Minimax Design of Adjustable-Bandwidth Linear-Phase FIR Filters

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***Abstract*—This paper considers the design of digital linear-phase finite-length impulse response (FIR) filters that have adjustable bandwidth(s) whereas the phase response is fixed. For this purpose, a structure is employed in which the overall transfer function is a weighted linear combination of fixed subfilters and where the weights are directly determined by the bandwidth(s). Minimax design techniques are introduced which generate globally optimal overall filters in the minimax (Chebyshev) sense over a whole set of filter specifications. The paper also introduces a new structure for bandstop and bandpass filters with individually adjustable upper and lower band edges, and with a substantially lower arithmetic complexity compared to structures that make use of two separate adjustable-bandwidth low-pass and high-pass filters in cascade or in parallel. Design examples are included in the paper.**

***Index Terms*—Adjustable-bandwidth linear-phase finite impulse response (FIR) filters, linear programming, minimax optimization.**

I. Introduction

MANY applications such as telecommunications, digital audio, medical, radar, sonar, and instrumentation require digital filters that have an adjustable frequency response [1]. This paper deals with linear-phase FIR filters that have adjustable bandwidth(s) whereas the phase response is fixed [1]–[5]. For low-pass and high-pass filters, the filter structure shown in Fig. 1 is employed for this purpose [6], [7][1]. The overall filter is a weighted linear combination of fixed linear-phase FIR subfilters. The main advantage of this structure is that the bandwidth b is directly given by the adjustable-bandwidth parameter b. This results in a very simple updating routine as b does not have to be computed. A drawback is however that the overall filter complexity may become high, i.e., it may require a large amount of arithmetic computations during normal operation. This is in contrast to those adjustable-bandwidth filters that have a more complex updating routine (where b must be computed through some trigonometric function), but lower overall filter complexity [1]–[5]. Which type of adjustable filter to choose depends on the application at hand. We also wish to point out that it was recently shown that the use of a number of fixed overdesigned linear-phase FIR filters in fact can be more efficient than that in Fig. 1 and also equally simple to update [9]. The advantage of the one in Fig. 1 is that its delay[2] is shorter. Hence this structure may still be of interest in applications where the delay is an important factor to consider.

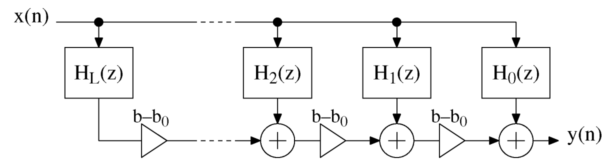

Fig. 1. Filter structure for low-pass and high-pass filters with an adjustable bandwidth b.

The main contributions of this paper are as follows.

1) Minimax design techniques are introduced which generate globally optimal overall filters in the minimax (Chebyshev) sense. Previous publications utilizing the structure in Fig. 1 have only considered least-squares design techniques [6], [7]. As is well known, the use of minimax instead of least-squares design techniques in applications that "require" minimax rather than least-squares solutions, enables one to improve the filter performance for a given filter order, or reduce the filter order for a given maximum allowable approximation error. The filters in this paper are thus superior to the ones in [6] and [7] for "minimax specifications" whereas the ones in [6] and [7] are preferred for "least-squares specifications." In other words, the two different filter design techniques complement each other rather than compete with each other. Which design criterion to use depends on the application at hand. It is often pointed out though that an advantage of least-squares design techniques is that they are usually much faster than optimization-based design techniques (like linear programming). However, in most practical cases the time it takes to optimize the filter is negligible to the time it takes to implement and manufacture the filter in hardware, and it is therefore usually not an important issue. In other words, the filter optimization complexity is usually irrelevant in a wider perspective. It is the resulting filter's implementation complexity and power consumption that are of importance.
2) It is shown how the structure in Fig. 1 can be modified, by incorporating another set of adjustable multiplier coefficients, in order to obtain bandstop and bandpass filters with individually adjustable upper and lower band edges. These cases have earlier been handled by making use of two separate adjustable-bandwidth low-pass and

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[1]The structure in Fig. 1 can also be used to realize adjustable fractional-delay filters [8], but this is not the topic of this paper.

[2]The delay is here the group and phase delay which for an Nth-order linear-phase FIR filter is $N/2$ samples.

high-pass filters in cascade or in parallel [7] which only yield suboptimal solutions and thereby a higher complexity than necessary. For the new structure proposed in this paper, globally optimal filters in the minimax sense are again obtained. It is demonstrated that the new structure requires substantially fewer arithmetic operations.

3) A properly chosen offset in the adjustable-bandwidth parameter is introduced. That is, instead of using the adjustable-bandwidth parameter b directly in the low-pass and high-pass filters (see Fig. 1), $b-b_0$ is used where $b_0 \neq 0$ is the offset. Previous publications utilizing the same filter structure have assumed that $b_0 = 0$. Although the use of a nonzero offset introduces the need to implement a few extra subtractions, it has the following advantages that makes it worthwhile: (a) it reduces the round-off noise emanating from rounding or truncating the intermediate data inside the filter; (b) it tends to make the filter coefficients smaller in magnitude which is advantageous in practical filter implementations where scaling of signal levels has to be considered; (c) it enables one to find in a simple manner a lower bound on the filter order required to meet the set of specifications considered in this paper (see later).

4) The filters in this paper are designed to meet each specification (for each value of b of interest) in a whole set of regular frequency-selective filter specifications. This means that, for all values of the adjustable-bandwidth parameter(s), the corresponding filter instances are guaranteed to meet their specified requirements on passband and stopband edges as well as passband and stopband ripples. This is different from how adjustable-bandwidth filters traditionally have been designed and implemented. For example, adjustable-bandwidth filters obtained through frequency transformations have been constructed in such a way that one cutoff frequency is controlled in the low-pass and high-pass filter cases and two cutoff frequencies in the bandpass and bandstop filter cases (see, e.g., [1], and references therein, and [2]–[4]). However, in practice, it is often desired to control two (for low-pass and high-pass filters) or four (for bandstop and bandpass filters) cutoff frequencies, namely the passband and stopband edges. This is precisely what is done in this paper. That is, the proposed design technique provides a complete control over the filter passband and stopband edges and ripples, and this in such a way that after each adjustment of the adjustable-bandwidth parameter(s), the filter instance is guaranteed to satisfy its specification. The problem of satisfying a whole set of specifications was considered recently in [10], [11] for IIR filters obtained analytically through frequency transformations, but the FIR filter case has not been treated before, except in [12], [13] where parts of the work in this paper have been presented.

The following sections (Sections II and III) concentrate on low-pass and bandstop filters. The reason why high-pass and bandpass filters are not dealt with in detail is that such filters are readily obtained after minor modifications.

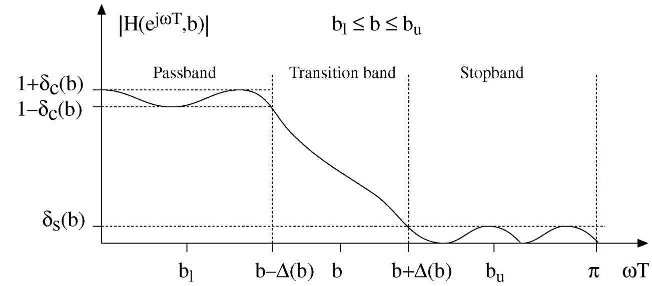

Fig. 2. Specification for a low-pass filter with an adjustable bandwidth b.

II. Low-Pass Filters

This section considers low-pass filters with an adjustable bandwidth b. Throughout this paper, the transfer function and frequency response of the low-pass filters are denoted as $H(z,b)$ and $H(e^{j\omega T},b)$, respectively.

A. Filter Specification

Let the set of low-pass filter specifications be

$$\begin{aligned} 1-\delta_c(b) \leq |H(e^{j\omega T},b)| \leq 1+\delta_c(b), \quad \omega T \in [0, b-\Delta(b)] \\ |H(e^{j\omega T},b)| \leq \delta_s(b), \quad \omega T \in [b+\Delta(b), \pi] \end{aligned} \quad (1)$$

for $b_l \leq b \leq b_u$ satisfying

$$b-\Delta(b) > 0, \quad b+\Delta(b) < \pi, \text{ and } \Delta(b) > 0. \quad (2)$$

The set of specifications is illustrated in Fig. 2. For each pair of values, b and $\Delta(b)$, the filter $H(z,b)$ should thus realize a low-pass filter having passband and stopband edges at $b-\Delta(b)$ and $b+\Delta(b)$, respectively, and fixed passband and stopband ripples of $\delta_c(b)$ and $\delta_s(b)$, respectively. In general, Δ, δ_c, and δ_s may thus depend on b; in many practical cases they are however constants.

B. Transfer Function and Frequency Response

Let the transfer function $H(z,b)$ be expressible as

$$H(z,b) = \sum_{k=0}^{L} (b-b_0)^k H_k(z) \quad (3)$$

where $H_k(z)$ are fixed N_kth-order linear-phase FIR filters with symmetrical impulse responses $h_k(n)$, i.e., $h_k(n) = h_k(N_k - n)$. The filters are referred to as Type I and Type II linear-phase FIR filters for even and odd N_k, respectively [14]. It is assumed here that all $H_k(z)$ are of the same type and order[3], i.e.,

$$N_k = N, \text{ for } k = 0, 1, \ldots, L. \quad (4)$$

This implies that $H(z)$ will be a linear-phase FIR filter of the same type and order as the subfilters $H_k(z)$. Further, b and b_0 are adjustable and fixed parameters, respectively, according to the

[3]If the orders N_k are not equal, extra delays must be introduced in (3) to make all terms have the same delay.

filter specification considered in the previous subsection. The realization corresponding to (3) was shown earlier in Fig. 1.

The frequency responses of $H_k(z)$ can be written as

$$H_k(e^{j\omega T}) = e^{-jN\omega T/2} H_{kR}(\omega T) \tag{5}$$

where $H_{kR}(\omega T)$ are the real zero-phase frequency responses of $H_k(e^{j\omega T})$ given by [14]

$$H_{kR}(\omega T) = \sum_{m=1}^{M} x_{km} \mathrm{trig}(m, \omega T) \tag{6}$$

where

$$\mathrm{trig}(m, \omega T) = \begin{cases} \cos(\omega T[m-1]), & \text{Type I} \\ \cos(\omega T[m-\frac{1}{2}]), & \text{Type II} \end{cases} \tag{7}$$

with

$$M = \begin{cases} \frac{N}{2}+1, & \text{Type I} \\ \frac{N+1}{2}, & \text{Type II} \end{cases} \tag{8}$$

and

$$\begin{aligned} x_{k1} &= h_k(M-1), \text{Type I} \\ x_{km} &= 2h_k(M-m), m = 2, 3, \ldots, M, \text{Type I} \\ x_{km} &= 2h_k(M-m), m = 1, 2, \ldots, M, \text{Type II.} \end{aligned} \tag{9}$$

The frequency response of $H(z, b)$ can consequently be written as

$$H(e^{j\omega T}, b) = e^{-jN\omega T/2} H_R(\omega T, b) \tag{10}$$

where

$$\begin{aligned} H_R(\omega T, b) &= \sum_{k=0}^{L} (b - b_0)^k H_{kR}(\omega T) \\ &= \sum_{k=0}^{L} \sum_{m=1}^{M} x_{km} (b-b_0)^k \mathrm{trig}(m, \omega T). \end{aligned} \tag{11}$$

C. Minimax Design

This section introduces minimax design of the overall filter $H(z, b)$. To this end, it is convenient to make use of $H_R(\omega T, b)$ instead of $H(e^{j\omega T}, b)$, which is possible due to (10). The specification of (1) is then restated according to

$$\begin{aligned} 1 - \delta_c(b) \le H_R(\omega T, b) \le 1 + \delta_c(b),\ \omega T \in [0, b - \Delta(b)] \\ -\delta_s(b) \le H_R(\omega T, b) \le \delta_s(b),\ \omega T \in [b + \Delta(b), \pi] \end{aligned} \tag{12}$$

for $b_l \le b \le b_u$ and with b and Δ satisfying (2).

In order to satisfy (12), we will solve the following *approximation problem*:

$$\text{minimize } \max |E(\omega T, b)| \tag{13}$$

on

$$\omega T \in [0, b - \Delta(b)] \cup [b + \Delta(b), \pi], \quad b \in [b_l, b_u] \tag{14}$$

where $E(\omega T, b)$ is the *weighted (Chebyshev) error function* given by

$$E(\omega T, b) = W(\omega T, b)[H_R(\omega T, b) - D(\omega T, b)]. \tag{15}$$

In the simplest case (conventional low-pass filter)

$$D(\omega T, b) = \begin{cases} 1, & \omega T \in [0, b - \Delta(b)] \\ 0, & \omega T \in [b + \Delta(b), \pi] \end{cases} \tag{16}$$

and

$$W(\omega T, b) = \begin{cases} 1, & \omega T \in [0, b - \Delta(b)] \\ \frac{\delta_c(b)}{\delta_s(b)}, & \omega T \in [b + \Delta(b), \pi] \end{cases}. \tag{17}$$

In general, $D(\omega T, b)$ and $W(\omega T, b)$ are arbitrary functions. The specification in (1) is satisfied if $\max |E(\omega T, b)| \le \delta_c(b)$ for all ωT and b in the specification.

The approximation problem above is a convex problem which guarantees that the solution is globally optimal in the minimax sense. The solution can be obtained using standard optimization techniques to this end. For example, by discretizing the problem, and making use of the real rotation theorem [15], the optimum solution (to the discretized problem) can be found by solving a finite-dimensional linear programming problem [16]. Discretization of the problem can be avoided by using semi-infinite programming [17] in which case the optimum solution to the original semi-infinite problem can be obtained. However, provided that the discrete grid is dense enough, the former technique will produce the same solution except for some very small numerical differences that can be neglected in practice.

In the example section (Section II-F), linear programming is used as it is readily available in MATLAB. To that end, ωT and b are first discretized into K_1 and K_2 grid points, respectively. The discrete approximation problem corresponding to (13) is then stated as

$$\text{minimize } \max |E(\omega_i T, b_j)| \tag{18}$$

for $i = 1, 2, \ldots, K_1$, $j = 1, 2, \ldots, K_2$. Using (11), and (15) – (17), the approximation problem of (18) can equivalently be written as the following linear programming problem:

$$\begin{aligned} &\text{minimize } \epsilon \text{ subject to} \\ &\quad \sum_{k=0}^{L} \sum_{m=1}^{M} x_{km}(b_j - b_0)^k \mathrm{trig}(m, \omega_i T) - \frac{\epsilon}{W(\omega_i T, b_j)} \\ &\quad \le D(\omega_i T, b_j) \\ &\quad -\sum_{k=0}^{L} \sum_{m=1}^{M} x_{km}(b_j - b_0)^k \mathrm{trig}(m, \omega_i T) - \frac{\epsilon}{W(\omega_i T, b_j)} \\ &\quad \le -D(\omega_i T, b_j) \end{aligned} \tag{19}$$

for $i = 1, 2, \ldots, K_1$, $j = 1, 2, \ldots, K_2$. This is a linear programming problem with $2K_1K_2$ constraints and P unknowns x_{km} and ϵ where

$$P = 1 + M(L+1). \tag{20}$$

The specification in (1) is met when $\epsilon \le \delta_c(b)$ for all b in the specification. A design example will be provided in Section II-F.

D. Choosing the Fixed Parameter b_0

In this paper, the fixed parameter b_0 is selected as

$$b_0 = \frac{b_l + b_u}{2}. \tag{21}$$

Although the chosen value for b_0 is not critical as to the filter order, the above selection has the following three advantages. First, b_0 lies in the interval $[b_l, b_u]$ which means that it is possible to derive a lower bound on the subfilter order required (see Section II-E). Second, this choice minimizes the maximum (over all b) of the roundoff noise introduced when quantizing the results of the multiplications in the structure in Fig. 1. The reason is that this maximum is determined by the maximum value of $|b-b_0|$ which apparently is minimized with the above selection of b_0. Third, selecting b_0 as in (21) instead of $b_0 = 0$, which one finds in the literature, has the advantage that the coefficients $h_k(n)$ tend to be smaller in magnitude. This is advantageous in practical implementations where scaling of signal levels must be taken into consideration. For example, using two's complement fixed-point arithmetic, it suffices to make sure that the inputs to the multipliers, and of course the output, are properly scaled, i.e, lie within their available number range [18]. The inputs to the multiplier coefficients $b-b_0$ in the structure in Fig. 1 are, in the most straightforward way, scaled by introducing a scaling constant at the input [which in practice can be merged with the filter impulse response coefficients $h_k(n)$]. The larger are the magnitudes of $h_k(n)$, the smaller is the value of this scaling constant which in turn means that the smaller is the signal-to-noise ratio at the output of the filter. In other words, it is beneficial to obtain a more "balanced structure" in which the filter coefficients are small in magnitude as in regular FIR filters. This is achieved by choosing b_0 according to (21) instead of $b_0 = 0$, as Example 1 in Section II-F will demonstrate.

E. Minimizing the Overall Complexity

This section discusses how to obtain the minimum complexity which amounts to finding the optimum values of L and N. For a fixed L, the minimum order, say N_L, is obtained as follows. First, a lower bound on the minimum order required to meet the specification in (1) is derived. This is possible due to the fact that, with the selection in (21), the transfer function in (3) must satisfy (1) with $b = b_0$, in which case one has $H(z,b) = H_0(z)$. That is, $H_0(z)$ is an ordinary low-pass filter that must satisfy the regular low-pass filter specification

$$\begin{aligned} 1-\delta_c(b_0) \le |H_0(e^{j\omega T})| \le 1+\delta_c(b_0), \quad & \omega T \in [0, b_0-\Delta(b_0)] \\ |H_0(e^{j\omega T})| \le \delta_s(b_0), \quad & \omega T \in [b_0+\Delta(b_0), \pi]. \end{aligned} \tag{22}$$

The minimum-order filter $H_0(z)$ meeting the above specification is easily found by first using the filter-order estimation given in, e.g., [19], and then by designing filters of orders in the neighborhood of these estimations using any standard method for optimizing in the minimax sense a conventional linear-phase FIR filter. For example, the efficient algorithm in [20] can be used for this purpose. Since the estimation in [19] is good, only a few designs need to be done. Once the minimum order for $H_0(z)$ alone has been found, say $N_{0,\min}$, the minimum order N_L of $H(z,b)$ is found by designing $H(z,b)$ for $N_L = N_{0,\min}, N_{0,\min}+1, N_{0,\min}+2, \ldots$, until the specification is met.

The remaining problem is to find the value of L that minimizes the overall complexity. For most practical cases, it turns out that only those cases where L is a small number are of interest. Hence, the optimum L and N are easily found by determining N_L for $1 \le L \le L_{\max}$, and then select the value of L that corresponds to the lowest overall complexity.

TABLE I
RESULTS OF THE LOW-PASS FILTER DESIGNS IN EXAMPLE 1

L	N	# of fixed multiplier coefficients	# of adjustable multiplier coefficients
1	260	262	1
2	100	153	2
3	36	76	3
4	26	70	4
5	26	84	5
6	26	98	6

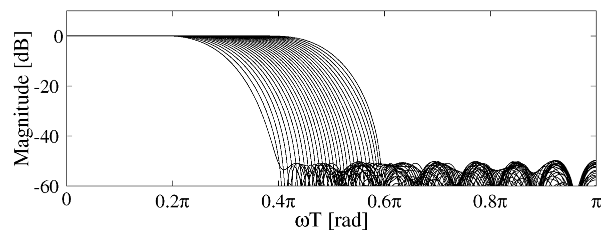

Fig. 3. Magnitude responses of the low-pass filter in Example 1 with an adjustable bandwidth b, for $L = 4$, $N = 26$, and 30 different values of b evenly distributed between 0.3π and 0.5π.

F. Design Example

Example 1: A Type 1 linear-phase FIR filter satisfying the low-pass filter specification in (1) with $\delta_c(b) = 0.01$, $\delta_s(b) = 0.003\,16$, $\Delta(b) = 0.1\pi$, $b_l = 0.3\pi$, and $b_u = 0.5\pi$ is considered.

First, according to Section II-E, $H_0(z)$ is designed to meet the specification in (22) by solving (19) with $b = b_0 = 0.4\pi$ and $L = 0$ for different filter orders N_0. The minimum filter order required is found to be $N_{0,\min} = 24$. Next, as outlined in Section II-E, the minimum order N of $H(z)$ satisfying (1) for each fixed L is found by solving (19) with $N = 24, 26, 28, \ldots$, until the specification is met. Table I shows the results for $L = 1, 2, \ldots, 6$. It can be seen that, for the filter with the lowest complexity ($L = 4$), the minimum order N is close to $N_{0,\min}$, implying that the delay is only slightly longer than that of the minimum-delay linear-phase FIR filter meeting one specification in (1). Fig. 3 shows the magnitude responses for the solution with lowest complexity. Tables II and III present the corresponding impulse response values $h_k(n) = h_k(26-n)$ for $n = 0, 1, \ldots, 13$. As discussed earlier in Section II-D, the selection of b_0 according to (21) instead of $b_0 = 0$ tend to make $h_k(n)$ smaller in magnitude, which is advantageous as to the signal-to-noise ratio. This is seen when comparing the impulse response values in Table II with those in Table III which presents $h_k(n)$ for the case where $b_0 = 0$.

As to the linear programming, MATLAB's function *lp.m* was used. On a standard PC computer, it took 24 seconds. The variables ωT and b were discretized to $K_1 = 180$ and $K_2 = 30$ grid points, respectively. The number of constraints was thus $2K_1K_2 = 10\,800$.

Example 2: Example 2 in [7] is considered in order to compare the minimax design with the least-squares design in that

TABLE II
IMPULSE RESPONSE VALUES $h_k(n) = h_k(N-n)$ FOR $L=4, N=26$, AND $b_0 = 0.5(b_l + b_u)$ IN EXAMPLE 1

n	$h_0(n)$	$h_1(n)$	$h_2(n)$	$h_3(n)$	$h_4(n)$
0	0.00028889692391	-0.02530542898039	-0.02850135265464	0.35828821529448	0.18344781507065
1	0.00519829718423	-0.01319732690517	-0.30045744205871	0.16330817349790	2.07869093360009
2	0.00598840999037	0.04762658834498	-0.27583538236326	-0.59438433707526	1.60164358904480
3	-0.00271761733541	0.08524479892026	0.10442538310704	-0.94258790133509	-0.33163526529612
4	-0.01440462697302	0.00515451552366	0.47570504621766	-0.07739480351367	-1.70301239056653
5	-0.00803106658102	-0.13607250359231	0.21782131878960	1.04329377758531	-0.69713693162115
6	0.01970186898968	-0.11996405932784	-0.39679587796735	0.77017126651960	0.56292527898233
7	0.03255666526422	0.09652720633197	-0.49826704545959	-0.41911879114708	0.44384881459363
8	-0.00644255818048	0.23232923173650	0.05566704255034	-0.76440700669646	0.13531350391549
9	-0.06633130667497	0.05435244851422	0.43630719378634	-0.17891533417009	0.55498775204578
10	-0.05106013985111	-0.22868793107958	0.20062278051533	0.06198611459602	0.41893777097009
11	0.09582435054472	-0.21751422026439	-0.13936070160502	-0.09757128319061	-0.35451055775894
12	0.29732955961515	0.09618483755692	-0.13299691637868	0.12284042493160	-0.31072063665430
13	0.39238720009063	0.28976834242620	-0.03444683579760	0.38668758927344	0.04081071672787

TABLE III
IMPULSE RESPONSE VALUES $h_k(n) = h_k(N-n)$ FOR $L=4, N=26$, AND $b_0 = 0$ IN EXAMPLE 1

n	$h_0(n)$	$h_1(n)$	$h_2(n)$	$h_3(n)$	$h_4(n)$
0	-0.26644782755688	0.28754786750535	0.35891896584705	-0.56382107777517	0.18344781507903
1	4.40682505031350	-14.98423897242072	18.77910814738971	-10.28533209219045	2.07869093359244
2	4.68402953269281	-14.78818294394957	17.14022584843490	-8.64512310983829	1.60164358903449
3	1.09854578770670	-2.01023992288259	0.51571353542100	0.72439275975398	-0.33163526531461
4	-3.36285186111343	11.9607691910679 7	-15.36825965998858	8.48287914073754	-1.70301239056106
5	-3.30181298556578	9.79259217474500	-10.32055066120831	4.54748619827518	-0.69713693162254
6	-0.58071954184458	0.05763035981393	2.03334266913716	-2.05939980708532	0.56292527899244
7	1.06293895057695	-4.15982610375209	5.28716129810543	-2.65014627145823	0.44384881458126
8	1.64382795296561	-4.60295218222678	4.21948468999600	-1.44456686238766	0.13531350390805
9	2.29335381378479	-6.29507701588077	6.36921129080072	-2.96858804559745	0.55498775204057
10	1.47481672987144	-3.76461764005810	3.93630069937964	-2.04382480321489	0.41893777098246
11	-0.54132150748309	2.48446766577804	-3.13044943248861	1.68439313901363	-0.35451055775987
12	-1.05216051161880	3.47876626501497	-3.54011658075750	1.68469269604055	-0.31072063665794
13	-0.69171764200270	1.88430095540323	-1.10555049915698	0.18155055268232	0.04081071673142

paper. Using the minimax design (for the same two subintervals used in that paper) the specification in [7] is met with $L = 2$ and $N = 30$. In [7], $N = 40$ is required. Hence, as expected and discussed in the introduction, the minimax filters are superior to the least-squares filters for "minimax specifications" (but, of course, the other way around for "least-squares specifications"). In this example, it took 11 seconds to design the filter. Again, the variables ωT and b were discretized to $K_1 = 180$ and $K_2 = 30$ grid points, respectively, and the number of constraints was $2K_1K_2 = 10\,800$. The optimization is slower than a least-squares design, but as discussed in the introduction, this is usually of no relevance in a wider perspective.

III. BANDSTOP FILTERS

This section considers bandstop filters with two individually adjustable bandwidths b_1 and b_2. Here, the transfer function and frequency response of the bandstop filters are denoted as $H(z, b_1, b_2)$ and $H(e^{j\omega T}, b_1, b_2)$, respectively.

A. Filter Specifications

Let the set of bandstop filter specifications be

$$\begin{aligned} 1 - \delta_c(b_1, b_2) \le |H(e^{j\omega T}, b_1, b_2)| \le 1 + \delta_c(b_1, b_2), \quad \omega T \in S_1 \\ |H(e^{j\omega T}, b_1, b_2)| \le \delta_s(b_1, b_2), \quad \omega T \in S_2 \end{aligned} \tag{23}$$

where

$$\begin{aligned} S_1 &= [0, b_1 - \Delta_1(b_1)] \cup [\pi - b_2 + \Delta_2(b_2), \pi] \\ S_2 &= [b_1 + \Delta_1(b_1), \pi - b_2 - \Delta_2(b_2)] \end{aligned} \tag{24}$$

for $b_{1l} \le b_1 \le b_{1u}$ and $b_{2l} \le b_2 \le b_{2u}$ satisfying

$$b_k - \Delta_k(b_k) > 0, \; \Delta_k(b_k) > 0 \tag{25}$$

for $k = 1, 2$, and

$$b_1 + \Delta_1(b_1) < \pi - b_2 - \Delta_2(b_2). \tag{26}$$

The set of specifications is illustrated in Fig. 4. For each set of values b_1, b_2, $\Delta_1(b_1)$, and $\Delta_2(b_2)$, the filter $H(z, b_1, b_2)$ is thus to realize a bandstop filter with passband edges at $b_1 - \Delta_1(b_1)$ and $\pi - b_2 + \Delta_2(b_2)$, stopband edges at $b_1 + \Delta_1(b_1)$ and $\pi - b_2 - \Delta_2(b_2)$, and passband and stopband ripples of $\delta_c(b_1, b_2)$ and $\delta_s(b_1, b_2)$, respectively.

B. Proposed Structure

This subsection proposes a new structure for implementing an adjustable-bandwidth bandstop filter meeting the specifications in (23). This structure is based on two observations. First, an adjustable bandstop filter can be obtained by summing an adjustable low-pass filter and an adjustable high-pass filter. Second, a high-pass filter can be obtained from a low-pass filter

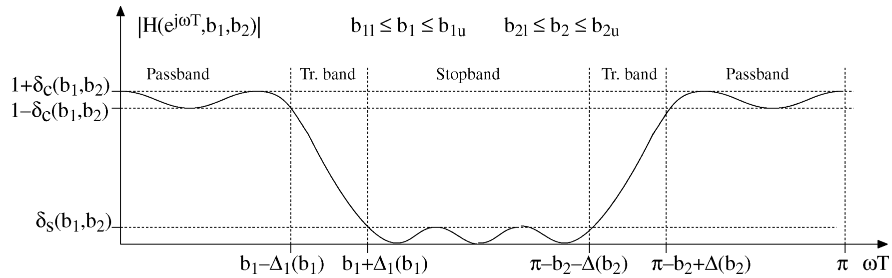

Fig. 4. Specification for a bandstop filter with two individually adjustable bandwidths b_1 and b_2.

through the frequency transformation $z \rightarrow -z$ which in the frequency domain corresponds to a shift of π radians with respect to ωT. That is, using a low-pass filter realized as in Fig. 1, a high-pass filter with the same bandwidth as the low-pass filter is obtained by replacing $H_k(z)$ with $H_k(-z)$[4]. To obtain a bandstop filter with individually adjustable upper and lower band edges (i.e., with individually controlled widths of the two passbands) it is however necessary to incorporate another set of adjustable multipliers. In other words, an adjustable-bandwidth bandstop filter can be obtained using one set of subfilters $H_k(z)$ and two sets of adjustable multipliers.

Based on the above, we propose an adjustable-bandwidth bandstop filter structure with a transfer function expressible as

$$H(z, b_1, b_2) = H_{\mathrm{LP}}(z, b_1) + aH_{\mathrm{HP}}(z, b_2) \tag{27}$$

where

$$H_{\mathrm{LP}}(z, b_1) = \sum_{k=0}^{L} (b_1 - b_{10})^k H_k(z) \tag{28}$$

and

$$H_{\mathrm{HP}}(z, b_2) = \sum_{k=0}^{L} (b_2 - b_{20})^k H_k(-z) \tag{29}$$

with b_{10} and b_{20} being chosen in this paper in accordance with (20), i.e.,

$$b_1 = \frac{b_{1l} + b_{1u}}{2}, \quad b_2 = \frac{b_{2l} + b_{2u}}{2} \tag{30}$$

and

$$a = \begin{cases} 1, & N = 4p \\ -1, & N = 4p + 2 \end{cases} \tag{31}$$

with p being a positive integer. When $N = 4p + 2$, $a = -1$ in order to make $H(z, b_1, b_2)$ have the same delay in both passbands. If a would be one in this case, there would be a difference in phase of π radians between the two passbands. Further, the filters are here restricted to be Type I filters, i.e., of even order. Otherwise, the low-pass and high-pass filters in (28) and (29) would be of different types (Type II and Type IV, respectively). The overall filter would then not have a linear phase response.

Realizing the overall filter straightforwardly would result in two copies of the fixed subfilters since $H_k(z)$ and $H_k(-z)$ are different. The filters $H_k(z)$ and $H_k(-z)$ do share, however, common parts which can be exploited to reduce the implementation complexity. This becomes evident if one utilizes polyphase decomposition [21] by which one can write

$$H_k(z) = H_{k0}(z^2) + z^{-1}H_{k1}(z^2) \tag{32}$$

and, thus,

$$H_k(-z) = H_{k0}(z^2) - z^{-1}H_{k1}(z^2) \tag{33}$$

where $H_{k0}(z)$ and $H_{k1}(z)$ are the polyphase components of $H_k(z)$. One can now rewrite $H(z, b_1, b_2)$ in (27) as

$$\begin{aligned} H(z, b_1, b_2) = &\sum_{k=0}^{L} (b_1 - b_{10})^k [H_{k0}(z^2) + z^{-1}H_{k1}(z^2)] \\ &+ a\sum_{k=0}^{L} (b_2 - b_{20})^k [H_{k0}(z^2) - z^{-1}H_{k1}(z^2)]. \end{aligned} \tag{34}$$

Since $(b_1 - b_{10})^0 = (b_2 - b_{20})^0 = 1$, the term for $k = 0$ in (34) becomes $2H_{00}(z^2)$ when $N = 4p$ $(a = 1)$ and $2z^{-1}H_{01}(z^2)$ when $N = 4p + 2$ $(a = -1)$. The corresponding efficient overall bandstop filter realization for $N = 4p$ is shown in Fig. 5. Apparently, only one copy of each subfilter is required in this structure. Finally, it is noted that the same structure can be used

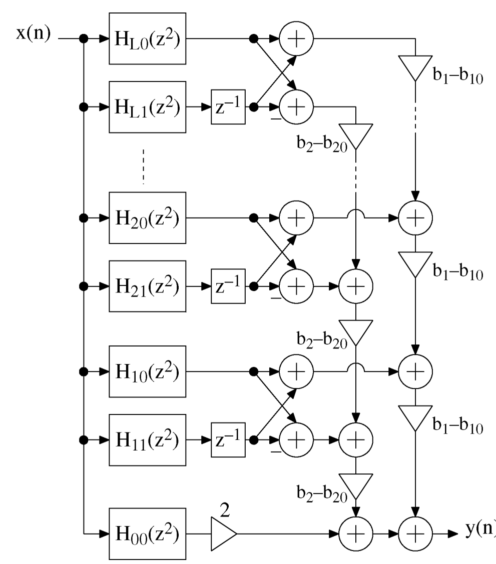

Fig. 5. Proposed filter structure for bandstop filters with two individually adjustable bandwidths b_1 and b_2 when the order is $N = 4p$, p being an integer. A bandpass filter structure is obtained by replacing $2H_{00}(z^2)$ with $z^{-N/2} - 2H_{00}(z^2)$ and changing the sign of the two downmost multiplier values $b_1 - b_{10}$ and $b_2 - b_{20}$.

[4]It should be noted that the transformation $z \rightarrow -z$ applied to a Type I (Type II) low-pass filter results in a Type I (Type IV) high-pass filter. Recall that Type III and Type IV linear-phase FIR filters have antisymmetrical impulse responses [14].

TABLE IV
IMPULSE RESPONSE VALUES $h_k(n) = h_k(N-n)$ FOR $L = 4$, $N = 26$, AND $b_0 = 0.5(b_l + b_u)$ IN EXAMPLE 2

n	$h_0(n)$	$h_1(n)$	$h_2(n)$	$h_3(n)$
0	-0.00146369160718	-0.01719549969242	0.05559947581645	0.20831547895298
1	0.00065496485051	-0.04687380240325	0.00395001572840	0.51362037920787
2	0.00607228672734	-0.05184214902737	-0.17320418848755	0.53476642583535
3	0.01326926517931	0.00899194466022	-0.33932212602334	-0.06617337730934
4	0.01050273304366	0.10744199863656	-0.23463812766681	-0.77109415969901
5	-0.00547174142398	0.16042825696770	0.09791944984440	-0.94147236507269
6	-0.03194980599235	0.08792487821578	0.45549000383878	-0.39376155948956
7	-0.04544429642238	-0.08258078330658	0.44790790913335	0.20694263076592
8	-0.02040530288921	-0.23082431912718	0.13944815521161	0.37379889865840
9	0.05036130144582	-0.22180904074388	-0.18957671994823	0.09021822146708
10	0.14998783406342	-0.04131655257896	-0.24304498355687	-0.04634852659909
11	0.23961036604442	0.18553709535704	-0.10925721942077	0.13033169845932
12	0.27430932244971	0.28423594608471	0.01130669439813	0.32171250964537

for bandpass filters after a slight modification. More precisely, using complementary techniques [14], a linear-phase bandpass filter with delay $N/2$ can be realized by subtracting a linear-phase bandstop filter from a delay term $z^{-N/2}$. A condition here is that N is even (as already assumed in this section; otherwise the filters would not be realizable.

C. Minimax Design

The bandstop filters can be designed in essentially the same way as the low-pass filters considered in Section II, after appropriate modifications, as pointed out in this section.

Again, it is convenient in the filter design to consider the zero-phase frequency responses. Here, the frequency response of $H(z, b_1, b_2)$ can be written as

$$H(e^{j\omega T}, b_1, b_2) = e^{-jN\omega T/2} H_R(\omega T, b_1, b_2) \tag{35}$$

where $H_R(\omega T, b_1, b_2)$ is the zero-phase frequency response given by

$$H_R(\omega T, b_1, b_2) = H_{\mathrm{LP}R}(\omega T, b_1) + H_{\mathrm{HP}R}(\omega T, b_2) \tag{36}$$

with

$$\begin{aligned} H_{\mathrm{LP}R}(\omega T, b_1) &= \sum_{k=0}^{L} (b_1 - b_{10})^k H_{kR}(\omega T) \\ &= \sum_{k=0}^{L} \sum_{m=1}^{M} x_{km} (b_1 - b_{10})^k \mathrm{trig}(m, \omega T) \end{aligned} \tag{37}$$

and

$$\begin{aligned} H_{\mathrm{HP}R}(\omega T, b_2) &= \sum_{k=0}^{L} (b_2 - b_{20})^k H_{kR}(\omega T - \pi) \\ &= \sum_{k=0}^{L} \sum_{m=1}^{M} x_{km} (b_2 - b_{20})^k \mathrm{trig}(m, \omega T - \pi). \end{aligned} \tag{38}$$

The specification in (23) can now be restated as

$$\begin{aligned} &1 - \delta_c(b_1, b_2) \le H_R(\omega T, b_1, b_2) \le 1 + \delta_c(b_1, b_2), \\ &\qquad \omega T \in S_1 \\ &-\delta_s(b_1, b_2) \le H_R(\omega T, b_1, b_2) \le \delta_s(b_1, b_2), \ \omega T \in S_2. \end{aligned} \tag{39}$$

In order to satisfy (39), we will solve the following approximation problem:

$$\text{minimize } \max |E(\omega T, b_1, b_2)| \tag{40}$$

on

$$\omega T \in S_1 \cup S_2, \ b_1 \in [b_{1l}, b_{1u}], \ b_2 \in [b_{2l}, b_{2u}] \tag{41}$$

where

$$\begin{aligned} E(\omega T, b_1, b_2) = W(\omega T, b_1, b_2)[&H_R(\omega T, b_1, b_2) \\ &- D(\omega T, b_1, b_2)]. \end{aligned} \tag{42}$$

Like the approximation problem in Section II, (42) is convex and the globally optimal solution in the minimax sense can in principle be obtained in the same way. The main difference is that the functions now depend on the three parameters, ωT, b_1, and b_2, instead of only two parameters, ωT, b, which makes the design problem more complex. Nevertheless, it is still possible to use, e.g., linear programming to solve the problem for many practical filter specifications. This will be demonstrated in Section III-D.

D. Design Example

Example 3: A Type 1 linear-phase FIR filter satisfying the bandstop filter specification in (23) with $\delta_c(b_1, b_2) = 0.01$, $\delta_s(b_1, b_2) = 0.01$, $\Delta_1(b_1) = 0.1\pi$, $\Delta_2(b_2) = 0.1\pi$, $b_{1l} = 0.2\pi$, $b_{1u} = 0.35\pi$, $b_{2l} = 0.2\pi$, and $b_{2u} = 0.35\pi$ is considered. The best solution using the proposed adjustable filter is obtained for $L = 3$ and $N = 24$, which corresponds to 45 fixed and six adjustable coefficients in an implementation. Fig. 3 shows the magnitude responses for this solution whereas Table IV presents the corresponding impulse response values $h_k(n) = h_k(24 - n)$ for $n = 0, 1, \ldots, 12$. Using instead two separate adjustable-bandwidth low-pass and high-pass filters in parallel, one obtains the best solution for $L = 3$ and $N = 26$, which corresponds to 112 fixed and 6 variable coefficients in an implementation. It should be noted however that the saving of using the proposed filter becomes smaller when the set of specifications is more asymmetric, the reason being that the overall adjustable bandwidth then becomes wider since it is basically the union of the adjustable low-frequency bandwidth

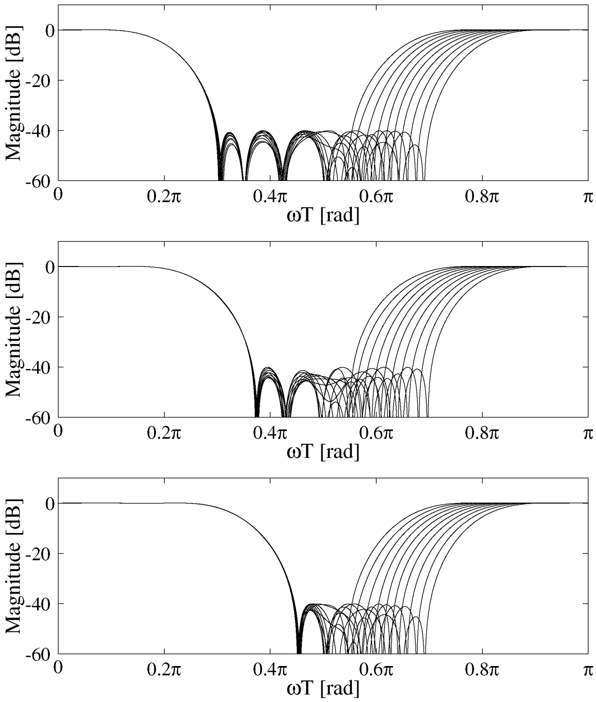

Fig. 6. Magnitude response of the bandstop filter in Example 3 with two individually adjustable bandwidths b_1 and b_2, for $L = 3$ and $N = 24$, and 30 different pairs of values of b_1 and b_2. Upper: $b_1 = 0.2\pi$. Middle: $b_1 = 0.275\pi$. Lower: $b_1 = 0.35\pi$. In all three figures, b_2 takes on values evenly distributed between 0.25π and 0.35π.

interval and adjustable high-frequency bandwidth interval[5]. Making the specification less symmetric thus increases the complexity of the proposed filter whereas the complexity using the traditional approach (with two separate filters) essentially remains the same.

In this example, the design took about 30 s. The variables ωT, b_1, and b_2 were discretized to 150, 10, and 10 grid points, respectively, and the number of constraints was thus 15 000.

IV. Concluding Remarks

The advantages of the proposed filters and design techniques were outlined in the introduction and demonstrated through design examples. As the approximation problems are convex, globally optimum filters in the minimax sense can always be obtained through conventional methods to this end (see Section II-C). In this paper, linear programming was used but other alternatives [17] can be used as well. The potential problem of using linear programming is that it tends to be rather slow for problems with many variables and constraints but filters with modest requirements, like those in the examples of this paper, can be designed relatively fast and without any numerical problems. Besides, as discussed in the introduction, the time it takes to optimize the filter is usually an irrelevant issue in a wider perspective.

[5]The set of specifications is symmetric if the adjustable low-frequency bandwidth interval $[b_{1l}, b_{1u}]$ coincides with the adjustable high-frequency bandwidth interval $[b_{2l}, b_{2u}]$ as is the case in this design example. It should however be stressed that even though the set of specifications is symmetric, the bandstop filter is *not* restricted to possess symmetry since b_1 and b_2 are *independently* adjustable parameters, as illustrated in Fig. 6.

The paper concentrated on low-pass and bandstop filters, as high-pass and bandpass filters are readily obtained after minor modifications. For example, the structure in Fig. 1 can be used for high-pass filters without modifications provided the design targets a high-pass filter specification instead of the low-pass filter specification in (1). Alternatively, a high-pass filter can be obtained from an appropriately designed low-pass filter through the frequency transformation $z \rightarrow -z$ as discussed in Section III-B. The bandpass filter can be obtained from an appropriately designed bandstop filter through the use of complementary techniques, as mentioned in Section III-B, which corresponds to the structure modification described in the figure text below Fig. 5.

Finally, it is pointed out that all subfilters have the same order in this paper. The generalized case allowing different subfilter orders was considered in [12] for the low-pass filter case. However, the results in that paper indicate that the use of different filter orders at best can offer only a marginal decrease of the overall complexity.

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