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(9665/FM03) Unit FP2 Pure Mathematics

Mark scheme

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Key to mark scheme abbreviations

M	Mark is for method
m	Mark is dependent on one or more M marks and is for method
A	Mark is dependent on M or m marks and is for accuracy
B	Mark is independent of M or m marks and is for method and accuracy
E	Mark is for explanation
√ **or ft**	Follow through from previous incorrect result
CAO	Correct answer only
CSO	Correct solution only
AWFW	Anything which falls within
AWRT	Anything which rounds to
ACF	Any correct form
AG	Answer given
SC	Special case
oe	Or equivalent
A2, 1	2 or 1 (or 0) accuracy marks
–*x* EE	Deduct *x* marks for each error
NMS	No method shown
PI	Possibly implied
SCA	Substantially correct approach
sf	Significant figure(s)
dp	Decimal place(s)

Q	Answer	Marks	Comments
1(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+(1+x)^2} + \frac{1}{2}\left(\frac{1}{1-\left(\frac{x}{2}\right)^2}\right)$	M1 A1	One term differentiated correctly Both terms differentiated correctly
		2	
1(b)	$y_P\ \left[= \tan^{-1} 1 + 0\right] = \frac{\pi}{4}$	B1	
	At *P*, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ Gradient of normal = –1	M1	Finds a value for their $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 0$ and its negative reciprocal seen
	Equation of normal: $y - \frac{\pi}{4} = -x$	A1	**CSO** **ACF** but must be exact
		3	
	Total	5	

Q	Answer	Marks	Comments
2(a)	Rotation through 60°, about the z-axis	M1 A1	**M0** if more than one transformation. **oe** **SC1** for 'rotate' or 'rotated' or 'rotates' and 60°, about the z-axis **oe**
		2	
2(b)	z-axis	B1	**oe** eg $x = y = 0$
		1	
	Total	**3**	

Q	Answer	Marks	Comments
3(a)	$6 = A(r+1) + B(r-1)$ $r = 1$: $6 = 2A$; $r = -1$: $6 = -2B$	M1	$6 = A(r+1) + B(r-1)$ used to form either a correct equation in A or B or a correct pair of simultaneous equations in A and B **PI**
	$A = 3$; $B = -3$. $\dfrac{3}{r-1} - \dfrac{3}{r+1}$	A1	$A = 3$; $B = -3$
		2	
3(b)	$\displaystyle\sum_{r=2}^{n} \frac{6}{(r-1)(r+1)} = \sum_{r=2}^{n} \frac{3}{r-1} - \frac{3}{r+1}$ $= \left(3 - \cancel{1}\right) + \left(\frac{3}{2} - \frac{3}{4}\right) + \left(\cancel{1} - \frac{3}{5}\right) + \ldots$ $+ \left(\frac{3}{n-3} - \cancel{\frac{3}{n-1}}\right) + \left(\frac{3}{n-2} - \frac{3}{n}\right) + \left(\cancel{\frac{3}{n-1}} - \frac{3}{n+1}\right)$	M1	Uses method of differences showing the first and last terms and at least two other terms so that a pair of values which cancel are seen
	$= 3 + \frac{3}{2} - \frac{3}{n} - \frac{3}{n+1}$	A1	
	$= \dfrac{9n(n+1) - 6(n+1) - 6n}{2n(n+1)}$	M1	Correctly writing $p + \dfrac{q}{n} + \dfrac{r}{n+1}$ as a single 'fraction' with denominator $2n(n+1)$
	$= \dfrac{9n^2 - 3n - 6}{2n(n+1)}$	A1	**CAO**
		4	
	Total	6	

Q	Answer	Marks	Comments
4	Aux. eqn $4m^2+4m+1=0$ $(2m+1)^2=0$ $m=-\frac{1}{2}$	M1	Factorises or applies other valid method to the correct Aux. eqn. **PI** by the correct value of m seen/used
	$y=(A+Bx)\mathrm{e}^{-\frac{1}{2}x}$	A1	$(A+Bx)\mathrm{e}^{-\frac{1}{2}x}$ **oe**
	$x=0 \quad y=4 \quad \Rightarrow A=4$	B1ft	**ft** on $y=(A+Bx)\mathrm{e}^{\pm\frac{1}{2}x}$
	$\frac{\mathrm{d}y}{\mathrm{d}x}=B\,\mathrm{e}^{-\frac{1}{2}x}+(A+Bx)\left(-\frac{1}{2}\mathrm{e}^{-\frac{1}{2}x}\right)$	M1	Product rule used correctly to find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$
	$x=0 \quad \frac{\mathrm{d}y}{\mathrm{d}x}=1 \quad \Rightarrow 1=B-\frac{1}{2}A$	A1	**PI**
	$y=(4+3x)\mathrm{e}^{-\frac{1}{2}x}$	A1	**ACF**
	Total	**6**	

Q	Answer	Marks	Comments
5(a)	Integrand, $\ln x$, is not defined at $x = 0$	E1	oe
		1	
5(b)	$u = \ln x \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x}$ $\int \ln x \, dx$ $dv = dx \Rightarrow v = x$	M1	$u = \ln x \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x}$ $dv = dx \Rightarrow v = x$ **PI**
	$\int \ln x \, dx = x \ln x - \int x\left(\frac{1}{x}\right) dx$	A1	**PI**
	$\int \ln x \, dx = x \ln x - x \quad [+c]$	A1	Correct integration of $\ln x$
	$\int_0^{e^2} \ln x \, dx = \lim_{a \to 0} \int_a^{e^2} \ln x \, dx$ $= e^2 \ln e^2 - e^2 - \lim_{a \to 0}(a \ln a - a)$	M1	Evidence of limit 0 replaced by a (**oe**), $\lim_{a \to 0}$ seen at any stage with no remaining lim relating to e^2
	$\lim_{a \to 0}(a \ln a) = 0$	E1	Accept if stated in the more general format.
	$\int_0^{e^2} \ln x \, dx = e^2 \ln e^2 - e^2 = e^2$	A1	First 4 marks must have been scored but can be awarded even if previous **E1** not scored provided limits clearly substituted and no errors seen.
		6	
	Total	7	

Q	Answer	Marks	Comments
6(a)	$[\mathbf{r}\times\mathbf{m}=\mathbf{n}\ \Rightarrow]$ $\mathbf{n}$ is perpendicular to $\mathbf{m}$	E1	
	$\mathbf{n}$ perpendicular to $\mathbf{m}$, $\mathbf{m}\cdot\mathbf{n}=0$	E1	
	[so $\mathbf{r}$ cannot be found such that both $\mathbf{r}\times\mathbf{m}=\mathbf{n}$ and $\mathbf{m}\cdot\mathbf{n}=12$ are true.]		
		2	
6(b)	$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}2 & p & -1\\ -p & 4 & -7\\ -4 & 2p & -9\end{vmatrix}$	M1	**oe** eg $\begin{bmatrix}2\\ p\\ -1\end{bmatrix}\cdot\begin{bmatrix}-36+14p\\ 28-9p\\ -2p^2+16\end{bmatrix}$ at least one term correct in the cross product
	$=2(-36+14p)-p(9p-28)-1(-2p^2+16)$	A1	**oe**
	$=-7p^2+56p-88$	A1	Condone at most one error up to this line of working
	$-7p^2+56p-88=17$ $7p^2-56p+105=0$; $7(p-5)(p-3)=0$ or $-7p^2+56p-88=-17$ $7p^2-56p+71=0$; $7(p-4)^2=41$	M1	Equating their quadratic expression for the scalar triple product to **either** 17 **or** –17
	$7(p-5)(p-3)=0 \Rightarrow p=3,\ p=5$	A1	**oe**
	$p=\dfrac{56\pm\sqrt{1148}}{14}$	A1	**oe** $p=\dfrac{28\pm\sqrt{287}}{7}$, $p=4\pm\dfrac{\sqrt{287}}{7}$ (6.4….; 1.5…)
		6	
	Total	8	

Q	Answer	Marks	Comments
7(a)	Ch Eqn. $\begin{vmatrix} 3-\lambda & -2 \\ 5 & p-\lambda \end{vmatrix} = 0$	M1	Seen or used
	When $\lambda = 1$, $\begin{vmatrix} 2 & -2 \\ 5 & p-1 \end{vmatrix} = 0$ $\Rightarrow 2p - 2 + 10 = 0 \quad \Rightarrow p = -4$	A1	$p = -4$
	$(3-\lambda)(-4-\lambda)+10=0$; $\quad \lambda^2 + \lambda - 2 = 0$	M1	Forms quadratic eqn
	$(\lambda-1)(\lambda+2)=0$; other eigenvalue = –2	A1	Correct other eigenvalue
		4	
7(b)	When $\lambda = 1$, $3x - 2y = x$, $5x - 4y = y$ When $\lambda = -2$, $3x - 2y = -2x$, $5x - 4y = -2y$	M1	Subst. **either** value of λ into $\mathbf{M}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$ to form two linear equations in x and y **oe** **PI** by at least one correct eigenvector provided no errors seen
	When $\lambda = 1$, $x = y$ Eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	A1	Eigenvector $\alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, for any $\alpha \neq 0$
	When $\lambda = -2$, $5x = 2y$ Eigenvector $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$	A1	Eigenvector $\beta\begin{bmatrix} 2 \\ 5 \end{bmatrix}$, for any $\beta \neq 0$
		3	
	Total	7	

Q	Answer	Marks	Comments
8(a)	det $= k\begin{vmatrix}3k-2 & 4\\ 3k & 5\end{vmatrix} - 2\begin{vmatrix}2k-2 & 4\\ 2k+3 & 5\end{vmatrix} + (k-4)\begin{vmatrix}2k-2 & 3k-2\\ 2k+3 & 3k\end{vmatrix}$	M1	Correctly expanding by any row or column **oe**
	$= k(3k-10) - 2(2k-22) + (k-4)(6-11k)$		
	$= -8k^2 + 36k + 20$	A1	Correct quadratic in k No errors seen
		2	
8(b)(i)	$-8k^2 + 36k + 20 = 0$	M1	Their answer to **(a)** $= 0$ and an attempt to solve or $-8k^2 + pk + q = 0$, with their p and q and attempt to solve
	$k = 5,\ -0.5$	A1	Correct two values
		2	
8(b)(ii)	$k = 5,$ $5x + 2y + z = a$ eqn1 $8x + 13y + 4z = b$ eqn2 $13x + 15y + 5z = c$ eqn3	B1ft	**ft** their integer value for k from **(b)(i)** **PI** by later work
	(eqn1) + (eqn2) – (eqn3)	M1	Relevant combination of arithmetical operations to eliminate x, y and z
	$a + b - c = 0 \Rightarrow\ b = c - a$	A1	$b = c - a$
		3	
	Total	**7**	

Q	Answer	Marks	Comments
9(a)	Since coefficients are all real, complex roots occur in conjugate pairs and so the cubic equation cannot have exactly one non-real root.	E1	**oe**
		1	
9(b)(i)	$\alpha+\beta+\gamma=-\frac{p}{2}$	M1	Any one correct equation seen or used
	$\alpha\beta+\alpha\gamma+\beta\gamma=2$ $\alpha\beta\gamma=3\mathrm{i}$	A1	All three correct equations seen or used
	$\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2=\alpha\beta\gamma(\alpha+\beta+\gamma)$	M1	Seen or used anywhere in the working
	$(\alpha\beta+2)(\alpha\gamma+2)(\beta\gamma+2)$ $=(\alpha\beta\gamma)^2+2\alpha\beta\gamma(\alpha+\beta+\gamma)+4(\alpha\beta+\alpha\gamma+\beta\gamma)+8$		
	$=(3\mathrm{i})^2+2(3\mathrm{i})\left(-\frac{p}{2}\right)+4(2)+8=7-3\mathrm{i}p$	A1	**CSO**
		4	
9(b)(ii)	$(\alpha\beta+2)(\alpha\gamma+2)+(\alpha\beta+2)(\beta\gamma+2)+(\alpha\gamma+2)(\beta\gamma+2)=3\mathrm{i}\left(-\frac{p}{2}\right)+4(2)+12$	B1	$3\mathrm{i}\left(-\frac{p}{2}\right)+4(2)+12$ or better
	Cubic eqn with roots $\alpha\beta+2,\ \alpha\gamma+2,\ \beta\gamma+2$ is $z^3-\left(\sum\alpha\right)z^2+\left(\sum\alpha\beta\right)z-\alpha\beta\gamma \quad [=0]$	M1	Used
	$z^3-8z^2+\left(20-\frac{3\mathrm{i}p}{2}\right)z-(7-3\mathrm{i}p)=0$	A1ft	**ft** their value of k from **(b)(i)**
		3	
	Total	**8**	

Q	Answer	Marks	Comments
10(a)	When $n=1$, LHS $=(\cos\theta+\mathrm{i}\sin\theta)^1=\cos\theta+\mathrm{i}\sin\theta$ RHS = $\cos 1\theta+\mathrm{i}\sin 1\theta=\cos\theta+\mathrm{i}\sin\theta$ so result is true for $n=1$	B1	Verifies LHS = RHS for $n=1$ and statement 'true for $n=1$' seen at any point
	Assume result true for $n=k$ **(*)** so $(\cos\theta+\mathrm{i}\sin\theta)^{k+1}=$ $=(\cos k\theta+\mathrm{i}\sin k\theta)(\cos\theta+\mathrm{i}\sin\theta)$	M1	Assumes true for $n=k$ and considers $(\cos\theta+\mathrm{i}\sin\theta)^{k+1}$
	$=\cos k\theta\cos\theta-\sin k\theta\sin\theta$ $+\mathrm{i}(\sin k\theta\cos\theta+\cos k\theta\sin\theta)$	A1	
	$=\cos(k\theta+\theta)+\mathrm{i}\sin(k\theta+\theta)$ $=\cos(k+1)\theta+\mathrm{i}\sin(k+1)\theta$	A1	Accept either form. Uses identities for $\cos(A+B)$ and $\sin(A+B)$
	Hence formula is true for $n=k+1$ **(**)** and since true for $n=1$, formula is true for $n=1,2,3\ldots$ by induction **(***)**	E1	Must have **(*)** and **(**)** present. previous 4 marks scored and concluding statement **(***)** must clearly indicate that it relates to positive integers.
		5	
10(b)	$2(\cos 3\theta+\mathrm{i}\ \sin 3\theta)=1-\sqrt{3}\ \mathrm{i}$ $\cos 3\theta=\frac{1}{2}$ and $\sin 3\theta=-\frac{\sqrt{3}}{2}$	M1	Uses result in **(a)** and equates real parts and imaginary parts **PI** $2(\cos 3\theta+\mathrm{i}\ \sin 3\theta)$ or $=2\left(\cos\left(-\frac{\pi}{3}\right)+\mathrm{i}\ \sin\left(-\frac{\pi}{3}\right)\right)$ seen
	Both eqns satisfied by solutions $3\theta=2N\pi-\frac{\pi}{3}$	M1	Finds the full set of general solutions for each eqn or considers signs of cos and sin to recognise that common solutions lie in the 4th quadrant **PI**
	[First two positive values of θ are] $\frac{5\pi}{9}$	A1	
	[and] $\frac{11\pi}{9}$	A1	
		4	
	Total	**9**	

Q	Answer	Marks	Comments
11(a)	$\mathbf{n} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$; $\mathbf{n} = \begin{bmatrix} -5 \\ 8 \\ 6 \end{bmatrix}$	M1 A1	Relevant vector product stated or used Correct $\mathbf{n}$
	$d = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot \mathbf{n} = (2)(-5)+(1)(8)+(2)(6)$	M1	**ft** on their $\mathbf{n}$
	$\mathbf{r} \cdot \begin{bmatrix} -5 \\ 8 \\ 6 \end{bmatrix} = 10$	A1	
		4	
11(b)	$\cos\theta = \dfrac{(-5\,\mathbf{i}+8\mathbf{j}+6\mathbf{k})\cdot(-3\mathbf{i}+\mathbf{j}+2\mathbf{k})}{\left(\sqrt{(-5)^2+8^2+6^2}\right)\left(\sqrt{(-3)^2+1^2+2^2}\right)}$	M1	**ft** on their $\mathbf{n}$
	Scalar product in numerator = 35	B1ft	Correct evaluation of scalar product **ft** on their $\mathbf{n}$ in **part (a)** **PI** by $\cos\theta = \dfrac{7}{\sqrt{70}}$ **oe**
	Denominator = $\sqrt{25+64+36}\ \sqrt{9+1+4}$	B1ft	Correct product of moduli ft on their $\mathbf{n}$ in **part (a)** **PI** by $\cos\theta = \dfrac{7}{\sqrt{70}}$ **oe**
	$\cos\theta = \dfrac{7}{\sqrt{70}}$; $\theta = 33.2^\circ$	A1	**CAO**
		4	
11(c)	$-3x+y+2z=5$	B1	
		1	

Q	Answer	Marks	Comments
11(d)	$\mathbf{b}=\begin{bmatrix}-5\\8\\6\end{bmatrix}\times\begin{bmatrix}-3\\1\\2\end{bmatrix}=\begin{bmatrix}16-6\\-18+10\\-5+24\end{bmatrix}=\begin{bmatrix}10\\-8\\19\end{bmatrix}$	M1 A1	**M1**: At least 2 components correct **ft** on **(a)** or $10\mathbf{i}+28\mathbf{j}-29\mathbf{k}$ **oe** seen **A1**: a correct **b** eg $\begin{bmatrix}10\\-8\\19\end{bmatrix}$
	$\Pi_1: -5x+8y+6z=10$ $\Pi_2: -3x+y+2z=5$ For common pt put eg x=0 and solve	M1	Valid method for finding a common point
	Common pt $(0,-1,3)$	A1	**oe** likely ones $\left(-\frac{30}{19},\frac{5}{19},0\right)$, $\left(-\frac{5}{4},0,\frac{5}{8}\right)$
	$(\mathbf{r}-(-\mathbf{j}+3\mathbf{k}))\times(10\mathbf{i}-8\mathbf{j}+19\mathbf{k})=\mathbf{0}$	A1ft	**oe** Both **M1**'s scored but must be in correct form
		5	
	Total	**14**	

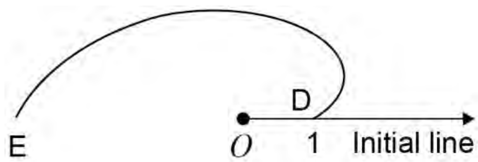
Q	Answer	Marks	Comments
12(a)(i)	eg $y = 2x + 2\ln(\sec x)$ or $\frac{dy}{dx} = \frac{2e^{2x}}{e^{2x}(1+\tan^2 x)}(1+\tan^2 x + \tan x \sec^2 x)$	M1	Writing y in an appropriate simplified form using log laws correctly or directly finding a correct unsimplified expression for the derivative
	$\frac{dy}{dx} = 2 + 2\left(\frac{1}{\sec^2 x}\right)(\sec^2 x \tan x)$ $= 2(1+\tan x)$	A1	Must see a relevant intermediate line of working **AG**
		2	
12(a)(ii)	$\frac{d^2y}{dx^2} = 2\sec^2 x$	B1	
	$\frac{d^3y}{dx^3} = (4\sec x)\sec x \tan x$	M1	$k\sec^2 x \tan x$ **oe**
	$\frac{d^4y}{dx^4} = (4\sec^4 x) + 8\sec^2 x\ \tan^2 x$	A1	**ACF**
		3	
12(b)	When $x = 0$: $y = 0,\ \frac{dy}{dx} = 2$	B1	**PI** by seeing 0 as the first term in the Maclaurin series and use of 2 in the second term of the Maclaurin series
	$\left[\frac{d^2y}{dx^2} = 2,\ \frac{d^3y}{dx^3} = 0,\ \frac{d^4y}{dx^4} = 4\right]$ Maclaurin series: $\ln\left[e^{2x}(1+\tan^2 x)\right]$ $= 0 + 2x + \frac{2}{2!}x^2 + 0x^3 + \frac{4}{4!}x^4$	M1	**ft** their derivatives
	$\ln\left[e^{2x}(1+\tan^2 x)\right] = 2x + x^2 + \frac{1}{6}x^4$	A1	**AG** Be convinced
		3	

Q	Answer	Marks	Comments
12(c)	$x\sin x = x^2 - \frac{1}{6}x^4 + \ldots$	**B1**	**oe**
	$2\ln(\cos x) = \ln(\cos^2 x)$ $= -\ln(\sec^2 x) = -x^2 - \frac{x^4}{6}$	**M1**	Valid method to find the correct first two non-zero terms in expansion of $2\ln(\cos x)$
	$\lim_{x\to 0}\left[\frac{2\ln(\cos x) + x\sin x}{2\sqrt{(x^8 + x^{10})}}\right]$ $= \lim_{x\to 0}\left[\frac{-\frac{x^4}{3} + O(x^6)}{2x^4\sqrt{(1+x^2)}}\right]$ $= \lim_{x\to 0}\left[\frac{-\frac{1}{3} + O(x^2)}{2\sqrt{(1+x^2)}}\right]$ so limit exists	**M1**	In place of $O(x^2)$ we may see terms in x^2 and higher powers of x
	$= -\frac{1}{6}$	**A1**	**CSO**
		4	
	Total	**12**	

Q	Answer	Marks	Comments
13(a)	$1+\sinh^2\theta = 1+\left[\frac{1}{2}\left(e^{\theta}-e^{-\theta}\right)\right]^2$	B1	Writing sinh or cosh correctly in terms of exponentials
	$1+\sinh^2\theta = 1+\frac{1}{4}\left(e^{2\theta}-2+e^{-2\theta}\right)$	M1	Correct expansion of either $\left(e^{\theta}-e^{-\theta}\right)^2$ or $\left(e^{\theta}+e^{-\theta}\right)^2$
	$=\frac{1}{4}\left(4+e^{2\theta}-2+e^{-2\theta}\right)=\frac{1}{4}\left(e^{2\theta}+2+e^{-2\theta}\right)$		
	$=\left[\frac{1}{2}\left(e^{\theta}+e^{-\theta}\right)\right]^2 = \cosh^2\theta$	A1	**AG** Be convinced
		3	

Q	Answer	Marks	Comments
13(b)	I.F. is $\exp\left(\int \frac{x}{\left(1+x^2\right)}\ [\mathrm{d}x]\right)$	M1	I.F. identified and integration attempted
	$= e^{\frac{1}{2}\ln\left(1+x^2\right)}$	A1	
	$=\sqrt{1+x^2}$	A1	
	$y\ \sqrt{1+x^2} = \int 2\sqrt{1+x^2}\ [\mathrm{d}x]$	A1	
	Let $x=\sinh\theta$	M1	Relevant substitution used
	$\int 2\sqrt{1+x^2}\,[\mathrm{d}x] = \int 2\sqrt{1+\sinh^2\theta}\ \cosh\theta\ \mathrm{d}\theta$	A1	
	$=\int 2\cosh^2\theta\ \mathrm{d}\theta$	m1	Identity in **(a)** used
	$=\int \left(1+\cosh 2\theta\right)\ \mathrm{d}\theta$	m1	Identity $\cosh 2\theta = 2\cosh^2\theta - 1$ used
	$=\theta + 0.5\sinh 2\theta\quad [+A]$	A1	
	$=\theta+\sinh\theta\cosh\theta\quad [+A]$		
	$=\sinh^{-1}x + x\sqrt{1+x^2}\quad [+A]$	A1	**oe**
	$y=\dfrac{\sinh^{-1}x + x\sqrt{1+x^2}+A}{\sqrt{1+x^2}}$	A1	**ACF**
		11	
	Total	**14**	

Q	Answer	Marks	Comments
13(b) ALT	I.F. is $\exp\left(\int \frac{x}{(1+x^2)}\ [\mathrm{d}x]\right)$	M1	I.F. identified and integration attempted
	$= \mathrm{e}^{\frac{1}{2}\ln(1+x^2)}$	A1	
	$= \sqrt{1+x^2}$	A1	
	$y\ \sqrt{1+x^2} = \int 2\sqrt{1+x^2}\ [\mathrm{d}x]$	A1	
	$\int\sqrt{x^2+1}\,\mathrm{d}x = x\sqrt{x^2+1} - \int x\frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x$	M1 A1	**M1**: Use of integration by parts with $u=\sqrt{x^2+1}$ and $\mathrm{d}v=\mathrm{d}x$ **A1**: All correct
	$\int\sqrt{x^2+1}\,\mathrm{d}x = x\sqrt{x^2+1} - \int\sqrt{x^2+1}-\frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$	m1	Use of $\frac{x^2}{\sqrt{x^2+1}}=\sqrt{x^2+1}-\frac{1}{\sqrt{x^2+1}}$
	$2\int\sqrt{x^2+1}\,\mathrm{d}x = x\sqrt{x^2+1} + \int\frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$	m1	
	$2\int\sqrt{x^2+1}\,\mathrm{d}x = x\sqrt{x^2+1}+\sinh^{-1}x\quad [+A]$	A1	
	$y\ \sqrt{1+x^2} = x\sqrt{x^2+1}+\sinh^{-1}x\quad [+A]$	A1	**oe**
	$y=\frac{\sinh^{-1}x + x\sqrt{1+x^2}+A}{\sqrt{1+x^2}}$	A1	**ACF**
		11	

Q	Answer	Marks	Comments
14(a)	$\frac{dx}{d\theta} = e^{0.5\theta}\cos\theta - 2e^{0.5\theta}\sin\theta$	**M1**	Product rule used at least once
	$\frac{dy}{d\theta} = e^{0.5\theta}\sin\theta + 2e^{0.5\theta}\cos\theta$	**A1**	Derivative of x or derivative of y correct
	$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(e^{0.5\theta}\right)^2\left(5\cos^2\theta + 5\sin^2\theta\right)$	**M1**	Finding an expression for $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$ in terms of θ
	$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(e^{0.5\theta}\right)^2(5)$	**A1**	$5\left(e^{0.5\theta}\right)^2$ seen or used
	$PQ = \int_0^{\pi} \sqrt{5}e^{0.5\theta}\ d\theta$	**M1**	$\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta$ used
	$= 2\sqrt{5}\left[e^{0.5\theta}\right]_0^{\pi}$	**A1**	Correct integration of $k\,e^{0.5\theta}$
	$= 2\sqrt{5}\left(e^{\frac{\pi}{2}} - 1\right)$	**A1**	**ACF** but must be exact
		7	
14(b)(i)		**B1**	Part of a spiral with distance of curve from O increasing as θ increases
		B1	Gradients of spiral at D and E being non-negative **and** indication that D is 1 on the initial line and E is the other end pt
		2	
14(b)(ii)	polar eqn of C_1 is $r = 2e^{0.5\theta}$	**M1 A1**	**M1**: $x = r\cos\theta$, $y = r\sin\theta$ or $x^2 + y^2 = r^2$ used **A1**: Correct polar eqn of C_1 seen or used
	[Area =] $\frac{1}{2}\int_0^{\pi}\left(2e^{0.5\theta}\right)^2\ d\theta - \frac{1}{2}\int_0^{\pi}\left(2e^{0.5\theta} - 1\right)^2\ d\theta$	**M1**	Condone subtraction in reverse order
	$= \frac{1}{2}\int_0^{\pi}\left(4e^{0.5\theta} - 1\right)\ d\theta = \frac{1}{2}\left[8e^{0.5\theta} - \theta\right]_0^{\pi}$	**A1**	$\frac{1}{2}\left[8e^{0.5\theta} - \theta\right]_0^{\pi}$ or $\frac{1}{2}\left[-8e^{0.5\theta} + \theta\right]_0^{\pi}$
	Area $= \frac{1}{2}\left(8\ e^{\frac{\pi}{2}} - 8 - \pi\right)$	**A1**	**CAO**
		5	

Q	Answer	Marks	Comments
14(b)(ii) ALT	[Area under C_1 =] $\int_{-2e^{\frac{\pi}{2}}}^{2} y \, dx$		
	$\int_{[\pi]}^{[0]} 2e^{0.5\theta}\sin\theta\left(e^{0.5\theta}\cos\theta - 2e^{0.5\theta}\sin\theta\right) d\theta$	**M1**	
	$\int_{[\pi]}^{[0]} e^{\theta}\left(\sin 2\theta + 2\cos 2\theta - 2\right) d\theta$		
	$= \int_{[\pi]}^{[0]} d\left(e^{\theta}\sin 2\theta - 2e^{\theta}\right)$		
	$= \left[e^{\theta}\sin 2\theta - 2e^{\theta}\right]_{[\pi]}^{[0]}$	**A1**	
	[Area under C_2 =] $\frac{1}{2}\int_0^{\pi}\left(2e^{0.5\theta} - 1\right)^2 d\theta$	**M1**	
	$= \frac{1}{2}\left(\left(4e^{\pi} - 8e^{\frac{\pi}{2}} + \pi\right) - \left(4 - 8 + 0\right)\right)$	**A1**	
	Area = $\left(-2 + 2e^{\pi}\right) - \frac{1}{2}\left(4e^{\pi} - 8e^{\frac{\pi}{2}} + \pi + 4\right)$		
	$= \frac{1}{2}\left(8\,e^{\frac{\pi}{2}} - 8 - \pi\right)$	**A1**	**CAO**
		5	
	Total	**14**	