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FM03

(9665/FM03) Unit FP2 Pure Mathematics

Mark scheme

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241XFM03/MS

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Key to mark scheme abbreviations

M	Mark is for method
m	Mark is dependent on one or more M marks and is for method
A	Mark is dependent on M or m marks and is for accuracy
B	Mark is independent of M or m marks and is for method and accuracy
E	Mark is for explanation
√ **or ft**	Follow through from previous incorrect result
CAO	Correct answer only
CSO	Correct solution only
AWFW	Anything which falls within
AWRT	Anything which rounds to
ACF	Any correct form
AG	Answer given
SC	Special case
oe	Or equivalent
A2, 1	2 or 1 (or 0) accuracy marks
–*x* EE	Deduct *x* marks for each error
NMS	No method shown
PI	Possibly implied
SCA	Substantially correct approach
sf	Significant figure(s)
dp	Decimal place(s)
ISW	Ignore subsequent working

Q	Answer	Marks	Comments
1	$(2t+1,\ 5t+4,\ 2t+1.5)$	**B1**	Correct general point on L seen or used **PI** by a correct linear equation in a single variable
	$2t+1+2(5t+4)+3(2t+1.5)=18$ $18t=4.5 \qquad t=0.25$	**M1**	Substituting their general point into the equation for Π to obtain an equation in a single variable and obtaining a value for the variable
	$A(1.5,\ 5.25,\ 2)$	**A1**	Correct coordinates **ACF** Do not condone answer given as a vector

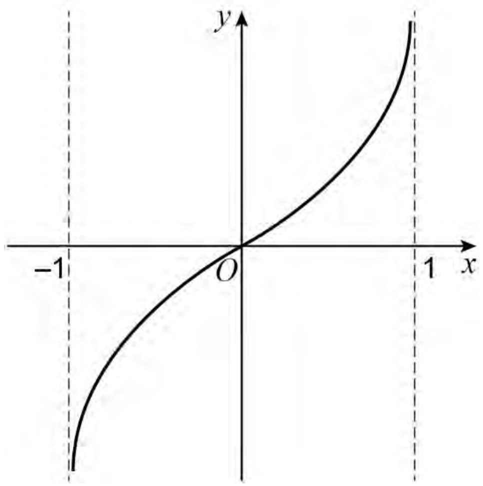
	Question 1 Total	**3**	

Q	Answer	Marks	Comments
2(a)	$\left[\tan\theta = -\frac{1}{\sqrt{3}}\right]$ $\left[-2\sqrt{3}+2\mathrm{i}=\right]\ 4\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$	**B1 B1**	**B1** for $r = 4$ **B1** for $\theta = \frac{5\pi}{6}$
		2	

Q	Answer	Marks	Comments
2(b)	$z^4 = 4\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{6}+2n\pi\right)}$		
	$z = \sqrt[4]{4}\ \mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{24}\left[+\frac{2n\pi}{4}\right]\right)}$	**B1ft M1**	**B1ft**: $r = \sqrt[4]{4}$ **oe ft** the fourth root of their value for r in **part (a)** **M1**: $z = \sqrt[4]{4}\ \mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{24}\right)}$ Divides their θ in **part (a)** by 4 to get their θ in **(b)**
	$z = \sqrt{2}\ \mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{24}\right)}$, $z = \sqrt{2}\ \mathrm{e}^{\mathrm{i}\left(\frac{17\pi}{24}\right)}$,	**M1**	Obtains at least three of the first four positive correct angles **ft** $\frac{\text{their } \theta \text{ in } (\mathbf{a})}{4} + \frac{n\pi}{2}$
	$z = \sqrt{2}\ \mathrm{e}^{\mathrm{i}\left(\frac{29\pi}{24}\right)}$, $z = \sqrt{2}\ \mathrm{e}^{\mathrm{i}\left(\frac{41\pi}{24}\right)}$	**A1**	**CAO**
		4	

	Question 2 Total	**6**	

Q	Answer	Marks	Comments
3(a)(i)	$y = \tanh^{-1} x$ $\dfrac{dy}{dx} = \dfrac{1}{1-x^2}$ [When $x = 0$] $\left[\dfrac{dy}{dx} = \right]$ 1	**B1**	**B0** if incorrect $\dfrac{dy}{dx}$ used
		1	

Q	Answer	Marks	Comments
3(a)(ii)		**B1**	Correct shaped graph with a positive gradient at the origin and asymptotic behaviour at $x = -1$ and $x = 1$
		1	

Q	Answer	Marks	Comments
3(b)	$\tanh^{-1}\left(\frac{1+x}{2}\right) = \frac{1}{2}\ln\left(\frac{3+x}{1-x}\right)$ $\tanh^{-1}\left(\frac{1-x}{2}\right) = \frac{1}{2}\ln\left(\frac{3-x}{1+x}\right)$	**M1**	$\tanh^{-1} X = \frac{1}{2}\ln\left(\frac{1+X}{1-X}\right)$ used at least once
	$\tanh^{-1}\left(\frac{1+x}{2}\right) + \tanh^{-1}\left(\frac{1-x}{2}\right)$ $= \frac{1}{2}\ln\left(\frac{(3+x)(3-x)}{(1-x)(1+x)}\right)$	**A1**	$\frac{1}{2}\ln\left(\frac{(3+x)(3-x)}{(1-x)(1+x)}\right)$ **oe** single logarithmic term
	$\frac{3}{2}\ln 3 - \frac{1}{2}\ln 2 = \frac{1}{2}\ln\frac{27}{2}$	**B1**	$\frac{3}{2}\ln 3 - \frac{1}{2}\ln 2 = \frac{1}{2}\ln\frac{27}{2}$ seen or used
	$\frac{(9-x^2)}{(1-x^2)} = \frac{27}{2} \Rightarrow 18-2x^2 = 27-27x^2$ $\Rightarrow 25x^2 = 9$	**M1**	Elimination of logs to obtain a single quadratic equation in x
	$\Rightarrow \quad x = \pm\frac{3}{5}$	**A1**	$[x =] \pm\frac{3}{5}$ **oe**
		5	

	Question 3 Total	**7**	

Q	Answer	Marks	Comments
4	$\dfrac{r^2+r+1}{r(r+1)}=1+\dfrac{1}{r(r+1)}$	B1	$A=1$ or $\dfrac{r^2+r+1}{r(r+1)}=1+\dfrac{1}{r(r+1)}$
	$\dfrac{1}{r(r+1)}=\dfrac{B}{r}+\dfrac{C}{r+1}\Rightarrow 1=B(r+1)+Cr$	M1	**PI** by either $B=1$ or $C=-1$
	$\dfrac{r^2+r+1}{r(r+1)}=1+\dfrac{1}{r}-\dfrac{1}{r+1}$	A1	$1+\dfrac{1}{r}-\dfrac{1}{r+1}$
	$\displaystyle\sum_{r=1}^{n}\frac{r^2+r+1}{r(r+1)}=\sum_{r=1}^{n}1+\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)$		
	$=n+\left(1-\cancel{\dfrac{1}{2}}\right)+\left(\cancel{\dfrac{1}{2}}-\cancel{\dfrac{1}{3}}\right)+\ldots$	B1	$\displaystyle\sum_{r=1}^{n}1=n$ seen or used
	$+\left(\cancel{\dfrac{1}{n-1}}-\cancel{\dfrac{1}{n}}\right)+\left(\cancel{\dfrac{1}{n}}-\dfrac{1}{n+1}\right)$	M1	Method of differences shown with at least the first two and last two terms of $\displaystyle\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)$ seen
	$\displaystyle\sum_{r=1}^{n}\frac{r^2+r+1}{r(r+1)}=1+n-\frac{1}{n+1}$	A1	**AG** Must be convincingly shown, including evidence of cancelling

	Question 4 Total	**6**	

Q	Answer	Marks	Comments
5	$\cos x \dfrac{dy}{dx} + y = \cos^2 x + \sin x$ $\Rightarrow \dfrac{dy}{dx} + y \sec x = \cos x + \tan x$		
	I.F. is $e^{\int \sec x \, dx}$	M1	$e^{\int \sec x \, dx}$
	I.F. $= e^{\ln(\sec x + \tan x)} = \sec x + \tan x$	A1	I.F. $= \sec x + \tan x$ or $\tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)$
	$\dfrac{d}{dx}\left(y(\sec x + \tan x)\right)$ $= (\cos x + \tan x)(\sec x + \tan x)$ $y(\sec x + \tan x)$ $= \int (\cos x + \tan x)(\sec x + \tan x)\, dx$	M1	Multiplying both sides of the rearranged DE by their I.F. and integrating LHS to get $y \times$ I.F.
	$y(\sec x + \tan x)$ $= \int \left(1 + \tan^2 x + \sin x + \sec x \tan x\right) dx$		
	$y(\sec x + \tan x) = \tan x - \cos x + \sec x \; [+A]$	A1	**oe** Condone absence of $+A$
	$y = 1 + \dfrac{A - \cos x}{\sec x + \tan x}$		
	when $x = \dfrac{\pi}{3}$, $y = 1$ $\Rightarrow A = \dfrac{1}{2}$	A1ft	
	$y = 1 + \dfrac{0.5 - \cos x}{\sec x + \tan x}$	A1	$y = f(x)$ with **ACF** for $f(x)$ eg $y = \sin(x) - \dfrac{1}{2}\tan(x) + \dfrac{1}{2}\sec(x)$

	Question 5 Total	**6**	

Q	Answer	Marks	Comments
6(a)	$\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4$	B1	
		1	

Q	Answer	Marks	Comments
6(b)	$e^{\cos(2x)-1} = e^{-2x^2+\frac{2}{3}x^4} = e^{-2x^2}e^{\frac{2}{3}x^4}$		
	$= \left(1 + \left(-2x^2\right) + \frac{\left(-2x^2\right)^2}{2} + \ldots\right)\left(1 + \frac{2}{3}x^4 + \ldots\right)$	M1	**ft** their $1 + ax^2 + bx^4$ in part **(a)** **oe** eg $e^{\cos(2x)-1} = e^{-2x^2+\frac{2}{3}x^4}$ $= 1 + \left(-2x^2 + \frac{2}{3}x^4\right) + \frac{1}{2}\left(4x^4 + \ldots\right)$
	$= \left(1 - 2x^2 + 2x^4 + \ldots\right)\left(1 + \frac{2}{3}x^4 + \ldots\right)$		
	$= 1 - 2x^2 + \frac{8}{3}x^4 + \ldots$	A1	$1 - 2x^2 + \frac{8}{3}x^4$
		2	

Q	Answer	Marks	Comments
6(c)	$e^{\cos(2x)} = e\left(1 - 2x^2 + \frac{8}{3}x^4 + \ldots\right)$	B1ft	**ft** their $1 + px^2 + qx^4$ in part **(b)**
	$\lim_{x\to 0}\left(\frac{e - e^{\cos(2x)}}{x^2}\right) = \lim_{x\to 0}\left(\frac{e - e\left(1 - 2x^2 + \frac{8}{3}x^4 + \ldots\right)}{x^2}\right)$ $= \lim_{x\to 0}\left(\frac{2e\,x^2 - \frac{8}{3}e\,x^4 + \ldots}{x^2}\right) = \lim_{x\to 0}\left(2e - \frac{8}{3}e\,x^2 + \ldots\right)$	M1	Substitutes series expansion and divides numerator and denominator by x^2 to reach the form $\lim_{x\to 0}\left[P + O\left(x^2\right)\right]$
	$\lim_{x\to 0}\left(\frac{e - e^{\cos(2x)}}{x^2}\right) = 2e$	A1	**A0** if not from $\lim_{x\to 0}\left(2e - \frac{8}{3}e\,x^2\right)$
		3	

	Question 6 Total	**6**	

Q	Answer	Marks	Comments
7(a)	The interval of integration is infinite	E1	oe
		1	

Q	Answer	Marks	Comments
7(b)	$[\text{I} =] \int \frac{x-3}{e^x}\,dx = \int (x-3)e^{-x}\,dx$		
	$u = x-3$, $\frac{dv}{dx} = e^{-x}$		
	$[\text{I} =] = -(x-3)e^{-x} - \int -e^{-x}(dx)$	M1	$\frac{du}{dx} = 1$, $v = \pm e^{-x}$ used in correct integration by parts formula **PI** by correct integration
	$= -(x-3)e^{-x} - e^{-x}$ $[+c]$	A1	**oe** $-xe^{-x} + 2e^{-x}$
	$\int_3^{\infty} \frac{x-3}{e^x}\,dx = \lim_{a\to\infty} \int_3^a (x-3)e^{-x}\,dx$ $= \lim_{a\to\infty}\left(-(a-3)e^{-a} - e^{-a}\right) - \left(-e^{-3}\right)$	M1	Evidence of limit ∞ having been replaced by a (oe) at any stage and $\lim_{a\to\infty}$ seen or taken at any stage with no remaining lim relating to 3
	$\left[= \lim_{a\to\infty}\left(-ae^{-a} + 2e^{-a}\right) - \left(-e^{-3}\right)\right]$		
	$\lim_{a\to\infty}\left(a\,e^{-a}\right) = 0$	E1	Must be explicitly stated Accept if stated in the more general format
	$\int_3^{\infty} \frac{x-3}{e^x}\,dx = e^{-3}$	A1	First 3 marks must have been scored but can be awarded even if previous **E1** not awarded provided limits clearly substituted and no errors seen
		5	

	Question 7 Total	**6**	

Q	Answer	Marks	Comments
8(a)	For $n=1$, $\sum_{r=1}^{n}\left(r^3+3r^5\right) = 1^3+3\times1^5=1+3=4$ and $\frac{1}{2}n^3(n+1)^3=\frac{1}{2}\times1\times2^3=\frac{1}{2}\times 8=4$	B1	Correct values to show formula true for $n=1$ Must see 4 preceded by a relevant unsimplified numerical expression for both the LHS and the RHS
	Assume formula true for $n=k$ **(*)**, so $\sum_{r=1}^{k+1}\left(r^3+3r^5\right)=\sum_{r=1}^{k}\left(r^3+3r^5\right)+(k+1)^3+3(k+1)^5$ $=\frac{1}{2}k^3(k+1)^3+(k+1)^3+3(k+1)^3(k+1)^2$	M1	Assumes formula true for $n=k$ and considers $\sum_{r=1}^{k+1}\left(r^3+3r^5\right)$
	$=\frac{1}{2}(k+1)^3\left(k^3+2+6(k+1)^2\right)$ $=\frac{1}{2}(k+1)^3\left(k^3+6k^2+12k+8\right)$	A1	
	$=\frac{1}{2}(k+1)^3(k+2)\left(k^2+4k+4\right)=\frac{1}{2}(k+1)^3(k+2)^3$	A1	Must be convincingly shown
	Hence formula is true for $n=k+1$ **(**)** and since true for $n=1$ **(***)**, formula $\sum_{r=1}^{n}\left(r^3+3r^5\right)=\frac{1}{2}n^3(n+1)^3$ is true for $n=1, 2, 3, \ldots.$ by induction (****)	E1	Must have **(*)** , **(**)**, **(***)**, present, previous 4 marks scored and a final statement **(****)** clearly indicating that it relates to positive integers
		5	

Q	Answer	Marks	Comments
8(b)	$\frac{n^2}{4}(n+1)^2+\sum_{r=1}^{n}3r^5=\frac{1}{2}n^3(n+1)^3$	M1	Splits the LHS of formula and uses the formula for sum of cubes given in the formulae booklet
	$\frac{(3N)^2}{4}(3N+1)^2+3\sum_{r=1}^{3N}r^5=\frac{1}{2}(3N)^3(3N+1)^3$	m1	Replaces n by $3N$ (condone $3n$ for $3N$)
	$\sum_{r=1}^{3N}r^5=\frac{1}{6}(3N)^3(3N+1)^3-\frac{(3N)^2}{12}(3N+1)^2$ $=N^2(3N+1)^2\left(\frac{9}{2}N(3N+1)-\frac{3}{4}\right)$		$\sum_{r=1}^{n}r^5=\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$ may be obtained after the **M1** line before replacing n by $3N$
	$=\frac{3}{4}N^2(3N+1)^2(6N(3N+1)-1)$ $=\frac{3}{4}N^2(3N+1)^2(18N^2+6N-1)$	A1	**AG** Must be convincingly shown Condone if N is written as n
		3	

	Question 8 Total	**8**	

Q	Answer	Marks	Comments
9(a)(i)	$\begin{bmatrix} -1 & 4 & k \\ 2 & 3 & 6 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\Rightarrow \begin{bmatrix} -x+4y+kz \\ 2x+3y+6z \\ x+3y-2z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$	M1	Use of **Mv** = **v** with at least two rows of **Mv** correct **oe** eg $\begin{vmatrix} -2 & 4 & k \\ 2 & 2 & 6 \\ 1 & 3 & -3 \end{vmatrix} = 0$
	$-2x+4y+kz=0$; $x+3y-3z=0$ $2x+2y+6z=0 \Rightarrow x+y+3z=0$;	A1	**oe** eg correct unsimplified expansion of determinant
	$\Rightarrow 2x+4y=0 \Rightarrow x=-2y$ $\Rightarrow 8y+kz=0$ and $y=3z$ $\Rightarrow k=-24$	A1	$k=-24$
		3	

Q	Answer	Marks	Comments
9(a)(ii)	$\dfrac{x}{-6} = \dfrac{y}{3} = z$	M1 A1	**M1**: Either a general point on the line or a direction vector $\lambda\begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}$ **PI** **A1**: Correct Cartesian equations Accept eg $-x=2y=6z$
		2	

Q	Answer	Marks	Comments
9(b)	det **M** $= 3k + 64$	B1	Correct expression for det **M**
	Cofactor matrix		
	$\begin{bmatrix} -24 & 10 & 3 \\ 3k+8 & 2-k & 7 \\ 24-3k & 2k+6 & -11 \end{bmatrix}$	M1	One complete row or column or diagonal correct
		A2	All nine entries correct else **A1** for at least six entries correct
	Inverse matrix $\mathbf{M}^{-1}$	M1	Transpose of their cofactors with no more than one further error **and** division by their det **M** $(\neq 0)$
	$= \dfrac{1}{3k+64}\begin{bmatrix} -24 & 3k+8 & 24-3k \\ 10 & 2-k & 2k+6 \\ 3 & 7 & -11 \end{bmatrix}$	A1	Correct $\mathbf{M}^{-1}$ scores 6 marks
		6	

	Question 9 Total	**11**	

Q	Answer	Marks	Comments
10(a)	Aux. equation $m^2+2m+1=0$; $m=-1,-1$	M1	Forming and solving the correct aux. equation. **PI** by correct value(s) of m seen/used
	$y_{\text{CF}}=(Ax+B)\text{e}^{-x}$	A1	Correct CF
	$y_{\text{PI}}=a\sin 4x+b\cos 4x$	M1	$y_{\text{PI}}=a\sin 4x+b\cos 4x$ seen or used
	$y_{\text{PI}}'=4a\cos 4x-4b\sin 4x$ $y_{\text{PI}}''=-16a\sin 4x-16b\cos 4x$ $16a\sin 4x-16b\cos 4x+8a\cos 4x-8b\sin 4x$ $+a\sin 4x+b\cos 4x=\sin 4x+38\cos 4x$	M1	y_{PI}' and y_{PI}'' both of the form $\pm c\sin 4x\pm d\cos 4x$ and substitution into the differential equation
	$-15a-8b=1$; $-15b+8a=38\Rightarrow a=1,\ b=-2$ $y_{\text{PI}}=\sin 4x-2\cos 4x$	A1	Correct particular integral
	$[y_{\text{GS}}=]\ (Ax+B)\text{e}^{-x}+\sin 4x-2\cos 4x$	B1ft	Their CF + their PI with exactly two arbitrary constants
		6	

Q	Answer	Marks	Comments
10(b)	When $x=0$, $y=4$ $\Rightarrow B-2=4$, $B=6$ When $x=0$, $\frac{dy}{dx}=0$ $\Rightarrow A-B+4=0$, $A=2$	**M1**	$f(0)=4$ or $f'(0)=0$ or $f''(0)=34$ or series expansions used with their GS as far as finding a value for one of the arbitrary constants
	When $x=0$ $\frac{d^2y}{dx^2}=34 \Rightarrow -2A+B+32=34, A=2$	**M1**	$f(0)=4$ or $f'(0)=0$ or $f''(0)=34$ or series expansions used with their GS as far as finding a value for the remaining arbitrary constant
	$[y=f(x)=]$ $(2x+6)e^{-x}+\sin 4x-2\cos 4x$	**A1**	$(2x+6)e^{-x}+\sin 4x-2\cos 4x$ **oe** **PI** by the correct $f\left(\frac{\pi}{16}\right)$
	$\left[f\left(\frac{\pi}{16}\right)=\right]$ $\left(\frac{\pi}{8}+6\right)e^{-\frac{\pi}{16}}-\frac{1}{\sqrt{2}}$	**A1**	**ACF** but must be exact with no trigonometric functions
		4	

	Question 10 Total	**10**	

Q	Answer	Marks	Comments
11(a)	Direction of normal to Π_1 is $\begin{bmatrix}-1\\1\\4\end{bmatrix} \times \begin{bmatrix}1\\-3\\4\end{bmatrix} = \begin{bmatrix}16\\8\\2\end{bmatrix}$	M1 A1	Forming correct cross product. **PI** by a correct direction vector **oe**
	Magnitude of $\begin{bmatrix}16\\8\\2\end{bmatrix} = \sqrt{16^2+8^2+2^2} = \sqrt{324}$	B1ft	**ft** their direction vector
	Direction cosines $\frac{16}{\sqrt{324}}, \frac{8}{\sqrt{324}}, \frac{2}{\sqrt{324}}$	M1	**ft**
	Required sum $= \frac{16}{18}+\frac{8}{18}+\frac{2}{18}=\frac{26}{18}=\frac{13}{9}$	A1ft	**oe** $-\frac{13}{9}$ **ft** one incorrect element in the direction vector for the normal
		5	

Q	Answer	Marks	Comments
11(b)	Direction of L is $\begin{bmatrix}16\\8\\2\end{bmatrix} \times \begin{bmatrix}6\\c\\2\end{bmatrix}$	M1	or $\begin{bmatrix}2\\-5\\4\end{bmatrix} \bullet \begin{bmatrix}6\\c\\2\end{bmatrix} = 0$
	$= \begin{bmatrix}16-2c\\-20\\16c-48\end{bmatrix} = t\begin{bmatrix}2\\-5\\4\end{bmatrix}$	M1	or $12-5c+8=0$
	$[t=4 \quad \Rightarrow]\ c=4$	A1	
	Equation of Π_1 is $\mathbf{r} \bullet \begin{bmatrix}16\\8\\2\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} \bullet \begin{bmatrix}16\\8\\2\end{bmatrix} = 38$	B1ft	**oe** **ft** their direction vector. If using the given equation for Π_1, award this mark for $\lambda=-2$, $\mu=3$
	$(p,q,7)$ lies in both planes so $16p+8q+14=38$ and $6p+qc+14=14$	M1	Substitutes $(p,q,7)$ into their equations of Π_1 and Π_2
	$\Rightarrow p=6$, $q=-9$	A1	$p=6$ and $q=-9$
		6	

	Question 11 Total	**11**	

Q	Answer	Marks	Comments
12(a)	$(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$	B1	or $\cos 5\theta = \mathrm{Re}\left((\cos\theta + \mathrm{i}\sin\theta)^5\right)$
	$\cos 5\theta$ $= \cos^5\theta + 10\cos^3\theta\,(\mathrm{i}\sin\theta)^2 + 5\cos\theta\,(\mathrm{i}\sin\theta)^4$	M1	
	$= \cos^5\theta + 10\mathrm{i}^2\cos^3\theta\,(1-\cos^2\theta)$ $+ 5\mathrm{i}^4\cos\theta\,(1-\cos^2\theta)^2$	M1	Replacing $\sin^2\theta$ with $(1-\cos^2\theta)$
	$= \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta)$ $+ 5\cos\theta\,(1-2\cos^2\theta+\cos^4\theta)$ $= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta$ $-10\cos^3\theta + 5\cos^5\theta$	A1	
	$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$	A1	
		5	

Q	Answer	Marks	Comments
12(b)	Roots of $\cos 5\theta = 1$ are $\theta = [0,]\ \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$	M1	
	So roots of $16x^5 - 20x^3 + 5x - 1 = 0$ are $[\cos 0\,(=1),]\ \cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{6\pi}{5}, \cos\frac{8\pi}{5}$	M1	
	$16x^5 - 20x^3 + 5x - 1 = 0$ $(x-1)(16x^4 + 16x^3 - 4x^2 - 4x + 1) = 0$	M1	$16x^5 - 20x^3 + 5x - 1$ $= (x-1)(16x^4 + 16x^3 - 4x^2 - 4x + 1)$
	The quartic equation whose roots are $\cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{6\pi}{5}, \cos\frac{8\pi}{5}$ is $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$	A1	
		4	

Q	Answer	Marks	Comments
12(c)	[Since $\cos(2\pi-\alpha)=\cos\alpha$] $\cos\frac{4\pi}{5}=\cos\frac{6\pi}{5}$; $\cos\frac{8\pi}{5}=\cos\frac{2\pi}{5}$	M1	Both results seen or used
	For the quartic equation in **part (b)** Sum of root $=2\cos\frac{2\pi}{5}+2\cos\frac{6\pi}{5}=-\frac{16}{16}=-1$ Product of roots $=\cos^2\frac{2\pi}{5}\cos^2\frac{6\pi}{5}=\frac{1}{16}$	m1	Uses sum of roots $=2\cos\frac{2\pi}{5}+2\cos\frac{6\pi}{5}=-\frac{16}{16}$ and Product of roots $=\cos^2\frac{2\pi}{5}\cos^2\frac{6\pi}{5}=\frac{1}{16}$
	Since $\cos\frac{2\pi}{5}>0$ and $\cos\frac{6\pi}{5}<0$, $\cos\frac{2\pi}{5}\cos\frac{6\pi}{5}=-\frac{1}{4}$	E1	
	$x^2-\left(-\frac{1}{2}\right)x+\left(-\frac{1}{4}\right)=0$ The quadratic equation with integer coefficients whose roots are $\cos\frac{2\pi}{5}$ and $\cos\frac{6\pi}{5}$ is $4x^2+2x-1=0$	A1	
		4	

	Question 12 Total	**13**	

Q	Answer	Marks	Comments
13(a)(i)	$\frac{dy}{dt} = -(\cosh t)^{-2} \sinh t = -\tanh t \operatorname{sech} t$	M1	Either form, condone sign error
	$\frac{dx}{dt} = \operatorname{sech}^2 t$	A1	Both derivatives correct
	$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \operatorname{sech}^4 t + \tanh^2 t \operatorname{sech}^2 t$	M1	Squaring and adding their derivatives
	$= \operatorname{sech}^2 t\left(\operatorname{sech}^2 t + \tanh^2 t\right) = \operatorname{sech}^2 t$	A1	
	$s = \int_{-1}^{1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{-1}^{1} \operatorname{sech} t \, dt$	A1	**AG** Must be convincingly shown
		5	

Q	Answer	Marks	Comments
13(a)(ii)	$s = \int_{-1}^{1} \frac{2}{e^t + e^{-t}}\, dt$	B1	$\operatorname{sech} t = \frac{2}{e^t + e^{-t}}$ or $\operatorname{sech} t = \frac{\cosh t}{1 + \sinh^2 t}$ or **PI**
	$s = \int_{-1}^{1} \frac{2e^t}{e^{2t} + 1}\, dt = \int_{e^{-1}}^{e} \frac{2}{u^2 + 1}\, du$	M1	Valid substitution seen or **PI**
	$s = \left[2 \tan^{-1} u\right]_{e^{-1}}^{e}$	M1	$k \tan^{-1} u$ or $k \tan^{-1}(\sinh t)$ **oe**
	$s = 2\tan^{-1}(e) - 2\tan^{-1}\left(e^{-1}\right)$	A1	**oe** eg condone $4\tan^{-1}(e) - \pi$
		4	

Q	Answer	Marks	Comments
13(b)	$y = \text{sech}\, t$, $\tanh^2 t + \text{sech}^2 t = 1$ $x^2 + y^2 = 1$ [Possible values of x] $-1 < x < 1$ [Possible values of y] $0 < y \leq 1$	 B1 B1 B1	
		3	

	Question 13 Total	**12**	

Q	Answer	Marks	Comments
14(a)	$9\left(x^2+y^2\right)=36-24x+4x^2$	M1	$x^2+y^2=r^2$ or $y=r\sin\theta$ used
	$9r^2=\left(6-2x\right)^2$; $9r^2=\left(6-2r\cos\theta\right)^2$	M1	$x=r\cos\theta$ used
	$\pm 3r=\left(6-2r\cos\theta\right)$	A1	Condone missing $\pm$
	$\frac{6}{r}=2\cos\theta\pm 3$; Since $r\geq 0$, take + sign		
	$\frac{6}{r}=2\cos\theta+3 \Rightarrow r=\frac{6}{3+2\cos\theta}$	A1	**AG** Must be convincingly shown
		4	

Q	Answer	Marks	Comments
14(b)	$5+4\cos\theta=\frac{6}{3+2\cos\theta}$		
	$8\cos^2\theta+22\cos\theta+9=0$ $\Rightarrow\cos\theta=-0.5$ [only as $\cos\theta\neq-2.25$]	M1 A1	**M1**: Forms a quadratic equation in $\cos\theta$ and solves to find a value for $\cos\theta$ **A1**: Correct value of $\cos\theta$
	When $\cos\theta=-0.5$, $r=3$; $P\left(3, \frac{2\pi}{3}\right)$, $Q\left(3, -\frac{2\pi}{3}\right)$, $\angle POQ=\frac{2\pi}{3}$	A1	Either correct coordinates of *P* and *Q* or $r=3$ and $\angle POQ=\frac{2\pi}{3}$ seen/used
	area of triangle *OPQ* $=\frac{1}{2}(3)^2\sin\left(\frac{2\pi}{3}\right)=\frac{9}{2}\left(\frac{\sqrt{3}}{2}\right)=\frac{9\sqrt{3}}{4}$	A1	**AG** Must be convincingly shown
		4	

Q	Answer	Marks	Comments
14(c)	Equation of *PQ* is $r\cos\theta = -1.5$	B1ft	ft their values for r and $\cos\theta$
	$(5+4\cos\theta)\cos\theta = -1.5$ $8\cos^2\theta + 10\cos\theta + 3 = 0$ $\cos\theta = -\frac{1}{2}$ [*P* and *Q*]; $\cos\theta = -\frac{3}{4}$ [*S* and *T*]	M1	Solving their $r\cos\theta = -k$ with $r = 5+4\cos\theta$ to obtain two possible values of either $\cos\theta$ or r
	when $\cos\theta = -\frac{3}{4}$, $r = 2$ coordinates of *S* and *T* are $(2,\ \pi-\alpha)$ and $(2,\ -\pi+\alpha)$ where $\cos\alpha = \frac{3}{4}$	A1	Correct polar coordinates of *S* and *T* **PI** by the correct area of *SOT*
	required area = $0.5(2)(2)\sin 2\alpha - 2(0.5)\int_{\pi-\alpha}^{\pi}(5+4\cos\theta)^2\,\mathrm{d}\theta$	M1	**oe** Representing the required area in any correct mathematical form
		B1	**B1** for correct exact value for area of triangle *SOT*
	$= \frac{3\sqrt{7}}{4} - \left[33\theta + 40\sin\theta + 4\sin 2\theta\right]_{\pi-\alpha}^{\pi}$	B1	**B1** for correct integration of $(5+4\cos\theta)^2$
	$= \frac{3\sqrt{7}}{4} - \{-(-33\alpha + 40\sin\alpha - 4\sin 2\alpha)\}$		
	$= \frac{3\sqrt{7}}{4} - 33\cos^{-1}\left(\frac{3}{4}\right) + 40\frac{\sqrt{7}}{4} - 8\frac{\sqrt{7}}{4}\frac{3}{4}$		
	$= \frac{37}{4}\sqrt{7} - 33\cos^{-1}\left(\frac{3}{4}\right)$	A1	
		7	

	Question 14 Total	**15**	