DYNAMICAL ZETA FUNCTIONS FOR AXIOM A FLOWS

SEMYON DYATLOV AND COLIN GUILLARMOU

ABSTRACT. We show that the Ruelle zeta function of any smooth Axiom A flow with orientable stable/unstable spaces has a meromorphic continuation to the entire complex plane. The proof uses the meromorphic continuation result of [DyGu16] together with [CoEa71, GMT17] which imply that every basic hyperbolic set can be put into the framework of [DyGu16].

Let \mathcal{M} be a compact manifold and X be a C^{∞} vector field on \mathcal{M} . Denote by $\varphi^t = \exp(tX)$ the corresponding flow. In this note we prove

Theorem. Assume that φ^t is an Axiom A flow (see Definition 2.3) with orientable stable/unstable spaces. Define the **Ruelle zeta function**

$$\zeta(\lambda) = \prod_{\gamma^{\sharp}} \left(1 - e^{-\lambda T_{\gamma^{\sharp}}} \right), \quad \text{Re } \lambda \gg 1,$$
 (1.1)

where the product is taken over the primitive closed orbits γ^{\sharp} of φ^{t} , with the exception of fixed points, and $T_{\gamma^{\sharp}}$ are the corresponding periods. Then $\zeta(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$.

We refer the reader to §2 for examples of Axiom A flows and for the dynamical systems terminology used here.

The above theorem was first conjectured by Smale [Sm67, bottom of page 802].† It was proved in the special case of Anosov flows by Giulietti–Liverani–Pollicott [GLP13]. Dyatlov–Zworski [DyZw16] gave a simple microlocal proof in the Anosov case. In our previous work [DyGu16], we generalized the microlocal techniques used in [DyZw16] to handle the case of open hyperbolic systems, defined as locally maximal hyperbolic sets with a smooth neighborhood on which the flow is strictly convex. We emphasize that the introduction of microlocal tools and the use of escape functions for the study of hyperbolic diffeomorphisms and flows appeared first in the works of Faure-Roy-Sjöstrand [FRS08] and Faure-Sjöstrand [FaSj11]; they were inspired by the works of Kitaev [Ki99], Blank-Keller-Liverani [BKL02], Liverani [Li04], Gouezel-Liverani [GoLi08], Baladi-Tsujii [BaTs07] that used anisotropic Sobolev/Hölder spaces adapted to the dynamics. The meromorphic extension of the zeta function for a particular case

[†]Strictly speaking, [Sm67] considers a subclass of Axiom A flows, namely perturbations of suspensions of certain Axiom A maps. He says 'I must admit a positive answer [to the question of meromorphy] would be a little shocking!'

of Axiom A flow, namely the Grassmanian extension of a contact Anosov flow, was also proved by Faure-Tsujii [FaTs17].

See the introduction to [DyGu16] for an overview of results on dynamical zeta functions and Pollicott–Ruelle resonances and the book by Baladi [Ba17] for the related case of hyperbolic maps.

The proof in this note combines [DyGu16] with the results of Conley–Easton [CoEa71] and Guillarmou–Mazzucchelli–Tzou [GMT17] which show that every locally maximal hyperbolic set has a smooth neighborhood with a strictly convex flow, as well as with Smale's spectral decomposition for Axiom A flows.

We note that just like in [DyGu16] one can consider a more general Ruelle zeta function with a potential. In fact, the methods of [DyGu16] apply to even more general dynamical traces and twisted zeta functions associated to the action of X on vector bundles, see [DyGu16, §5.1, Theorem 4]. Moreover, the assumption that X is C^{∞} can be relaxed to C^k for large $k = k(C_0)$ if we only want to continue the Ruelle zeta function to a half-plane {Re $\lambda \geq -C_0$ }. Finally, the orientability hypothesis holds in many natural cases (such as geodesic flows on orientable negatively curved manifolds) and can be removed under certain topological assumptions by using a twisted zeta function, see [GLP13, Appendix B].

2. Review of hyperbolic dynamics

In this section we briefly review several standard definitions and facts from the theory of hyperbolic dynamical systems. We refer the reader to [KaHa97] for a comprehensive treatment of hyperbolic dynamics.

Assume that \mathcal{M} is a compact C^{∞} manifold without boundary and $\varphi^t = \exp(tX)$ is a C^{∞} flow generated by a vector field X.

Definition 2.1. Let $K \subset M$ be a compact φ^t -invariant set. We say that K is **hyperbolic** for the flow φ^t , if the generator X of the flow does not vanish on K and each tangent space T_xM , $x \in K$, admits a flow/stable/unstable decomposition

$$T_x \mathcal{M} = E_0(x) \oplus E_s(x) \oplus E_u(x), \quad x \in \mathcal{K}$$

such that:

- $E_0(x) = \mathbb{R}X(x)$;
- $E_s(x), E_u(x)$ depend continuously on the point x;
- $d\varphi^t(E_s(x)) = E_s(\varphi^t(x))$ and $d\varphi^t(E_u(x)) = E_u(\varphi^t(x))$ for all $x \in \mathcal{K}$, $t \in \mathbb{R}$;

• for any choice of a continuous norm $| \bullet |$ on the fibers of TM, there exist constants $C, \theta > 0$ such that for all $x \in \mathcal{K}$,

$$|d\varphi^{t}(x)v| \leq Ce^{-\theta|t|}|v| \quad when \quad \begin{cases} t \geq 0, & v \in E_{s}(x); \\ t \leq 0, & v \in E_{u}(x). \end{cases}$$
 (2.1)

We say φ^t is an **Anosov flow** if the whole \mathcal{M} is hyperbolic.

A fixed point $x \in \mathcal{M}$, X(x) = 0, is called *hyperbolic* if the differential $\nabla X(x)$ has no eigenvalues on the imaginary axis. Hyperbolic fixed points are nondegenerate and thus isolated.

We also define the *nonwandering set*:

Definition 2.2. [Sm67, p. 796] We call $x \in \mathcal{M}$ a nonwandering point if for every neighborhood V of x and every T > 0 there exists $t \in \mathbb{R}$ such that $|t| \geq T$ and $\varphi^t(V) \cap V \neq \emptyset$. The set of all nonwandering points is called the **nonwandering set**.

The nonwandering set is closed and φ^t -invariant, see [KaHa97, Proposition 3.3.4]. Note that each closed orbit of φ^t lies in the nonwandering set.

We now give the definition of Axiom A flows:

Definition 2.3. [Sm67, §II.5, (5.1)] The flow φ^t is **Axiom A** if:

- (1) the nonwandering set is the disjoint union of the set \mathcal{F} of fixed points and the closure \mathcal{K} of the union of all closed orbits;
- (2) all fixed points of φ^t are hyperbolic;
- (3) the set K is hyperbolic for the flow φ^t .

We also define locally maximal sets and basic hyperbolic sets:

Definition 2.4. We say that a compact φ^t -invariant set $K \subset \mathcal{M}$ is **locally maximal** for the flow φ^t , if there exists a neighborhood V of K such that

$$K = \bigcap_{t \in \mathbb{R}} \varphi^t(V).$$

Definition 2.5. [PaPo83, Chapter 9] A compact φ^t -invariant set $K \subset \mathcal{M}$ is called a basic hyperbolic set[†] for φ^t if

- (1) K is locally maximal for φ^t ;
- (2) K is hyperbolic for φ^t ;
- (3) the flow $\varphi^t|_K$ topologically transitive (i.e. contains a dense orbit); and
- (4) K is the closure of the union of all closed orbits of $\varphi^t|_K$.

[†]Strictly speaking, a single closed orbit is not considered a basic set, but we ignore this minor detail here.

The nonwandering set of an Axiom A flow has the following spectral decomposition:

Proposition 2.6. [Sm67, §II.5, Theorem 5.2] Assume that φ^t is an Axiom A flow and let \mathcal{K} be given by Definition 2.3. Then we can write \mathcal{K} as a finite disjoint union

$$\mathcal{K} = K_1 \sqcup \cdots \sqcup K_N$$

where each K_i is a basic hyperbolic set.

See also [KaHa97, Exercise 18.3.7] for a proof of the spectral decomposition, and [Bo75, Lemma 3.9] for the local maximality of the nonwandering set.

We finally give a few examples:

Example 2.7. Let $\mathcal{M} = \mathbb{R}^2_{x,y}/2\pi\mathbb{Z}^2$ be the torus and $\varphi_t = \exp(tX)$ where $X = (\sin x)\partial_x + \partial_y$. The nonwandering set \mathcal{K} has two connected components, each being a single closed orbit:

$$\mathcal{K} = \{x = 0\} \sqcup \{x = \pi\}. \tag{2.2}$$

The spectral decomposition is given by (2.2). Note however that the entire \mathcal{M} is not hyperbolic for φ^t .

Example 2.8. Let (M,g) be a (possibly noncompact) complete Riemannian manifold of negative sectional curvature which is asymptotically hyperbolic. Put $\mathcal{M} := SM$, the sphere bundle of M, and let $\varphi^t : \mathcal{M} \to \mathcal{M}$ be the geodesic flow. Let \mathcal{K} be the union of all geodesics which are trapped, that is their closures in SM are compact. Then \mathcal{K} is a locally maximal hyperbolic set for φ^t . (The noncompactness of \mathcal{M} is not an issue since \mathcal{K} is compact and the behavior of the flow outside of a neighborhood of \mathcal{K} is irrelevant.) See [DyGu16, §6.3] for details and more general examples.

3. Proof of the theorem

We first show meromorphic continuation of the Ruelle zeta function ζ_K for a locally maximal hyperbolic set:

Proposition 3.1. Let $K \subset \mathcal{M}$ be a locally maximal hyperbolic set for φ^t . Define the Ruelle zeta function ζ_K by (1.1) where we only take the closed trajectories of φ^t which lie in K. Then ζ_K continues to a meromorphic function on \mathbb{C} .

The proof of Proposition 3.1 relies on [DyGu16]. To reduce to the case considered in [DyGu16] we use

Lemma 3.2. Let $K \subset \mathcal{M}$ be a locally maximal hyperbolic set for φ^t . Then there exists a neighborhood \mathcal{U} of K in \mathcal{M} with C^{∞} boundary and a C^{∞} vector field X_0 on $\overline{\mathcal{U}}$ such that:

(1) the boundary $\partial \mathcal{U}$ is strictly convex with respect to X_0 in the following sense:

$$\rho(x) = 0, \quad X_0 \rho(x) = 0 \implies X_0^2 \rho(x) < 0$$
 (3.1)

where $\rho \geq 0$ is any boundary defining function on \mathcal{U} ;

- (2) $X_0 = X$ in a neighborhood of K;
- (3) $K = \bigcap_{t \in \mathbb{R}} \varphi_0^t(\mathcal{U})$ where $\varphi_0^t := \exp(tX_0)$ is the flow generated by X_0 .

Proof. The proof follows Guillarmou–Mazzucchelli–Tzou [GMT17, Lemma 2.3]. We first use a result of Conley–Easton [CoEa71, Theorem 1.5]. Since K is locally maximal for φ^t , there exists an open set $V \subset \mathcal{M}$ containing K such that $K = \bigcap_{t \in \mathbb{R}} \varphi^t(\overline{V})$. If $x \in \partial V$, then in particular $x \notin K$, thus there exists $t \in \mathbb{R}$ such that $\varphi^t(x) \notin \overline{V}$. Therefore V is an isolating neighborhood for φ^t in the sense of [CoEa71, Definition 1.1] and K is an isolated invariant set in the sense of [CoEa71, Definition 1.2]. Therefore by [CoEa71, Theorem 1.5] there exists an isolating block, which is an open subset $\mathcal{U} \subset \mathcal{M}$ with $K \subset \mathcal{U}$ with the following properties:

- \mathcal{U} has compact closure and C^{∞} boundary, that is $\mathcal{U} = \{x \in \mathcal{M} \mid \rho(x) > 0\}$ for some function $\rho \in C^{\infty}(\mathcal{M}; \mathbb{R})$ such that $d\rho \neq 0$ on $\partial \mathcal{U} = \{x \in \mathcal{M} \mid \rho(x) = 0\}$.
- the set $\partial_0 \mathcal{U} := \{ x \in \partial \mathcal{U} \mid X_0 \rho(x) = 0 \}$ is a codimension 1 smooth submanifold of $\partial \mathcal{U}$;
- for each $x \in \partial_0 \mathcal{U}$, there exists $\varepsilon > 0$ such that

$$\varphi^t(x) \notin \overline{\mathcal{U}} \quad \text{for all} \quad t \in (-\varepsilon, \varepsilon) \setminus \{0\}.$$
 (3.2)

Now the vector field X_0 is obtained by modifying X slightly near $\partial_0 \mathcal{U}$ so that we still have $K = \bigcap_{t \in \mathbb{R}} \varphi_0^t(\overline{\mathcal{U}})$ and the topological (local) convexity condition (3.2) is replaced by the quadratic differential convexity condition (3.1). We refer to the proof of [GMT17, Lemma 2.3] for details.

We now give

Proof of Proposition 3.1. By part (3) of Lemma 3.2, every closed trajectory of φ_0^t in $\overline{\mathcal{U}}$ lies in K and thus is a closed trajectory of φ^t . Therefore the Ruelle zeta function ζ_K is equal to the Ruelle zeta function of φ_0^t on $\overline{\mathcal{U}}$. Now $(\mathcal{U}, \varphi_0^t)$ is an open hyperbolic system in the sense of [DyGu16, Assumptions (A1)–(A4)]. Therefore [DyGu16, Theorem 3] applies (with potential set to 0) and gives a meromorphic continuation of ζ_K to \mathbb{C} . \square

The main theorem now follows using the spectral decomposition theorem, Proposition 2.6. Indeed, if $K = K_1 \sqcup \cdots \sqcup K_N$ is the spectral decomposition of φ^t , then each closed trajectory of φ^t is contained in one of the sets K_j . Therefore the Ruelle zeta function (1.1) factorizes as

$$\zeta(\lambda) = \zeta_{K_1}(\lambda) \cdots \zeta_{K_N}(\lambda).$$

Since each ζ_{K_j} admits a meromorphic continuation to \mathbb{C} by Proposition 3.1, the function ζ admits a meromorphic continuation to \mathbb{C} as well.

Acknowledgements. This research was conducted during the period SD served as a Clay Research Fellow. CG was partially supported by the ANR project ANR-13-JS01-0006 and by the ERC consolidator grant IPFLOW.

References

- [Ba17] Viviane Baladi, Dynamical zeta functions and dynamical determinants for hyperbolic maps: a functional approach, to appear in Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer.
- [BaTs07] Viviane Baladi and Masato Tsujii, Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms, Ann. Inst. Fourier 57(2007), 127–154.
- [BKL02] Michael Blank, Gerhard Keller, and Carlangelo Liverani, Ruelle-Perron-Frobenius spectrum for Anosov maps, Nonlinearity 15(2002), 1905–1973.
- [Bo75] Rufus Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, Lect. Notes in Math. 470, Springer, 1975.
- [BuLi07] Oliver Butterley and Carlangelo Liverani, Smooth Anosov flows: correlation spectra and stability, J. Mod. Dyn. 1(2007), 301–322.
- [CoEa71] Charles Conley and Robert Easton, *Isolated invariant sets and isolating blocks*, Trans. Amer. Math. Soc. **158**(1971), 35–61.
- [DyGu16] Semyon Dyatlov and Colin Guillarmou, *Pollicott–Ruelle resonances for open systems*, Ann. Henri Poincaré **17**(2016), 3089–3146.
- [DyZw16] Semyon Dyatlov and Maciej Zworski, Dynamical zeta functions for Anosov flows via microlocal analysis, Ann. Sc. Ec. Norm. Sup. 49(2016), 543–577.
- [FRS08] Frédéric Faure, Nicolas Roy, and Johannes Sjöstrand, A semiclassical approach for Anosov diffeomorphisms and Ruelle resonances, Open Math. J. 1(2008), 35–81.
- [FaSj11] Frédéric Faure and Johannes Sjöstrand, Upper bound on the density of Ruelle resonances for Anosov flows, Comm. Math. Phys. **308**(2011), 325–364.
- [FaTs17] Frédéric Faure, Masato Tsujii, The semiclassical zeta function for geodesic flows on negatively curved manifolds, Invent. math. 208 (2017), no3, 851–998.
- [GLP13] Paolo Giulietti, Carlangelo Liverani, and Mark Pollicott, Anosov flows and dynamical zeta functions, Ann. of Math. (2) 178(2013), 687–773.
- [GoLi08] S. Gouëzel, C. Liverani, Compact locally hyperbolic sets for smooth maps: fine statistical properties, J. Diff. Geom. 79 (2008) 433–477.
- [GMT17] Colin Guillarmou, Marco Mazzucchelli, and Leo Tzou, Boundary and lens rigidity for non-convex manifolds, preprint, arXiv:1711.10059.
- [KaHa97] Anatole Katok and Boris Hasselblatt, Introduction to the modern theory of dynamical systems, Cambridge Univ. Press, 1997.
- [Ki99] Alexey Kitaev, Fredholm determinants for hyperbolic diffeomorphisms of finite smoothness, Nonlinearity 12(1999), 141–179.
- [Li04] Carlangelo Liverani, On contact Anosov flows, Ann. of Math. (2) 159(2004), 1275–1312.
- [PaPo83] William Parry and Mark Pollicott, An analogue of the prime number theorem for closed orbits of Axiom A flows, Ann. of Math. (2) 118(1983), 573–591.
- [Sm67] Steven Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc. 73(1967), 747–817.

E-mail address: dyatlov@math.berkeley.edu

Department of Mathematics, University of California, Berkeley, CA 94720

E-mail address: cguillar@math.cnrs.fr

Laboratoire de Mathématiques d'Orsay, Faculté des Sciences d'Orsay Université Paris-Sud, F-91405 Orsay Cedex