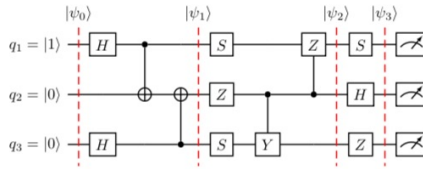2025_02.pdf

EX1. QUANTUM CIRCUIT

- Consider the following quantum circuit defined on 3 qubits. Compute the probability of measuring each possible state at the end of the circuit.

The S gate is as follows:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Important: Motivate your answer by showing all stages of the computation.

$|\psi_0\rangle = |1\rangle \otimes |0\rangle \otimes |0\rangle$

$|\psi_1\rangle = C_{x,q_3 q_2} \, C_{x,q_1 q_2} \, (H \otimes I \otimes H) \, |\psi_0\rangle$

$\quad = C_{x,q_3 q_2} \, C_{x,q_1 q_2} \, |-\rangle \otimes |0\rangle \otimes |+\rangle$

$\quad = C_{x,q_3 q_2} \, C_{x,q_1 q_2} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \otimes |0\rangle \otimes |+\rangle$

$\quad = C_{x,q_3 q_2} \, \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |0\rangle \otimes |+\rangle - |1\rangle \otimes |1\rangle \otimes |+\rangle \right)$

$\quad = C_{x,q_3 q_2} \, \frac{1}{2} \left(|0\rangle \otimes |0\rangle \otimes (|0\rangle + |1\rangle) - |1\rangle \otimes |1\rangle \otimes (|0\rangle + |1\rangle) \right)$

$\quad = \frac{1}{2} \left(|000\rangle + |011\rangle - |110\rangle - |101\rangle \right)$

$|\psi_2\rangle = C_{z,q_2 q_1} \, C_{y,q_3 q_1} \, (S \otimes Z \otimes S) \, |\psi_1\rangle$

$\quad = C_{z,q_2 q_1} \, C_{y,q_3 q_1} \, \frac{1}{2} \left(S|0\rangle \otimes Z|0\rangle \otimes S|0\rangle + S|0\rangle \otimes Z|1\rangle \otimes S|1\rangle - S|1\rangle \otimes Z|1\rangle \otimes S|0\rangle - S|1\rangle \otimes Z|0\rangle \otimes S|1\rangle \right)$

$\quad = C_{z,q_2 q_1} \, C_{y,q_3 q_1} \, \frac{1}{2} \left(|000\rangle + |0\rangle \otimes -|1\rangle \otimes i|1\rangle + i|1\rangle \otimes |1\rangle \otimes |0\rangle - i|1\rangle \otimes |0\rangle \otimes i|1\rangle \right)$

$\quad = C_{z,q_2 q_1} \, \frac{1}{2} \left(|000\rangle - i|01\rangle \otimes Y|1\rangle + i|11\rangle \otimes Y|0\rangle + |101\rangle \right)$

$\quad = C_{z,q_2 q_1} \, \frac{1}{2} \left(|000\rangle - |010\rangle - |111\rangle + |101\rangle \right)$

$\quad = \frac{1}{2} \left(|000\rangle - Z|0\rangle \otimes |10\rangle - Z|1\rangle \otimes |11\rangle + |101\rangle \right)$

$\quad = \frac{1}{2} \left(|000\rangle - |010\rangle + |111\rangle + |101\rangle \right)$

$|\psi_3\rangle = (S \otimes H \otimes Z) \, |\psi_2\rangle$

$\quad = \frac{1}{2} \left(S|0\rangle \otimes H|0\rangle \otimes Z|0\rangle - S|0\rangle \otimes H|1\rangle \otimes Z|0\rangle + S|1\rangle \otimes H|1\rangle \otimes Z|1\rangle + S|1\rangle \otimes H|0\rangle \otimes Z|1\rangle \right)$

$\quad = \frac{1}{2} \left(|0\rangle \otimes |+\rangle \otimes |0\rangle - |0\rangle \otimes |-\rangle \otimes |0\rangle + i|1\rangle \otimes |-\rangle \otimes -|1\rangle + i|1\rangle \otimes |+\rangle \otimes -|1\rangle \right)$

$\quad = \frac{1}{2\sqrt{2}} \left(|0\rangle \otimes (|0\rangle + |1\rangle) \otimes |0\rangle - |0\rangle \otimes (|0\rangle - |1\rangle) \otimes |0\rangle - i|1\rangle \otimes (|0\rangle - |1\rangle) \otimes |1\rangle - i|1\rangle \otimes (|0\rangle + |1\rangle) \otimes |1\rangle \right)$

$\quad = \frac{1}{2\sqrt{2}} \left(|000\rangle + |010\rangle - |000\rangle + |010\rangle - i|101\rangle + i|111\rangle - i|101\rangle - i|111\rangle \right)$

$\quad = \frac{1}{2\sqrt{2}} \left(2|010\rangle - 2i|101\rangle \right)$

$\quad = \frac{1}{\sqrt{2}} \left(|010\rangle - i|101\rangle \right)$

$\mathbb{P}(q_1=0, q_2=1, q_3=0) = \left| \frac{1}{\sqrt{2}} \right|^2 = 0,5 \qquad \mathbb{P}(q_1=1, q_2=0, q_3=1) = \left| -\frac{i}{\sqrt{2}} \right|^2 = 0,5 \qquad \mathbb{P}\left(\substack{\text{ALL THE OTHER} \\ \text{COMBINATIONS}} \right) = 0$

EX3 · QUBO

- Compute the QUBO formulation of the following SAT problem. Show its coefficient matrix in binary variables and spin variables, compute the Problem Hamiltonian and use it to identify which is the best solution.

$$\bar{x}_1 \vee x_2$$

- Compute the QUBO formulation of the Max-Cut problem related to this graph. Show its coefficient matrix in binary variables and spin variables. Draw the circuit required to solve this problem with QAOA using 1 layer.
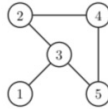
Important: Motivate your answer by showing all stages of the computation.

• SAT   PROBLEM

$$\min \quad E(x) = x_1 - x_1 x_2$$

$$Q = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \, C = 0 \qquad S = \frac{1}{4}\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \, \tilde{C} = 0,25$$

$$h = (-0,25, \, 0,25) \qquad J_{12} = -0,25$$

$$\text{DIAG}\left(\sigma_z^{(1)}\right) = (1, 1, -1, -1) \qquad \text{DIAG}\left(\sigma_z^{(1)}\sigma_z^{(2)}\right) = (1, -1, -1, 1)$$

$$\text{DIAG}\left(\sigma_z^{(2)}\right) = (1, -1, 1, -1)$$

$$\text{DIAG}\left(H_{PROB}\right) = -0,25 \, (1, 1, -1, -1) + 0,25 \, (1, -1, 1, -1) - 0,25 \, (1, -1, -1, 1)$$

$$= (-0,25, -0,25, 0,75, -0,25)$$

OPTIMAL   SOLUTION = -0,25

SOLUTION   INDEX = 0, 1, 3

SOLUTION   BITSTRING = 00, 01, 11

$$E(x) = E(s) = -0,25 + 0,25 = 0$$

- MAX-CUT PROBLEM

min $\quad -(x_1+x_3-2x_1x_3)-(x_2+x_3-2x_2x_3)-(x_2+x_4-2x_2x_4)-(x_3+x_5-2x_3x_5)-(x_4+x_5-2x_4x_5)$

min $\quad -x_1-x_3+2x_1x_3-x_2-x_3+2x_2x_3-x_2-x_4+2x_2x_4-x_3-x_5+2x_3x_5-x_4-x_5+2x_4x_5$

min $\quad E(x) = -x_1-2x_2-3x_3-2x_4-2x_5+2x_1x_3+2x_2x_3+2x_2x_4+2x_3x_5+2x_4x_5$

$$Q = \begin{bmatrix} -1 & 0 & 2 & 0 & 0 \\ 0 & -2 & 2 & 2 & 0 \\ 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \qquad \tilde{Q} = S = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$