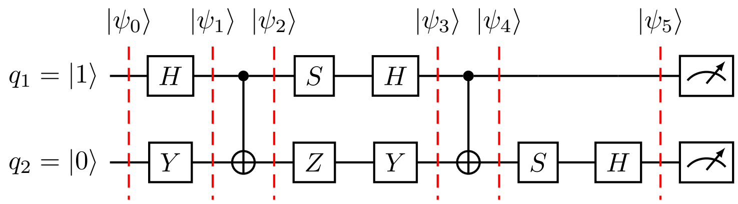
Course:	QUANTUM COMPUTING		
Professor:	P. Cremonesi M. Ferrari Dacrema	**Date:**	17/01/2025
Last Name:		**Codice Persona:**	
First Name:		**Grade:**	

Exam Rules:

- Exam duration: 2 h
- Write your answers on this exam paper by using a pen. Please do not use pencils or pens with red ink.
- You are NOT allowed to use notes, books or any other materials as well as electronic devices.
- You are NOT allowed to copy someone else's answers or let others copy from you. Those in violation of this rule will receive a failing grade and will be excluded from the next exam call.

1 Quantum Circuits (10 points)

- Consider the following quantum circuit defined on 2 qubits. Compute the probability of measuring each possible state at the end of the circuit.



The S gate is as follows:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Important: Motivate your answer by showing all stages of the computation.

The quantum states of the circuit are as follows:

$$|\psi_4\rangle = (S \otimes H)\, C_{X,q_2q_1}\, (I \otimes Z)\, C_{X,q_1q_2}\, (H \otimes S)\, |01\rangle$$

$$|\psi_0\rangle = |10\rangle$$

$$|\psi_1\rangle = (H \otimes Y)\, |\psi_0\rangle = H\,|1\rangle \otimes Y\,|0\rangle = |-\rangle \otimes \cancel{(-i)}\,|1\rangle = \frac{1}{\sqrt{2}}\,(|01\rangle - |11\rangle)$$

$$|\psi_2\rangle = C_{X,q_1q_2}\,|\psi_1\rangle = \frac{1}{\sqrt{2}}\,(|01\rangle - |10\rangle)$$

$$|\psi_3\rangle = (H \otimes Y)\,(S \otimes Z)\,|\psi_2\rangle = (H \otimes Y)\,(S \otimes Z)\,\frac{1}{\sqrt{2}}\,(|01\rangle - |10\rangle)$$

$$= (H \otimes Y)\,\frac{1}{\sqrt{2}}\,(S\,|0\rangle\, Z\,|1\rangle - S\,|1\rangle\, Z\,|0\rangle) = (H \otimes Y)\,\frac{1}{\sqrt{2}}\,(\cancel{-}\,|01\rangle \cancel{-} i\,|10\rangle) = (H \otimes Y)\,\frac{1}{\sqrt{2}}\,(|01\rangle + i\,|10\rangle)$$

$$= \frac{1}{\sqrt{2}}\,(H\,|0\rangle\, Y\,|1\rangle + iH\,|1\rangle\, Y\,|0\rangle) = \frac{1}{\sqrt{2}}\left(-\cancel{i}\,|+\rangle\,|0\rangle + i^{\cancel{2}}\,|-\rangle\,|1\rangle\right) = \frac{1}{\sqrt{2}}\left(\frac{-|00\rangle - |10\rangle + i\,|01\rangle - i\,|11\rangle}{\sqrt{2}}\right)$$

$$|\psi_4\rangle = C_{X,q_1q_2}\,|\psi_3\rangle = \frac{1}{\sqrt{2}}\left(\frac{-|00\rangle - |11\rangle + i\,|01\rangle - i\,|10\rangle}{\sqrt{2}}\right)$$

$$|\psi_5\rangle = (I \otimes H)\,(I \otimes S)\,|\psi_4\rangle = (I \otimes H)\,(I \otimes S)\,\frac{1}{\sqrt{2}}\left(\frac{-|00\rangle - |11\rangle + i\,|01\rangle - i\,|10\rangle}{\sqrt{2}}\right)$$

$$= (I \otimes H)\,\frac{1}{\sqrt{2}}\left(\frac{\cancel{-}\,|00\rangle \cancel{-} i\,|11\rangle + i^{\cancel{2}}\,|01\rangle \cancel{-} i\,|10\rangle}{\sqrt{2}}\right) = (I \otimes H)\,\frac{1}{\sqrt{2}}\,(|0\rangle\,|+\rangle + i\,|1\rangle\,|+\rangle)$$

$$= \frac{1}{\sqrt{2}}\,(|00\rangle + i\,|10\rangle)$$

There are only two possible outcomes with equal probability:

$$P(|00\rangle) = P(|10\rangle) = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^{\dagger} = \left(\frac{i}{\sqrt{2}}\right)\left(\frac{i}{\sqrt{2}}\right)^{\dagger} = \frac{1}{2}$$

2 Qubit States and Operations (10 points)

- Describe the state space of a single qubit in terms of the two basis states $|0\rangle$ and $|1\rangle$. Explain what the Block Sphere is, how a qubit can be represented on it, and the meaning of the global phase.
- Describe the mathematical formalism used to represent a qubit gate and the properties it must have. Show that gates σ_X and σ_Y satisfy those properties.
- Provide a brief description of the no-cloning and no-delete principles.

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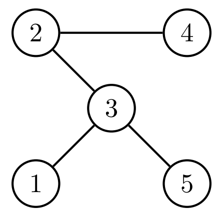
3 QUBO (11 points)

- Compute the QUBO formulation of the following SAT problem. Show its coefficient matrix in binary variables and spin variables, compute the Problem Hamiltonian and use it to identify which is the best solution.

$$\bar{x}_1 \vee \bar{x}_2$$

- Compute the QUBO formulation of the Max-Cut problem related to this graph. Show its coefficient matrix in binary variables and spin variables. Draw the circuit required to solve this problem with QAOA using 1 layer.



Important: Motivate your answer by showing all stages of the computation.

Solution 1

Objective function:

$$\min_x \quad E(x) = +x_1x_2$$

$$\min_s \quad E(s) = 0.25 - 0.25s_1 + 0.25s_1s_2 - 0.25s_2$$

Coefficient matrix:

$$Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} -0.25 & 0.25 \\ 0 & -0.25 \end{pmatrix}$$

Pauli σ_z applied on all qubits:

$$\text{diag}(\sigma_z^1) = (1, 1, -1, -1)$$

$$\text{diag}(\sigma_z^2) = (1, -1, 1, -1)$$

Problem Hamiltonian:

$$\text{diag}(H_{prob}) = (-0.25, -0.25, -0.25, 0.75)$$

Optimal Solution:
min(H_{prob}): -0.25
Solution index: 0, 1, 2
Solution bitstring: 00, 01, 10
E(x)=E(s)=0.0

Solution 2

Objective function:

$$\min_x \quad E(x) = +2x_2x_3 - 3x_3 + 2x_3x_5 - x_1 + 2x_3x_1 - 2x_2 - x_5 + 2x_2x_4 - x_4$$

$$\min_s \quad E(s) = -2 + 0.5s_5s_3 + 0.5s_3s_1 + 0.5s_4s_2 + 0.5s_3s_2$$

Coefficient matrix:

$$Q = \begin{pmatrix} -1 & 0 & 2 & 0 & 0 \\ 0 & -2 & 2 & 2 & 0 \\ 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

QAOA Circuit with 1 layer: