

# The Demography of Kinship (2)

## The formal demography of kinship

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European Doctoral School of Demography 2022-23  
25 April 2023



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# Agenda

1. The Goodman-Keyfitz-Pullum kinship equations
2. The Caswell kinship universe
3. Implementations

# What are kinship models?

- ① Kinship is an *emergent property* of demographic systems
- ② Simplified representation of interaction between reproduction and death
- ③ Not restricted to humans<sup>1</sup>

→ What are emergent properties?

→ Can you think of other emergent properties in nature, society, or demography?

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<sup>1</sup>Coste, C. F. D., Bienvenu, F., Ronget, V., Ramirez-Loza, J.-P., Cubaynes, S., & Pavard, S. (2021). The kinship matrix: Inferring the kinship structure of a population from its demography (T. Coulson, Ed.). *Ecology Letters*, 24(12), 2750–2762. <https://doi.org/10.1111/ele.13854>

# Formal models of kinship

Given a set of:

- ▶ age-specific fertility rates
- ▶ survival probabilities
- ▶ simplifying assumptions

The models produce:

- 1 Number of (living/dead) kin
- 2 Age distribution of relatives
- 3 From the point of view of an average member of the population ('Focal')

Focal: an average member of the population



# Typology of kinship models

time	sex	state	reference
invariant	female	age	2 3
variant	female	age	4
invariant	both	age	5
variant	both	age	DemoKin
invariant	female	multiple	6

<sup>2</sup>Goodman, L. A. (1974). Family Formation and the Frequency of Various Kinship Relationships. *Theoretical Population Biology*, 27

<sup>3</sup>Caswell, H. (2019). The formal demography of kinship: A matrix formulation. *Demographic Research*, 41, 679–712

<sup>4</sup>Caswell, H., & Song, X. (2021). The formal demography of kinship. III. kinship dynamics with time-varying demographic rates. *Demographic Research*, 45, 517–546

<sup>5</sup>Caswell, H. (2022). The formal demography of kinship IV: Two-sex models and their approximations. *Demographic Research*, 47, 359–396

<sup>6</sup>Caswell, H. (2020). The formal demography of kinship II: Multistate models, parity, and sibship. *Demographic Research*, 42, 1097–1146

# Model characteristics

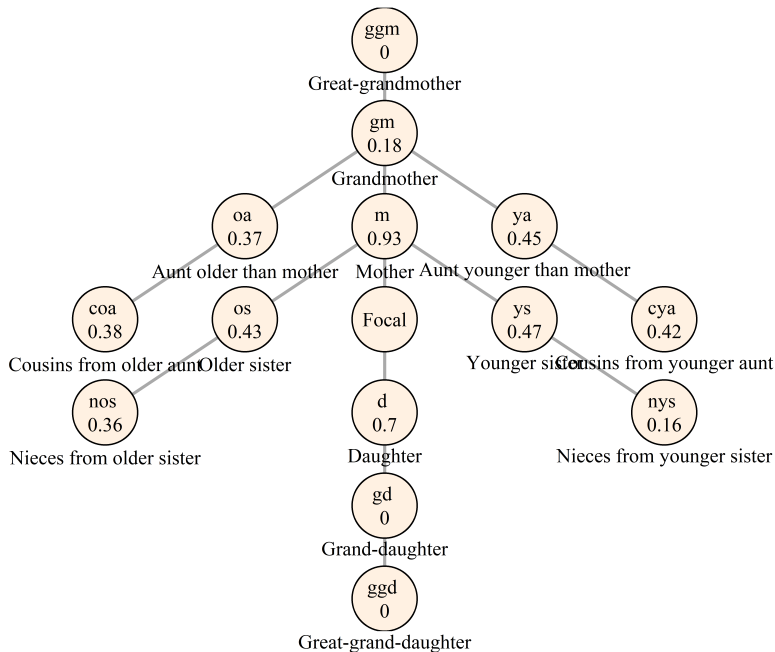
Define the following model characteristics:

- ① Time-(in)variance
- ② One/two-sex models
- ③ (Multi)state models

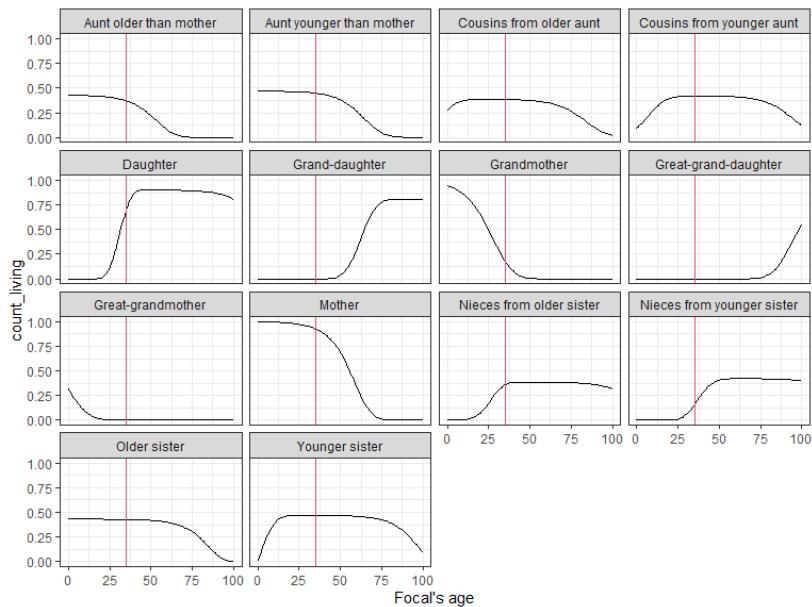
## The Goodman-Keyfitz-Pullum kinship equations



# The tree of life



# Expected number of kin



# Daughters

$B_1(a)$  is the expected number of living daughters in a time-invariant female-only population<sup>7</sup>:

$$B_1(a) = \int_{\alpha}^a m(x)l(a-x) dx \quad (1)$$

where:

- ▶  $m(x)$  are fertility rates of mothers
- ▶  $l(a-x)$  are survival probabilities of daughters

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<sup>7</sup>Goodman, L. A. (1974). Family Formation and the Frequency of Various Kinship Relationships. *Theoretical Population Biology*, 27.

# Daughters

If  $a = 20$  and  $\alpha = 15$ ; then:

$$B_1(20) \approx \sum_{15}^{20} m(x)l(20 - x)$$

So...

$$B_1(20) \approx m(15)l(0) + m(16)l(1) + m(17)l(2) \dots$$

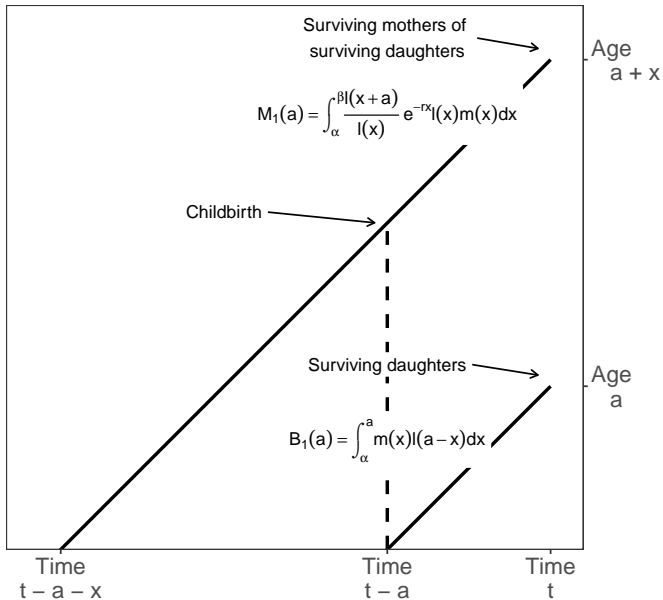
# Granddaughters

$B_1(a)$  is the expected number of living granddaughters in a time-invariant female-only population<sup>8</sup>:

$$B_2(a) = \int_{\alpha}^a m(x) \int_{\alpha}^{a-x} l(y) m(y) l(a-x-y) dy dx \quad (2)$$

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<sup>8</sup>Goodman, L. A. (1974). Family Formation and the Frequency of Various Kinship Relationships. *Theoretical Population Biology*, 27.



# Mothers

$M_1(a)$  is the probability of having a living mother in a time-invariant female-only population<sup>9</sup>:

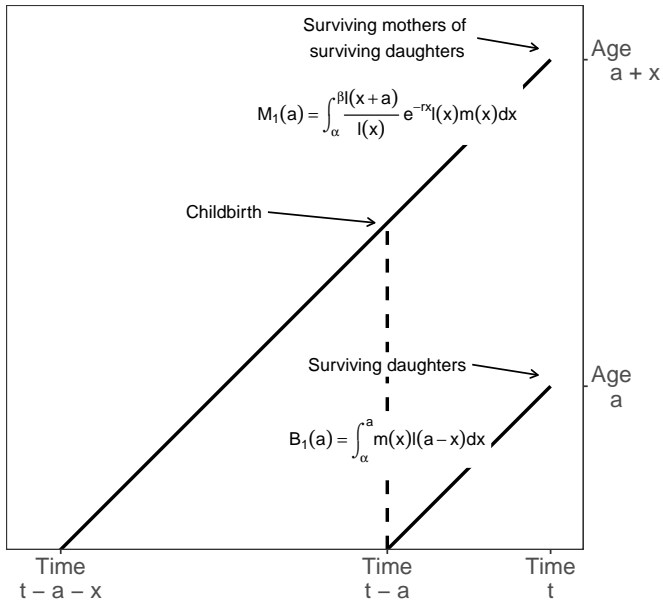
$$M_1(a) = \int_{\alpha}^{\beta} \underbrace{\frac{l(x+a)}{l(x)}}_{\text{prob of surviving from } x \text{ to } a+x} \times \underbrace{W(x)}_{\text{age distribution of mothers}} dx. \quad (3)$$

where:

- ▶  $W(x) = e^{-rx}l(x)m(x)$  is the age distribution of mothers
- ▶  $l(x)$  are survival probabilities
- ▶  $m(x)$  are fertility rates
- ▶  $r$  is the population growth rate
- ▶  $\alpha$ - $\beta$  is the reproductive period

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<sup>9</sup>Goodman, L. A. (1974). Family Formation and the Frequency of Various Kinship Relationships. *Theoretical Population Biology*, 27.





# Grandmothers

$M_2(a)$  is the expected number of living grandmothers in a time-invariant female-only population<sup>10</sup>:

$$M_2(a) = \int_{\alpha}^{\beta} \underbrace{M_1(a)}_{\text{prob of having living mother}} \times \underbrace{W(x)}_{\text{age distribution of mothers}} dx. \quad (4)$$

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<sup>10</sup>Goodman, L. A. (1974). Family Formation and the Frequency of Various Kinship Relationships. *Theoretical Population Biology*, 27.

# Great-grandmothers

$M_3(a)$  is the expected number of living great-grandmothers in a time-invariant female-only population<sup>11</sup>:

$$M_3(a) = \int_{\alpha}^{\beta} \underbrace{M_2(a)}_{\text{number of grandmother}} \times \underbrace{W(x)}_{\text{age distribution of mothers}} dx. \quad (5)$$

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<sup>11</sup>Goodman, L. A. (1974). Family Formation and the Frequency of Various Kinship Relationships. *Theoretical Population Biology*, 27.

# Sisters

$B_1(a)$  is the expected number of living older sisters in a time-invariant female-only population<sup>12</sup>:

$$S^{old}(a) = \int_{\alpha}^{\beta} \int_{\alpha}^x m(y) l(a+x-y) W(x) dy dx \quad (6)$$

$$S^{young}(a) = \int_{\alpha}^{\beta} \int_0^a \left[ \frac{l(x+u)}{l(x)} \right] m(x+u) l(a-u) du W(x) dx \quad (7)$$

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<sup>12</sup>Goodman, L. A. (1974). Family Formation and the Frequency of Various Kinship Relationships. *Theoretical Population Biology*, 27.



# Demographic subsidy

“New members of the population arise not from reproduction of current members, but from elsewhere”<sup>13</sup>

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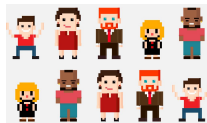
<sup>13</sup>Caswell, H. (2019). The formal demography of kinship: A matrix formulation. *Demographic Research*, 41, 679–712

# Break

## The Caswell kinship universe

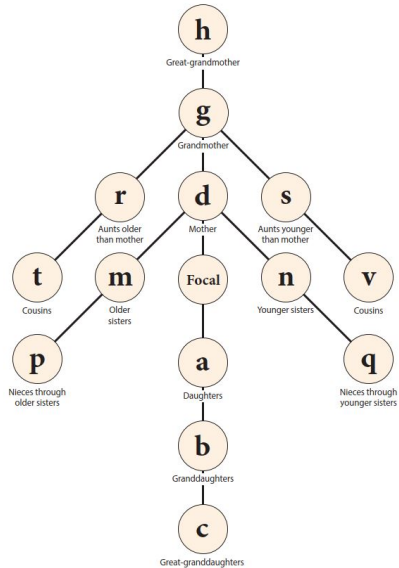
# From recursive equations to matrix operations

- 1 The relatives of Focal constitute a population
- 2 They can be modelled using traditional projection methods
- 3 Matrix operations provide an efficient implementation





# The tree of life (2)



## Implementation: time-invariant, one-sex models<sup>14</sup>

The models are of the general form:

$$\underbrace{\mathbf{k}(x+1)}_{\text{age structure of kin at Focal's age } x+1} = \underbrace{\mathbf{U}\mathbf{k}(x)}_{\text{ageing and survival of existing kin}} + \underbrace{\begin{Bmatrix} 0 \\ \mathbf{F}\mathbf{k}^*(x) \end{Bmatrix}}_{\text{new kin members added to the population}}.$$

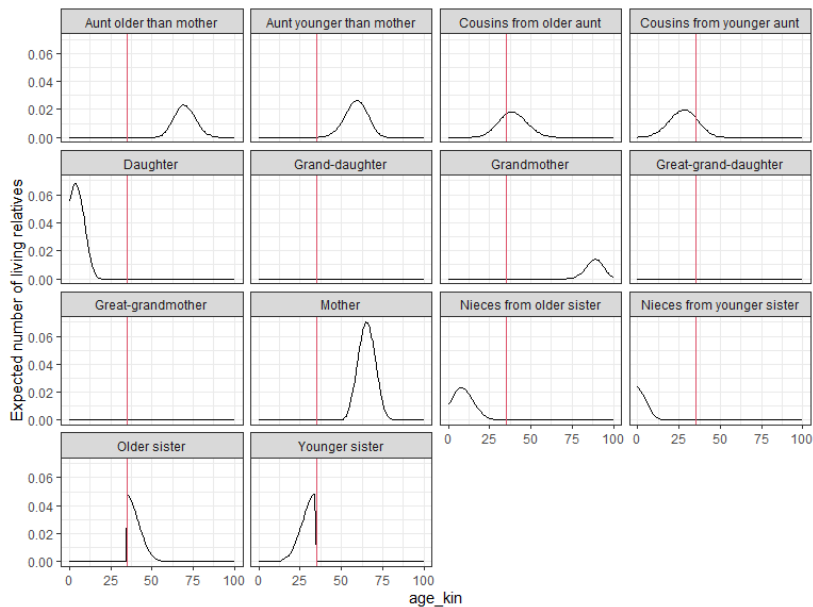
where:

- ▶ **U** a matrix with survival probabilities in the subdiagonal
- ▶ **F** a matrix with fertility rates in the first row

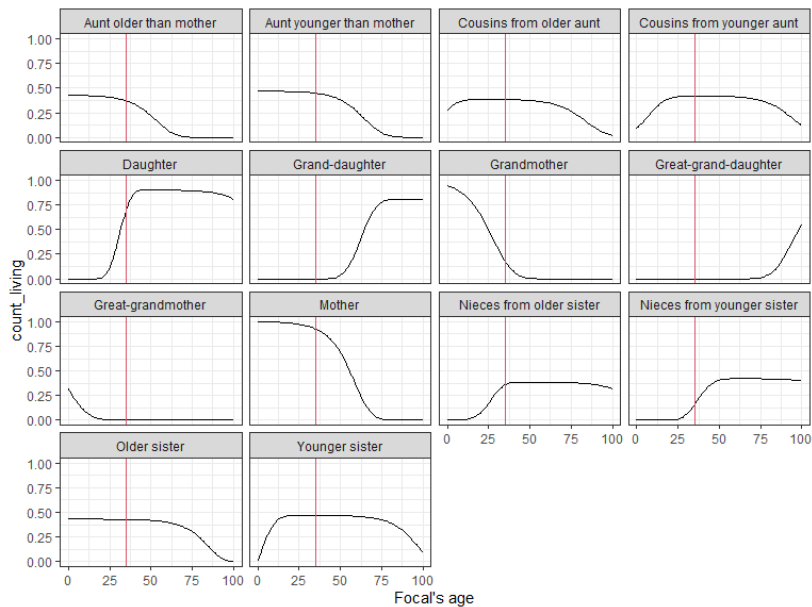
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<sup>14</sup>Caswell, H. (2019). The formal demography of kinship: A matrix formulation. *Demographic Research*, 41, 679–712

# Age distributions of kin



# Expected number of kin



# Daughters

Daughters (**a**) are the result of the reproduction of Focal:

$$\underbrace{\mathbf{a}(x+1)}_{\text{age structure of daughters at Focal's age } x+1} = \underbrace{\mathbf{U}\mathbf{a}(x)}_{\text{ageing and survival of existing daughters}} + \underbrace{\mathbf{F}\mathbf{e}_x}_{\text{new daughters (subsidy)}} \quad (8)$$

$$b(0) = \mathbf{0}.$$

where:

- ▶ **U** is a matrix with survival probabilities in the subdiagonal
- ▶ **F** is a matrix with fertility rates in the first row
- ▶ **F e<sub>x</sub>** is the subsidy vector
- ▶ **e<sub>x</sub>** is the unit vector for age *x*
- ▶ **b(0)** is the distribution of daughters at Focal's birth

# Mothers

The population of mothers ( $\mathbf{d}$ ) of Focal consists of at most a single individual:

$$\underbrace{\mathbf{d}(x+1)}_{\text{age structure of mothers at Focal's age } x+1} = \underbrace{\mathbf{U} \mathbf{d}(x)}_{\text{ageing and survival of existing mothers}} + \underbrace{\mathbf{0.}}_{\text{new mothers (subsidy)}} \quad (9)$$

$$d(0) = \pi.$$

where:

- ▶  $b(0)$  is the distribution of mothers at Focal's birth
- ▶  $\pi$  is the distribution of ages of mothers in the population

**Table 1:** Summary of the components of the kin model given in equations (4) and (5)

Symbol	Kin	Initial condition	Subsidy $\beta(x)$
<b>a</b>	daughters	<b>0</b>	<b><math>\mathbf{F}e_x</math></b>
<b>b</b>	granddaughters	<b>0</b>	<b><math>\mathbf{F}a(x)</math></b>
<b>c</b>	great-granddaughters	<b>0</b>	<b><math>\mathbf{F}b(x)</math></b>
<b>d</b>	mothers	<b><math>\pi</math></b>	<b>0</b>
<b>g</b>	grandmothers	$\sum_i \pi_i \mathbf{d}(i)$	<b>0</b>
<b>h</b>	great-grandmothers	$\sum_i \pi_i \mathbf{g}(i)$	<b>0</b>
<b>m</b>	older sisters	$\sum_i \pi_i \mathbf{a}(i)$	<b>0</b>
<b>n</b>	younger sisters	<b>0</b>	<b><math>\mathbf{F}d(x)</math></b>
<b>p</b>	nieces via older sisters	$\sum_i \pi_i \mathbf{b}(i)$	<b><math>\mathbf{F}m(x)</math></b>
<b>q</b>	nieces via younger sisters	<b>0</b>	<b><math>\mathbf{F}n(x)</math></b>
<b>r</b>	aunts older than mother	$\sum_i \pi_i \mathbf{m}(i)$	<b>0</b>
<b>s</b>	aunts younger than mother	$\sum_i \pi_i \mathbf{n}(i)$	<b><math>\mathbf{F}g(x)</math></b>
<b>t</b>	cousins from aunts older than mother	$\sum_i \pi_i \mathbf{p}(i)$	<b><math>\mathbf{F}r(x)</math></b>
<b>v</b>	cousins from aunts younger than mother	$\sum_i \pi_i \mathbf{q}(i)$	<b><math>\mathbf{F}s(x)</math></b>

<sup>15</sup>Caswell, H. (2019). The formal demography of kinship: A matrix formulation. *Demographic Research*, 41, 679–712

## Consider a baby born in Spain in 1950...

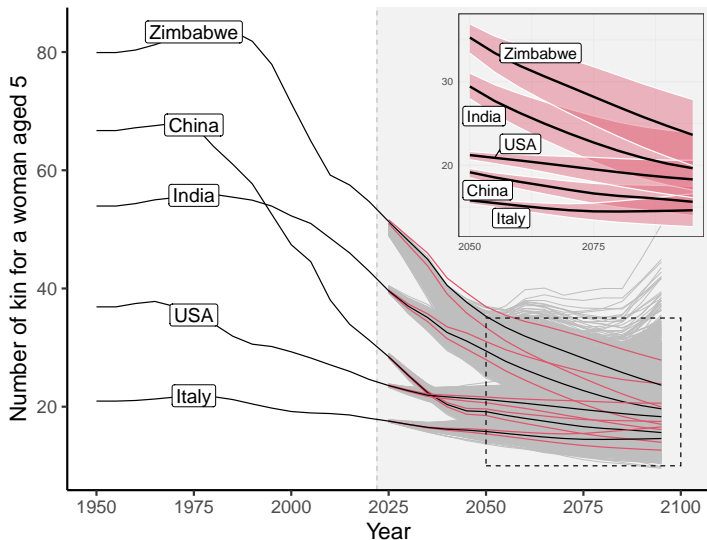
- ① How old were her grandparents when she was born, on average?
- ② How many living children did she have on her 70th birthday?
- ③ How many grandchildren?



# Break

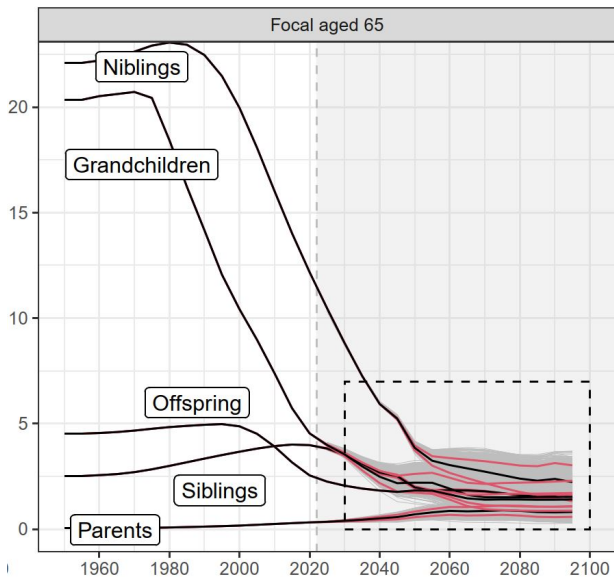
# Implementations

# Total number of kin (all kin combined) for a 5yo woman<sup>16</sup>



<sup>16</sup>Alburez-Gutierrez, D., Williams, I., and Hal Caswell (unpublished).  
Projections of kinship for all countries.

# Number of living kin for a 65yo in China<sup>17</sup>



<sup>17</sup>Alburez-Gutierrez, D., Williams, I., and Hal Caswell (unpublished).  
Projections of kinship for all countries.

# DemoKin: matrix kinship models in R

- ▶ Time-(in)variant models
- ▶ One/two-sex models
- ▶ Multistate models
- ▶ <https://github.com/IvanWilli/DemoKin>
- ▶ More in the lab session...



Hal Caswell



Ivan Williams



Xi Song

# Models vs reality

Discuss:

- ① What is the relationship between demographic models and reality?
- ② Would we expect kinship models to agree with 'empirical' observations of kinship?
- ③ Where can we find empirical data on kinship availability?