INTERNATIONAL A-LEVEL

FURTHER MATHEMATICS

FM03

(9665/FM03) Unit FP2 Pure Mathematics

Mark scheme

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231XFM03/MS

Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Examiner.

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Key to mark scheme abbreviations

M	Mark is for method
m	Mark is dependent on one or more M marks and is for method
A	Mark is dependent on M or m marks and is for accuracy
B	Mark is independent of M or m marks and is for method and accuracy
E	Mark is for explanation
√ **or ft**	Follow through from previous incorrect result
CAO	Correct answer only
CSO	Correct solution only
AWFW	Anything which falls within
AWRT	Anything which rounds to
ACF	Any correct form
AG	Answer given
SC	Special case
oe	Or equivalent
A2, 1	2 or 1 (or 0) accuracy marks
–*x* EE	Deduct *x* marks for each error
NMS	No method shown
PI	Possibly implied
SCA	Substantially correct approach
sf	Significant figure(s)
dp	Decimal place(s)

Q	Answer	Marks	Comments
1(a)	$6+2\sin\theta=3 \Rightarrow \sin\theta=-1.5$ No solutions as $-1\le\sin\theta\le 1$ so C_1 and C_2 do not intersect	**E1**	Must show $\sin\theta=-1.5$ followed by some justification for no solutions **oe**, e.g. C_1: minimum value of r is $6+2(-1)=4$ followed by some justification for no solutions Condone omission of concluding statement
		1	

Q	Answer	Marks	Comments
1(b)	For C_1 : Area = $\frac{1}{2}\int_{[0]}^{[2\pi]}(6+2\sin\theta)^2\,[\mathrm{d}\theta]$	**M1**	Use of $\frac{1}{2}\int\ r^2[\mathrm{d}\theta]$
	$=\int_{[0]}^{[2\pi]}(18+12\sin\theta+1-\cos 2\theta)\,[\mathrm{d}\theta]$	**M1**	Use of $\cos 2\theta=\pm1\pm2\sin^2\theta$ with $k\int r^2[\mathrm{d}\theta]$ **PI** by correct integration of r^2
	$=\left[18\theta-12\cos\theta+\theta-0.5\sin 2\theta\right]_0^{2\pi}$ $=38\pi$	**A1**	38π after correct integration
	For C_2 : Area $=9\pi$ Required area $=38\pi-9\pi=29\pi$	**A1**	**AG** Must be convincingly shown
		4	

	Question 1 Total	**5**	

Q	Answer	Marks	Comments
2	When $n=1$, LHS $=\begin{bmatrix}-3 & 1\\ -16 & 5\end{bmatrix}$ RHS $=\begin{bmatrix}1-4 & 1\\ -16 & 4+1\end{bmatrix}=\begin{bmatrix}-3 & 1\\ -16 & 5\end{bmatrix}$	B1	Correct values to show formula true for $n=1$
	Assume formula true for $n=k$ **(*)**, [integer $k \geq 1$], so $\begin{bmatrix}-3 & 1\\ -16 & 5\end{bmatrix}^k=\begin{bmatrix}1-4k & k\\ -16k & 4k+1\end{bmatrix}$ Consider $\begin{bmatrix}-3 & 1\\ -16 & 5\end{bmatrix}^{k+1}=\begin{bmatrix}1-4k & k\\ -16k & 4k+1\end{bmatrix}\begin{bmatrix}-3 & 1\\ -16 & 5\end{bmatrix}$	M1	Assumes formula true for $n=k$ and considers $\begin{bmatrix}1-4k & k\\ -16k & 4k+1\end{bmatrix}\begin{bmatrix}-3 & 1\\ -16 & 5\end{bmatrix}$ **oe**
	$=\begin{bmatrix}-3+12k-16k & 1-4k+5k\\ 48k-64k-16 & -16k+20k+5\end{bmatrix}$	A1	**oe**
	$=\begin{bmatrix}-3-4k & k+1\\ -16k-16 & 4k+5\end{bmatrix}$ $=\begin{bmatrix}1-4(k+1) & k+1\\ -16(k+1) & 4(k+1)+1\end{bmatrix}$	A1	Must be convincingly shown
	Hence formula is true for $n=k+1$ **(**)** and since true for $n=1$ **(***)**, formula is true for $n=1, 2, 3, \ldots$ by induction **(****)**	E1	Must have **(*)**, **(**)**, **(***)**, present, previous 4 marks scored and final statement **(****)** clearly indicating that it relates to positive integers
		5	

	Question 2 Total	**5**	

Q	Answer	Marks	Comments
3	$u = \tan^{-1} x;\quad \mathrm{d}v = 2x\,\mathrm{d}x$ $\mathrm{d}u = \dfrac{1}{1+x^2}\mathrm{d}x;\quad v = x^2$	M1	Use of integration by parts **PI**
	$\int 2x\tan^{-1}x\,\mathrm{d}x$ $= x^2\tan^{-1}x - \int x^2\left(\dfrac{1}{1+x^2}\right)\mathrm{d}x$	A1	
	$= x^2\tan^{-1}x - \int\left(1-\dfrac{1}{1+x^2}\right)\mathrm{d}x$	M1	**PI** Writing $\left(\dfrac{x^2}{1+x^2}\right)$ as $\left(1-\dfrac{1}{1+x^2}\right)$
	$= x^2\tan^{-1}x - x + \tan^{-1}x \quad [+c]$	A1	
	$\int_1^{\sqrt{3}} 2x\tan^{-1}x\,\mathrm{d}x$ $= \left(\pi - \sqrt{3} + \dfrac{\pi}{3}\right) - \left(\dfrac{\pi}{4} - 1 + \dfrac{\pi}{4}\right)$ $= \dfrac{5\pi}{6} + 1 - \sqrt{3}$	A1	**AG** Must be convincingly shown
		5	

	Question 3 Total	5	

Q	Answer	Marks	Comments
4(a)	**u**, **v** and **w** are coplanar vectors if eg $\mathbf{u}.(\mathbf{v}\times\mathbf{w})=0$ $\begin{bmatrix}4\\3\\8\end{bmatrix}\cdot\left(\begin{bmatrix}-1\\n\\n\end{bmatrix}\times\begin{bmatrix}5\\-1\\n\end{bmatrix}\right)=0$ or $\begin{vmatrix}4 & 3 & 8\\-1 & n & n\\5 & -1 & n\end{vmatrix}=0$	**M1**	Equates a relevant scalar triple product to zero or Expresses scalar triple product as a relevant determinant and equates to 0 **PI** by later work
	$4n^2-18n+8=0$	**A1**	Correct quadratic equation
	$n=4$, $n=0.5$	**A1**	Correct two values of n
		3	

Q	Answer	Marks	Comments
4(b)	[When $n=4$] $[\mathbf{u}=]\ \mathbf{v}+\mathbf{w}$	**B1**	**oe**
	[When $n=0.5$] $[\mathbf{u}=]\ \dfrac{38}{3}\mathbf{v}+\dfrac{10}{3}\mathbf{w}$	**B1**	**oe**
		2	

	Question 4 Total	**5**	

Q	Answer	Marks	Comments
5	$4y^2 = 4 - 4x - 3x^2 \Rightarrow 4x^2 + 4y^2 = (2-x)^2$ $4r^2 = (2 - r\cos\theta)^2$	M2,1	**M2**: Correctly uses two of $x = r\cos\theta$ $x^2 + y^2 = r^2$, $y = r\sin\theta$ If not **M2** then award **M1** if only uses one of the three correctly
	$2r = -(2 - r\cos\theta)$ or $2r = (2 - r\cos\theta)$ $\Rightarrow r(2-\cos\theta) = -2$ or $r(2+\cos\theta) = 2$ As $(2-\cos\theta)$ and r are both positive, $r(2-\cos\theta) \neq -2$	E1	Justification for rejecting the invalid root , $r(2-\cos\theta) = -2$ **oe**
	$r = \dfrac{2}{2+\cos\theta}$	A1	**ACF** (**M2 E0 A1** is possible)
		4	

	Question 5 Total	**4**	

Q	Answer	Marks	Comments
6	I.F. is $\mathrm{e}^{\int \tan x \, \mathrm{d}x} = \mathrm{e}^{-\ln \cos x}$	M1	I.F. identified and integration attempted
	$= \sec x$	A1	Correct integrating factor
	$y \sec x = \int \tan^2 x (\sec x \tan x) \mathrm{d}x$	m1	Multiplying both sides of the given DE by the I.F. and integrating LHS to get $y \times$ I.F.
	Let $u = \sec x$ $y \sec x = \int (u^2 - 1) \mathrm{d}u$	M1	Valid method to integrate $\tan^n x \sec x$ **PI** by later work
	$= \frac{1}{3}\sec^3 x - \sec x + A$	A1	Correct integration of $\tan^3 x \sec x$ Condone missing arbitrary constant
	$y \sec x = \frac{1}{3}\sec^3 x - \sec x + A$ $y = \frac{1}{3}\sec^2 x - 1 + A\cos x$	A1	Accept **oe** of either form
		6	

	Question 6 Total	**6**	

Q	Answer	Marks	Comments
7(a)	$[\alpha+\beta+\gamma+\delta=]\ 0$	B1	
		1	

Q	Answer	Marks	Comments
7(b)(i)	$\delta=-2+\mathrm{i}$	B1	
	$(-2+\mathrm{i})^4+p(-2+\mathrm{i})+q=0$	M1	Substitutes their δ (or its conjugate) in the given quartic equation to obtain a non-real equation in p and q
	$-7-24\mathrm{i}-2p+\mathrm{i}p+q=0$ **(*)**		
	Equating imaginary parts: $-24+p=0$	m1	Equates imaginary parts to find p
	$p=24$	A1	
7(b)(i) ALT	$\gamma=-2-\mathrm{i}$, $\gamma+\delta=-4$, $\gamma\delta=5$, $\alpha+\beta=4$	B1	$\delta=-2+\mathrm{i}$ or any one of the four listed
	$\alpha\beta+(\alpha+\beta)(\gamma+\delta)+\gamma\delta=0$, $\alpha\beta-16+5=0$	M1	$\alpha\beta+(\alpha+\beta)(\gamma+\delta)+\gamma\delta=0$ used
	$p=-\alpha\beta(\gamma+\delta)-\gamma\delta(\alpha+\beta)=-11(-4)-5(4)$	m1	$p=-\alpha\beta(\gamma+\delta)-\gamma\delta(\alpha+\beta)$ used
	$p=24$	A1	
		4	

Q	Answer	Marks	Comments
7(b)(ii)	$\alpha^4 + p\alpha + q = 0$	M1	$\alpha^4 + p\alpha + q = 0$ **oe**
	$\sum\alpha^4 + p\sum\alpha + 4q = 0$	M1	**oe**
	$\sum\alpha^4 = -4q$; from (*), $q = 2p + 7 = 55$		
	$\sum\alpha^4 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 = -4(55) = -220$	A1	**AG** Must be convincingly shown
7(b)(ii) ALT 1	α, β are roots of $z^2 - 4z + 11 = 0$ so $\alpha, \beta = 2 \pm \mathrm{i}\sqrt{7}$	M1	
	$\delta^4 = -7 - 24\mathrm{i}$, $\gamma^4 = -7 + 24\mathrm{i}$, $\alpha^4, \beta^4 = -103 \pm 24\sqrt{7}\,\mathrm{i}$	M1	Any three of the four correct
	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = -14 - 206 = -220$	A1	**AG** Must be convincingly shown
7(b)(ii) ALT 2	$\gamma^4 + \delta^4 = -14$	M1	
	$\alpha^4 + \beta^4 = -206$	M1	
	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = -14 - 206 = -220$	A1	**AG** Must be convincingly shown
		3	

	Question 7 Total	**8**	

Q	Answer	Marks	Comments
8(a)(i)	**M** singular so det **M** $= 0$ det **M** $= 1(4) - c(-2) = 2c + 4 = 0$ $c = -2$	M1 A1	Uses det **M** $= 0$ to form an equation in c
		2	

Q	Answer	Marks	Comments
8(a)(ii)	Cofactor matrix $\begin{bmatrix} 4 & -1 & -2 \\ -2c & 3+2c & -2 \\ 2c & -1-c & 2 \end{bmatrix}$ Inverse matrix $\mathbf{M}^{-1} =$ $\dfrac{1}{2c+4}\begin{bmatrix} 4 & -2c & 2c \\ -1 & 3+2c & -1-c \\ -2 & -2 & 2 \end{bmatrix}$	M1 A2 M1 A1	One complete row, column or diagonal correct All nine entries correct else **A1** for at least six entries correct **M1**: Transpose of their cofactors with no more than one further error **and** division by their det **M** in terms of c **A1**: Correct $\mathbf{M}^{-1}$ scores 5 marks
		5	

Q	Answer	Marks	Comments
8(b)	$\det(\mathbf{M}-\lambda\mathbf{I})=\begin{vmatrix}1-\lambda & 0 & -c\\ 1 & 2-\lambda & 1\\ 2 & 2 & 3-\lambda\end{vmatrix}\ [=0]$ $(1-\lambda)\big((2-\lambda)(3-\lambda)-2\big)-c(-2+2\lambda)[=0]$	M1	Sets up and expands $\det(\mathbf{M}-\lambda\mathbf{I})$
	$(1-\lambda)\big((2-\lambda)(3-\lambda)-2+2c\big)=0$	m1	Reduces to $(1-\lambda)(\text{quadratic in }\lambda)$ **PI** by later work
	$(1-\lambda)(\lambda^2-5\lambda+4+2c)=0$	A1	**PI** by later work
	$\lambda=1 \quad \lambda^2-5\lambda+4+2c=0$ If $\lambda=1$ is the only real eigenvalue then roots of $\lambda^2-5\lambda+4+2c=0$ are non-real so $25-4(4+2c)<0$	m1	Considers $b^2-4ac<0$ for their quadratic equation in λ
	$c>1.125$	A1	$c>1.125$ **oe**
		5	

	Question 8 Total	**12**	

Q	Answer	Marks	Comments
9(a)	Aux. equation $m^2+2m+2=0$ $\quad m=\dfrac{-2\pm\sqrt{4-8}}{2}$	M1	Using quadratic formula **oe** on correct aux. equation. **PI** by correct values of m seen/used
	$[y_{\text{CF}}=]\quad e^{-x}(A\sin x+B\cos x)$	A1	Correct CF
	$y_{\text{PI}}=ax+b \Rightarrow 2a+2(ax+b)=2x$	M1	**PI** by correct y_{PI}
	$y_{\text{PI}}=x-1$	A1	Correct y_{PI} seen/used
	$[y_{\text{GS}}=]\quad e^{-x}(A\sin x+B\cos x)+x-1$	B1ft	Their CF $+$ their PI but must have exactly two arbitrary constants
	$y'=-e^{-x}(A\sin x+B\cos x)+e^{-x}(A\cos x-B\sin x)+1$	M1	Clear use of the product rule on their CF $+$ their PI
	$x=0\ \ y=-2 \Rightarrow B=-1$; $\ x=0,\ \ y'=2\Rightarrow A=0$	A1	Correct values for A and B
	$[f(x)=]-e^{-x}\cos x+x-1$	A1	
		8	

Q	Answer	Marks	Comments
9(b)	$f(x)=-e^{-x}\cos x+x-1$	M1	Uses correct series expansions throughout for their $f(x)$
	$=-\left(1-x+\dfrac{x^2}{2}-\dfrac{x^3}{6}+\dfrac{x^4}{24}\right)\left(1-\dfrac{x^2}{2}+\dfrac{x^4}{24}\right)+x-1$	A1ft	Correct **ft** expansions up to x^4 and attempt to multiply out brackets
	$=-2+2x-\dfrac{1}{3}x^3+\dfrac{1}{6}x^4$	A1	
9(b) ALT	$f(0)=-2$; $f'(0)=2$; $f''(0)=0$; $f'''(0)+2f''(0)+2f'(0)=2$; $f^{(4)}(0)+2f'''(0)+2f''(0)=0$	M1	Four of the five seen or used
	$f(x)=-2+2x+\dfrac{0}{2}x^2+\dfrac{2-4-0}{3!}x^3+\dfrac{0-0-2(-2)}{4!}x^4$	A1ft	**ft** on one numerical error
	$=-2+2x-\dfrac{1}{3}x^3+\dfrac{1}{6}x^4$	A1	
		3	

	Question 9 Total	**11**	

Q	Answer	Marks	Comments
10(a)	$\cosh x \cosh y + \sinh x \sinh y$ $= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$	M1 A1	Correct substitution of at least three of the four hyperbolic functions in terms of exponentials All correct
	$= \frac{e^{x+y} + e^{-(x+y)} + e^{x-y} + e^{-(x-y)} + e^{x+y} + e^{-(x+y)} - e^{x-y} - e^{-(x-y)}}{4}$	B1	Correct expansion of brackets
	$= \frac{2e^{x+y} + 2e^{-(x+y)}}{4} = \frac{e^{x+y} + e^{-(x+y)}}{2} = \cosh(x+y)$	A1	**AG** Must be convincingly shown
		4	

Q	Answer	Marks	Comments
10(b)	$\frac{dy}{dx} = 8\cosh(x + \ln 4) + 4\sinh x - 7$	M1	At least two terms in x differentiated correctly
	$= 8\left(\cosh x \cosh(\ln 4) + \sinh x \sinh(\ln 4)\right) + 4\sinh x - 7$	M1	Use of **part (a)** **oe**
	$= 17\cosh x + 15\sinh x + 4\sinh x - 7$	B1	$\cosh(\ln 4) = \frac{17}{8}$; $\sinh(\ln 4) = \frac{15}{8}$ seen or used
	For st pt $y'(x) = 0 \Rightarrow 18e^{2x} - 7e^x - 1 = 0$	M1 A1	Forming a quadratic in e^x Correct quadratic in e^x
	$(9e^x + 1)(2e^x - 1) = 0$; $e^x = \frac{1}{2}$, $e^x = -\frac{1}{9}$	A1	Solving the correct quadratic eqn.
	$e^x = -\frac{1}{9}$ is not possible [as $e^x > 0$]	E1ft	Eliminating a negative root of an exponential
	$e^x = \frac{1}{2} \Rightarrow x = -\ln 2 \Rightarrow y = 11 + 7\ln 2$ 'Curve has exactly one stationary point' $[(-\ln 2,\ 11 + 7\ln 2)]$	A2,1	Statement (could be seen earlier) and $y = 11 + 7\ln 2$ obtained convincingly (**A1** if correct y-coordinate but statement missing)
		9	

Q	Answer	Marks	Comments
10(b) ALT	$\frac{\mathrm{d}y}{\mathrm{d}x} = 8\cosh(x+\ln 4) + 4\sinh x - 7$	**M1**	At least two terms in x differentiated correctly
	$= 8\left(\cosh x\cosh(\ln 4) + \sinh x\sinh(\ln 4)\right) + 4\sinh x - 7$	**M1**	Use of **part (a) oe**
	$= 17\cosh x + 15\sinh x + 4\sinh x - 7$	**B1**	$\cosh(\ln 4) = \frac{17}{8}$; $\sinh(\ln 4) = \frac{15}{8}$ seen or used
	For st pt $y'(x) = 0 \Rightarrow x + \tanh^{-1}\left(\frac{17}{19}\right) = \sinh^{-1}\left(\frac{7\sqrt{2}}{12}\right)$	**M1** **A1** **A1**	Forming an equation in x and two numerical real inverse hyperbolic expressions Correct numerical inverse hyperbolic expressions
	$x = \sinh^{-1}\left(\frac{7\sqrt{2}}{12}\right) - \tanh^{-1}\left(\frac{17}{19}\right) = \ln\left(\frac{3\sqrt{2}}{2}\right) - \ln\left(\sqrt{18}\right) = \ln\left(\frac{1}{2}\right)$	**E1**	Finding or justifying that there is only one value of x
	$e^x = \frac{1}{2} \quad \Rightarrow \quad x = -\ln 2 \quad \Rightarrow \quad y = 11 + 7\ln 2$ 'Curve has exactly one stationary point' $[(-\ln 2,\ 11+7\ln 2)]$	**A2,1**	Statement (could be seen earlier) and $y = 11+7\ln 2$ obtained convincingly (**A1** if correct y-coordinate but statement missing)
		9	

	Question 10 Total	**13**	

Q	Answer	Marks	Comments
11(a)(i)	$\mathbf{r} \cdot \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} = 3$	**B1**	Correct vector equation for Π_2 in the required form
		1	

Q	Answer	Marks	Comments
11(a)(ii)	$\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$	**M1**	Use of scalar product on the two normal vectors, **ft** on their $\mathbf{n}$ in **(a)(i)**
	$= 1 - 12 - 9 \ [= -20]$	**A1ft**	Correct evaluation, unsimplified or simplified, of scalar product ft on their $\mathbf{n}$ in **(a)(i)**. Accept its modulus value.
	$\cos\theta = \dfrac{-20}{\sqrt{26}\sqrt{19}}$	**B1ft**	Correct product of moduli in the denominator, **ft** on their $\mathbf{n}$ in **(a)(i)**
	Acute angle = 25.9°	**A1**	**CAO** Final value must be 25.9
		4	

Q	Answer	Marks	Comments
11(b)(i)	$\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \times \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -7 \end{bmatrix}$	**M1**	Vector product of the two normal vectors attempted **OR** applying a correct method to obtain and use two common points
	$3 : -6 : -7$	**A1**	A correct set of direction ratios Do not condone left as column vector
		2	

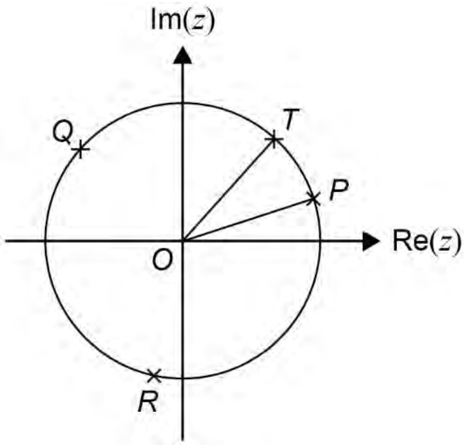
Q	Answer	Marks	Comments
11(b)(ii)	eg $(0, a, b)$ solving simultaneously $4a-3b=5$ and $-3a+3b=3$	**M1**	Valid method to find a common point
	Common point $(0, 8, 9)$	**A1**	Any correct common point eg $\left(4, 0, -\frac{1}{3}\right)$, $\left(\frac{27}{7}, \frac{2}{7}, 0\right)$
	line L : $\frac{x}{3}=\frac{y-8}{-6}=\frac{z-9}{-7}[=\lambda]$	**A1ft**	**oe ft** their **(b)(i)** direction ratios
		3	

Q	Answer	Marks	Comments
11(c)	$3\lambda-6\lambda+8=5 \Rightarrow \lambda=1$	**M1**	Substitute general point on their L into $x+y=5$ **or** $x+4y-3z=5$, $x-3y+3z=3$, $x+y=5$ solved simultaneously
	Point of intersection $(3, 2, 2)$	**A1**	Do not condone answer given as vector
		2	

	Question 11 Total	**12**	

Q	Answer	Marks	Comments
12(a)	$r\,e^{i\phi} = -n + i m = i(m + i n) = i\,r\,e^{i\theta}$ $r\,e^{i\phi} = e^{i\left(\frac{\pi}{2}\right)} r\,e^{i\theta} = r\,e^{i\left(\theta+\frac{\pi}{2}\right)}$	M1	$i = e^{i\left(\frac{\pi}{2}\right)}$ seen or used. **PI**
	$\Rightarrow \phi = \theta + \dfrac{\pi}{2}$	A1	**NMS** scores 2/2 Condone $\phi = \theta + \dfrac{\pi}{2} + 2n\pi$
		2	

Q	Answer	Marks	Comments
12(b)(i)	Radius $= \lvert z\rvert = \left(\sqrt{\left(a^2+b^2\right)}\right)^{\frac{1}{3}} = \left(a^2+b^2\right)^{\frac{1}{6}}$	B2,1	Accept either form. If **B2** not scored, award **B1** for $\lvert a+ib\rvert = \sqrt{\left(a^2+b^2\right)}$ seen or used
		2	

Q	Answer	Marks	Comments
12(b)(ii)		B1	Plots the point *T* on the arc of the circle in the 1st quadrant above *P* so that its distance along the arc is closer to *P* than to the top point of the circle.
		1	

Q	Answer	Marks	Comments
12(c)	Area of triangle *OTP* $=\frac{1}{2}r^2\sin(\angle TOP)$	M1	Seen or used
	$\frac{1}{2}r^2\sin\left(\frac{\pi}{6}\right)=16\Rightarrow r=8$	A1	Correct value for the radius, seen or used
	<u>For root at *P*</u>: $\arg(z_P)=\frac{1}{3}\tan^{-1}(\sqrt{3})=\frac{\pi}{9}$ <u>For root at *T*</u>: $\arg(z_T)=\frac{\pi}{9}+\frac{\pi}{6}=\frac{5\pi}{18}$	M1	Either correct Condone angle in degrees for **M1**
	Let *M* be the midpoint of chord *TP*: $\arg(z_M)=\frac{\pi}{9}+\frac{1}{2}\left(\frac{\pi}{6}\right)=\frac{7\pi}{36}$	A1	Correct $\arg(z_M)$
	OM $=\lvert z_M\rvert=8\cos\left(\frac{\pi}{12}\right)$ or $2\left(\sqrt{6}+\sqrt{2}\right)$	B1ft	**ft** on c's $r\cos\left(\frac{1}{2}\angle TOP\right)$
	$z_M=8\cos\left(\frac{\pi}{12}\right)\mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{36}\right)}=2\left(\sqrt{6}+\sqrt{2}\right)\mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{36}\right)}$	A1	$8\cos\left(\frac{\pi}{12}\right)\mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{36}\right)}$ **or** $2\left(\sqrt{6}+\sqrt{2}\right)\mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{36}\right)}$ **oe** but must be in exponential form
		6	

	Question 12 Total	**11**	

Q	Answer	Marks	Comments
13(a)	$u=\sinh^{-1}\left(\frac{1}{x}\right)$ Let $v=\left(\frac{1}{x}\right)$ then $u=\sinh^{-1}v$ $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{1}{\sqrt{1+v^2}}\left(-\frac{1}{x^2}\right)$	M1 A1	Chain rule used with no more than one incorrect derivative $\frac{1}{\sqrt{1+v^2}}\left(-\frac{1}{x^2}\right)$ **oe**
	$\frac{\mathrm{d}u}{\mathrm{d}x}=-\frac{1}{x\sqrt{x^2+x^2v^2}}$		
	$\frac{\mathrm{d}u}{\mathrm{d}x}=-\frac{1}{x\sqrt{x^2+1}}=-\frac{1}{x\sqrt{1+x^2}}$	A1	**AG** Must be convincingly shown
		3	

Q	Answer	Marks	Comments
13(b)	$y = \ln x \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{x}$	B1	
	$s = \displaystyle\int_{\left[\frac{7}{24}\right]}^{\left[\frac{12}{5}\right]} \sqrt{1+\frac{1}{x^2}}\ [\mathrm{d}x]$	M1	
	$s = \displaystyle\int_{\left[\frac{7}{24}\right]}^{\left[\frac{12}{5}\right]} \left(\frac{x}{\sqrt{x^2+1}} + \frac{1}{x\sqrt{x^2+1}}\right) \mathrm{d}x$	m1	$\sqrt{x^2+1} = \dfrac{x^2}{\sqrt{x^2+1}} + \dfrac{1}{\sqrt{x^2+1}}$ used or using a relevant substitution as far as writing the integral in a form which can be integrated directly
	$s = \left[\sqrt{x^2+1} - \sinh^{-1}\left(\dfrac{1}{x}\right)\right]_{\left[\frac{7}{24}\right]}^{\left[\frac{12}{5}\right]}$	A1 A1	Correct integration of each term
	$s = \dfrac{13}{5} - \sinh^{-1}\left(\dfrac{5}{12}\right) - \left[\dfrac{25}{24} - \sinh^{-1}\left(\dfrac{24}{7}\right)\right]$		
	$s = \dfrac{187}{120} - \ln\left(\dfrac{5}{12} + \dfrac{13}{12}\right) + \ln\left(\dfrac{24}{7} + \dfrac{25}{7}\right)$	m1	Correct substitution of correct limits in a two term expression and uses $\sinh^{-1}(k) = \ln\left(k + \sqrt{k^2+1}\right)$
	$s = \dfrac{187}{120} + \ln\left(\dfrac{14}{3}\right)$	A1	$s = \dfrac{187}{120} + \ln\left(\dfrac{14}{3}\right)$
		7	

	Question 13 Total	**10**	

Q	Answer	Marks	Comments
14(a)	$(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta$	B1	PI $\cos 4\theta = \mathrm{Re}\left[(\cos\theta + \mathrm{i}\sin\theta)^4\right]$
	$\cos 4\theta = \mathrm{Re}\left[(\cos\theta + \mathrm{i}\sin\theta)^4\right]$ Using $c = \cos\theta$ and $s = \sin\theta$	M1	With expansion attempted
	$\cos 4\theta = c^4 + 6c^2(-s^2) + s^4$ $= (1-s^2)^2 - 6s^2(1-s^2) + s^4$	A1	
	$\cos 4\theta = 8\sin^4\theta - 8\sin^2\theta + 1$	A1	
		4	

Q	Answer	Marks	Comments
14(b)	$\cos\left(\frac{\pi}{2} - 3\theta\right) = \sin 3\theta$	B1	
	$\sin 3\theta = \sin(2\theta + \theta)$		
	$\sin 3\theta = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$	M1	**PI oe** eg de Moivre using $\sin 3\theta = \mathrm{Im}\left[(\cos\theta + \mathrm{i}\sin\theta)^3\right]$
	$= 2\sin\theta(1-\sin^2\theta) + \sin\theta(1-2\sin^2\theta)$ $\cos 4\theta = \sin 3\theta$	A1	A correct expression for $\sin 3\theta$ in terms of $\sin\theta$
	$8\sin^4\theta - 8\sin^2\theta + 1 = 3\sin\theta - 4\sin^3\theta$		
	$8\sin^4\theta + 4\sin^3\theta - 8\sin^2\theta - 3\sin\theta + 1 = 0$ (*)	A1	Must be convincingly shown
		4	

Q	Answer	Marks	Comments
14(c)	$4\theta = 2n\pi \pm \left(\frac{\pi}{2} - 3\theta\right)$, (for integer n)	M1	oe
	$\theta = 2n\pi - \frac{\pi}{2}$, $\theta = \frac{2n\pi}{7} + \frac{\pi}{14}$	A1	$\theta = 2n\pi - \frac{\pi}{2}$ **PI** by $\sin\left(-\frac{7\pi}{14}\right)$ in the next step.
	Roots of the quartic eqn (*) in **(b)** are $\sin\left(\frac{\pi}{14}\right)$, $\sin\left(\frac{5\pi}{14}\right)$, $\sin\left(-\frac{3\pi}{14}\right)$ and $\sin\left(-\frac{7\pi}{14}\right) = -1$	B1	States/uses the four roots $\sin\left(\frac{\pi}{14}\right)$, $\sin\left(\frac{5\pi}{14}\right)$, $\sin\left(-\frac{3\pi}{14}\right)$, $\sin\left(-\frac{7\pi}{14}\right)$
	Sum of roots of eqn (*) = $-\frac{4}{8}$ $\sin\left(\frac{\pi}{14}\right) + \sin\left(\frac{5\pi}{14}\right) + \sin\left(-\frac{3\pi}{14}\right) - 1 = -\frac{4}{8}$	M1	Equates the sum of the four correct roots to $-\frac{4}{8}$ **oe**
	$\sin\left(\frac{\pi}{14}\right) + \sin\left(\frac{5\pi}{14}\right) - \sin\left(\frac{3\pi}{14}\right) - 1 = -\frac{4}{8}$ $\Rightarrow \sin\left(\frac{\pi}{14}\right) + \sin\left(\frac{5\pi}{14}\right) = \frac{1}{2} + \sin\left(\frac{3\pi}{14}\right)$	A1	**AG** Must be convincingly shown
		5	

	Question 14 Total	**13**	