Cardiff University School of Mathematics

MA0000

Test Module

Notes

2014-15

Version: 4/2/2015

Contents

Lecture 1 Divisions

Some initial text, before the first section

1.1 First Section

This is the first section.

1.2 Another Section

This is the second section.

1.2.1 A subsection

This is a subsection.

Lecture 2 Exercises

- Doesn't work across chapters yet: Lemma 4.3.

2.1 Diagnostic exercises (multiple choice)

Exercise 2.1 [**Diagnositic**]

1. Which among the following distributions is the odd one out?
   A. Binomial
   B. Exponential
   C. Geometric
   D. Poisson
2. Which is the definition of a random variable?
   A. A function $X : \Omega \to \mathbb{R}$.
   B. A function $X : \Omega \to [0, 1]$.
   C. A function $X : \mathbb{R} \to [0, 1]$.

2.2 Formative exercises

Exercise 2.2 [**Formative**]
This is the introduction.

1. This is the first question.
2. This is the second question.
3. This is the third question.
   (a) This is the first part of the third question
   (b) This is the second part of the third question

Lecture 3 Figures and Tables

3.1 Figures

Not done yet.

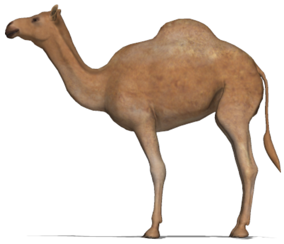

Figure 3.1: Humpty the Camel.

3.2 Tables

Here are some tables:

Notation	$X \sim \text{Uniform}\{1, 2, \ldots, n\}$
Parameter(s)	$n \in \mathbb{N}$
Range	$\{1, 2, \ldots, n\}$
PMF	$f(k) = 1/n$ for all $k = 1, 2, \ldots, n$

Table 3.1: The uniform distribution

Notation	$X \sim \text{Bernoulli}(p)$
Parameter(s)	$p \in [0,1]$ (probability of success)
Range	$\{0,1\}$
PMF	$f(0) = 1 - p$ and $f(1) = p$

Table 3.2: The Bernoulli distribution

Lecture 4 Theorems

Probability is a *function* that assigns numerical value to random events.

Definition 4.1
Let Ω be the sample space of some random experiment, and let $\mathcal{F}$ be a field of sets over Ω. A *probability measure* on $(\Omega, \mathcal{F})$ is a function

$$\begin{array}{rccc} \mathbb{P}: & \mathcal{F} & \to & [0,1] \\ & A & \mapsto & \mathbb{P}(A) \end{array}$$

such that $\mathbb{P}(\Omega) = 1$, and for any countable collection of pairwise disjoint events $\{A_1, A_2, \ldots\}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space.*

Theorem 4.2 (Properties of probability measures)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A, B \in \mathcal{F}$.

(1) Complementarity: $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

(2) $\mathbb{P}(\emptyset) = 0$,

(3) Monotonicity: if $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

(4) Addition rule: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Proof:

(1) Since $A \cup A^c = \Omega$ is a disjoint union and $\mathbb{P}(\Omega) = 1$, it follows by additivity that

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c).$$

(2) Since $\emptyset = \Omega^c$ and $\mathbb{P}(\Omega) = 1$, it follows by complemenarity that

$$\mathbb{P}(\emptyset) = \mathbb{P}(\Omega^c) = 1 - \mathbb{P}(\Omega) = 1 - 1 = 0.$$

(3) Let $A \subseteq B$ and let us write $B = A \cup (B \setminus A)$.
Since A and $B \setminus A$ are disjoint sets, it follows by additivity that

$$\mathbb{P}(B) = \mathbb{P}\big[A \cup (B \setminus A)\big] = \mathbb{P}(A) + \mathbb{P}(B \setminus A).$$

Hence, because $\mathbb{P}(B \setminus A) \geq 0$, it follows that $\mathbb{P}(B) \geq \mathbb{P}(A)$.

(4)
- $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$
- $A = (A \setminus B) + (A \cap B)$
- $B = (B \setminus A) + (A \cap B)$

These are disjoint unions, so by additivity,

- $\mathbb{P}(A \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)$

- $\mathbb{P}(A) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B)$
- $\mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)$

Hence $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$, as required.

Here is a lemma:

Lemma 4.3 (Zorn's lemma)
If every totally ordered subset of a partially ordered set A has an upper bound in A, then A contains at least one maximal element.

Lecture 5 Lists

Here are nested itemize environments

- First item
  - First sub-item
  - Second sub-item
- Second item

Here are nested enumerate environments

(1) First item
   (a) First sub-item
   (b) Second sub-item

(2) Second item

Lecture 6 Labels, References and Citations

- Here is a reference to Lemma 4.3 – does it work?
- Here is a reference to Definition 4.1 – does it work?