Course:	QUANTUM COMPUTING		
Professor:	P. Cremonesi M. Ferrari Dacrema	**Date:**	Examples
Last Name:		**Codice Persona:**	
First Name:		**Grade:**	

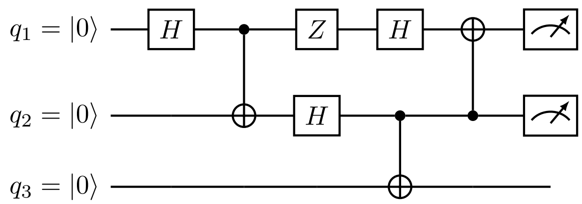
Exam Rules:

- Exam duration: TBD
- Students must use a pen to write answers. Please do not use a pencil or write answers in red.
- Students are NOT allowed to copy someone else's answers or let others copy from them, as well as use any electronic devices. Those in violation of this rule will receive a zero grade for the exam and will be excluded from the next exam call.
- Write your answers using these sheets.

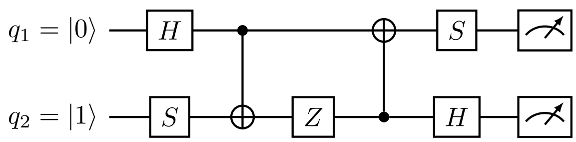
Note: This document contains a few examples of the type of questions that may occur in the exam.

1 Quantum Circuits

1. Consider the following quantum circuit defined on 3 qubits. Compute the probability of measuring state 0 for q_1 and the probability of measuring state 1 for q_2.



2. Consider the following quantum circuit defined on 2 qubits. Compute the probability of measuring each possible state at the end of the circuit. Remember that gate S corresponds to a 90° rotation around the Z axis.



Important: Motivate your answer by showing all stages of the computation.

Question 1

The quantum states of the circuit are as follows:

$$|\psi_{final}\rangle = (C_{X,q_2q_1} \otimes I)\,(H \otimes C_{X,q_2q_3})\,(Z \otimes H \otimes I)\,(C_{X,q_1q_2} \otimes I)\,(H \otimes I \otimes I)\,|000\rangle$$

$$\begin{aligned}
|\psi_1\rangle &= (H \otimes I \otimes I)\,|000\rangle = \frac{1}{\sqrt{2}}\,(|0\rangle + |1\rangle) \otimes |00\rangle \\
|\psi_2\rangle &= (C_{X,q_1q_2} \otimes I)\,|\psi_1\rangle = (C_{X,q_1q_2} \otimes I)\,\frac{1}{\sqrt{2}}\,(|00\rangle + |10\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}\,(|00\rangle + |1\rangle \otimes X\,|0\rangle) \otimes I\,|0\rangle \\
&= \frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle) \otimes |0\rangle \\
|\psi_3\rangle &= (Z \otimes H \otimes I)\,|\psi_2\rangle = \frac{1}{\sqrt{2}}\,(Z\,|0\rangle \otimes H\,|0\rangle + Z\,|1\rangle \otimes H\,|1\rangle) \otimes I\,|0\rangle \\
&= \frac{1}{2}\,(|0\rangle \otimes (|0\rangle + |1\rangle) - |1\rangle \otimes (|0\rangle - |1\rangle)) \otimes |0\rangle \\
&= \frac{1}{2}\,(|00\rangle + |01\rangle - |10\rangle + |11\rangle) \otimes |0\rangle \\
|\psi_4\rangle &= (H \otimes C_{X,q_2q_3})\,|\psi_3\rangle = (H \otimes C_{X,q_2q_3})\,\frac{1}{2}\,(|0\rangle \otimes (|00\rangle + |10\rangle) - |1\rangle \otimes (|00\rangle - |10\rangle)) \\
&= \frac{1}{2}\,(H\,|0\rangle \otimes (|00\rangle + |1\rangle \otimes X\,|0\rangle) - H\,|1\rangle \otimes (|00\rangle - |1\rangle \otimes X\,|0\rangle)) \\
&= \frac{1}{2\sqrt{2}}\,((|0\rangle + |1\rangle) \otimes (|00\rangle + |11\rangle) - (|0\rangle - |1\rangle) \otimes (|00\rangle - |11\rangle)) \\
&= \frac{1}{2\sqrt{2}}\,(\cancel{|000\rangle} + |011\rangle + |100\rangle + \cancel{|111\rangle} - \cancel{|000\rangle} + |011\rangle + |100\rangle - \cancel{|111\rangle}) = \frac{1}{\sqrt{2}}\,(|011\rangle + |100\rangle) \\
|\psi_5\rangle &= (C_{X,q_2q_1} \otimes I)\,|\psi_4\rangle = \frac{1}{\sqrt{2}}\,(X\,|0\rangle \otimes |11\rangle + |100\rangle) = \frac{1}{\sqrt{2}}\,(|111\rangle + |100\rangle)
\end{aligned}$$

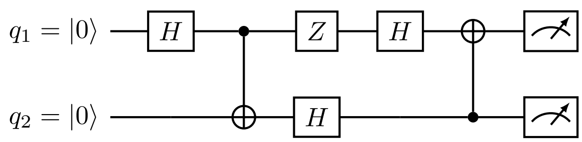
The probability of measuring state 0 for q_1 is:

$$P(q_1 = 0) = 0$$

The probability of measuring state 1 for q_2 is:

$$P(q_2 = 1) = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^{\dagger} = \frac{1}{2}$$

It should be noted that among the operations performed in the circuit there is C_{X,q_2q_3} which will change the state of q_3. Since q_3 is never actually used to control anything nor it is measured we can simplify the circuit by removing q_3 and the operations performed on it. The simplified circuit and the corresponding states are as follows:



$$|\psi_{final}\rangle = C_{X,q_2q_1}\,(H \otimes I)\,(Z \otimes H)\,C_{X,q_1q_2}\,(H \otimes I)\,|00\rangle$$

$$\begin{aligned}
|\psi_1\rangle &= (H \otimes I)\,|00\rangle = \frac{1}{\sqrt{2}}\,(|0\rangle + |1\rangle) \otimes |0\rangle \\
|\psi_2\rangle &= C_{X,q_1q_2}\,|\psi_1\rangle = C_{X,q_1q_2}\frac{1}{\sqrt{2}}\,(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\,(|00\rangle + |1\rangle \otimes X\,|0\rangle) = \frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle) \\
|\psi_3\rangle &= (Z \otimes H)\,|\psi_2\rangle = \frac{1}{\sqrt{2}}\,(Z\,|0\rangle \otimes H\,|0\rangle + Z\,|1\rangle \otimes H\,|1\rangle) \\
&= \frac{1}{2}\,(|0\rangle \otimes (|0\rangle + |1\rangle) - |1\rangle \otimes (|0\rangle - |1\rangle)) \\
|\psi_4\rangle &= (H \otimes I)\,|\psi_3\rangle = \frac{1}{2}\,(H\,|0\rangle \otimes (|0\rangle + |1\rangle) - H\,|1\rangle \otimes (|0\rangle - |1\rangle)) \\
&= \frac{1}{2\sqrt{2}}\,((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) - (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle)) \\
&= \frac{1}{2\sqrt{2}}\,(\cancel{|00\rangle} + |01\rangle + |10\rangle + \cancel{|11\rangle} - \cancel{|00\rangle} + |01\rangle + |10\rangle - \cancel{|11\rangle}) = \frac{1}{\sqrt{2}}\,(|01\rangle + |10\rangle) \\
|\psi_5\rangle &= C_{X,q_2q_1}\,|\psi_4\rangle = \frac{1}{\sqrt{2}}\,(X\,|0\rangle \otimes |1\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\,(|11\rangle + |10\rangle)
\end{aligned}$$

The probability of measuring state 0 for q_1 is:

$$P(q_1 = 0) = 0$$

The probability of measuring state 1 for q_2 is:

$$P(q_2 = 1) = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^{\dagger} = \frac{1}{2}$$

Question 2

The quantum states of the circuit are as follows:

$$|\psi_{final}\rangle = (S \otimes H)\, C_{X,q_2q_1}\, (I \otimes Z)\, C_{X,q_1q_2}\, (H \otimes S)\, |01\rangle$$

$$|\psi_1\rangle = (H \otimes S)\, |01\rangle = H\, |0\rangle \otimes S\, |1\rangle = \frac{1}{\sqrt{2}}\, (|0\rangle + |1\rangle) \otimes i\, |1\rangle = \frac{1}{\sqrt{2}}\, (|0\rangle + |1\rangle) \otimes |1\rangle$$

Notice how the S gate only creates a global phase i that can be removed.

$$|\psi_2\rangle = C_{X,q_1q_2}\, |\psi_1\rangle = C_{X,q_1q_2} \frac{1}{\sqrt{2}}\, (|01\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\, (|01\rangle + |1\rangle \otimes X\, |1\rangle) = \frac{1}{\sqrt{2}}\, (|01\rangle + |10\rangle)$$

$$|\psi_3\rangle = (I \otimes Z)\, |\psi_2\rangle = \frac{1}{\sqrt{2}}\, (I\, |0\rangle \otimes Z\, |1\rangle + I\, |1\rangle \otimes Z\, |0\rangle) = \frac{1}{\sqrt{2}}\, (|0\rangle \otimes (-\, |1\rangle) + |10\rangle) = \frac{1}{\sqrt{2}}\, (-\, |01\rangle + |10\rangle)$$

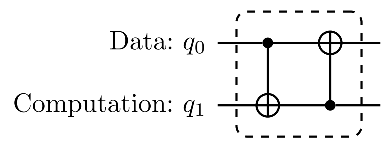
$$|\psi_4\rangle = C_{X,q_2q_1}\, |\psi_3\rangle = \frac{1}{\sqrt{2}}\, (-X\, |0\rangle \otimes |1\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\, (-\, |11\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\, |1\rangle \otimes (|0\rangle - |1\rangle)$$

$$|\psi_5\rangle = (S \otimes H)\, |\psi_4\rangle = \frac{1}{\sqrt{2}} S\, |1\rangle \otimes (H\, |0\rangle - H\, |1\rangle) = i\frac{1}{2}\, |1\rangle \otimes \left(\cancel{|0\rangle} + |1\rangle - \cancel{|0\rangle} + |1\rangle\right) = i\, |11\rangle = |11\rangle$$

The only state with a non-zero amplitude is $|11\rangle$ and therefore this is the only state that will be measured at the end of the computation.

2 Deutsch Algorithm

1. Briefly state the definition of the problem that is solved by the Deutsch algorithm, and then by the Deutsch-Jozsa algorithm. Draw the quantum circuits used by both algorithms and state their speedup compared to a classical algorithm.
2. Write a full description of the Deutsch algorithm, list the existing oracle functions, draw their corresponding quantum circuits and show the outcome of the algorithm for each of them.
3. Write a full description of the Deutsch-Jozsa algorithm, provide a formal definition of the problem it solves and show the outcome of the algorithm in the general case.
4. Consider a function implemented with the following quantum circuit. Apply the Deutsch algorithm to this function, show its outcome and state, based on that, whether the function is constant or balanced.



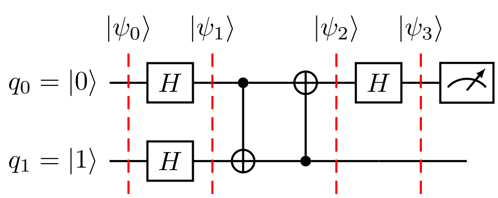
Important: Motivate your answer by showing all stages of the computation.

Question 1,2,3

We refer to the lecture notes. Note that in Question 1 we expect a concise answer with the problem statement, the circuits and a comment on their outcome in particular how to conclude if the function is constant or balanced; we *do not expect* a proof of the circuit outcome or an extended in-depth discussion. In Questions 2 and 3 we instead expect a full description and proof.

Question 4

The full circuit for the Deutsch algorithm is as follows:



Note that the function does not follow the formalism $U_f |x, y\rangle = |x, y \oplus f(x)\rangle$ and it does not meet the formal definition of the functions the Deutsch algorithm expects, however it is still a valid quantum circuit and therefore we can use it in a quantum computation. The outcome of the circuit is computed as follows:

$$
\begin{aligned}
|\psi_0\rangle &= |0,1\rangle \\
|\psi_1\rangle &= (H \otimes H)\,|0,1\rangle = |+,-\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_2\rangle &= C_{X,q_1q_0} C_{X,q_0q_1}\,|+,-\rangle \\
&= \frac{1}{2} C_{X,q_1q_0} C_{X,q_0q_1}\,(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \\
&= \frac{1}{2} C_{X,q_1q_0}\,(|00\rangle - |01\rangle + |11\rangle - |10\rangle) \\
&= \frac{1}{2}\,|00\rangle - |11\rangle + |01\rangle - |10\rangle \\
&= \frac{1}{2}\,|0\rangle \otimes (|0\rangle + |1\rangle) - |1\rangle \otimes (|0\rangle + |1\rangle) \\
&= \frac{1}{2}\,(|0\rangle - |1\rangle) \otimes (|0\rangle + |1\rangle)
\end{aligned}
$$

$$
\begin{aligned}
|\psi_3\rangle &= (H \otimes I)\,\frac{1}{2}\,(|0\rangle - |1\rangle) \otimes (|0\rangle + |1\rangle) \\
&= \frac{1}{2\sqrt{2}}\,(|0\rangle + |1\rangle - |0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \\
&= \frac{1}{2} 2\,|1\rangle \otimes \frac{1}{\sqrt{2}}\,(|0\rangle + |1\rangle) \\
&= |1,+\rangle
\end{aligned}
$$

To assess whether the function is constant or balanced we have to measure the first qubit, q_0. In this case we see its state is $|1\rangle$ so the measurement will return 1 with 100% probability. The Deutsch algorithm returns 1, and therefore its conclusion is that the function is *balanced*.

3 Simon's Algorithm

1. State the problem that is solved by the Simon's algorithm. Draw its quantum circuit and state its speedup compared to a classical algorithm.
2. Write the full description of the Simon's algorithm, showing its result for a generic function defined over n qubits and of period a.
3. Consider a function $f(\{0,1\}^2) \to \{0,1\}^2$ such that $f(00) = f(01), f(10) = f(11)$. Use the Simon's algorithm to compute its period.

Important: Motivate your answer by showing all stages of the computation.

Question 1,2

We refer to the lecture notes. Note that in Question 1 we expect a concise answer with the problem statement, the circuits and a comment on how to use their outcome; we *do not expect* a proof of the circuit outcome or an extended in-depth discussion. In Question 2 we expect the full proof of the algorithm in the general case.

Question 3

We refer to the lecture notes for the full solution.

4 Grover's Algorithm

1. State the problem that is solved by the Grover's algorithm. Draw its quantum circuit and state its speedup compared to a classical algorithm.
2. Write the full description of the Grover's algorithm, provide an example with $n = 2$ (alternatively a longer question might ask for an example with $n = 3$), show how the amplitudes change and compute the probability of reading the correct state once the algorithm converges.
3. Apply the Grover's algorithm on function $f(x)$ defined over $n = 3$ bits with $x_0 = 101$, show how the amplitudes change and compute the probability of reading the correct state once the algorithm converges.

Important: Motivate your answer by showing all stages of the computation.

Question 1

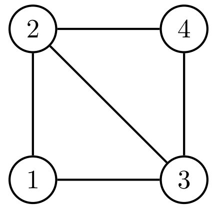
We refer to the lecture notes.

Question 2,3

We refer to the lecture notes. Note that in Question 2 we expect that you to define function $f(x)$ so that it is clear which is the state you are looking for, i.e., x_0. For both questions we expect that you then show how the amplitudes of the superposition change at each iteration, as done in the lecture notes, as well as state when convergence is reached.

5 QUBO

1. Compute the QUBO formulation of the Max-Cut problem related to this graph. Show its coefficient matrix in binary variables and spin variables. Draw the circuit required to solve this problem with QAOA using 1 layer.



2. Compute the QUBO formulation of the following SAT problem. Show its coefficient matrix in binary variables and spin variables. Draw the circuit required to solve this problem with QAOA using 1 layer.

$x_1 \vee x_2$	$x_2 \vee x_4$
$x_3 \vee x_4$	$\bar{x}_1 \vee x_2$
$\bar{x}_1 \vee x_3$	$\bar{x}_2 \vee x_3$
$x_1 \vee \bar{x}_2$	$x_2 \vee \bar{x}_3$
$\bar{x}_1 \vee \bar{x}_2$	$\bar{x}_1 \vee \bar{x}_3$
$\bar{x}_2 \vee \bar{x}_3$	$\bar{x}_3 \vee \bar{x}_4$

3. Compute the QUBO formulation of the following SAT problem. Show its coefficient matrix in binary variables and spin variables. Draw the circuit required to solve this problem with QAOA using 1 layer.

$x_1 \vee x_2$	$x_1 \vee x_3$
$\bar{x}_1 \vee \bar{x}_3$	$\bar{x}_2 \vee \bar{x}_3$

4. Compute the QUBO formulation of the following SAT problem. Show its coefficient matrix in binary variables and spin variables, compute the Problem Hamiltonian and use it to identify which is the best solution. Draw the circuit required to solve this problem with QAOA using 1 layer.

$\bar{x}_1 \vee x_2$	$x_1 \vee \bar{x}_2$
$\bar{x}_1 \vee \bar{x}_2$	

Important: Motivate your answer by showing all stages of the computation.

Question 1

Objective function:

$$\min_x \quad E(x) = +2x_3x_4 - 3x_2 + 2x_3x_1 + 2x_3x_2 + 2x_4x_2 - 2x_1 + 2x_2x_1 - 3x_3 - 2x_4$$

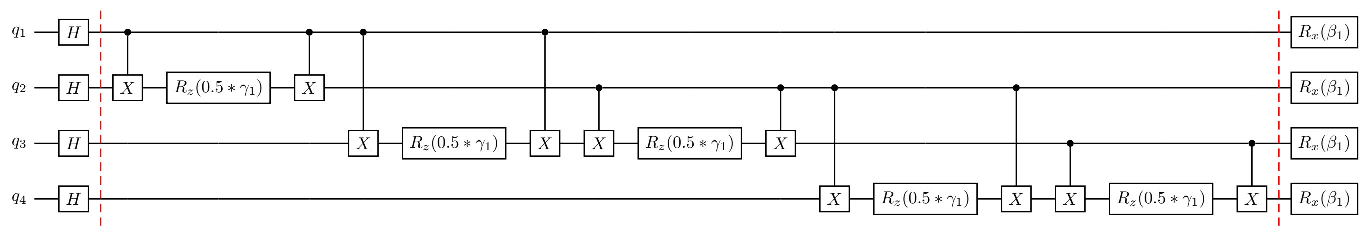
$$\min_s \quad E(s) = -2.5 + 0.5s_3s_1 + 0.5s_3s_2 + 0.5s_4s_2 + 0.5s_2s_1 + 0.5s_3s_4$$

Coefficient matrix:

$$Q = \begin{pmatrix} -2 & 2 & 2 & 0 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

QAOA Circuit with 1 layer:



Note: the following part was not requested, it is left here just for completeness
Pauli σ_z applied on all qubits:

$$\text{diag}(\sigma_z^1) = (1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1)$$
$$\text{diag}(\sigma_z^2) = (1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1)$$
$$\text{diag}(\sigma_z^3) = (1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1)$$
$$\text{diag}(\sigma_z^4) = (1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1)$$

Problem Hamiltonian:

$$\text{diag}(H_{prob}) = (2.5, 0.5, -0.5, -0.5, -0.5, -0.5, -1.5, 0.5, 0.5, -1.5, -0.5, -0.5, -0.5, -0.5, 0.5, 2.5)$$

Optimal Solution:
min(H_{prob}): -1.5
Solution index: 6, 9
Solution bitstring: 0110, 1001
E(x)=E(s)=-4.0

Question 2

Objective function:

$$\min_{x} \quad E(x) = 3 - x_3x_2 - 2x_4 + x_4x_2 + 2x_3x_4 + x_1$$

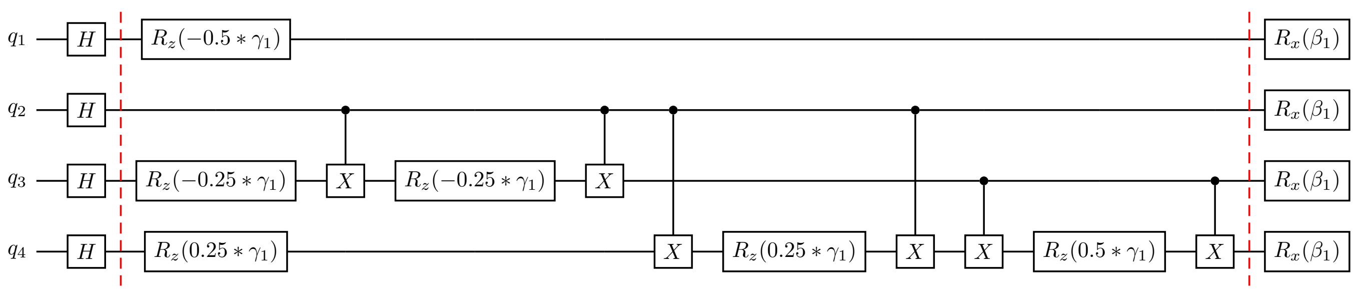
$$\min_{s} \quad E(s) = 3 - 0.25s_3s_2 + 0.25s_4s_2 - 0.25s_3 + 0.5s_3s_4 + 0.25s_4 - 0.5s_1$$

Coefficient matrix:

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$S = \begin{pmatrix} -0.5 & 0 & 0 & 0 \\ 0 & 0 & -0.25 & 0.25 \\ 0 & 0 & -0.25 & 0.5 \\ 0 & 0 & 0 & 0.25 \end{pmatrix}$$

QAOA Circuit with 1 layer:



Note: the following part was not requested, it is left here just for completeness

Pauli σ_z applied on all qubits:

$$\text{diag}(\sigma_z^1) = (1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1)$$

$$\text{diag}(\sigma_z^2) = (1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, -1, -1, -1, -1)$$

$$\text{diag}(\sigma_z^3) = (1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1)$$

$$\text{diag}(\sigma_z^4) = (1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1)$$

Problem Hamiltonian:

$$\text{diag}(H_{prob}) = (0, -2, 0, 0, 0, -1, -1, 0, 1, -1, 1, 1, 1, 0, 0, 1)$$

Optimal Solution:
$\min(H_{prob})$: -2
Solution index: 1
Solution bitstring: 0001
E(x)=E(s)=1.0

Question 3

Objective function:

$$\min_x \quad E(x) = 2 - 2x_1 + x_2x_1 + 2x_3x_1 - x_2 + x_2x_3 - x_3$$

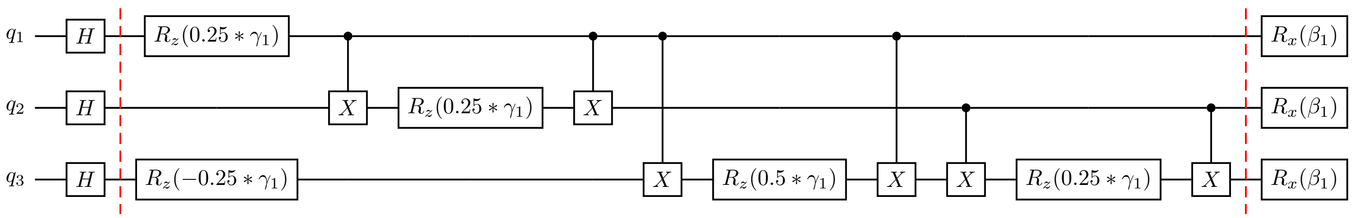
$$\min_s \quad E(s) = 1 + 0.25s_1 + 0.25s_2s_1 + 0.5s_3s_1 + 0.25s_2s_3 - 0.25s_3$$

Coefficient matrix:

$$Q = \begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$S = \begin{pmatrix} 0.25 & 0.25 & 0.5 \\ 0 & 0 & 0.25 \\ 0 & 0 & -0.25 \end{pmatrix}$$

QAOA Circuit with 1 layer:



Note: the following part was not requested, it is left here just for completeness

Pauli σ_z applied on all qubits:

$$\mathrm{diag}(\sigma_z^1) = (1, 1, 1, 1, -1, -1, -1, -1)$$
$$\mathrm{diag}(\sigma_z^2) = (1, 1, -1, -1, 1, 1, -1, -1)$$
$$\mathrm{diag}(\sigma_z^3) = (1, -1, 1, -1, 1, -1, 1, -1)$$

Problem Hamiltonian:

$$\mathrm{diag}(H_{prob}) = (1, 0, 0, 0, -1, 0, -1, 1)$$

Optimal Solution:
min(H_{prob}): -1
Solution index: 4, 6
Solution bitstring: 100, 110
E(x)=E(s)=0.0

Question 4

Objective function:

$$\min_{x} \quad E(x) = +x_1 - x_1x_2 + x_2$$

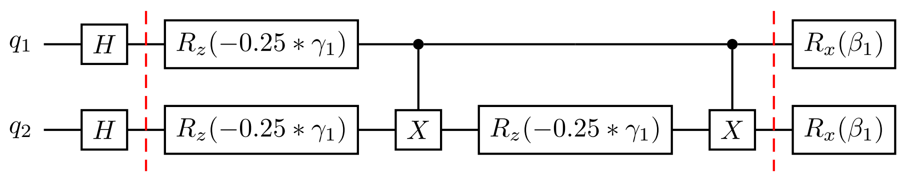
$$\min_{s} \quad E(s) = 0.75 - 0.25s_1 - 0.25s_1s_2 - 0.25s_2$$

Coefficient matrix:

$$Q = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} -0.25 & -0.25 \\ 0 & -0.25 \end{pmatrix}$$

QAOA Circuit with 1 layer:



Pauli σ_z applied on all qubits:

$$\mathrm{diag}(\sigma_z^1) = (1, 1, -1, -1)$$

$$\mathrm{diag}(\sigma_z^2) = (1, -1, 1, -1)$$

Problem Hamiltonian:

$$\mathrm{diag}(H_{prob}) = (-0.75, 0.25, 0.25, 0.25)$$

Optimal Solution:
min(H_{prob}): -0.75
Solution index: 0
Solution bitstring: 00
E(x)=E(s)=0.0