&= |1,+\rangle
\end{aligned}
$$

To assess whether the function is constant or balanced we have to measure the first qubit, q_0. In this case we see its state is $|1\rangle$ so the measurement will return 1 with 100% probability. The Deutsch algorithm returns 1, and therefore its conclusion is that the function is *balanced*.

2 Variational Quantum Algorithms (11 points)

- Describe the general idea behind Variational Quantum Algorithms, explain how they work, their advantages and disadvantages.
- Describe QAOA in detail and explain how to construct the ansatz with one or more layers.

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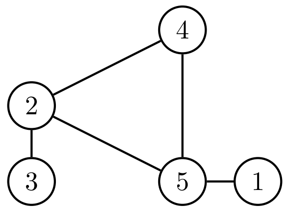
3 QUBO (10 points)

- Compute the QUBO formulation of the following SAT problem. Show its coefficient matrix in binary variables and spin variables, compute the Problem Hamiltonian and use it to identify which is the best solution.

$$\bar{x}_1 \vee \bar{x}_2$$

- Compute the QUBO formulation of the Max-Cut problem related to this graph. Show its coefficient matrix in binary variables and spin variables. Draw the circuit required to solve this problem with QAOA using 1 layer.



Important: Motivate your answer by showing all stages of the computation.

Solution 1

Objective function:

$$\min_x \quad E(x) = +x_1x_2$$

$$\min_s \quad E(s) = 0.25 + 0.25s_1s_2 - 0.25s_1 - 0.25s_2$$

Coefficient matrix:

$$Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} -0.25 & 0.25 \\ 0 & -0.25 \end{pmatrix}$$

Pauli σ_z applied on all qubits:

$$\text{diag}(\sigma_z^1) = (1, 1, -1, -1)$$

$$\text{diag}(\sigma_z^2) = (1, -1, 1, -1)$$

Problem Hamiltonian:

$$\text{diag}(H_{prob}) = (-0.25, -0.25, -0.25, 0.75)$$

Optimal Solution:
$\min(H_{prob})$: -0.25
Solution index: 0, 1, 2
Solution bitstring: 00, 01, 10
E(x)=E(s)=0.0

Solution 2

Objective function:

$$\min_x \quad E(x) = +2x_5x_1 - 2x_4 + 2x_2x_4 - x_3 + 2x_2x_3 + 2x_4x_5 + 2x_2x_5 - 3x_2 - 3x_5 - x_1$$

$$\min_s \quad E(s) = -2.5 + 0.5s_5s_4 + 0.5s_5s_1 + 0.5s_5s_2 + 0.5s_3s_2 + 0.5s_4s_2$$

Coefficient matrix:

$$Q = \begin{pmatrix} -1 & 0 & 0 & 0 & 2 \\ 0 & -3 & 2 & 2 & 2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

QAOA Circuit with 1 layer: