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(9665/FM03) Unit FP2 Pure Mathematics

Mark scheme

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2 2 1 X F M 0 3 / M S

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Key to mark scheme abbreviations

M	Mark is for method
m	Mark is dependent on one or more M marks and is for method
A	Mark is dependent on M or m marks and is for accuracy
B	Mark is independent of M or m marks and is for method and accuracy
E	Mark is for explanation
✓ or ft	Follow through from previous incorrect result
CAO	Correct answer only
CSO	Correct solution only
AWFW	Anything which falls within
AWRT	Anything which rounds to
ACF	Any correct form
AG	Answer given
SC	Special case
oe	Or equivalent
A2, 1	2 or 1 (or 0) accuracy marks
–x EE	Deduct x marks for each error
NMS	No method shown
PI	Possibly implied
SCA	Substantially correct approach
sf	Significant figure(s)
dp	Decimal place(s)

Q	Answer	Marks	Comments
1(a)	Reflection in [the plane] $x = y$	M1 A1	Reflection $x = y$ oe M0 If more than one transformation SC1 for 'reflect' or 'reflected' and $x = y$
		2	

Q	Answer	Marks	Comments
1(b)	y -axis	B1	oe eg $x = z = 0$
		1	

	Question 1 Total	3	
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Q	Answer	Marks	Comments
2	$8\mathbf{u} \times \mathbf{u} - 16\mathbf{u} \times \mathbf{v} + 12\mathbf{u} \times \mathbf{w}$ $+ 6\mathbf{v} \times \mathbf{u} - 12\mathbf{v} \times \mathbf{v} + 9\mathbf{v} \times \mathbf{w}$ $+ 12\mathbf{w} \times \mathbf{u} - 24\mathbf{w} \times \mathbf{v} + 18\mathbf{w} \times \mathbf{w}$ $= -16\mathbf{u} \times \mathbf{v} + 12\mathbf{u} \times \mathbf{w} + 6\mathbf{v} \times \mathbf{u} + 9\mathbf{v} \times \mathbf{w}$ $+ 12\mathbf{w} \times \mathbf{u} - 24\mathbf{w} \times \mathbf{v}$ $= -16\mathbf{u} \times \mathbf{v} - 12\mathbf{w} \times \mathbf{u} - 6\mathbf{u} \times \mathbf{v} + 9\mathbf{v} \times \mathbf{w}$ $+ 12\mathbf{w} \times \mathbf{u} + 24\mathbf{v} \times \mathbf{w}$ $= -22\mathbf{u} \times \mathbf{v} + 33\mathbf{v} \times \mathbf{w}$ $= 165\mathbf{i} - 44\mathbf{j}$	<p>M1</p> <p>M1</p> <p>M1</p> <p>B1</p> <p>A1CSO</p>	<p>Expansion of brackets at least five correct PI</p> <p>Condone lack of \times or \wedge except for the final mark but check order of letters in each term.</p> <p>$\mathbf{u} \times \mathbf{u} = \mathbf{v} \times \mathbf{v} = \mathbf{w} \times \mathbf{w} = \mathbf{0}$ stated or used</p> <p>Clear evidence of the consistent use of any two of: $\mathbf{u} \times \mathbf{w} = -\mathbf{w} \times \mathbf{u}$ $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$ $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$</p> <p>Either $165\mathbf{i}$ or $-44\mathbf{j}$</p> <p>$165\mathbf{i} - 44\mathbf{j}$ with no other terms and correct notation used throughout</p>
		5	

	Question 2 Total	5	
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Q	Answer	Marks	Comments
3	<p>When $n = 1$, $u_1 = \frac{5+1}{5-3} = \frac{6}{2} = 3$</p> <p>Assume formula true for $n = k$ (*), integer $k \geq 1$, so</p> $u_{k+1} = \frac{9\left(\frac{5k+1}{5k-3}\right) - 5}{5\left(\frac{5k+1}{5k-3}\right) - 1}$ $u_{k+1} = \frac{9(5k+1) - 5(5k-3)}{5(5k+1) - 1(5k-3)}$ $u_{k+1} = \frac{20k+24}{20k+8}$ $u_{k+1} = \frac{5k+6}{5k+2}$ $u_{k+1} = \frac{5(k+1)+1}{5(k+1)-3}$ <p>Hence formula is true for $n = k+1$ (**) and since true for $n = 1$ (***), formula is true for $n = 1, 2, 3, \dots$ by induction (****)</p>	<p>B1</p> <p>M1</p> <p>m1</p> <p>A1</p> <p>A1</p> <p>E1</p>	<p>Correct values to show formula true for $n = 1$</p> <p>Assumes formula true for $n = k$ and considers $u_{k+1} = \frac{9\left(\frac{5k+1}{5k-3}\right) - 5}{5\left(\frac{5k+1}{5k-3}\right) - 1}$ oe</p> <p>Simplifies into a single fraction</p> <p>Be convinced</p> <p>Must have (*), (**), (***), present, previous 5 marks scored and a final statement (****) clearly indicating that it relates to positive integers</p>
		6	
	Question 3 Total	6	

Q	Answer	Marks	Comments
4	<p>Aux equation</p> $m^2 + 3m + 2 = 0 \quad m = \frac{-3 \pm \sqrt{9-8}}{2}$ <p>$[y_{CF} =] \quad Ae^{-x} + Be^{-2x}$</p> <p>$y_{PI} = a \sin 2x + b \cos 2x$</p> <p>$(-4a-6b+2a)\sin 2x + (-4b+6a+2b)\cos 2x$ $= 2\sin 2x + 14\cos 2x$</p> <p>$-2a-6b=2 \quad 6a-2b=14$</p> <p>$y_{PI} = 2\sin 2x - \cos 2x$</p> <p>$[y_{GS} =] \quad Ae^{-x} + Be^{-2x} + 2\sin 2x - \cos 2x$</p>	<p>M1</p> <p>A1</p> <p>M1</p> <p>m1</p> <p>A1</p> <p>A1</p> <p>B1ft</p>	<p>Using quadratic formula oe on correct aux. equation. PI by correct values of m seen/used</p> <p>Correct CF</p> <p>PI by correct y_{PI}</p> <p>y'_{PI} and y''_{PI} both of the form $p \sin 2x + q \cos 2x$ and substitution into the DE to form an equation in x PI</p> <p>Seen or used, PI by $a=2$, $b=-1$</p> <p>Correct y_{PI} seen/used</p> <p>c's CF + c's PI but must have exactly two arbitrary constants</p>
		7	
	Question 4 Total	7	

Q	Answer	Marks	Comments
5(a)	$\sin(2r+1)x - \sin(2r-1)x = 2\cos\frac{4rx}{2}\sin\frac{2x}{2}$ $\frac{1}{2}[\sin(2r+1)x - \sin(2r-1)x] = \cos 2rx \sin x$	B1	Replaces A by $(2r+1)x$ and B by $(2r-1)x$ and obtains $\cos 2rx \sin x$
		1	

Q	Answer	Marks	Comments
5(b)	$\sum_{r=1}^n \cos 2rx \sin x$ $= \sum_{r=1}^n \frac{1}{2}[\sin(2r+1)x - \sin(2r-1)x]$ $\sum_{r=1}^n \sin x (1 - 2 \sin^2 rx)$ $= \frac{1}{2} \left[\begin{array}{l} \sin 3x - \sin x + \sin 5x - \sin 3x + \dots \\ + \sin(2n-1)x - \sin(2n-3)x \\ + \sin(2n+1)x - \sin(2n-1)x \end{array} \right]$ $\sin x \left(n - 2 \sum_{r=1}^n \sin^2 rx \right)$ $= \frac{1}{2} [\sin(2n+1)x - \sin x]$ $\sum_{r=1}^n \sin^2 rx = \frac{n}{2} - \frac{1}{4 \sin x} (\sin(2n+1)x - \sin x)$ $= \frac{n}{2} - \frac{1}{4 \sin x} \left(2 \cos \left[\frac{(2n+2)x}{2} \right] \sin \left(\frac{2nx}{2} \right) \right)$ $\sum_{r=1}^n \sin^2 rx = \frac{n}{2} - \frac{\sin nx \cos(n+1)x}{2 \sin x}$	<p>M1</p> <p>M1</p> <p>M1</p> <p>B1</p> <p>A1</p> <p>A1</p>	<p>Uses result in part (a)</p> <p>Uses $\cos 2rx = \pm 1 \pm 2 \sin^2 rx$</p> <p>Uses method of differences showing at least terms that cancel</p> <p>$\sum_{r=1}^n 1 = n$ seen/used at any stage</p> <p>$\sin(2n+1)x - \sin x$</p> <p>AG Printed result convincingly obtained</p>
		6	

	Question 5 Total	7	
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Q	Answer	Marks	Comments
6	$u = x^2 + 1 \Rightarrow du = 2x dx$ $dv = e^{-x} dx \Rightarrow v = -e^{-x}$ $\int (x^2 + 1)e^{-x} dx$ $= (x^2 + 1)(-e^{-x}) - \int -e^{-x} 2x dx$ $= -(x^2 + 1)e^{-x} + 2 \left(-xe^{-x} + \int e^{-x} dx \right)$ $= -(x^2 + 1)e^{-x} - 2xe^{-x} - 2e^{-x} \quad [+c]$ $I = \int_0^{\infty} (x^2 + 1)e^{-x} dx$ $= \lim_{a \rightarrow \infty} \int_0^a (x^2 + 1)e^{-x} dx$ $= \lim_{a \rightarrow \infty} (-a^2 e^{-a} - 2a e^{-a} - 3e^{-a}) - (-3)$ $\lim_{a \rightarrow \infty} (a^n e^{-a}) = 0$ $\int_0^{\infty} (x^2 + 1)e^{-x} dx = 3$	<p>B1</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>M1</p> <p>A1</p> <p>E1</p> <p>A1</p>	<p>Applies integration by parts twice</p> <p>Correct integration of $(x^2 + 1)e^{-x}$</p> <p>Evidence of limit ∞ replaced by a (oe) $\lim_{a \rightarrow \infty}$ seen at any stage with no remaining lim relating to 0</p> <p>$F(a) - F(0)$</p> <p>ACF</p> <p>Stated in general form or for $n = 1$ or for $n = 2$</p> <p>First 6 marks must have been scored but can be awarded even if previous E1 not scored</p>
		8	
	Question 6 Total	8	

Q	Answer	Marks	Comments
7(a)	I.F. is $e^{\int (x^{-1} - (x+2)^{-1}) dx} = e^{\ln x - \ln(x+2)}$	M1	I.F. identified and integration attempted
		A1	Correct integration
	I.F. = $\frac{x}{x+2}$	A1	Correct integrating factor
	$\frac{x}{x+2} y = \int \frac{x^2}{x+2} dx$	M1	Multiplying both sides of the given DE by their I.F. and integrating LHS to get $y \times$ I.F. RHS must not remain as x
	$\frac{x}{x+2} y = \int x - 2 + \frac{4}{x+2} dx$	M1	Expressing $\frac{x^2}{x+2}$ in the form $x + a + \frac{b}{x+2}$ PI
	$\frac{x}{x+2} y = \frac{x^2}{2} - 2x + 4\ln(x+2) [+A]$	A1ft	ft on candidate's non-zero numerical values for a and b ; condone absence of constant of integration
	$\frac{x}{x+2} y = \frac{x^2}{2} - 2x + 4\ln(x+2) + A$	A1	Correct GS. Apply ISW if incorrect further rearrangement
		7	

Q	Answer	Marks	Comments
7(b)	Substituting $\frac{dy}{dx} = 0$ and $x = 2$ into the DE	M1	Uses $\frac{dy}{dx} = 0$ when $x = 2$ to obtain either a non-zero value for y or a value for the constant in their GS
	$0 + \left(\frac{1}{2} - \frac{1}{4}\right)y = 2 \Rightarrow y = 8$		
	$\frac{2}{4} \times 8 = 2 - 4 + 4\ln(4) + A \Rightarrow A = 6 - 4\ln(4)$		
	$y = \frac{x+2}{x} \left(\frac{x^2}{2} - 2x + 4\ln(x+2) + 6 - 4\ln 4 \right)$	A1	$y = f(x)$ with ACF for $f(x)$
		3	

	Question 7 Total	10	
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Q	Answer	Marks	Comments
8(a)	[distance =] $\frac{14}{\sqrt{1^2 + 1^2 + 1^2}}$	M1	PI
	$= \frac{14}{\sqrt{3}}$	A1	ACF but must be exact eg $\frac{14\sqrt{3}}{3}$
		2	

Q	Answer	Marks	Comments
8(b)(i)	General point on L $\left(3\lambda + 2, 2\lambda - 1, \frac{\lambda + 4}{2} \right)$	B1	ACF
	$\begin{bmatrix} 3\lambda + 2 \\ 2\lambda - 1 \\ \frac{1}{2}\lambda + 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 14$	M1	Substitutes general point for \mathbf{r} in the equation of Π_1 and evaluates the scalar product to form an equation for λ oe
	$\Rightarrow (3\lambda + 2) + (2\lambda - 1) + \frac{\lambda}{2} + 2 = 14$		
	$\Rightarrow \frac{11}{2}\lambda + 3 = 14 \Rightarrow \lambda = 2 \Rightarrow P(8, 3, 3)$	A1	(8, 3, 3) Condone $\begin{bmatrix} 8 \\ 3 \\ 3 \end{bmatrix}$
		3	

Q	Answer	Marks	Comments
8(b)(ii)	Direction vector for L: $\mathbf{d} = \begin{bmatrix} 3 \\ 2 \\ 0.5 \end{bmatrix}$	B1	Seen or used
	$\mathbf{n} \cdot \mathbf{d} = 3 + 2 + 0.5$	M1	Finds a numerical expression for the scalar product of the normal to Π_1 and \mathbf{d} ; ft one error in \mathbf{d}
	$\sqrt{1^2 + 1^2 + 1^2} \sqrt{3^2 + 2^2 + 0.5^2} \cos \theta = 5.5$	m1	ft one error in \mathbf{d}
	Angle = $90^\circ - \cos^{-1}\left(\frac{11}{\sqrt{159}}\right) = 60.7^\circ$	A1	CAO 60.7° (Condone missing $^\circ$)
		4	

Q	Answer	Marks	Comments
8(c)	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ [Direction cosines] $\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$	 M1 A1 A1ft	$\text{or } \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ Vector product of the two normal vectors, in either order, attempted Correct evaluation of the vector product $\text{or } \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$ oe exact form ft on incorrect evaluation of the vector product
		3	
	Question 8 Total	12	

	Answer	Marks	Comments
9(a)	$\det \mathbf{M}$ $= 4(12 - k - 1) - 3(15 - k - 1) + k(5 - 4)$ $= 44 - 4k - 42 + 3k + k$	M1	Correct method to expand $\det \mathbf{M}$
	$\det \mathbf{M} = 2 \neq 0$ so \mathbf{M} is a non-singular matrix	A1	$\det \mathbf{M} = 2 \neq 0$ with conclusion
		2	

Q	Answer	Marks	Comments
9(b)	Cofactor matrix		
	$\begin{bmatrix} 11-k & k-14 & 1 \\ k-9 & 12-k & -1 \\ 3-k & k-4 & 1 \end{bmatrix}$	M1	One complete row or column correct
		A2	All nine entries correct else A1 for at least six entries correct
	Inverse matrix $\mathbf{M}^{-1} =$ $\frac{1}{2} \begin{bmatrix} 11-k & k-9 & 3-k \\ k-14 & 12-k & k-4 \\ 1 & -1 & 1 \end{bmatrix}$	M1	Transpose of their cofactors with no more than one further error and division by their $\det \mathbf{M} (\neq 0)$
		A1	Correct \mathbf{M}^{-1} scores 5 marks
		5	

Q	Answer	Marks	Comments
9(c)	<p>When $k = 5$, $\mathbf{M}^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -4.5 & 3.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \end{bmatrix}$</p> <p>$\begin{bmatrix} 3 & -2 & -1 \\ -4.5 & 3.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} \lambda \\ -1+\lambda \\ 4-2\lambda \end{bmatrix}$</p> <p>$= \begin{bmatrix} 3\lambda + 2 - 2\lambda - 4 + 2\lambda \\ -4.5\lambda - 3.5 + 3.5\lambda + 2 - \lambda \\ 0.5\lambda + 0.5 - 0.5\lambda + 2 - \lambda \end{bmatrix}$</p> <p>$= \begin{bmatrix} 3\lambda - 2 \\ -2\lambda - 1.5 \\ -\lambda + 2.5 \end{bmatrix}$</p> <p>$\left(\mathbf{r} - \begin{bmatrix} -2 \\ -1.5 \\ 2.5 \end{bmatrix} \right) \times \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} = \mathbf{0}$</p>	<p>M1</p> <p>M1</p> <p>A1ft</p> <p>m1</p> <p>A1</p>	<p>Substitutes $k = 5$ into their \mathbf{M}^{-1} from (b) or correct \mathbf{M}^{-1} from a calculator</p> <p>Product of candidate's \mathbf{M}^{-1} when $k = 5$ and RHS of vector equation for L. Condone one miscopy.</p> <p>At least two of the three components correct; accept unsimplified versions</p> <p>ft on candidate's three components each of the form $a\lambda + b$ with no more than one zero value for the numerical constants a, b.</p> <p>Condone r and/or 0 not underlined</p>
		5	
	Question 9 Total	12	

Q	Answer	Marks	Comments
10(a)	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$ $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots$ $\tanh^{-1}x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ $= \frac{1}{2} [\ln(1+x) - \ln(1-x)]$ $\tanh^{-1}x = \frac{1}{2} \left(2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \dots \right)$ $\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$	<p>B1</p> <p>M1</p> <p>A1</p>	<p>Series expansion for $\ln(1-x)$ at least up to x^3</p> <p>AG printed answer obtained convincingly with no relevant terms missing in other series above.</p>
		3	

Q	Answer	Marks	Comments
10(b)(i)	$y = \tan x; \frac{dy}{dx} = \sec^2 x;$ $\frac{d^2 y}{dx^2} = 2 \sec^2 x \tan x = 2 \tan^3 x + 2 \tan x$ $\frac{d^3 y}{dx^3} = 2 \sec^2 x + 6 \tan^2 x \sec^2 x$ $\frac{d^3 y}{dx^3} = 6 \tan^4 x + 8 \tan^2 x + 2$ $\frac{d^4 y}{dx^4} = 24 \tan^3 x \sec^2 x + 16 \tan x \sec^2 x$ $\frac{d^4 y}{dx^4} = 24 \tan^5 x + 40 \tan^3 x + 16 \tan x$ $\frac{d^5 y}{dx^5} =$ $120 \tan^4 x \sec^2 x + 120 \tan^2 x \sec^2 x + 16 \sec^2 x$ $\Rightarrow \frac{d^5 y}{dx^5} = 16 \text{ when } x = 0$	<p>B1</p> <p>B1</p> <p>M1</p> <p>A1</p>	<p>ACF Correct first two derivatives eg $y' = 1 + y^2; y'' = 2y y'$</p> <p>ACF Correct third derivative eg $y''' = 2(y')^2 + 2y y''$</p> <p>Correct method to differentiate two powers of $\tan x$ or two powers of $\sec x$ or a term of the form $\tan^2 x \sec^2 x$ oe or both $(y')^2$ and $y y''$ terms when finding the fourth derivative eg $y^{(iv)} = 6y' y'' + 2y y'''$</p> <p>eg $y^{(v)} = 6(y'')^2 + 8y' y''' + 2y y^{(iv)}$</p> <p>AG Printed answer obtained convincingly.</p>
		4	

Q	Answer	Marks	Comments
10(b)(ii)	<p>From part (b)(i) $y(0) = 0$; $y'(0) = 1$; $y''(0) = 0$; $y'''(0) = 2$; $y^{(iv)}(0) = 0$; $y^{(v)}(0) = 16$</p> $\tan x = 0 + x + \frac{0x^2}{2!} + \frac{2x^3}{3!} + \frac{0x^4}{4!} + \frac{16x^5}{5!} + \dots$ $\tanh^{-1}x - \tan x =$ $x + \frac{x^3}{3} + \frac{x^5}{5} + \dots - \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots \right)$ $\tanh^{-1}x - \tan x = \frac{x^5}{15} + \dots$	<p>B1</p> <p>M1</p> <p>A1</p>	<p>All six stated/used with at least four correct. Accept if seen in (b)(i)</p> <p>Maclaurin series applied for the candidate's at least three non-zero values</p> <p>AG Printed answer obtained convincingly</p>
		3	

Q	Answer	Marks	Comments
10(c)	$\cos 2x = 1 - \frac{4x^2}{2} + O(x^4)$ $\left[\frac{\tan x + \tanh^{-1}x - 2x}{x(1 - \cos 2x)} \right]$ $= \frac{2x + \frac{2}{3}x^3 + \frac{1}{3}x^5 + \dots - 2x}{x\left(1 - 1 + \frac{4}{2}x^2 + O(x^4)\right)}$ $\lim_{x \rightarrow 0} \left[\frac{\tan x + \tanh^{-1}x - 2x}{x(1 - \cos 2x)} \right]$ $= \lim_{x \rightarrow 0} \left[\frac{\frac{2}{3}x^3 + O(x^5)}{\frac{4}{2}x^3 + O(x^5)} \right]$ $= \lim_{x \rightarrow 0} \left[\frac{\frac{2}{3} + O(x^2)}{\frac{4}{2} + O(x^2)} \right] \quad [\text{so the limit exists}]$ $= \frac{1}{3}$	<p>B1</p> <p>M1</p> <p>m1</p> <p>A1</p>	<p>$\cos 2x = 1 - \frac{4x^2}{2} + \dots$ oe seen or used</p> <p>Substitutes series expansions</p> <p>Divides numerator and denominator by x^3 to reach the form $\lim_{x \rightarrow 0} \left[\frac{P + O(x^2)}{Q + O(x^2)} \right]$, so limit exists</p> <p>$= \frac{P}{Q}$</p> <p>In place of $O(\quad)$ we may see equivalent term(s)</p> <p>$= \frac{1}{3}$</p>
		4	
	Question 10 Total	14	

Q	Answer	Marks	Comments
11(a)	$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 2$ $S = 2\pi \int_0^{\sqrt{3}} 2t \sqrt{4t^2 + 4} \, dt$ $S = 8\pi \left[\frac{1}{3} (t^2 + 1)^{1.5} \right]_0^{\sqrt{3}}$ $S = \frac{8\pi}{3} (8 - 1) = \frac{56\pi}{3}$	M1 M1 A1 M1 A1	Both attempted, at least one correct oe $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$ oe Substitutes into a correct formula for the surface area S . Condone incorrect/absent limits and absent dt Correct definite integral Integrates $\lambda t \sqrt{t^2 + 1}$ or $\beta \sqrt{x + 1}$ correctly AG Printed answer obtained convincingly
		5	

Q	Answer	Marks	Comments
11(b)	$s = \int_0^{\sqrt{3}} \sqrt{4t^2 + 4} \, dt = 2 \int_0^{\sqrt{3}} \sqrt{t^2 + 1} \, dt$ $\int \sqrt{t^2 + 1} \, dt = t \sqrt{t^2 + 1} - \int t \frac{t}{\sqrt{t^2 + 1}} \, dt$ $\int \sqrt{t^2 + 1} \, dt = t \sqrt{t^2 + 1} - \int \sqrt{t^2 + 1} - \frac{1}{\sqrt{t^2 + 1}} \, dt$ $\int \sqrt{t^2 + 1} \, dt = t \sqrt{t^2 + 1} + \sinh^{-1} t - \int \sqrt{t^2 + 1} \, dt$ $2 \int_0^{\sqrt{3}} \sqrt{t^2 + 1} \, dt = \left[t \sqrt{t^2 + 1} + \sinh^{-1} t \right]_0^{\sqrt{3}}$ $[\text{Arc length } OP] = 2\sqrt{3} + \sinh^{-1}(\sqrt{3})$	M1 M1 A1 m1 A1 A1 A1	oe Substitutes into a correct formula for the arc length s . ft derivatives from part (a) if incorrect. Condone incorrect/absent limits and absent dt Uses integration by parts with $u = \sqrt{t^2 + 1}$ and $dv = dt$ $\int \sqrt{t^2 + 1} \, dt = t \sqrt{t^2 + 1} - \int t \frac{t}{\sqrt{t^2 + 1}} \, dt$ Uses $\frac{t^2}{\sqrt{t^2 + 1}} = \sqrt{t^2 + 1} - \frac{1}{\sqrt{t^2 + 1}}$ AG Printed answer obtained convincingly
		7	

Q	Answer	Marks	Comments
11(b) ALT	$s = \int_0^{\sqrt{3}} \sqrt{4t^2 + 4} \, dt = 2 \int_0^{\sqrt{3}} \sqrt{t^2 + 1} \, dt$ <p>Let $t = \sinh u \Rightarrow$</p> $s = 2 \int_0^{\sinh^{-1}\sqrt{3}} \sqrt{\sinh^2 u + 1} \cosh u \, du$ $s = 2 \int_0^{\sinh^{-1}\sqrt{3}} \cosh^2 u \, du$ $s = \int_0^{\sinh^{-1}\sqrt{3}} (1 + \cosh 2u) \, du$ $s = \left[u + \frac{1}{2} \sinh 2u \right]_0^{\sinh^{-1}\sqrt{3}} =$ $\sinh^{-1}\sqrt{3} + \frac{2}{2} \sinh(\sinh^{-1}\sqrt{3}) \cosh(\sinh^{-1}\sqrt{3})$ $= \sinh^{-1}(\sqrt{3}) + \sqrt{3} \sqrt{1 + (\sqrt{3})^2}$ <p>[Arc length OP] $= 2\sqrt{3} + \sinh^{-1}(\sqrt{3})$</p>	<p>M1</p> <p>M1</p> <p>A1</p> <p>m1</p> <p>A1</p> <p>A1</p> <p>A1</p>	<p>oe Substitutes into a correct formula for the arc length s. ft derivatives from part (a) if incorrect. Condone incorrect/absent limits and absent dt</p> <p>Uses a suitable substitution to integrate. Ignore limits</p> <p>Ignore limits</p> <p>Ignore limits</p> <p>Correct integration with correct limits</p> <p>PI by the next line</p> <p>AG Printed answer obtained convincingly</p>
		7	
	Question 11 Total	12	

Q	Answer	Marks	Comments
12(a)(i)	$z^n = \cos n\theta + i \sin n\theta$ $z^{-n} = \cos(-n\theta) + i \sin(-n\theta)$ $= \cos(n\theta) - i \sin(n\theta)$ $z^n + \frac{1}{z^n} = 2 \cos n\theta$	M1 B1 A1	$z^n = \cos n\theta + i \sin n\theta$ <p>Need both lines or multiplies both numerator and denominator of $\frac{1}{z^n}$ by $\cos(n\theta) - i \sin(n\theta)$</p> <p>AG [M1B0A1 is possible eg for those who just quote the result $z^{-n} = \cos(n\theta) - i \sin(n\theta)$]</p>
		3	

Q	Answer	Marks	Comments
12(a)(ii)	$(-64 \sin^6 \theta)(4 \cos^2 \theta) = \left(z - \frac{1}{z}\right)^4 \left(z^2 - \frac{1}{z^2}\right)^2$ $= \left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right) \left(z^4 - 2 + \frac{1}{z^4}\right)$ $-256 \sin^6 \theta \cos^2 \theta = z^8 + \frac{1}{z^8} - 4 \left(z^6 + \frac{1}{z^6}\right) +$ $+ 4 \left(z^4 + \frac{1}{z^4}\right) + 4 \left(z^2 + \frac{1}{z^2}\right) - 10$ $= 2 \cos 8\theta - 8 \cos 6\theta + 8 \cos 4\theta + 8 \cos 2\theta - 10$ $128 \sin^6 \theta \cos^2 \theta$ $= 5 - 4 \cos 2\theta - 4 \cos 4\theta + 4 \cos 6\theta - \cos 8\theta$	M1 A1 A1 A1	<p>M1 for expansions of both brackets on RHS attempted with at least one correct.</p> <p>A1 for all four brackets raised to powers expanded correctly</p> <p>RHS of previous line multiplied out and result in part (a)(i) used</p> <p>AG Printed answer obtained convincingly</p>
		4	

Q	Answer	Marks	Comments
12(b)(i)	$r = 32 \sin^3 \theta \cos \theta$ $\frac{dr}{d\theta} = 96 \sin^2 \theta \cos^2 \theta - 32 \sin^4 \theta$ At P $32 \sin^2 \theta (3 \cos^2 \theta - \sin^2 \theta) = 0$ $\Rightarrow \tan \theta = \sqrt{3}$ $\theta = \frac{\pi}{3}$ $\Rightarrow r = 6\sqrt{3} \quad \Rightarrow P\left(6\sqrt{3}, \frac{\pi}{3}\right)$	M1 m1 A1 A1	$\frac{dr}{d\theta} = \pm a \sin^2 \theta \cos^2 \theta \pm b \sin^4 \theta$ Solves $\frac{dr}{d\theta} = 0$ using a valid method to obtain a positive value for either $\tan \theta$ or $\cos \theta$ or $\sin \theta$ PI if $\theta = \frac{\pi}{3}$ $\theta = \frac{\pi}{3}$ $r = 6\sqrt{3}$
		4	

Q	Answer	Marks	Comments
12(b)(ii)	$\text{Area} = \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (32 \sin^3 \theta \cos \theta)^2 [d\theta]$ $= 4 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 128 \sin^6 \theta \cos^2 \theta [d\theta]$ $= 4 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left\{ \begin{array}{l} 5 - 4 \cos 2\theta - 4 \cos 4\theta \\ + 4 \cos 6\theta - \cos 8\theta \end{array} \right\} [d\theta]$ $= 4 \left[\begin{array}{l} 5\theta - 2 \sin 2\theta - \sin 4\theta \\ + \frac{2}{3} \sin 6\theta - \frac{1}{8} \sin 8\theta \end{array} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$ $= 4 \left[\frac{5\pi}{2} - \left(\frac{5\pi}{3} - \sqrt{3} + \frac{\sqrt{3}}{2} + 0 - \frac{\sqrt{3}}{16} \right) \right]$ $= \frac{20\pi}{6} + 4 \left(\frac{16\sqrt{3} - 8\sqrt{3} + \sqrt{3}}{16} \right) = \frac{10\pi}{3} + \frac{9\sqrt{3}}{4}$	M1 M1 A1 A1	Use of $\frac{1}{2} \int r^2 [d\theta]$ Uses the printed result from part (a)(ii) Correct integration Correct answer in the requested form $a\pi + b\sqrt{n}$
		4	

	Question 12 Total	15	
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Q	Answer	Marks	Comments
13(a)	$tx^3 + ux^2 + vx + w = 0$ <p>Let roots be $a - d$, a, $a + d$</p> <p>so $a - d + a + a + d = -\frac{u}{t} \Rightarrow a = -\frac{u}{3t}$</p> $ta^3 + ua^2 + va + w = 0$ $t\left(\frac{-u}{3t}\right)^3 + u\left(\frac{-u}{3t}\right)^2 + v\left(\frac{-u}{3t}\right) + w = 0$ $-u^3 + 3u^3 - 9tuv + 27t^2w = 0$ $2u^3 - 9tuv + 27t^2w = 0$	<p>M1</p> <p>M1</p> <p>A1</p>	<p>$\alpha + \beta + \gamma = -\frac{u}{t}$ used</p> <p>Substitutes their middle root for x in the given cubic equation or substitutes $u = -3at$, $v = t(3a^2 - d^2)$ and $w = -at(a^2 - d^2)$ into the LHS of the equation in 13(a)</p> <p>AG Printed result convincingly obtained</p>
		3	

Q	Answer	Marks	Comments
13(a) ALT	$tx^3 + ux^2 + vx + w = 0$ <p>Let roots be $a, a + d, a + 2d$, so</p> $a + a + d + a + 2d = -\frac{u}{t} \Rightarrow 3(a + d) = -\frac{u}{t}$ $a(a + d)(a + 2d) = -\frac{w}{t} \Rightarrow a(a + 2d) = \frac{3w}{u}$ $a(a + d) + (a + d)(a + 2d) + a(a + 2d) = \frac{v}{t}$ $2(a + d)^2 + a(a + 2d) = \frac{v}{t}$ $\frac{2}{9} \left(\frac{-u}{t} \right)^2 + \frac{3w}{u} = \frac{v}{t}$ $2u^3 + 27t^2w = 9tvu$ $2u^3 - 9tuv + 27t^2w = 0$	<p>M1</p> <p>M1</p> <p>A1</p>	<p>$\alpha + \beta + \gamma = -\frac{u}{t}$ used</p> <p>$\alpha\beta\gamma = -\frac{w}{t}$ and $\sum \alpha\beta = \frac{v}{t}$ used to form an equation which requires just substitution of known expressions and simplification to reach the printed answer</p> <p>AG Printed result convincingly obtained</p>
		3	

Q	Answer	Marks	Comments
13(b)(i)	<p>Comparing $kx^3 - 36x^2 + mx - 3 = 0$ with $tx^3 + ux^2 + vx + w = 0$ gives $t = k, u = -36, v = m, w = -3$</p> <p>$\alpha + \beta + \gamma = \frac{36}{k} \Rightarrow a = \frac{12}{k}$</p> <p>$\alpha\beta\gamma = \frac{3}{k} \Rightarrow a(a^2 - d^2) = \frac{3}{k}$</p> <p>$\Rightarrow \frac{12}{k} \left(\frac{144}{k^2} - d^2 \right) = \frac{3}{k}$</p> <p>$\Rightarrow d^2 = \frac{576 - k^2}{4k^2}$</p>	<p>M1</p> <p>A1</p>	<p>$\alpha\beta\gamma = \frac{3}{k}$ and $\alpha + \beta + \gamma = \frac{36}{k}$ both used to form an equation in d and k only</p> <p>ACF eg $d^2 = \frac{144}{k^2} - \frac{1}{4}$</p>
		2	

Q	Answer	Marks	Comments
13(b)(ii)	<p>Comparing $kx^3 - 36x^2 + 38x - 3 = 0$ with $tx^3 + ux^2 + vx + w = 0$ gives $t = k, u = -36, v = m, w = -3$</p> <p>so $2(-36)^3 - 9k(38)(-36) + 27k^2(-3) = 0$</p> <p>$\Rightarrow k^2 - 152k + 1152 = 0$</p> <p>$\Rightarrow k = 8, k = 144$</p> <p>When $k = 8, d = \pm\sqrt{2}$</p> <p>When $k = 144, d = \pm\sqrt{\frac{35}{144}}$ i</p>	<p>M1</p> <p>A1</p> <p>A2,1</p>	<p>Using the result in part (a) to form a quadratic equation in k</p> <p>Both correct values for k</p> <p>oe A2 all four correct exact values for d otherwise A1 for any two correct</p>
		4	

	Question 13 Total	9	
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