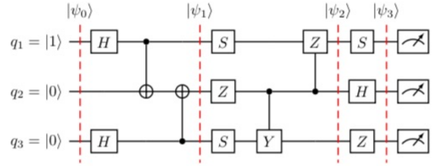
2025_02.pdf

EX1. QUANTUM CIRCUIT

- Consider the following quantum circuit defined on 3 qubits. Compute the probability of measuring each possible state at the end of the circuit.



The S gate is as follows:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Important: Motivate your answer by showing all stages of the computation.

$|\psi_0\rangle = |1\rangle \otimes |0\rangle \otimes |0\rangle$

$|\psi_1\rangle = C_{X,q_3q_2}\, C_{X,q_1q_2}\, (H \otimes I \otimes H)\, |\psi_0\rangle$

$= C_{X,q_3q_2}\, C_{X,q_1q_2}\, |-\rangle \otimes |0\rangle \otimes |+\rangle$

$= C_{X,q_3q_2}\, C_{X,q_1q_2} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \otimes |0\rangle \otimes |+\rangle$

$= C_{X,q_3q_2} \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |0\rangle \otimes |+\rangle - |1\rangle \otimes |1\rangle \otimes |+\rangle\right)$

$= C_{X,q_3q_2} \frac{1}{2} \left(|0\rangle \otimes |0\rangle \otimes (|0\rangle + |1\rangle) - |1\rangle \otimes |1\rangle \otimes (|0\rangle + |1\rangle)\right)$

$= \frac{1}{2} \left(|000\rangle + |011\rangle - |110\rangle - |101\rangle\right)$

$|\psi_2\rangle = C_{Z,q_2q_1}\, C_{Y,q_2q_3}\, (S \otimes Z \otimes S)\, |\psi_1\rangle$

$= C_{Z,q_2q_1}\, C_{Y,q_2q_3} \frac{1}{2} \left(S|0\rangle \otimes Z|0\rangle \otimes S|0\rangle + S|0\rangle \otimes Z|1\rangle \otimes S|1\rangle - S|1\rangle \otimes Z|1\rangle \otimes S|0\rangle - S|1\rangle \otimes Z|0\rangle \otimes S|1\rangle\right)$

$= C_{Z,q_2q_1}\, C_{Y,q_2q_3} \frac{1}{2} \left(|000\rangle + |0\rangle \otimes -|1\rangle \otimes i|1\rangle + i|1\rangle \otimes |1\rangle \otimes |0\rangle - i|1\rangle \otimes |0\rangle \otimes i|1\rangle\right)$

$= C_{Z,q_2q_1} \frac{1}{2} \left(|000\rangle - i|01\rangle \otimes Y|1\rangle + i|11\rangle \otimes Y|0\rangle + |101\rangle\right)$

$= C_{Z,q_2q_1} \frac{1}{2} \left(|000\rangle - |010\rangle - |111\rangle + |101\rangle\right)$

$= \frac{1}{2} \left(|000\rangle - Z|0\rangle \otimes |10\rangle - Z|1\rangle \otimes |11\rangle + |101\rangle\right)$

$= \frac{1}{2} \left(|000\rangle - |010\rangle + |111\rangle + |101\rangle\right)$

$|\psi_3\rangle = (S \otimes H \otimes Z)\, |\psi_2\rangle$

$= \frac{1}{2} \left(S|0\rangle \otimes H|0\rangle \otimes Z|0\rangle - S|0\rangle \otimes H|1\rangle \otimes Z|0\rangle + S|1\rangle \otimes H|1\rangle \otimes Z|1\rangle + S|1\rangle \otimes H|0\rangle \otimes Z|1\rangle\right)$

$= \frac{1}{2} \left(|0\rangle \otimes |+\rangle \otimes |0\rangle - |0\rangle \otimes |-\rangle \otimes |0\rangle + i|1\rangle \otimes |-\rangle \otimes -|1\rangle + i|1\rangle \otimes |+\rangle \otimes -|1\rangle\right)$

$= \frac{1}{2\sqrt{2}} \left(|0\rangle \otimes (|0\rangle + |1\rangle) \otimes |0\rangle - |0\rangle \otimes (|0\rangle - |1\rangle) \otimes |0\rangle - i|1\rangle \otimes (|0\rangle - |1\rangle) \otimes |1\rangle - i|1\rangle \otimes (|0\rangle + |1\rangle) \otimes |1\rangle\right)$

$= \frac{1}{2\sqrt{2}} \left(|0\cancel{00}\rangle + |010\rangle - |0\cancel{00}\rangle + |010\rangle - i|101\rangle + i|1\cancel{11}\rangle - i|101\rangle - i|11\cancel{1}\rangle\right)$

$= \frac{1}{2\sqrt{2}} \left(2|010\rangle - 2i|101\rangle\right)$

$= \frac{1}{\sqrt{2}} \left(|010\rangle - i|101\rangle\right)$

$\mathbb{P}(q_1 = 0, q_2 = 1, q_3 = 0) = \left|\frac{1}{\sqrt{2}}\right|^2 = 0{,}5$ $\quad$ $\mathbb{P}(q_1 = 1, q_2 = 0, q_3 = 1) = \left|-\frac{i}{\sqrt{2}}\right|^2 = 0{,}5$ $\quad$ $\mathbb{P}(\text{ALL THE OTHER COMBINATIONS}) = 0$

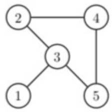
EX2 · THEORY

EX3 · QUBO

- Compute the QUBO formulation of the following SAT problem. Show its coefficient matrix in binary variables and spin variables, compute the Problem Hamiltonian and use it to identify which is the best solution.

$$\bar{x}_1 \vee x_2$$

- Compute the QUBO formulation of the Max-Cut problem related to this graph. Show its coefficient matrix in binary variables and spin variables. Draw the circuit required to solve this problem with QAOA using 1 layer.



Important: Motivate your answer by showing all stages of the computation.

- SAT PROBLEM

$\min \quad E(x) = x_1 x_2$

$\min \quad E(s) = \frac{1}{4} s_1 s_2 - \frac{1}{4} s_1 - \frac{1}{4} s_2 + \frac{1}{4}$

$\left[x_i = -\frac{1}{2} s_i + \frac{1}{2} \;,\; x_i x_j = \frac{1}{4} s_i s_j - \frac{1}{4} s_i - \frac{1}{4} s_j + \frac{1}{4} \right]$

$Q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, c = 0 \qquad S = \tilde{Q} = \begin{bmatrix} -0{,}25 & 0{,}25 \\ 0 & -0{,}25 \end{bmatrix}, c = 0{,}25$

$H_c = c + \sum_i h_i s_i + \sum_{i<j} J_{ij} s_i s_j$

$c = 0{,}25 \qquad h_1 = h_2 = -0{,}25 \qquad J_{12} = 0{,}25$

COEFFICIENTS ARE OBTAINED FROM $E(s)$

$H_c = \sum_i h_i \sigma_z^{(i)} + \sum_{i<j} J_{ij} \sigma_z^{(i)} \sigma_z^{(j)}$

$\{ \text{DIAG}(\sigma_z^{(1)}) = Z \otimes I = (1, 1, -1, -1) \qquad \text{DIAG}(\sigma_z^{(2)}) = I \otimes Z = (1, -1, 1, -1) \qquad \text{DIAG}(\sigma_z^{(1)} \sigma_z^{(2)}) = \text{DIAG}(\sigma_z^{(1)}) \cdot \text{DIAG}(\sigma_z^{(2)}) = (1, -1, -1, 1) \}$

$= \underbrace{-0{,}25}_{h_1} \sigma_z^{(1)} \underbrace{-0{,}25}_{h_2} \sigma_z^{(2)} + \underbrace{0{,}25}_{J_{12}} \sigma_z^{(1)} \sigma_z^{(2)}$

$= \begin{bmatrix} -0{,}25 & 0 & 0 & 0 \\ 0 & -0{,}25 & 0 & 0 \\ 0 & 0 & 0{,}25 & 0 \\ 0 & 0 & 0 & 0{,}25 \end{bmatrix} + \begin{bmatrix} -0{,}25 & 0 & 0 & 0 \\ 0 & 0{,}25 & 0 & 0 \\ 0 & 0 & -0{,}25 & 0 \\ 0 & 0 & 0 & 0{,}25 \end{bmatrix} + \begin{bmatrix} 0{,}25 & 0 & 0 & 0 \\ 0 & -0{,}25 & 0 & 0 \\ 0 & 0 & -0{,}25 & 0 \\ 0 & 0 & 0 & 0{,}25 \end{bmatrix}$

$= \text{DIAG}(-0{,}25, -0{,}25, -0{,}25, 0{,}75)$

BEST SOLUTION: $\min(\overset{H_c}{\overset{\shortparallel}{H_{PROB}}}) = -0{,}25$

SOLUTION INDEX: $0, 1, 2$

SOLUTION BITSTRING: $00, 01, 10$

$E(x) = E(s) = 0$

- MAX-CUT PROBLEM

$$\min \quad -(x_1+x_3-2x_1x_3)-(x_2+x_3-2x_2x_3)-(x_2+x_4-2x_2x_4)-(x_3+x_5-2x_3x_5)-(x_4+x_5-2x_4x_5)$$

$$\min \quad -x_1-x_3+2x_1x_3-x_2-x_3+2x_2x_3-x_2-x_4+2x_2x_4-x_3-x_5+2x_3x_5$$

$$\min \quad E(x) = -x_1-2x_2-3x_3-2x_4-2x_5+2x_1x_3+2x_2x_3+2x_2x_4+2x_3x_5$$

$$\min \quad -\left(-\tfrac{1}{2}s_1+\tfrac{1}{2}\right)-2\left(-\tfrac{1}{2}s_2+\tfrac{1}{2}\right)-3\left(-\tfrac{1}{2}s_3+\tfrac{1}{2}\right)-2\left(-\tfrac{1}{2}s_4+\tfrac{1}{2}\right)-2\left(-\tfrac{1}{2}s_5+\tfrac{1}{2}\right)$$
$$+2\left(\tfrac{1}{4}s_1s_3-\tfrac{1}{4}s_1-\tfrac{1}{4}s_3+\tfrac{1}{4}\right)+2\left(\tfrac{1}{4}s_2s_3-\tfrac{1}{4}s_2-\tfrac{1}{4}s_3+\tfrac{1}{4}\right)+2\left(\tfrac{1}{4}s_2s_4-\tfrac{1}{4}s_2-\tfrac{1}{4}s_4+\tfrac{1}{4}\right)+2\left(\tfrac{1}{4}s_3s_5-\tfrac{1}{4}s_3-\tfrac{1}{4}s_5+\tfrac{1}{4}\right)$$

$$\min \quad E(s) = -2+\tfrac{1}{2}s_4+\tfrac{1}{2}s_5+\tfrac{1}{2}s_1s_3+\tfrac{1}{2}s_2s_3+\tfrac{1}{2}s_2s_4+\tfrac{1}{2}s_3s_5$$

$$Q = \begin{bmatrix} -1 & 0 & 2 & 0 & 0 \\ 0 & -2 & 2 & 2 & 0 \\ 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \qquad \tilde{Q} = S = \begin{bmatrix} 0 & 0 & 0{,}5 & 0 & 0 \\ 0 & 0 & 0{,}5 & 0{,}5 & 0 \\ 0 & 0 & 0 & 0 & 0{,}5 \\ 0 & 0 & 0 & 0{,}5 & 0 \\ 0 & 0 & 0 & 0 & 0{,}5 \end{bmatrix}$$