Cardiff University School of Mathematics

MA2500

Foundations of Probability and Statistics

Reading Material

2014-15

Version: 31/1/2015

Contents

Lecture 1 Set Theory

1.1 Elementary set theory

A set is a collection of distinct *elements*.

- If a is an element of the set A, we denote this by $a \in A$.
- If a is *not* an element of A, we denote this by $a \notin A$.
- The *cardinality* of a set is the number of elements it contains.
- The *empty set* contains no elements, and is denoted by $\emptyset$.

1.1.1 Set relations

Let A, B be sets.

- If $a \in B$ for every $a \in A$, we say that A is a *subset* of B, denoted by $A \subseteq B$.
- If $A \subseteq B$ and $B \subseteq A$, we say that A is *equal* to B, denoted by $A = B$,
- If $A \subseteq B$ and $A \neq B$, we say that A is a *proper subset* of B, denoted by $A \subset B$.

1.1.2 Set operations

Let A, B and Ω be sets, with $A, B \subseteq \Omega$.

- The *union* of A and B is the set $A \cup B = \{a \in \Omega : a \in A \text{ or } a \in B\}$.
- The *intersection* of A and B is the set $A \cap B = \{a \in \Omega : a \in A \text{ and } a \in B\}$.
- The *complement* of A (relative to Ω) is the set $A^c = \{a \in \Omega : a \notin A\}$.

1.1.3 Set algebra

Commutative property:	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associative property:	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive property:	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

1.2 Sample space, outcomes and events

Definition 1.1

(1) Any process of observation or measurement whose outcome is uncertain is called a *random experiment*.

(2) A random experiment has a number of possible *outcomes*.

(3) Each time a random experiment is performed, *exactly one* of its outcomes will occur.

(4) The set of all possible outcomes is called the *sample space*, denoted by Ω.

(5) Outcomes are also called *elementary events*, and denoted by $\omega \in \Omega$.

Example 1.2

- $\{1, 2, \ldots, n\}$ is a finite sample space,
- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is a countably infinite sample space,
- $[0, 1]$ is an uncountable sample space.

Definition 1.3

(1) An *event* A is a subset of the sample space, Ω.

(2) If outcome ω occurs, we say that event A *occurs* if and only if $\omega \in A$.

(3) Two events A and B with $A \cap B = \emptyset$ are called *disjoint* or *mutually exclusive.*

(4) The empty set $\emptyset$ is called the *impossible event.*

(5) The sample space itself is called the *certain event.*

Remark 1.4

- If A occurs and $A \subseteq B$, then B occurs.
- If A occurs and $A \cap B = \emptyset$, then B does not occur.

1.3 Countable unions and intersections

Definition 1.5
Let Ω be any set. The set of all subsets Ω is called its *power set.*

- If Ω is a finite set, its power set is also finite.
- If Ω is a countably infinite set, its power set is uncountable set (Cantor's Theorem).
- If Ω is an uncountable set, its power set is also uncountable.

Definition 1.6
Let $A_1, A_2, \ldots$ be subsets of Ω.

(1) The (countable) *union* of $A_1, A_2, \ldots$ is the set

$$\bigcup_{i=1}^{\infty} A_i = \{\omega : \omega \in A_i \text{ for some } A_i\}.$$

(2) The (countable) *intersection* of $A_1, A_2, \ldots$ is the set

$$\bigcap_{i=1}^{\infty} A_i = \{\omega : \omega \in A_i \text{ for all } A_i\}.$$

Theorem 1.7 (De Morgan's laws)
For a countable collection of sets $\{A_1, A_2, \ldots\}$,

(1) $\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$,

(2) $\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c$.

Proof:

(1) Let $a \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c$. Then $a \notin \bigcup_{i=1}^{\infty} A_i$, and so $a \in A_i^c$ for all A_i.
Hence $\left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c$.
Let $a \in \bigcap_{i=1}^{\infty} A_i^c$. Then $a \notin A_i$ for all A_i, and so $a \notin \bigcup_{i=1}^{\infty} A_i$.
Hence $\bigcap_{i=1}^{\infty} A_i^c \subseteq \left(\bigcup_{i=1}^{\infty} A_i\right)^c$.

(2) Applying part (1) to the collection of sets $\{A_1^c, A_2^c, \ldots\}$,
$\left(\bigcup_{i=1}^{\infty} A_i^c\right)^c = \bigcap_{i=1}^{\infty} \left(A_i^c\right)^c = \bigcap_{i=1}^{\infty} A_i$. Taking the complement of both sides,
$\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$.

1.4 Collections of sets

Definition 1.8
Let Ω be any set. Any subset of its power set is called a *collection of sets over* Ω.

Let Ω be the sample space of some random experiment. If we are interested whether the events A and B occur, we must also be interested in

- the event $A \cup B$: whether event A occurs *or* event B occurs;
- the event $A \cap B$: whether event A occurs *and* event B occurs;
- the event A^c: whether the event A does *not* occur.

Thus we can not use arbitrary collections of sets over Ω as the basis for investigating random experiments. Instead, we allow only collections which are *closed* under certain set operations.

Definition 1.9
A collection of sets $\mathcal{C}$ over Ω is said to be

(1) *closed under complementation* if $A^c \in \mathcal{C}$ for every $A \in \mathcal{C}$,

(2) *closed under pairwise unions* if $A \cup B \in \mathcal{C}$ for every $A, B \in \mathcal{C}$,

(3) *closed under finite unions* if $\bigcup_{i=1}^{n} A_i \in \mathcal{C}$ for every $A_1, A_2, \ldots A_n \in \mathcal{C}$,

(4) *closed under countable unions* if $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$ for every $A_1, A_2, \ldots \in \mathcal{C}$.

Definition 1.10
A collection of sets $\mathcal{F}$ over Ω is called a *field* over Ω if

(1) $\Omega \in \mathcal{F}$,

(2) $\mathcal{F}$ is closed under complementation, and

(3) $\mathcal{F}$ is closed under pairwise unions.

Theorem 1.11 (Properties of fields)
Let $\mathcal{F}$ be a field over Ω. Then

(1) $\emptyset \in \mathcal{F}$,

(2) $\mathcal{F}$ is closed under set differences,

(3) $\mathcal{F}$ is closed under finite unions,

(4) $\mathcal{F}$ is closed under finite intersections.

Proof: See exercises.

Definition 1.12
A collection of sets $\mathcal{F}$ over Ω is called a *σ-field* ("sigma-field") over Ω if

(1) $\Omega \in \mathcal{F}$,

(2) $\mathcal{F}$ is closed under complementation, and

(3) $\mathcal{F}$ is closed under countable unions.

Theorem 1.13 (Properties of σ-fields)
Let $\mathcal{F}$ be a σ-field over Ω. Then

(1) $\emptyset \in \mathcal{F}$,

(2) $\mathcal{F}$ is closed under set differences,

(3) $\mathcal{F}$ is closed under finite unions,

(4) $\mathcal{F}$ is closed under finite intersections,

(5) $\mathcal{F}$ is closed under countable intersections.

Proof: See exercises.

1.5 Borel sets

In many situations of interest, random experiments yield outcomes that are *real numbers*.

Definition 1.14
- The *open interval* (a, b) is the set $\{x \in \mathbb{R} : a < x < b\}$.
- The *closed interval* $[a, b]$ is the set $\{x \in \mathbb{R} : a \leq x \leq b\}$.

Definition 1.15
The *Borel* σ-field over $\mathbb{R}$ is defined to be the smallest σ-field over $\mathbb{R}$ that contains all open intervals.

Remark 1.16
- The Borel σ-field is usually denoted by $\mathcal{B}$, and includes all closed interval, all half-open intervals, all finite sets and all countable sets.
- The elements of $\mathcal{B}$ are called *Borel sets* over $\mathbb{R}$.
- Borel sets can be thought of as the "nice" subsets of $\mathbb{R}$.

Proposition 1.17
The Borel σ-field over $\mathbb{R}$ contains all closed intervals.

Proof: Any closed interval $[a, b]$ can be written as a countable intersection of open intervals:

$$[a, b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n},\ b + \frac{1}{n}\right).$$

Hence $[a, b] \in \mathcal{B}$, because

- for every $n \in \mathbb{N}$, $\left(a - \dfrac{1}{n}, b + \dfrac{1}{n}\right) \in \mathcal{B}$, and

- by Theorem 1.13, $\mathcal{B}$ is closed under countable intersections.

1.6 Exercises

Exercise 1.1

1. Let $\mathcal{F}$ be a field over Ω. Show that
   (a) $\emptyset \in \mathcal{F}$,
   (b) $\mathcal{F}$ is closed under set differences,
   (c) $\mathcal{F}$ is closed under pairwise intersections,
   (d) $\mathcal{F}$ is closed under finite unions,
   (e) $\mathcal{F}$ is closed under finite intersections.
2. Let $\mathcal{F}$ be a σ-field over Ω. Show that
   (a) $\mathcal{F}$ is closed under finite unions,
   (b) $\mathcal{F}$ is closed under finite intersections.
   (c) $\mathcal{F}$ is closed under countable intersections.

Exercise 1.2

1. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$.
   (a) What is the smallest σ-field containing the event $A = \{1, 2\}$?
   (b) What is the smallest σ-field containing the events $A = \{1, 2\}$, $B = \{3, 4\}$ and $C = \{5, 6\}$?
2. Let $\mathcal{F}$ and $\mathcal{G}$ be σ-fields over Ω.
   (a) Show that $\mathcal{H} = \mathcal{F} \cap \mathcal{G}$ is a σ-field over Ω.
   (b) Find a counterexample to show that $\mathcal{H} = \mathcal{F} \cup \mathcal{G}$ is not necessarily a σ-field over Ω.

Lecture 2 Probability Spaces

2.1 Probability measures

Definition 2.1
Let Ω be a sample space, and let $\mathcal{F}$ be a σ-field over Ω. A *probability measure* on $(\Omega, \mathcal{F})$ is a function

$$\begin{aligned} \mathbb{P}: \quad \mathcal{F} &\to [0,1] \\ A &\mapsto \mathbb{P}(A) \end{aligned}$$

such that $\mathbb{P}(\Omega) = 1$, and for any countable collection of pairwise disjoint events $\{A_1, A_2, \ldots\}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space.*

Remark 2.2

- The second property is called *countable additivity.*

Remark 2.3
In the more general setting of measure theory:

- The elements of $\mathcal{F}$ are called *measurable sets.*
- The pair $(\Omega, \mathcal{F})$ is called a *measurable space.*
- The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *measure space.*

Example 2.4
A fair six-sided die is rolled once. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for the experiment is given by

- $\Omega = \{1, 2, 3, 4, 5, 6\}$,
- $\mathcal{F} = \mathcal{P}(\Omega)$, where $\mathcal{P}(\Omega)$ denotes the power set of Ω,
- $\mathbb{P}(A) = |A|/|\Omega|$ for every $A \in \mathcal{F}$ (where $|A|$ denotes the cardinality of A).

If we are only interested in odd and even numbers, we can instead take

- $\Omega = \{1, 2, 3, 4, 5, 6\}$,
- $\mathcal{F} = \big\{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, \{1, 2, 3, 4, 5, 6\}\big\}$
- $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\{1, 3, 5\}) = 1/2$, $\mathbb{P}(\{2, 4, 6\}) = 1/2$, $\mathbb{P}(\{1, 2, 3, 4, 5, 6\}) = 1$.

2.2 Null and almost-certain events

Definition 2.5
(1) If $\mathbb{P}(A) = 0$, we say that A is a *null event.*

(2) If $\mathbb{P}(A) = 1$, we say that A occurs *almost surely* (or "*with probability 1*").

Remark 2.6

- A null event is not the same as the impossible event ($\emptyset$).
- An event that occurs almost surely is not the same as the certain event (Ω).

Example 2.7
A dart is thrown at a dartboard.

- The probability that the dart hits a given point of the dartboard is 0.
- The probability that the dart does not hit a given point of the dartboard is 1.

2.3 Properties of probability measures

Theorem 2.8 (Properties of probability measures)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A, B \in \mathcal{F}$.

(1) Complementarity: $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

(2) $\mathbb{P}(\emptyset) = 0$,

(3) Monotonicity: if $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

(4) Addition rule: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Proof:

(1) Since $A \cup A^c = \Omega$ is a disjoint union and $\mathbb{P}(\Omega) = 1$, it follows by additivity that

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c).$$

(2) Since $\emptyset = \Omega^c$ and $\mathbb{P}(\Omega) = 1$, it follows by complemenarity that

$$\mathbb{P}(\emptyset) = \mathbb{P}(\Omega^c) = 1 - \mathbb{P}(\Omega) = 1 - 1 = 0.$$

(3) Let $A \subseteq B$ and let us write $B = A \cup (B \setminus A)$.
Since A and $B \setminus A$ are disjoint sets, it follows by additivity that

$$\mathbb{P}(B) = \mathbb{P}\big[A \cup (B \setminus A)\big] = \mathbb{P}(A) + \mathbb{P}(B \setminus A).$$

Hence, because $\mathbb{P}(B \setminus A) \geq 0$, it follows that $\mathbb{P}(B) \geq \mathbb{P}(A)$.

(4) Let us write:

- $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$
- $A = (A \setminus B) + (A \cap B)$
- $B = (B \setminus A) + (A \cap B)$

These are disjoint unions, so by additivity,

- $\mathbb{P}(A \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)$
- $\mathbb{P}(A) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B)$
- $\mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)$

Hence $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$, as required.

2.4 Continuity of probability measures

Theorem 2.9 (Continuity of probability measures)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(1) For an increasing sequence of events $A_1 \subseteq A_2 \subseteq \ldots$ in $\mathcal{F}$,

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

(2) For a decreasing sequence of events $B_1 \supseteq B_2 \supseteq \ldots$ in $\mathcal{F}$,

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mathbb{P}(B_n).$$

Proof: To prove the first part, let $A_1 \subseteq A_2 \subseteq \ldots$ be an increasing sequence of events, and

$$A = \bigcup_{i=1}^{\infty} A_i.$$

We can write A as a disjoint union

$$A = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \ldots$$

Since the sets $A_{i+1} \setminus A_i$ are disjoint, by countable additivity we have

$$\mathbb{P}(A) = \mathbb{P}(A_1) + \mathbb{P}(A_2 \setminus A_1) + \mathbb{P}(A_3 \setminus A_2) + \ldots$$

Furthermore, $A_i \subseteq A_{i+1}$ means that $A_{i+1} = (A_{i+1} \setminus A_i) \cup A_i$ is a disjoint union, so

$$\mathbb{P}(A_{i+1} \setminus A_i) = \mathbb{P}(A_{i+1}) - \mathbb{P}(A_i).$$

Hence

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(A_1) + \big[\mathbb{P}(A_2) - \mathbb{P}(A_1)\big] + \big[\mathbb{P}(A_3) - \mathbb{P}(A_2)\big] + \ldots \\
&= \big[\mathbb{P}(A_1) - \mathbb{P}(A_1)\big] + \big[\mathbb{P}(A_2) - \mathbb{P}(A_2)\big] + \big[\mathbb{P}(A_3) - \mathbb{P}(A_3)\big] + \ldots \\
&= \lim_{n\to\infty} \mathbb{P}(A_n).
\end{aligned}$$

To prove the second part, let $B_1 \supseteq B_2 \supseteq \ldots$ be a decreasing sequence of events, and

$$B = \bigcap_{i=1}^{\infty} B_i.$$

Let $A_i = B_i^c$ and $A = B^c$. Then $A_1 \subseteq A_2 \subseteq \ldots$ is an increasing sequence, and

$$A = \bigcup_{i=1}^{\infty} A_i.$$

Hence by the first part of the theorem,

$$\begin{aligned}
\mathbb{P}(B) &= 1 - \mathbb{P}(A) \\
&= 1 - \lim_{n\to\infty} \mathbb{P}(A_n) \\
&= \lim_{n\to\infty} (1 - \mathbb{P}(A_n)) \\
&= \lim_{n\to\infty} \mathbb{P}(B_n).
\end{aligned}$$

2.5 Exercises

Exercise 2.1

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A, B, C \in \mathcal{F}$. Show that

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$

   This is called the *inclusion-exclusion principle.*

2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
   (a) Show that $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$ for all $A, B \in \mathcal{F}$. This is called *subadditivity.*
   (b) Show that for any sequence $A_1, A_2, \ldots$ of events in $\mathcal{F}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

   This is called *countable subadditivity.*

Exercise 2.2

1. Let A and B be events with probabilities $\mathbb{P}(A) = 3/4$ and $\mathbb{P}(B) = 1/3$.
   (a) Show that $\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq \frac{1}{3}$, and construct examples to show that both extremes are possible.
   (b) Find corresponding bounds for $\mathbb{P}(A \cup B)$.

2. A roulette wheel consists of 37 slots of equal size. The slots are numbered from 0 to 36, with odd-numbered slots coloured red, even-numbered slots coloured black, and the slot labelled 0 coloured green. The wheel is spun in one direction and a ball is rolled in the opposite direction along a track running around the circumference of the wheel. The ball eventually falls on to the wheel and into one of the 37 slots. A player bets on the event that the ball stops in a red slot, and another player bets on the event that the ball stops in a black slot.
   (a) Define a suitable sample space Ω for this random experiment, and identify the events of interest.
   (b) Find the smallest field $\mathcal{F}$ over Ω that contains the events of interest.
   (c) Define a suitable probability measure $(\Omega, \mathcal{F})$ to represent the game.

Exercise 2.3

1. A biased coin has probability p of showing heads. The coin is tossed repeatedly until a head occurs. Describe a suitable probability space for this experiment.

2. A fair coin is tossed repeatedly.
   (a) Show that a head eventually occurs with probability one.
   (b) Show that a sequence of 10 consecutive tails eventually occurs with probability one.
   (c) Show that any finite sequence of heads and tails eventually occurs with probability one.

Lecture 3 Conditional Probability

3.1 Conditional probability

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $B \in \mathcal{F}$.

Definition 3.1
If $\mathbb{P}(B) > 0$, the *conditional probability of A given B* is defined to be

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

3.2 Bayes' theorem

Definition 3.2
A countable collection of sets $\{A_1, A_2, \dots\}$ is said to form a *partition* of a set B if

(1) $A_i \cap A_j = \emptyset$ for all $i \neq j$, and

(2) $B \subseteq \bigcup_{i=1}^{\infty} A_i$.

Theorem 3.3 (The Law of Total Probability)
If $\{A_1, A_2, \dots\}$ is a partition of B, then

$$\mathbb{P}(B) = \sum_{i=1}^{\infty} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{\infty} \mathbb{P}(B|A_i)\mathbb{P}(A_i)$$

Theorem 3.4 (Bayes' Theorem)
If $\{A_1, A_2, \dots\}$ is a partition of B where $\mathbb{P}(B) > 0$, then

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_j \mathbb{P}(B|A_j)\mathbb{P}(A_j)}$$

3.3 Independence

Definition 3.5
Two events A and B are said to be *independent* if $\mathbb{P}(A|B) = \mathbb{P}(A)$, or equivalently,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Definition 3.6
A collection of events $\{A_1, A_2, \dots\}$ is said to be

(1) *pairwise independent* if $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$ for all $i \neq j$.

(2) *totally independent* if, for every finite subset $\{B_1, B_2, \dots, B_m\} \subset \{A_1, A_2, \dots\}$,

$$\mathbb{P}(B_1 \cap B_2 \cap \dots \cap B_m) = \mathbb{P}(B_1)\mathbb{P}(B_2)\cdots\mathbb{P}(B_m).$$

This can also be written as $\mathbb{P}\left(\bigcap_{j=1}^m B_j\right) = \prod_{j=1}^m \mathbb{P}(B_j)$.

Remark 3.7
Total independence implies pairwise independence, but not vice versa.

3.4 Conditional probability spaces

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $B \in \mathcal{F}$.

Theorem 3.8
The family of sets $\mathcal{G} = \{A \cap B : A \in \mathcal{F}\}$ is a σ-field over B.

Remark 3.9
$\mathcal{G}$ contains all sets of the form $A \cap B$, where A is some element of $\mathcal{F}$. This means that $A' \in \mathcal{G}$ if and only if there is some $A \in \mathcal{F}$ for which $A' = A \cap B$.

Proof: To show that $\mathcal{G}$ is a σ-field over B, we need to show that

(1) $B \in \mathcal{G}$,

(2) if $A' \in \mathcal{G}$ then $B \setminus A' \in \mathcal{G}$, and

(3) if $A'_1, A'_2, \ldots \in \mathcal{G}$ then $\cup_{i=1}^\infty A'_i \in \mathcal{G}$.

(1) Clearly, $B \in \mathcal{G}$ because there is a set $A \in \mathcal{F}$ for which $B = A \cap B$, namely the set B istelf.

(2) Let $A' \in \mathcal{G}$. Then there exists a set $A \in \mathcal{F}$ for which $A' = A \cap B$.
The complement of A' relative to B can be written as

$$B \setminus A' = B \setminus (A \cap B) = \left[(A \cap B)^c\right] \cap B.$$

- $\mathcal{F}$ is closed under pairwise unions and complementation.
- Since $A, B \in \mathcal{F}$, it thus follows that $(A \cap B)^c \in \mathcal{F}$.
- Hence $B \setminus A'$ can be written as $\left[(A \cap B)^c\right] \cap B$ where $\left[(A \cap B)^c\right] \in \mathcal{F}$
- This shows that $B \setminus A \in \mathcal{G}$.

(3) Let A'_1, A'_2, be elements of $\mathcal{G}$. Then for each A'_i there exists some $A_i \in \mathcal{F}$ such that $A'_i = A_i \cap B$. Using the fact that set intersection is distributive over set union,

$$\cup_i A'_i = \cup_i (A_i \cap B) = \big(\cup_i A_i\big) \cap B.$$

- $\mathcal{F}$ is closed under countable unions.
- Since $A_1, A_2, \ldots \in \mathcal{F}$, it thus follows that $\cup_i A_i \in \mathcal{F}$.
- Hence $\cup_i A'_i$ can be written in the form $\big(\cup_i A_i\big) \cap B$ where $\cup_i A_i \in \mathcal{F}$.
- This shows that $\cup_i A'_i \in \mathcal{G}$.

Thus we have shown that $\mathcal{G}$ is a σ-field over B, as required.

Theorem 3.10
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $B \in \mathcal{F}$, and let $\mathcal{G} = \{A \cap B : A \in \mathcal{F}\}$.
If $\mathbb{P}(B) > 0$, then

$$\begin{aligned} \mathbb{Q} : \quad \mathcal{G} &\to [0,1] \\ A' &\mapsto \mathbb{P}(A'|B) \end{aligned}$$

is a probability measure on $(B, \mathcal{G})$.

Remark 3.11
$(B, \mathcal{G}, \mathbb{Q})$ is called a *conditional probability space.*.

Proof: To show that $\mathbb{Q}$ is a probability measure on $(B, \mathcal{G})$, we need to show that

- $\mathbb{Q}(B) = 1$,
- $\mathbb{Q}\left(\bigcup_i A'_i\right) = \sum_i \mathbb{Q}(A'_i)$ whenever the $A'_i \in \mathcal{G}$ are pairwise disjoint.

First,

$$\mathbb{Q}(B) = \mathbb{P}(B|B) = \frac{\mathbb{P}(B \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1.$$

To prove countable additivity, let $A'_1, A'_2, \ldots$ be pairwise disjoint events in $\mathcal{G}$. Then, using the fact that set intersection is distributive over set union,

$$\begin{aligned}
\mathbb{Q}(\cup_i A'_i) = \mathbb{P}(\cup_i A'_i \,|\, B) &= \frac{\mathbb{P}\big[(\cup_i A'_i) \cap B\big]}{\mathbb{P}(B)} \\
&= \frac{\mathbb{P}\big[\cup_i (A'_i \cap B)\big]}{\mathbb{P}(B)} \\
&= \frac{\sum_i \mathbb{P}(A'_i \cap B)}{\mathbb{P}(B)} \quad \text{because the } A'_i \text{ are disjoint,} \\
&= \sum_i \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} \\
&= \sum_i \mathbb{Q}(A'_i).
\end{aligned}$$

Thus we have shown that $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{G})$, as required.

Remark 3.12
We have shown that $\mathbb{Q}$ is a probability measure on $(B, \mathcal{G})$. Using an almost identical argument, it can be shown that $\mathbb{Q}$ is also a probability measure on $(\Omega, \mathcal{F})$.

- In the probability space $(B, \mathcal{G}, \mathbb{Q})$, outcomes $\omega \notin B$ are excluded from consideration.
- In the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, outcomes $\omega \notin B$ are assigned probability zero.

3.5 Exercises

Exercise 3.1 [Revision]

1. Let Ω be a sample space, and let $A_1, A_2, \ldots$ be a partition of Ω with the property that $\mathbb{P}(A_i) > 0$ for all i.

   (a) Show that $\mathbb{P}(B) = \sum_{i=1}^{\infty} \mathbb{P}(B|A_i)\mathbb{P}(A_i)$.

   (b) Show that $\mathbb{P}(A_i|B) = \dfrac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_{j=1}^{\infty} \mathbb{P}(B|A_j)\mathbb{P}(A_j)}$.

Exercise 3.2

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$, and consider the function $\mathbb{Q} : \mathcal{F} \to [0, 1]$ defined by $\mathbb{Q}(A) = \mathbb{P}(A|B)$.

   (a) Show that $(\Omega, \mathcal{F}, \mathbb{Q})$ is a probability space.

   (b) If $C \in \mathcal{F}$ and $\mathbb{Q}(C) > 0$, show that $\mathbb{Q}(A|C) = \mathbb{P}(A|B \cap C)$.

2. A random number N of dice are rolled. Let A_k be the event that $N = k$, and suppose that $\mathbb{P}(A_k) = 2^{-k}$ for $k \in \{1, 2, \ldots\}$ (and zero otherwise). Let S be the sum of the scores shown on the dice. Find the probability that:

   (a) $N = 2$ given that $S = 4$,

   (b) $S = 4$ given that N is even,

   (c) $N = 2$ given that $S = 4$ and the first die shows 1,

   (d) the largest number shown by any dice is r (where S is unknown).

3. Let $\Omega = \{1, 2, \ldots, p\}$ where p is a prime number. Let $\mathcal{F}$ be the power set of Ω, and let $\mathbb{P} : \mathcal{F} \to [0, 1]$ be the probability measure on $(\Omega, \mathcal{F})$ defined by $\mathbb{P}(A) = |A|/p$, where $|A|$ denotes the cardinality of A. Show that if A and B are independent events, then at least one of A and B is either $\emptyset$ or Ω.

Lecture 4 Random Variables

4.1 Random variables

Random variables are functions that transform abstract sample spaces to the real numbers.

Definition 4.1
Let Ω be the sample space of some random experiment, and let $\mathcal{F}$ be a σ-field of events over Ω. A *random variable* on $(\Omega, \mathcal{F})$ is a function

$$\begin{aligned} X: \quad \Omega &\to \mathbb{R} \\ \omega &\mapsto X(\omega) \end{aligned}$$

with the property that $\{\omega : X(\omega) \in B\} \in \mathcal{F}$ for every $B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-field over $\mathbb{R}$.

Remark 4.2

- The set $\{\omega : X(\omega) \in B\}$ contains precisely those outcomes that are mapped by X into the set B.
- X is a random variable only if every set of this form is an element of the σ-field $\mathcal{F}$.
- This condition means that, for any Borel set B, the probability that X takes a value in B is well-defined.

Let us define the following notation:

$$\{X \in B\} = \{\omega : X(\omega) \in B\}$$

- The expression $\{X \in B\}$ should not be taken literally: X is a function, while B is a subset of the real numbers.
- Instead, think of $\{X \in B\}$ as the event that X takes a value in B.
- The condition $\{X \in B\} \in \mathcal{F}$ ensures that the probability of this event is well-defined.

We denote the probability of $\{X \in B\}$ by $\mathbb{P}(X \in B)$, by which we mean

$$\mathbb{P}(X \in B) = \mathbb{P}\big(\{\omega : X(\omega) \in B\}\big)$$

Proposition 4.3
A function $X : \Omega \to \mathbb{R}$ is a random variable if and only if $\{X \leq x\} \in \mathcal{F}$ for every $x \in \mathbb{R}$.

[*Proof omitted.*]

Remark 4.4
To check whether or not a function $X : \Omega \to \mathbb{R}$ is a random variable, by the proposition we need not verify that $\{X \in B\} \in \mathcal{F}$ for all Borel sets $B \in \mathcal{B}$. Instead, it is enough to verify only that the sets $\{\omega : X(\omega) \leq x\}$ are included in $\mathcal{F}$ (for every $x \in \mathbb{R}$).

4.2 Indicator variables

The elementary random variable is the *indicator variable* of an event A.

Definition 4.5
The *indicator variable* of an event A is the random variable $I_A : \Omega \to \mathbb{R}$ defined by

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Theorem 4.6
Let A and B be any two events. Then

(1) $I_{A^c} = 1 - I_A$

(2) $I_{A \cap B} = I_A I_B$

(3) $I_{A \cup B} = I_A + I_B - I_{A \cap B}$

Proof: Exercise. Note that for two functions to be equal, they must be equal at every point of their common domain, so for the first part we need to show that $I_{A^c}(\omega) = 1 - I_A(\omega)$ for every $\omega \in \Omega$, and similarly for parts (2) and (3).

4.3 Simple random variables

Definition 4.7
A *simple random variable* is one that takes only finitely many values.

If $X : \Omega \to \mathbb{R}$ is a simple random variable, it can be represented as:

$$X(\omega) = \sum_{i=1}^{n} a_i I_{A_i}(\omega)$$

where

- $\{a_1, a_2, \ldots, a_n\} \subset \mathbb{R}$ is the range of X, and
- $\{A_1, A_2, \ldots, A_n\}$ is a partition of the sample space, Ω.

4.4 Probability on $\mathbb{R}$

Definition 4.8
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a random variable on $(\Omega, \mathcal{F})$. The function

$$\begin{array}{rccl} \mathbb{P}_X : & \mathcal{B} & \to & [0,1] \\ & B & \mapsto & \mathbb{P}(X \in B). \end{array}$$

is called the *distribution* of X.

Theorem 4.9
$\mathbb{P}_X$ is a probability measure on $(\mathbb{R}, \mathcal{B})$.

Proof: First we need to show that $\mathbb{P}_X(\mathbb{R}) = 1$:

$$\mathbb{P}_X(\mathbb{R}) = \mathbb{P}(X \in \mathbb{R}) = \mathbb{P}\big(\{\omega : X(\omega) \in \mathbb{R}\}\big) = 1.$$

We also need to show that $\mathbb{P}_X$ is countably additive. If $B_1, B_2, \dots$ is a sequence of pairwise disjoint sets in $\mathcal{B}$, then

$$\begin{aligned}
\mathbb{P}_X\big(\textstyle\bigcup_{i=1}^\infty B_i\big) &= \mathbb{P}\big(\{\omega : X(\omega) \in \textstyle\bigcup_{i=1}^\infty B_i\}\big) \\
&= \mathbb{P}\big(\textstyle\bigcup_{i=1}^\infty \{\omega : X(\omega) \in B_i\}\big) \\
&= \sum_{i=1}^\infty \mathbb{P}\big(\{\omega : X(\omega) \in B_i\}\big) \quad \text{because the } B_i \text{ are disjoint,} \\
&= \sum_{i=1}^\infty \mathbb{P}_X(B_i),
\end{aligned}$$

which concludes the proof.

Remark 4.10
A random variable X transforms an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$ into a more tractable probability space $(\mathbb{R}, \mathcal{B}, \mathbb{P}_X)$, where we can apply the methods of *real analysis*.

4.5 Exercises

Exercise 4.1

1. Let Ω be the sample space of some random experiment, and let $\mathcal{F}$ be a σ-field over Ω.
   (a) For any $A \in \mathcal{F}$, show that the function $X : \Omega \to \mathbb{R}$, defined by

   $$X(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A, \end{cases}$$

   is a random variable on $(\Omega, \mathcal{F})$.
   (b) Let $A_1, A_2, \dots, A_n \in \mathcal{F}$ be a partition of Ω and let $a_1, a_2, \dots, a_n \in \mathbb{R}$. Show that the function $X : \Omega \to \mathbb{R}$, defined by

   $$X(\omega) = \sum_{i=1}^n a_i I_{A_i}(\omega) \qquad \text{where} \qquad I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A, \end{cases}$$

   is a random variable on $(\Omega, \mathcal{F})$.

Lecture 5 Distributions

5.1 Probability on the real line

Let $X : \Omega \to \mathbb{R}$ be a random variable, and recall the probability measure on $(\mathbb{R}, \mathcal{B})$, defined by

$$\mathbb{P}_X(B) = \mathbb{P}(X \in B) = \mathbb{P}\big(\{\omega : X(\omega) \in B\}\big),$$

where $\mathcal{B}$ is the Borel σ-field over $\mathbb{R}$.

Definition 5.1

(1) The *distribution* of X is the probability measure $\mathbb{P}_X(B) = \mathbb{P}(X \in B)$.

(2) The *cumulative distribution function* (CDF) of X is the function $F(x) = \mathbb{P}(X \leq x)$.

(3) The *survival function* (SF) of X is the function $S(t) = \mathbb{P}(X > t)$.

Remark 5.2

The survival function is also called the *complementary* distribution function. If X represents the *lifetime* of some random system, then $S(t) = \mathbb{P}(X > t)$ is the probability that the system survives beyond time t. In this context, $F(t) = 1 - S(t)$ is called the *lifetime distribution function.*

5.2 Cumulative distribution functions (CDFs)

Proposition 4.3 states that $X : \Omega \to \mathbb{R}$ is a random variable if and only if the sets $\{X \leq x\}$ are *events* over Ω:

$$\{X \leq x\} = \big\{\omega : X(\omega) \leq x\big\} \in \mathcal{F} \quad \text{for all} \quad x \in \mathbb{R}.$$

It can be shown that the probability measure

$$\mathbb{P}_X(B) = \mathbb{P}(X \in B) = \mathbb{P}\big(\{\omega : X(\omega) \in B\}\big),$$

is uniquely defined by the values it takes on the events $\{X \leq x\}$ for $x \in \mathbb{R}$. Consequently, the distribution of a random variable is uniquely determined by its *cumulative distribution function* (CDF):

Definition 5.3

The *cumulative distribution function* (CDF) of a random variable $X : \Omega \to \mathbb{R}$ is the function

$$\begin{array}{rccl} F : & \mathbb{R} & \longrightarrow & [0,1] \\ & x & \mapsto & \mathbb{P}(X \leq x). \end{array}$$

Theorem 5.4

Let $F : \mathbb{R} \to [0, 1]$ be a CDF. Then there is a unique probability measure $\mathbb{P}_F : \mathcal{B} \to [0, 1]$ on the real line with the property that

$$\mathbb{P}_F\big((a, b]\big) = F(b) - F(a)$$

for every such half-open interval $(a, b] \in \mathcal{B}$.

[*Proof omitted.*]

- The triple $(\mathbb{R}, \mathcal{B}, \mathbb{P}_F)$ is sometimes called the *probability space induced by F*.

Remark 5.5
Compare the probability measure $\mathbb{P}_F$ of the interval $(a, b] \subset \mathbb{R}$ to the usual measure of its *length*:

- Length: $\mathbb{L}\big((a, b]\big) = b - a$
- Probability measure: $\mathbb{P}_F\big((a, b]\big) = F(b) - F(a)$.

Thus $\mathbb{P}_F\big((a, b]\big)$ quantifies the "amount of probability" in any given interval $(a, b]$.

5.3 Properties of CDFs

Theorem 5.6
A cumulative distribution function $F : \mathbb{R} \to [0, 1]$ has the following properties:

(1) if $x < y$ then $F(x) \leq F(y)$,

(2) $F(x) \to 0$ as $x \to -\infty$,

(3) $F(x) \to 1$ as $x \to +\infty$, and

(4) $F(x + h) \to F(x)$ as $h \downarrow 0$ (right continuity).

Proof:

(1) To show that F is increasing, let $x < y$ and consider the events

$$\begin{aligned} A &= \{X \leq x\} &&= \{\omega : X(\omega) \leq x\}, \\ B &= \{X \leq y\} &&= \{\omega : X(\omega) \leq y\}. \end{aligned}$$

By construction, $F(x) = \mathbb{P}(A)$ and $F(y) = \mathbb{P}(B)$ so by the monotonicity of probability measures (Theorem 2.8),

$$x < y \Rightarrow A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B) \Rightarrow F(x) \leq F(y).$$

(2) To show that $F(x) \to 0$ as $x \to -\infty$, let

$$B_n = \{X \leq -n\} = \{\omega : X(\omega) \in (-\infty, -n]\} \quad \text{for } n = 1, 2, \ldots$$

so that $F(-n) = \mathbb{P}(X \leq -n) = \mathbb{P}(B_n)$.
The sequence $B_1, B_2, \ldots$ is decreasing ($B_{n+1} \subseteq B_n$), with

$$\bigcap_{n=1}^{\infty} B_n = \emptyset,$$

because for any x, there exists an n such that $x \notin (-\infty, -n]$.
By the continuity of probability measures (Theorem 2.9),

$$\lim_{n\to\infty} F(-n) = \lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}\left(\bigcap_{n=1}^{n} B_n\right) = \mathbb{P}(\emptyset) = 0,$$

and because $F(x)$ is an increasing function,

$$\lim_{n\to\infty} F(-n) = 0 \quad \Leftrightarrow \quad \lim_{x\to-\infty} F(x) = 0.$$

(3) To show that $F(x) \to 1$ as $x \to \infty$, let

$$A_n = \{X \leq n\} = \{\omega : X(\omega) \in (-\infty, n]\} \quad \text{for } n = 1, 2, \ldots,$$

so that $F(n) = \mathbb{P}(X \leq n) = \mathbb{P}(A_n)$.

The sequence $A_1, A_2, \dots$ is increasing ($A_n \subseteq A_{n+1}$), with

$$\bigcup_n A_{n=1}^{\infty} = \Omega,$$

because for any x, there exists an n such that $x \in (-\infty, n]$.

By the continuity of probability measures,

$$\lim_{n\to\infty} F(n) = \lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \mathbb{P}(\Omega) = 1,$$

and because $F(x)$ is an increasing function,

$$\lim_{n\to\infty} F(n) = 1 \quad \Leftrightarrow \quad \lim_{x\to\infty} F(x) = 1.$$

(4) To show that $F(x)$ is right-continuous, let

$$B_n = \left\{\omega : X(\omega) \in \left(-\infty, x + \frac{1}{n}\right]\right\}$$

so that $F\left(x + 1/n\right) = \mathbb{P}\left(X \leq x + 1/n\right) = \mathbb{P}(B_n)$.

The sequence $B_1, B_2, \dots$ is decreasing ($B_{n+1} \subseteq B_n$), with

$$\bigcap_{n=1}^{\infty} B_n = (-\infty, x],$$

so

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right) = \mathbb{P}\big(\{\omega : X(\omega) \in (-\infty, x]\}\big) = \mathbb{P}\big(\{\omega : X(\omega) \leq x\}\big) = F(x).$$

By the continuity of probability measures,

$$F(x) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mathbb{P}(B_n) = \lim_{n\to\infty} F\left(x + \frac{1}{n}\right).$$

which concludes the proof.

Theorem 5.7
Let $F : \mathbb{R} \to [0, 1]$ be a function with properties (i)-(iv) of Theorem 5.6. Then F is a cumulative distribution function.

[*Proof omitted.*]

Remark 5.8
The last two theorems make no explicit reference to random variables:

- many different random variables can have the same distribution function;
- a distribution function can represent many different random variables.

5.4 Discrete distributions and PMFs

The *range* of a random variable $X : \Omega \to \mathbb{R}$ is the set of all possible values it can take:

$$\text{Range}(X) = \{x \in \mathbb{R} : X(\omega) = x \text{ for some } \omega \in \Omega\}.$$

Definition 5.9

- $X : \Omega \to \mathbb{R}$ is called a *discrete random variable* if its range is a countable subset of $\mathbb{R}$.
- A discrete random variable is described by its *probability mass function* (PMF),

$$\begin{aligned} f: \quad \mathbb{R} &\to [0,1] \\ k &\mapsto \mathbb{P}(X=k), \end{aligned}$$

  which must have the property that $\sum_k f(k) = 1$.
- A probability mass function defines a *discrete probability measure* on $\mathbb{R}$,

$$\begin{aligned} \mathbb{P}_X: \quad \mathcal{B} &\to [0,1] \\ B &\mapsto \sum_{k\in B} \mathbb{P}(X=k), \end{aligned}$$

- The triple $(\mathbb{R}, \mathcal{B}, \mathbb{P}_X)$ is called a *discrete probability space* over $\mathbb{R}$.

5.5 Continuous distributions and PDFs

Definition 5.10

- A cumulative distribution function $F : \mathbb{R} \to [0,1]$ is said to be *absolutely continuous* if there exists an integrable function $f : \mathbb{R} \to [0,\infty)$ such that

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \quad \text{for all} \quad x \in \mathbb{R}.$$

- The function $f : \mathbb{R} \to [0,\infty)$ is called the *probability density function* (PDF) of F.
- The triple $(\mathbb{R}, \mathcal{B}, \mathbb{P}_F)$ is called a *continuous probability space* over $\mathbb{R}$.

Definition 5.11

A *continuous random variable* is one whose distribution function is absolutely continuous.

If $X : \Omega \to \mathbb{R}$ is a continuous random variable, then

- $f(x) = F'(x)$ for all $x \in \mathbb{R}$.
- Probabilities correspond to areas under the curve $f(x)$:

$$\mathbb{P}_X\big((a,b]\big) = \mathbb{P}(a < X \le b) = F(b) - F(a) = \int_a^b f(x)\,dx.$$

- Note that $\mathbb{P}(X = x) = 0$ for all $x \in \mathbb{R}$.

Remark 5.12

The continuity of a random variable $X : \Omega \to \mathbb{R}$ refers to the continuity of its distribution function, and *not* to the continuity (or otherwise) of itself as a function on Ω.

5.6 Exercises

Exercise 5.1

1. Let F and G be CDFs, and let $0 < \lambda < 1$ be a constant. Show that $H = \lambda F + (1-\lambda)G$ is also a CDF.
2. Let X_1 and X_2 be the numbers observed in two independent rolls of a fair die. Find the PMF of each of the following random variables:
   (a) $Y = 7 - X_1$,

(b) $U = \max(X_1, X_2)$,
(c) $V = X_1 - X_2$.
(d) $W = |X_1 - X_2|$.

3. The PDF of a continuous random variable X is given by $f(x) = \begin{cases} cx^2 & 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$
   (a) Find the value of the constant c, and sketch the PDF of X.
   (b) Find the value of $P(X > 3/2)$.
   (c) Find the CDF of X.

4. The PDF of a continuous random variable X is given by $f(x) = \begin{cases} cx^{-d} & \text{for } x > 1, \\ 0 & \text{otherwise.} \end{cases}$
   (a) Find the range of values of d for which $f(x)$ is a probability density function.
   (b) If $f(x)$ is a density function, find the value of c, and the corresponding CDF.

5. Let $f(x) = \dfrac{ce^x}{(1+e^x)^2}$ be a PDF, where c is a constant. Find the value of c, and the corresponding CDF.

6. Let $X_1, X_2, \ldots$ be independent and identically distributed observations, and let F denote their common CDF. If F is unknown, describe and justify a way of estimating F, based on the observations. [Hint: consider the indicator variables of the events $\{X_j \leq x\}$.]