Matrix models: Markov chains for individual stochasticity

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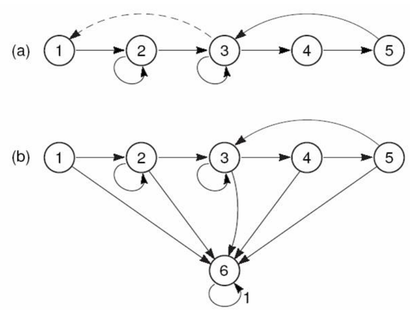
The demography of individuals

- individuals move through the life cycle
- eventual departure (death or otherwise)
- transitions among stages are stochastic
  - life table functions (survivorship, mortality, distribution of age at death)
  - discrete Markov chains
- stochasticity creates variation among individuals ("individual stochasticity")

Individual stochasticity

“Statistically speaking, human life is a random experiment and its outcome, survival or death, is subject to chance. If two people were subjected to the same risk of dying (force of mortality) during a calendar year, one might die during the year and the other survive. If a person was allowed to relive the year he survived the first time, he might not survive the second time."
Chiang (1979; Life Table and Mortality Analysis)

Individual Markov chain



stages: 1=calf, 2=immature female, 3=mature female, 4=mom, 5=resting post-mom, 6=dead

States: transient and absorbing

- transient state: will return only a finite number of times
- absorbing state: once you are in it you can't get out. Death.
- a chain with one or more absorbing states is an **absorbing Markov chain**

Markov chain formulation

Transition matrix[1]

$$\mathbf{P} = \left(\begin{array}{c|c} \mathbf{U} & \mathbf{0} \\ \hline \mathbf{M} & \mathbf{I} \end{array}\right) \qquad \text{transition matrix}$$

$$\begin{aligned} \mathbf{U} &= \text{transient matrix} \qquad \tau \times \tau \\ \mathbf{M} &= \text{mortality matrix} \qquad \alpha \times \tau \\ \mathbf{I} &= \text{identity matrix} \qquad \alpha \times \alpha \end{aligned}$$

[1] The Markov chain literature often writes **P** in transposed form. Always check.

Projection matrices and transition matrices

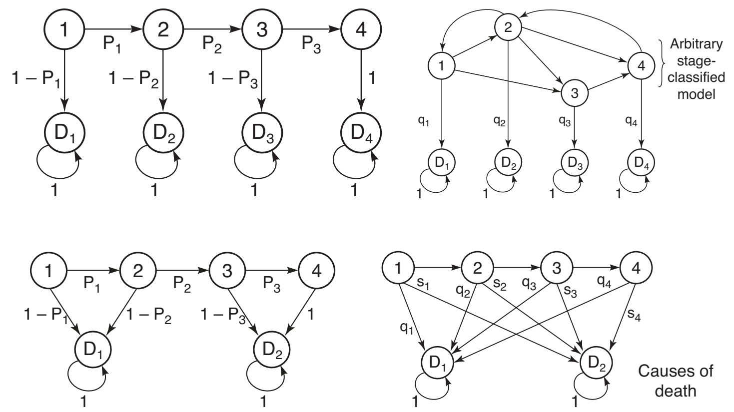
- population projection matrix projects *numbers of individuals* in each stage

$$\begin{aligned} \mathbf{n}(t) &= \text{vector of numbers} \\ \mathbf{n}(t+1) &= (\mathbf{U} + \mathbf{F})\,\mathbf{n}(t) \end{aligned}$$

- Markov chain transition matrix projects *probability distribution of individual state*

$$\begin{aligned} \mathbf{z}(t) &= \text{probability distribution of i-state} \\ \mathbf{z}(t+1) &= \left(\begin{array}{c|c} \mathbf{U} & \mathbf{0} \\ \hline \mathbf{M} & \mathbf{I} \end{array}\right) \mathbf{z}(t) \end{aligned}$$

Absorbing chain structures



Generality of the approach

transition/survival matrix **U**

- age-classified
- stage-classified
- multistate (age$\times$stage)

mortality (or absorption) matrix **M**

- death
- age or stage at death
- cause of death
- other than death (permanent emigration, . . .)

Basic concepts

- individual trajectories are stochastic
- trajectories pass through transient states
- eventual absorption is certain
- stochasticity in life trajectories: moments, means, variances, etc. over the trajectories

Pioneers of Markov chains in demography

Jan M. Hoem (former Director, MPIDR)

Gustav Feichtinger (Vienna Technical University)

Outcomes of interest

- occupancy (how much time spent in each transient state)
- longevity (how long until reaching an absorbing state)
- eventual fate (probability of each absorbing state)
- life lost (how much does a death in each stage cost)
- net reproductive rate (lifetime reproductive output)
- more

Occupancy and the fundamental matrix

Until you die, you move among transient states. How much time do you spend in each? Define:

$$\nu_{ij} = \text{time spent in } i, \text{ starting in } j$$

Define $\mathbf{N}_1$ as the matrix of *expected* occupancy time

$$\mathbf{N}_1(i,j) = E\left(\nu_{ij}\right)$$

and

$$\mathbf{N}_1 = (\mathbf{I} - \mathbf{U})^{-1}$$

The **fundamental matrix**

A digression: statistical moments

- suppose X is a random variable
- $E(X)$ = mean
- moments are the expectations of powers of the random variable

$$\begin{aligned} E(X) &= \text{first moment} \\ E(X^2) &= \text{second moment} \\ &\vdots \\ E(X^k) &= k\text{th moment} \end{aligned}$$

From moments to other things

$$\begin{aligned} m_i &= i\text{th moment of } X \\ V(X) &= m_2 - m_1^2 \\ SD(X) &= \sqrt{V(X)} \\ CV(X) &= \frac{SD(X)}{m_1} \end{aligned}$$

Individual stochasticity in occupancy

Moments of time spent in each transient stage

$$\mathbf{N}_k = \left(\; E\left(\nu_{ij}^k\right) \; \right)$$

$$\begin{aligned} \mathbf{N}_1 &= (\mathbf{I}_s - \mathbf{U})^{-1} \\ \mathbf{N}_2 &= \left(2\mathbf{N}_{\rm dg} - \mathbf{I}_s\right)\mathbf{N}_1 \\ \mathbf{N}_3 &= \left(6\mathbf{N}_{\rm dg}^2 - 6\mathbf{N}_{\rm dg} + \mathbf{I}_s\right)\mathbf{N}_1 \\ \mathbf{N}_4 &= \left(24\mathbf{N}_{\rm dg}^3 - 36\mathbf{N}_{\rm dg}^2 + 14\mathbf{N}_{\rm dg} - \mathbf{I}_s\right)\mathbf{N}_1 \end{aligned}$$

where

$\mathbf{N}_{\rm dg}$ = matrix with diagonal of $\mathbf{N}_1$ on the diagonal, zeros elsewhere start in j start in j

$$U = \begin{pmatrix} U(i,j) \end{pmatrix} \leftarrow \text{move to } i \qquad N = \begin{pmatrix} N(i,j) \end{pmatrix} \leftarrow \text{occupy } i$$

Some statistics of occupancy times

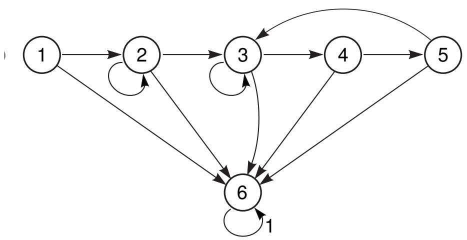
$$\left(V(\nu_{ij}) \right) = \mathbf{N}_2 - \mathbf{N}_1 \circ \mathbf{N}_1$$

$$\left(SD(\nu_{ij}) \right) = \left(\sqrt{V(\nu_{ij})} \right)$$

$$\left(CV(\nu_{ij}) \right) = \left(\frac{SD(\nu_{ij})}{E(\nu_{ij})} \right)$$

Right whale (1980)

stages: 1=calf, 2=immature female, 3=mature female, 4=mom, 5=resting post-mom, 6=dead



$$\mathbf{P} = \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ 0.9281 & 0.8461 & 0 & 0 & 0 & 0 \\ 0 & 0.1245 & 0.5930 & 0 & 0.9895 & 0 \\ 0 & 0 & 0.3964 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9683 & 0 & 0 \\ \hline 0.0719 & 0.0295 & 0.0105 & 0.0317 & 0.0105 & 1 \end{array} \right).$$

Right whale

$$\mathbf{N} = \begin{pmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 6.03 & 6.50 & 0.00 & 0.00 & 0.00 \\ 27.62 & 29.76 & 36.81 & 35.27 & 36.42 \\ 10.95 & 11.80 & 14.59 & 14.98 & 14.44 \\ 10.60 & 11.43 & 14.13 & 14.51 & 14.98 \end{pmatrix},$$

$$V\left(\ \nu_{ij}\ \right) = \begin{pmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 35.95 & 35.70 & 0.00 & 0.00 & 0.00 \\ 1243.09 & 1275.68 & 1318.32 & 1317.48 & 1318.56 \\ 197.26 & 202.53 & 209.71 & 209.47 & 209.72 \\ 194.68 & 200.38 & 209.60 & 209.72 & 209.47 \end{pmatrix},$$

$$\left(\ SD(\nu_{ij})\ \right) = \begin{pmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 6.00 & 5.97 & 0.00 & 0.00 & 0.00 \\ 35.26 & 35.72 & 36.31 & 36.30 & 36.31 \\ 14.04 & 14.23 & 14.48 & 14.47 & 14.48 \\ 13.95 & 14.16 & 14.48 & 14.48 & 14.47 \end{pmatrix}$$

Longevity

Longevity — the time to absorption

$$\eta_j = \text{time to death} \mid \text{start in stage } j$$

$$\mathbf{N}_1 = E\begin{pmatrix} \nu_{11} & \cdots & \nu_{1s} \\ \vdots & & \vdots \\ \nu_{s1} & \cdots & \nu_{ss} \end{pmatrix}$$

$$\text{column sums} = E\begin{pmatrix} \eta_1 & \cdots & \eta_s \end{pmatrix}$$

Mean longevity (life expectancy)

$$E(\eta_j) = \text{sum of column } j \text{ of } \mathbf{N}_1$$

or in vector form

$$\eta_1^{\mathsf{T}} = \mathbf{1}^{\mathsf{T}}\mathbf{N}_1$$

Individual stochasticity in longevity

Moments of time to absorption $\eta_k = E\left(\ \eta_1^k, \cdots, \eta_s^k\ \right)^{\mathsf{T}}$.

$$\begin{aligned} \eta_1^{\mathsf{T}} &= \mathbf{1}^{\mathsf{T}}\mathbf{N}_1 \\ \eta_2^{\mathsf{T}} &= \eta_1^{\mathsf{T}}\left(2\mathbf{N}_1 - \mathbf{I}_s\right) \\ \eta_3^{\mathsf{T}} &= \eta_1^{\mathsf{T}}\left(6\mathbf{N}_1^2 - 6\mathbf{N}_1 + \mathbf{I}_s\right) \\ \eta_4^{\mathsf{T}} &= \eta_1^{\mathsf{T}}\left(24\mathbf{N}_1^3 - 36\mathbf{N}_1^2 + 14\mathbf{N}_1 - \mathbf{I}_s\right). \end{aligned}$$

Statistics of longevity

$$\begin{aligned} V(\eta)^{\mathsf{T}} &= \eta_2 - \eta_1 \circ \eta_1 \\ SD(\eta) &= \sqrt{V(\eta)} \\ CV(\eta) &= \operatorname{diag}(\eta_1)\, SD(\eta) \\ Sk(\eta) &= \operatorname{diag}(V(\eta))^{-3/2}\left[\eta_3 - 3(\eta_1 \circ \eta_2) + 2(\eta_1 \circ \eta_1 \circ \eta_1)\right] \end{aligned}$$

start in j

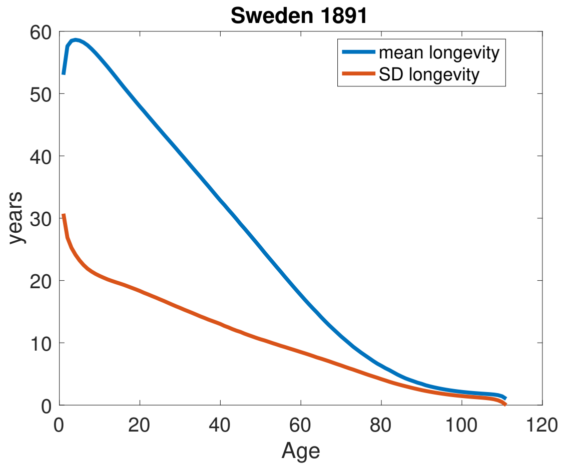
$$U = \begin{pmatrix} u(i,j) \end{pmatrix}$$ move to i

start in j

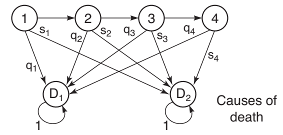
$$N = \begin{pmatrix} N(i,j) \end{pmatrix}$$ occupy i

$$\eta = \begin{pmatrix} \eta(j) \end{pmatrix}$$ longevity starting in j

Sweden 1891



Ultimate fates: eventual absorption



4 transient states, 2 absorbing states

$$\mathbf{U} = 4 \times 4 \qquad \mathbf{M} = 2 \times 2$$

$$\mathbf{P} = \left(\begin{array}{c|c} \mathbf{U} & \mathbf{0} \\ \hline \mathbf{M} & \mathbf{I} \end{array} \right) \qquad 6 \times 6$$

$$\mathbf{N} = (\mathbf{I} - \mathbf{U})^{-1} \qquad 4 \times 4$$

$$\mathbf{B} = \mathbf{MN} \qquad 2 \times 4$$

$$\mathbf{B}(i,j) = Pr\,(\text{absorbption in } i, \text{ starting in } j)$$

start in j

$U = \big(u(i,j) \big) \leftarrow$ move to i

start in j

$N = \big(N(i,j) \big) \leftarrow$ occupy i

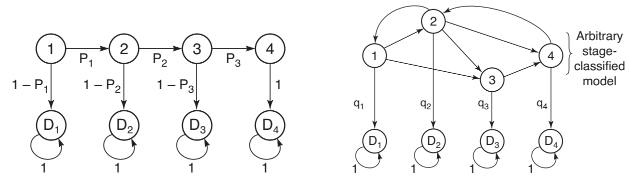
$\eta = \big(\eta(j) \big) \leftarrow$ longevity starting in j

start in j

$B = \big(B(i,j) \big) \leftarrow$ absorbed in i

The distribution of age (or stage) at death

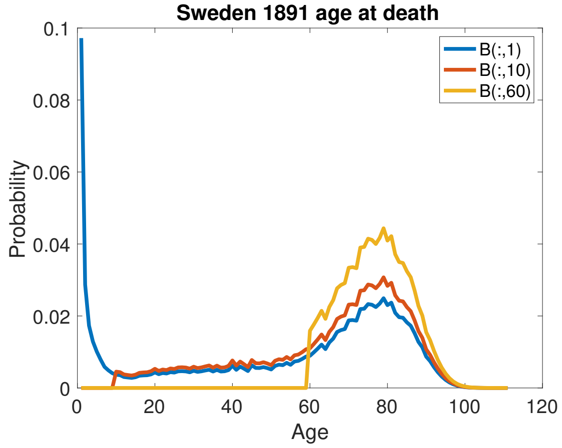
1. create absorbing states corresponding to stage at death



2. mortality matrix

$$\mathbf{M} = \begin{pmatrix} q_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & q_s \end{pmatrix}$$

Sweden 1891



The distribution of age (or stage) at death

3. transition matrix

$$\mathbf{P} = \left(\begin{array}{c|c} \mathbf{U} & \mathbf{0} \\ \hline \mathbf{M} & \mathbf{I} \end{array} \right)$$

4. fundamental matrix

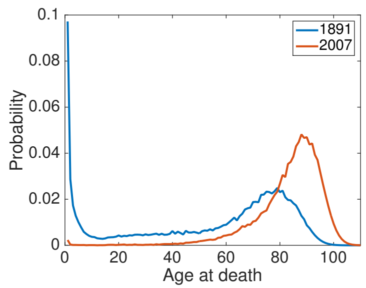
$$\mathbf{N} = (\mathbf{I} - \mathbf{U})^{-1}$$

5. distribution of age or stage at death

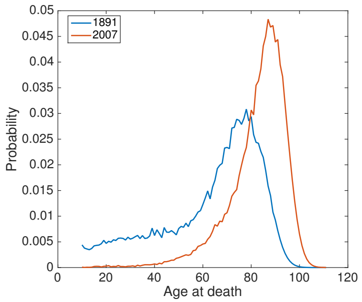
$$\begin{aligned} \mathbf{B} &= \mathbf{MN} \\ \mathbf{B}(i,j) &= Pr\,(\text{death in stage } i, \text{ starting in } j) \end{aligned}$$

Distribution of age at death

starting at birth $\mathbf{B}(:,1)$



starting at age 10 $\mathbf{B}(:,10)$



"Life lost" due to mortality

- How much life is lost due to a death in (st)age j?
- define

$$\mathbf{f} = \mathbf{B}(:,1)$$

distribution of age at death starting in stage 1

- *Expected* life lost[2]

$$\begin{aligned} \eta_1^{\dagger} &= \sum_x Pr(\text{death at } x)\ (\text{life exp at } x) \\ &= \mathbf{f}^{\mathsf{T}} \eta_1 \end{aligned}$$

[2] Vaupel (1986). Sometimes called life disparity.

"Life lost" due to mortality

We could calculate:

- *Moments* of life lost

$$\eta_k^\dagger = \mathbf{f}^\mathsf{T} \boldsymbol{\eta}_k \qquad k = 1, 2, \ldots$$

- Variance of life lost

$$V(\eta^\dagger) = \eta_2^\dagger - \left(\eta_1^\dagger\right)^2$$

Sweden 1891: Life lost

mean life lost

$$\eta_1^\dagger = 23.7y$$

variance in life lost

$$V(\eta^\dagger) = 602y^2$$

Implementation: how to do it?

1. find and partition a projection matrix $\mathbf{A} = \mathbf{U} + \mathbf{F}$
2. calculate the fundamental matrix $\mathbf{N}_1$
3. calculate higher moments of occupancy, $\mathbf{N}_2, \ldots$
4. calculate moments of longevity $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3$
5. calculate, as desired, statistics of longevity or occupancy (variance, SD, CV, skewness,...)
6. calculate distribution of age or stage at death

$$\begin{aligned} \mathbf{M} &= \mathrm{diag}\left[\mathbf{1}_s^\mathsf{T} (\mathbf{I}_s - \mathbf{U})\right] \\ \mathbf{B} &= \mathbf{M}\mathbf{N}_1 \end{aligned}$$

7. calculate moments of life lost
8. and more

Net reproductive rate R_0

- mean lifetime reproductive output
- per-generation growth rate
- indicator function for population growth

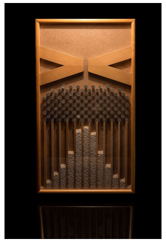
$$\begin{aligned} R_0 &< 1 \rightarrow \text{population decline} \\ R_0 &> 1 \rightarrow \text{population growth} \end{aligned}$$

In general

$$R_0 = \max \mathrm{eig}\, \mathbf{F}\mathbf{N}$$

Stochasticity

- the matrix **P** contains the transition probabilities
- they apply to every one, exactly the same
- the variation among individuals is due to stochastic outcomes of the transitions



https://upload.wikimedia.org/wikipedia/commons/d/dc/Galton_box.webm

Notation

Matrices are bold face upper case (**U**), vectors are bold face lower case (**x**). $\mathbf{X}^{\mathsf{T}}$ is the transpose of **X**. The product $\mathbf{A} \circ \mathbf{B}$ is the element-by-element product of **A** and **B**. Some special vectors and matrices:

$$
\begin{aligned}
\mathbf{1}_n &= \text{vector of ones, } n \times 1 \\
\boldsymbol{e}_i &= i\text{th unit vector} \\
\boldsymbol{I}_n &= \text{identity matrix, } n \times n \\
\mathbf{0}_{m\times n} &= m \times n \text{ matrix of zeros}
\end{aligned}
$$