OXFORD
INTERNATIONAL
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# INTERNATIONAL A-LEVEL
# FURTHER MATHEMATICS
# FM03

(9665/FM03) Unit FP2 Pure Mathematics

Mark scheme

June 2023

Version: 1.0 Final

236XFM03/MS

Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Examiner.

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**Key to mark scheme abbreviations**

| | |
|---|---|
| **M** | Mark is for method |
| **m** | Mark is dependent on one or more M marks and is for method |
| **A** | Mark is dependent on M or m marks and is for accuracy |
| **B** | Mark is independent of M or m marks and is for method and accuracy |
| **E** | Mark is for explanation |
| √ **or ft** | Follow through from previous incorrect result |
| **CAO** | Correct answer only |
| **CSO** | Correct solution only |
| **AWFW** | Anything which falls within |
| **AWRT** | Anything which rounds to |
| **ACF** | Any correct form |
| **AG** | Answer given |
| **SC** | Special case |
| **oe** | Or equivalent |
| **A2, 1** | 2 or 1 (or 0) accuracy marks |
| **–*x* EE** | Deduct $x$ marks for each error |
| **NMS** | No method shown |
| **PI** | Possibly implied |
| **SCA** | Substantially correct approach |
| **sf** | Significant figure(s) |
| **dp** | Decimal place(s) |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 1 | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ | B2,1,0 | If not **B2** then award **B1** for either<br>$[\mathbf{N}=]\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $[\mathbf{M}=]\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$<br><br>**B2** cannot be awarded if the correct matrix for **NM** is obtained by using an incorrect matrix for **N** or **M** or if found by calculating **MN** |
| | | **2** | |

| | | | |
|---|---|---|---|
| | **Question 1 Total** | **2** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 2(a) | $\alpha+\beta+\gamma=4$<br>$\alpha\beta+\beta\gamma+\gamma\alpha=3$ | B1 | For either $\alpha+\beta+\gamma=4$ or $\alpha\beta+\beta\gamma+\gamma\alpha=3$ seen or used |
| | $\alpha^2+\beta^2+\gamma^2$<br>$=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$<br>$=4^2-2(3)$ | M1 | Correct formula seen/used |
| | $\alpha^2+\beta^2+\gamma^2=16-6=10$ | A1 | **CSO AG** Must be convincingly shown |
| | | **3** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 2(b) | $\beta$ is a root of the [cubic] equation | E1 | **oe** |
| | | **1** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 2(c) | $\alpha^3+\beta^3+\gamma^3$<br>$=4(\alpha^2+\beta^2+\gamma^2)-3(\alpha+\beta+\gamma)-3c$ | M1 | **oe** in terms of $c$ and expressions whose values have been stated/found<br>eg<br>$\alpha^3+\beta^3+\gamma^3=(\alpha+\beta+\gamma)^3$<br>$-3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\alpha\gamma)+3(-c)$ |
| | $=4(10)-3(4)-3c=40-12-3c=28-3c$ | A1 | **AG** Must be convincingly shown |
| | | **2** | |

| | | | |
|---|---|---|---|
| | **Question 2 Total** | **6** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 3 | $10 = \det(\mathbf{A})\det(\mathbf{B})$<br>$\det(\mathbf{A})\det(\mathbf{A}^{-1}) = 1$; $5\det(\mathbf{A}) = 1$ | **M1** | Either $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ seen/used<br>or $\det(\mathbf{A})\det(\mathbf{A}^{-1}) = 1$ seen/used |
| | $\det(\mathbf{A}) = \frac{1}{5}$ | **A1** | Correct value for $\det(\mathbf{A})$ seen/used<br>**PI** by use of $\det(\mathbf{A})\det(\mathbf{A}^{-1}) = 1$ |
| | $\det(\mathbf{B}) = 50$ | **A1ft** | **ft** $10 \div$ their value for $\det(\mathbf{A})$ |
| | Volume of $S_2 = 6 \times 50 = 300$ [cm$^3$] | **B1ft** | **ft** $6 \times$ their $\lvert\det(\mathbf{B})\rvert$ |
| | | **4** | |

| | | | |
|---|---|---|---|
| | **Question 3 Total** | **4** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 4(a) | $y = x\sqrt{x} - \frac{1}{3}\sqrt{x} \quad \Rightarrow \frac{dy}{dx} = \frac{3}{2}\sqrt{x} - \frac{1}{6\sqrt{x}}$ | M1 | **ACF** At least one correct term for $\frac{dy}{dx}$ |
| | $1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{9}{4}x-\frac{1}{2}+\frac{1}{36x}$ <br> $=\left(\frac{3}{2}\sqrt{x}+\frac{1}{6\sqrt{x}}\right)^2 \quad =\left(\frac{3}{2}x^{\frac{1}{2}}+\frac{1}{6}x^{-\frac{1}{2}}\right)^2$ | A1 | A correct expression for $1+\left(\frac{dy}{dx}\right)^2$ in the form $\left(px^n+qx^{-n}\right)^2$ |
| | | 2 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 4(b) | $S = 2\pi\int_{[1]}^{[4]} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$ | | |
| | $= 2\pi\int_{[1]}^{[4]}\left(x\sqrt{x}-\frac{1}{3}\sqrt{x}\right)\left(\frac{3}{2}\sqrt{x}+\frac{1}{6\sqrt{x}}\right)\ [dx]$ | M1 | Substitutes into a correct formula for the surface area $S$; **ft** their derivative. Condone missing brackets |
| | $= 2\pi\int_1^4\left(\frac{3}{2}x^2-\frac{1}{3}x-\frac{1}{18}\right)\ [dx]$ | M1 | Integrand $ax^2+bx+c$ with at least one correct coefficient |
| | $= 2\pi\left[\frac{x^3}{2}-\frac{x^2}{6}-\frac{x}{18}\right]_1^4$ | A1 | |
| | $= 2\pi\left[\left(\frac{64}{2}-\frac{16}{6}-\frac{4}{18}\right)-\left(\frac{1}{2}-\frac{1}{6}-\frac{1}{18}\right)\right]$ | | |
| | $= 2\pi\left(\frac{63}{2}-\frac{15}{6}-\frac{3}{18}\right)=\pi\left(63-\frac{16}{3}\right)=\frac{173}{3}\pi$ | A1 | **AG** Must be convincingly shown |
| | | 4 | |

| | | | |
|---|---|---|---|
| | **Question 4 Total** | **6** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 5 | $\int 9x^2 \ln x \, dx = 3x^3 \ln x - \int 3x^3 \left(\frac{1}{x}\right) dx$ | M1 | $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$<br>$\frac{dv}{dx} = 9x^2 \Rightarrow v = 3x^3$ **PI** |
| | $\int 9x^2 \ln x \, dx = 3x^3 \ln x - x^3 \quad [+c]$ | A1 | Correct integration of $9x^2 \ln x$ |
| | $\int_0^e 9x^2 \ln x \, dx = \lim_{a \to 0} \int_a^e 9x^2 \ln x \, dx$<br>$= \left(3e^3 - e^3\right) - \lim_{a \to 0}\left(3a^3 \ln a - a^3\right)$ | M1 | Evidence of limit 0 having been replaced by $a$ (**oe**) at any stage and $\lim_{a \to 0}$ seen or taken at any stage with no remaining lim relating to e |
| | $\lim_{a \to 0}\left(a^3 \ln a\right) = 0$ | B1 | Accept if stated in the more general format. |
| | $\int \frac{4}{1+4x^2} dx = \int \frac{1}{\frac{1}{4}+x^2} dx$<br>$= 2\tan^{-1}(2x) \quad [+c]$ | M1 A1 | **M1**: $k\tan^{-1}(kx)$ or $\lambda\tan^{-1}(kx)$<br>**A1**: $2\tan^{-1}(2x)$ |
| | $\int_0^e \left(9x^2 \ln x + \frac{4}{1+4x^2}\right) dx$<br>$= 2e^3 + 2\tan^{-1}(2e)$ | A1 | $2e^3 + 2\tan^{-1}(2e)$<br>Must have scored all previous **M** and **A** marks and no errors when substituting limits. |
| | | 7 | |

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|---|---|---|---|
| | **Question 5 Total** | 7 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 6 | I.F. is $\mathrm{e}^{\int\left(8x\left(x^2+2\right)^{-1}\right)\,\mathrm{d}x}=\mathrm{e}^{4\ln\left(x^2+2\right)}$ | M1 | I.F. identified and integration attempted |
| | I.F. $=\left(x^2+2\right)^4$ | A1 | Correct integrating factor |
| | $\left(x^2+2\right)^4\dfrac{\mathrm{d}y}{\mathrm{d}x}+8x\left(x^2+2\right)^3y$ $=2x^3\left(x^2+2\right)^4+\dfrac{\left(x^2+2\right)^4}{\left(x^2+2\right)^{\frac{9}{2}}}$ | | |
| | $\left(x^2+2\right)^4y=\displaystyle\int\left(2x^3\left(x^2+2\right)^4+\frac{1}{\sqrt{x^2+2}}\right)\mathrm{d}x$ | M1 | Multiplying both sides of the given DE by their I.F. and integrating LHS to get $y\times$ I.F. |
| | Let $u=x^2+2$ <br> $\Rightarrow\int 2x^3\left(x^2+2\right)^4\mathrm{d}x=\int(u-2)u^4\mathrm{d}u$ | M1 | A relevant substitution or relevant integration by parts used to find an expression for the integral of $2x^3\left(x^2+2\right)^4$ <br> **PI** by correct integration |
| | $\displaystyle\int 2x^3\left(x^2+2\right)^4\mathrm{d}x=\frac{\left(x^2+2\right)^6}{6}-\frac{2\left(x^2+2\right)^5}{5}\;[+A]$ | A1 | **oe** Correct integration of $2x^3\left(x^2+2\right)^4$ <br> eg $\dfrac{x^2\left(x^2+2\right)^5}{5}-\dfrac{\left(x^2+2\right)^6}{30}\;[+k]$ or <br> $\dfrac{1}{6}x^{12}+\dfrac{8}{5}x^{10}+6x^8+\dfrac{32}{3}x^6+8x^4\;[+c]$ |
| | $\displaystyle\int\left(\frac{1}{\sqrt{x^2+2}}\right)\mathrm{d}x=\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)\;[+B]$ | B1 | Correct integration of $\dfrac{1}{\sqrt{x^2+2}}$ <br> **oe**, such as $\ln\left(x+\sqrt{x^2+2}\right)\;[+B]$ |
| | $y=\dfrac{\left(x^2+2\right)^2}{6}-\dfrac{2\left(x^2+2\right)}{5}+\dfrac{\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)+c}{\left(x^2+2\right)^4}$ | A1 | Correct GS <br> $y=\mathrm{f}(x)$ with **ACF** for $\mathrm{f}(x)$ |
| | | **7** | |

| | | | |
|---|---|---|---|
| | **Question 6 Total** | **7** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 7(a) | $\sum_{r=1}^{n}(u_{r+1}-u_r)=\sum_{r=1}^{n}(2u_r)+\sum_{r=1}^{n}4$<br>LHS =<br>$\cancel{u_2}-u_1+\cancel{u_3}-\cancel{u_2}+\cancel{u_4}-\cancel{u_3}+\ldots+\cancel{u_n}-\cancel{u_{n-1}}+u_{n+1}-\cancel{u_n}$<br>$=u_{n+1}-u_1$ | M1 | Uses method of differences with $\sum_{r=1}^{n}(u_{r+1}-u_r)=\sum_{r=1}^{n}(2u_r)+\sum_{r=1}^{n}4$ |
| | RHS $=\sum_{r=1}^{n}(2u_r)+4n$ | B1 | $\sum_{r=1}^{n}4=4n$ or $\sum_{r=1}^{n}2=2n$ |
| | $2\sum_{r=1}^{n}(u_r)+4n=u_{n+1}-u_1$; $2\sum_{r=1}^{n}(u_r)=u_{n+1}-4n-3$<br>$\Rightarrow\sum_{r=1}^{n}u_r=\frac{1}{2}u_{n+1}-2n-\frac{3}{2}$ | A1 | **AG** Must be convincingly shown |
| | | 3 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 7(b) | When $n=1$, $u_1=5\times3^0-2=5\times1-2=5-2=3$<br>[Formula is true for $n=1$] | B1 | Correct values to show formula true for $n=1$ |
| | Assume formula true for $n=k$ **(*)**,<br>[integer $k\geq1$,] so<br>$u_{k+1}=3(5\times3^{k-1}-2)+4$ | M1 | Assumes formula true for $n=k$ and considers<br>$u_{k+1}=3(5\times3^{k-1}-2)+4$ **oe** |
| | $u_{k+1}=5\times3^k-6+4$ ; $u_{k+1}=5\times3^{(k+1)-1}-2$ | A1 | Be convinced |
| | Hence formula is true for $n=k+1$ **(**)** and since true for $n=1$ **(***)**, formula $u_n=5\times3^{n-1}-2$ is true for $n=1, 2, 3, \ldots.$ by induction (****) | E1 | Must have **(*)** , **(**)**, **(***)**, present, previous 3 marks scored and a final statement **(****)** clearly indicating that it relates to positive integers and 'induction' |
| | | 4 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 7(c) | $\sum_{r=1}^{n}u_r=\frac{1}{2}(5\times3^n-2)-2n-\frac{3}{2}$<br>$=\frac{5}{2}\times(3^n-1)-2n$ | B1 | **ACF** Accept unsimplified |
| | | 1 | |

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|---|---|---|---|
| | **Question 7 Total** | **8** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 8(a) | Aux. equation $m^2+2m=0$; $m=-2,\ 0$ | M1 | Forming and solving the correct aux. equation.<br>**PI** by correct values of $m$ seen/used |
| | $y=A+B\,e^{-2x}$ | A1 | Correct general solution |
| | | **2** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 8(b) | $[\,y_{CF}=A+B\,e^{-2x}\,]$; $y_{PI}=a\,x\,e^{-2x}$ | M1 | $y_{PI}=a\,x\,e^{-2x}$ seen or used |
| | $y'_{PI}=a\,e^{-2x}-2a\,x\,e^{-2x}$<br>$y''_{PI}=-4a\,e^{-2x}+4a\,x\,e^{-2x}$ | M1 | $y'_{PI}$ and $y''_{PI}$ both of the form<br>$\pm c\,e^{-2x}\pm d\,x\,e^{-2x}$ |
| | $-4ae^{-2x}+4axe^{-2x}+2ae^{-2x}-4ax\,e^{-2x}=6e^{-2x}$<br>$\Rightarrow a=-3$ | M1 | Substitution into the DE to form an equation in $x$ and solve to find a value for $a$ |
| | $[y_{PI}=]-3\,x\,e^{-2x}$ | A1 | |
| | $[y_{GS}=]\ A+B\,e^{-2x}-3\,x\,e^{-2x}$ | B1ft | Their CF + their PI with exactly two arbitrary constants |
| | $A+B=0$ ; $4B+12=4$ (or $-2B-3=1$ ) | | |
| | $A=2$ and $B=-2$ | A1 | $A=2$ and $B=-2$ |
| | $y=2-2\,e^{-2x}-3\,x\,e^{-2x}$<br>When $x=3$ $y=2-11e^{-6}$ | A1 | $2-11e^{-6}$ **oe** |
| | | **7** | |

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|---|---|---|---|
| | **Question 8 Total** | **9** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| **9(a)** | det **A** $= 2k+12$ | **B1** | Correct expression for det **A** |
| | Cofactor matrix | | |
| | $\begin{bmatrix} 1 & -3 & 4 \\ 3k+12 & -3k & 2k-12 \\ -4k-17 & 4k+3 & 16-2k \end{bmatrix}$ | **M1** | One complete row or column or diagonal correct |
| | | **A2** | All nine entries correct<br>else **A1** for at least six entries correct |
| | Inverse matrix $\mathbf{A}^{-1} =$ | **M1** | Transpose of their cofactors with no more than one further error **and** division by their det **A** $(\neq 0)$ |
| | $= \dfrac{1}{2k+12}\begin{bmatrix} 1 & 3k+12 & -4k-17 \\ -3 & -3k & 4k+3 \\ 4 & 2k-12 & 16-2k \end{bmatrix}$ | **A1** | Correct $\mathbf{A}^{-1}$ scores 6 marks |
| | | **6** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| **9(b)(i)** | $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ | **M1** | $\mathbf{A}^{-1}$ **v** for their $\mathbf{A}^{-1}$ with at least one ft component correct. |
| | $= \dfrac{1}{2k+12}\begin{bmatrix} 1+3k+12-4k-17 \\ -3-3k+4k+3 \\ 4+2k-12+16-2k \end{bmatrix}$ | **A1ft** | At least two **ft** components correct. |
| | $x = -\left(\dfrac{k+4}{2k+12}\right) \quad y = \dfrac{k}{2k+12} \quad z = \dfrac{8}{2k+12}$ | **A1** | **ACF** All three correct |
| | | **3** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| **9(b)(ii)** | $0 < x+y+z < \dfrac{1}{3}$ | **B1** | |
| | | **1** | |

| | | | |
|---|---|---|---|
| | **Question 9 Total** | **10** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 10(a)(i) | $e^{\frac{i\theta}{2}}+e^{-\frac{i\theta}{2}}=2\cos\left(\frac{\theta}{2}\right)$ | B1 | |
| | | 1 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 10(a)(ii) | $\frac{1}{e^{i\theta}+1}=\frac{e^{-\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}}+e^{-\frac{i\theta}{2}}}=$ | M1 | **Either** $\frac{1}{e^{i\theta}+1}=\frac{e^{-\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}}+e^{-\frac{i\theta}{2}}}$ **or** $i\tan\left(\frac{\theta}{2}\right)=\left(\frac{e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}}+e^{-\frac{i\theta}{2}}}\right)$ seen/used $\frac{1}{2}-\frac{i}{2}\tan\left(\frac{\theta}{2}\right)=\frac{1}{2}-\frac{1}{2}\left[\frac{e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}}+e^{-\frac{i\theta}{2}}}\right]$ |
| | $=\frac{\cos\left(\frac{\theta}{2}\right)-i\sin\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{\theta}{2}\right)}=\frac{1}{2}-\frac{i}{2}\tan\left(\frac{\theta}{2}\right)$ | A1 | **AG** Must be convincingly shown |
| | | 2 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 10(b) | $\frac{1}{e^{i(\pi-\theta)}+1}=\frac{1}{2}-\frac{i}{2}\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=\frac{1}{2}-\frac{i}{2}\cot\left(\frac{\theta}{2}\right)$ | M1 | |
| | $\frac{1}{-e^{-i\theta}+1}=\frac{1}{2}-\frac{i}{2}\cot\left(\frac{\theta}{2}\right)$ | B1 | $e^{i\pi}=-1$ seen or used at any stage |
| | $\frac{1}{-e^{-i\theta}+1}-1=-\frac{1}{2}-\frac{i}{2}\cot\left(\frac{\theta}{2}\right)$ <br> $\frac{e^{-i\theta}}{1-e^{-i\theta}}=-\frac{1}{2}-\frac{i}{2}\cot\left(\frac{\theta}{2}\right)$ | | |
| | $\frac{1}{e^{i\theta}-1}=-\frac{1}{2}-\frac{i}{2}\cot\left(\frac{\theta}{2}\right)$ | A1 | **AG** Must be convincingly shown |
| | | 3 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 10(c) | $\left(\frac{1}{e^{i\theta}+1}\right)\left(\frac{1}{e^{i\theta}-1}\right)=\frac{1}{e^{2i\theta}-1}$ $=\frac{1}{\cos 2\theta+i\sin 2\theta-1}$ <br><br> $\left(\frac{1}{e^{i\theta}+1}\right)\left(\frac{1}{e^{i\theta}-1}\right)=$ $\left(\frac{1}{2}-\frac{i}{2}\tan\left(\frac{\theta}{2}\right)\right)\left(-\frac{1}{2}-\frac{i}{2}\cot\left(\frac{\theta}{2}\right)\right)$ $=-\frac{1}{2}+\frac{i}{4}\left(\tan\left(\frac{\theta}{2}\right)-\cot\left(\frac{\theta}{2}\right)\right)$ <br><br> $\frac{1}{\cos 2\theta-1+i\sin 2\theta}$ $=-\frac{1}{2}+i\frac{1}{4}\left(\tan\left(\frac{\theta}{2}\right)-\cot\left(\frac{\theta}{2}\right)\right)$ | M1 <br><br> A1 | Considers a relevant combination of $\frac{1}{e^{i\theta}+1}$ and $\frac{1}{e^{i\theta}-1}$ to obtain either $\frac{1}{\cos 2\theta+i\sin 2\theta-1}$ or $a+i\,b\left(\tan\left(\frac{\theta}{2}\right)-\cot\left(\frac{\theta}{2}\right)\right)$ with $a$ or $b$ correct <br><br> **oe** eg $\frac{1}{2}\left(\frac{1}{e^{i\theta}-1}-\frac{1}{e^{i\theta}+1}\right)=\frac{1}{e^{2i\theta}-1}$ $=\frac{1}{\cos 2\theta+i\sin 2\theta-1}$ <br><br> $\frac{1}{2}\left(\frac{1}{e^{i\theta}-1}-\frac{1}{e^{i\theta}+1}\right)=$ $\frac{1}{2}\left\{-\frac{1}{2}-\frac{i}{2}\cot\left(\frac{\theta}{2}\right)-\left(\frac{1}{2}-\frac{i}{2}\tan\left(\frac{\theta}{2}\right)\right)\right\}$ $=-\frac{1}{2}+\frac{i}{4}\left(\tan\left(\frac{\theta}{2}\right)-\cot\left(\frac{\theta}{2}\right)\right)$ <br><br> $\frac{1}{\cos 2\theta-1+i\sin 2\theta}$ $=-\frac{1}{2}+i\frac{1}{4}\left(\tan\left(\frac{\theta}{2}\right)-\cot\left(\frac{\theta}{2}\right)\right)$ |
| | | 2 | |

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|---|---|---|---|
| | **Question 9 Total** | **8** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| **11(a)** | $\begin{bmatrix}-2\\1\\2\end{bmatrix}\times\begin{bmatrix}1\\-3\\4\end{bmatrix}=\begin{bmatrix}10\\10\\5\end{bmatrix}$ | **B1** | $\begin{bmatrix}10\\10\\5\end{bmatrix}$ **oe** |
| | | **1** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| **11(b)** | $\left(\begin{bmatrix}-2\\1\\2\end{bmatrix}\times\begin{bmatrix}1\\-3\\4\end{bmatrix}\right)\bullet\begin{bmatrix}-4\\3\\-2\end{bmatrix}=\begin{bmatrix}10\\10\\5\end{bmatrix}\bullet\begin{bmatrix}-4\\3\\-2\end{bmatrix}=$ | **M1** | Correct method to evaluate a relevant scalar triple product; ft their **(a)**<br>Condone a miscopy of sign from given vectors |
| | $=-40+30-10\,[=-20]\neq 0$ so they are not coplanar vectors | **A1ft** | Must see correct ft evaluation and explicit comparison with 0, or relevant initial statement involving 0, followed by a correct ft conclusion. |
| | | **2** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| **11(c)** | Direction vector for $L$: $\mathbf{d}=\begin{bmatrix}2\\3\\-6\end{bmatrix}$<br>Direction of normal to the plane: $\mathbf{n}=\begin{bmatrix}2\\2\\1\end{bmatrix}$ | **B1ft** | Seen or used; **d** correct and **ft** their answer to part (a) for **n** in a simplified or unsimplified form |
| | **n . d** $=4+6-6$ | **M1** | Finds a numerical expression for the scalar product of their normal to $\Pi$ and their **d** **PI** |
| | $\sqrt{2^2+2^2+1^2}\sqrt{2^2+3^2+(-6)^2}\cos\theta=4$<br>$\cos\theta=\dfrac{4}{21}$ | **m1** | **ft** their **n** and **d** **PI** |
| | Angle between $L$ and $\Pi=90^\circ-\cos^{-1}\left(\dfrac{4}{21}\right)$<br>$=11^\circ$ to nearest degree | **A1** | **CAO** $11^\circ$ (Condone missing $^\circ$) |
| | | **4** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 11(d) | Line $L$: $\mathbf{r} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ | | |
| | General point on $L$ is $(1+2t, 3t, 2-6t)$ | M1 | Finding a general point on $L$ |
| | $1+2t = 1-2\lambda+\mu$<br>$3t = -2+\lambda-3\mu$; $8t = -4-5\mu$<br>$2-6t = 3+2\lambda+4\mu$; $3-4t = 4+5\mu$<br>$t = -\frac{3}{4}$ | M1 | Eliminating **r** for $L$ and $\Pi$, forming and solving three equations in three unknowns (or substituting general pt on $L$ into $2x+2y+z=1$ ) to find a value for $t$ |
| | $P\left(-\frac{1}{2}, -\frac{9}{4}, \frac{13}{2}\right)$ | A1 | Correct coordinates or position vector for $P$ **PI** or $\left\lvert-\frac{3}{4}\right\rvert \times \sqrt{2^2+3^2+(-6)^2}$ |
| | $PQ = \sqrt{1.5^2+2.25^2+4.5^2} = 5.25$ | A1 | Correct distance for $PQ$ |
| | | **4** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| **11(d) ALT** | Plane $\Pi$: $\mathbf{r} = \begin{bmatrix} 1-2\lambda+\mu \\ -2+\lambda-3\mu \\ 3+2\lambda+4\mu \end{bmatrix}$ | | |
| | Sub in $L$: $\begin{bmatrix} -2\lambda+\mu \\ -2+\lambda-3\mu \\ 1+2\lambda+4\mu \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} = \mathbf{0}$ | | |
| | $9-12\lambda+6\mu=0$<br>$2-8\lambda+14\mu=0$<br>$4-8\lambda+9\mu=0$ | M1 | Eliminating **r** for $L$ and $\Pi$, forming at least two equations in two unknowns |
| | $\mu=\frac{2}{5}$ $\lambda=\frac{19}{20}$ | M1 | Solving three correct equations to obtain a value for each of the unknowns |
| | $P\left(-\frac{1}{2}, -\frac{9}{4}, \frac{13}{2}\right)$ | A1 | Correct coordinates or position vector for $P$ **PI** or $\left\lvert-\frac{3}{4}\right\rvert \times \sqrt{2^2+3^2+(-6)^2}$ |
| | $PQ = \sqrt{1.5^2+2.25^2+4.5^2} = 5.25$ | A1 | Correct distance for $PQ$ |
| | | **4** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| **11(e)** | Let $T$ be the point on $\Pi$ such that $QT$ is perpendicular to $\Pi$. $Q$ is a point on line $L$ so angle $PQT$, is the angle $\theta$ from **part (c)** between normal to $\Pi$ and line $L$.<br>$QT = PQ\cos\theta$ | M2 | **oe** eg $QT = PQ\sin\left(90^\circ - \theta\right)$<br>condoning their rounded answer for $90^\circ - \theta$ found in **part (c)** |
| | Shortest distance $= 5.25 \times \dfrac{4}{21} = 1$ | A1 | **CAO** Do not accept 1 from non-exact values |
| | | **3** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| **11(e) ALT** | $1+2t = 1-2\lambda+\mu$<br>$2t = -2+\lambda-3\mu$; $6t = -4-5\mu$<br>$2+t = 3+2\lambda+4\mu$; $3+3t = 4+5\mu$<br>$t = -\dfrac{1}{3}$ | M1 | Finds a general point on $QT$ and eliminates **r** for $QT$ and $\Pi$, forming and solving three equations in three unknowns **oe** to find a value for $t$ |
| | $T\left(\dfrac{1}{3}, -\dfrac{2}{3}, \dfrac{5}{3}\right)$ $Q(1, 0, 2)$<br>$QT = \sqrt{\left(\dfrac{2}{3}\right)^2 + \left(\dfrac{2}{3}\right)^2 + \left(\dfrac{1}{3}\right)^2}$ | m1 | Finds coordinates of $T$ and uses distance formula to find a value for the distance $QT$ **oe** |
| | Shortest distance $= 1$ | A1 | **CAO** Must be from exact values. |
| | | **3** | |

| | | | |
|---|---|---|---|
| | **Question 11 Total** | **14** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 12(a) | $y = e^{\frac{7}{25}x}(\cosh x)^{-1}$ ,<br>$\frac{dy}{dx} = \frac{7}{25}e^{\frac{7}{25}x}(\cosh x)^{-1} - e^{\frac{7}{25}x}(\cosh x)^{-2}\sinh x$ | M1<br>A1 | Use of the product formula or quotient formula.<br>**ACF** |
| | At *P*, $\frac{dy}{dx} = 0 \Rightarrow \frac{7}{25}\cosh x - \sinh x = 0$ | M1 | |
| | $\tanh x = \frac{7}{25}$ | A1 | |
| | $x = \tanh^{-1}\left(\frac{7}{25}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{7}{25}}{1-\frac{7}{25}}\right)$ | M1 | Using $\tanh^{-1}(n) = \frac{1}{2}\ln\left(\frac{1+n}{1-n}\right)$ |
| | $x = \ln\left(\frac{4}{3}\right)$ | A1 | $x = \ln\left(\frac{4}{3}\right)$ |
| | | 6 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 12(b) | $y_P = e^{\frac{7}{25}(\ln k)}\operatorname{sech}(\ln k)$ | M1 | Attempts to find the $y$-coordinate of *P*, **ft** their value of $k$ from **part (a)** |
| | $\operatorname{sech}(\ln k) = \frac{2k}{k^2+1}$ ; $e^{\frac{7}{25}(\ln k)} = k^{\frac{7}{25}}$ | B1ft | $e^{\frac{7}{25}(\ln k)} = k^{\frac{7}{25}}$ and $\operatorname{sech}(\ln k) = \frac{2k}{k^2+1}$<br>**oe ft** their $k$ value |
| | Shortest distance = $\left(\frac{24}{25}\right)\left(\frac{4}{3}\right)^{\frac{7}{25}}$ | A1 | **oe** but must be in the required printed form eg $\left(\frac{18}{25}\right)\left(\frac{4}{3}\right)^{\frac{32}{25}}$ |
| | | 3 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 12(c) | Shortest distance $\left(\frac{24}{25}\right)\left(\frac{4}{3}\right)^{\frac{7}{25}} = 1.04\ldots$ | B1 | 1.04… |
| | Since $[-1<]\tanh x < 1$ , and line *L* ,<br>$y = 1.04\ldots$ is above $y = 1$, *L* does not intersect the curve $y = \tanh x$ | E1ft | States $\tanh x < 1$ and<br>if (their shortest distance) < 1 states *L* intersects $y = \tanh x$ **oe**<br>if (their shortest distance) $\geq 1$ states *L* does not intersect $y = \tanh x$ **oe** |
| | | 2 | |

| | | | |
|---|---|---|---|
| | **Question 12 Total** | **11** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 13(a) | $\left[\ln(1+4x)=\right]\ \ 4x-8x^2+\frac{64}{3}x^3$ | B1 | Ignore higher order terms |
| | [valid for] $-\frac{1}{4}<x\le\frac{1}{4}$ | B1 | |
| | | 2 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 13(b)(i) | $y=\ln(\cos x-\sin x)$; $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\sin x-\cos x}{\cos x-\sin x}=1-\frac{2\cos x}{\cos x-\sin x}$; | B1 | **ACF** for the first derivative |
| | $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=$ $\frac{2\sin x(\cos x-\sin x)+2\cos x(-\sin x-\cos x)}{(\cos x-\sin x)^2}$ | M1 | Quotient rule used |
| | $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{-2(\sin^2x+\cos^2x)}{\cos^2x+\sin^2x-2\sin x\cos x}$ | | |
| | $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{-2}{1-\sin 2x}$ | A1 | **AG** Must be convincingly shown |
| | | 3 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 13(b)(ii) | $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{0+2(-2\cos 2x)}{(1-\sin 2x)^2}$ | B1 | **ACF** for the third derivative |
| | $y(0)=0; y'(0)=-1; y''(0)=-2; y'''(0)=-4$ | M1 | All 4 attempted with at least 2 correct |
| | $\ln(\cos x-\sin x)=0-1x+\frac{(-2)x^2}{2!}+\frac{(-4)x^3}{3!}$ | | |
| | $\ln(\cos x-\sin x)=-x-x^2-\frac{2}{3}x^3$ | A1 | **AG** Must be convincingly shown |
| | | 3 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 13(c) | $\ln\left[(1-\sin 2x)\sqrt{1+4x}\right] = \ln(1-\sin 2x)+\ln\sqrt{1+4x}$ | M1 | Seen or used |
| | $= 2\ln(\cos x-\sin x)+\frac{1}{2}\ln(1+4x)$ | M1 | $\ln(1-\sin 2x)=2\ln(\cos x-\sin x)$ seen or used |
| | $= 2\left(-x-x^2-\frac{2}{3}x^3\right)+\frac{1}{2}\left(4x-8x^2+\frac{64}{3}x^3\right)$ | A1ft | **ft** their 3-term expansion in **part (a)** |
| | $\lim_{x\to 0}\left[\frac{\ln\left((1-\sin 2x)\sqrt{1+4x}\right)}{5x^2+6x^3}\right]$ $=\lim_{x\to 0}\left[\frac{-6x^2+\frac{28}{3}x^3\ldots}{5x^2+6x^3}\right]$ $=\lim_{x\to 0}\left[\frac{-6+\frac{28}{3}x\ldots}{5+6x}\right]$ [so the limit exists] | M1 | Substitutes series expansions and divides numerator and denominator by $x^2$ to reach the form $\lim_{x\to 0}\left[\frac{P+O(x)}{Q+O(x)}\right]$, so limit exists $=\frac{P}{Q}$ |
| | $\lim_{x\to 0}\left[\frac{\ln\left((1-\sin 2x)\sqrt{1+4x}\right)}{5x^2+6x^3}\right]=-\frac{6}{5}$ | A1 | **CSO** |
| | | **5** | |

| | | | |
|---|---|---|---|
| | **Question 13 Total** | **13** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 14(a) | $B\left(6+3\sqrt{2}\ ,\ \dfrac{7\pi}{4}\right)$; $OA = OB = 6+3\sqrt{2}$<br>Angle $AOB = \dfrac{\pi}{4}+\dfrac{\pi}{4}=\dfrac{\pi}{2}$ | M1 | Uses polar coordinates of $A$ and $B$ **oe** to find two relevant lengths and an angle to use in finding the required area **PI** |
| | Area of triangle $AOB$ =<br>$\dfrac{1}{2}\left(6+3\sqrt{2}\right)^2=\dfrac{1}{2}\left(54+36\sqrt{2}\right)=27+18\sqrt{2}$ | A1 | **AG** Must be convincingly shown |
| | | **2** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 14(b) | $r=\dfrac{3}{2}\text{cosec}^2\left(\dfrac{\theta}{2}\right)$; $r=\dfrac{3}{1-\cos\theta}$ | M1 | $r=\dfrac{3}{1-\cos\theta}$ condone sign error |
| | $r-r\cos\theta=3$ ; $r=3+x$ | M1 | Use of $r\cos\theta=x$ to eliminate $\theta$ |
| | $r^2=(3+x)^2$ ; $x^2+y^2=(3+x)^2$ | M1 | $r^2=x^2+y^2$ used at any stage |
| | $y^2=6x+9$ | A1 | $y^2=6x+9$ **oe** for $\mathrm{f}(x)$ |
| | | **4** | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 14(c)(i) | $\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}, \quad \theta = \frac{4\pi}{3}$<br>When $\theta = \frac{\pi}{3}$, $r = 6$ When $\theta = \frac{4\pi}{3}$, $r = 2$ | B2,1,0 | **B2** any three correct;<br>else **B1** for any two correct |
| | Polar coordinates of $P$ and $Q$ are<br>$\left(6, \frac{\pi}{3}\right)$, $\left(2, \frac{4\pi}{3}\right)$ | B1 | $\left(6, \frac{\pi}{3}\right)$, $\left(2, \frac{4\pi}{3}\right)$ |
| | | 3 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| 14(c)(ii) | $\int \left(1+\cot^2\left(\frac{\theta}{2}\right)\right)\operatorname{cosec}^2\left(\frac{\theta}{2}\right) \mathrm{d}\theta$<br>Let $u = \cot\left(\frac{\theta}{2}\right)$, $\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{1}{2}\operatorname{cosec}^2\left(\frac{\theta}{2}\right)$<br>$\int \operatorname{cosec}^4\left(\frac{\theta}{2}\right) \mathrm{d}\theta = \int \left(1+u^2\right)(-2)\ \mathrm{d}u$ | M1 | Uses a relevant substitution or integration by parts so as to require a single step to find the integral.<br><br>**PI** by the **A1** form below |
| | $= -2\cot\left(\frac{\theta}{2}\right) - \frac{2}{3}\cot^3\left(\frac{\theta}{2}\right) \quad [+c]$ | A2,1,0 | If not **A2**, award **A1** for the form<br>$k\left(\cot\left(\frac{\theta}{2}\right) + \frac{1}{3}\cot^3\left(\frac{\theta}{2}\right)\right) \quad [+c]$<br>where $k = 2$ or $\frac{1}{2}$ or $-\frac{1}{2}$ |
| | | 3 | |

| Q | Answer | Marks | Comments |
|---|---|---|---|
| **14(c)(iii)** | Area $= \dfrac{1}{2}\displaystyle\int_{\left[\frac{\pi}{3}\right]}^{\left[\frac{4\pi}{3}\right]} \dfrac{9}{4}\text{cosec}^4\left(\dfrac{\theta}{2}\right)[\mathrm{d}\theta]$ | **M1** | Use of $\dfrac{1}{2}\displaystyle\int r^2\,[\mathrm{d}\theta]$ oe with integration attempted.<br>Condone missing or incorrect limits |
| | $=\dfrac{9}{8}\left[-2\cot\left(\dfrac{\theta}{2}\right)-\dfrac{2}{3}\cot^3\left(\dfrac{\theta}{2}\right)\right]_{\frac{\pi}{3}}^{\frac{4\pi}{3}}$ | | |
| | $=\dfrac{9}{8}\left(\left\{-2\cot\left(\dfrac{2\pi}{3}\right)-\dfrac{2}{3}\cot^3\left(\dfrac{2\pi}{3}\right)\right\}-\left\{-2\cot\left(\dfrac{\pi}{6}\right)-\dfrac{2}{3}\cot^3\left(\dfrac{\pi}{6}\right)\right\}\right)$ | **M1** | Uses their answer to **part (c)(ii)** and substitutes their non-zero values for $\theta$ found in **part (c)(i)** as limits with the appropriate subtraction included.<br>**PI** by the line above in the Answer column followed by the correct final answer. |
| | $=\dfrac{9}{8}\left(\left\{-2\left(-\dfrac{1}{\sqrt{3}}\right)-\dfrac{2}{3}\left(-\dfrac{1}{\sqrt{3}}\right)^3\right\}-\left\{-2\left(\sqrt{3}\right)-\dfrac{2}{3}\left(\sqrt{3}\right)^3\right\}\right)$ | | |
| | $=\dfrac{9}{8}\left(\left\{\dfrac{2}{\sqrt{3}}+\dfrac{2}{9\sqrt{3}}\right\}-\left\{-2\sqrt{3}-2\sqrt{3}\right\}\right)$ | | |
| | $=\dfrac{16}{3}\sqrt{3}$ | **A1** | **CAO** |
| | | **3** | |

| | | | |
|---|---|---|---|
| | **Question 14 Total** | **15** | |