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abstract	Introduction Analytic vectors and their analytic continuation. Let G be a Lie group and (Ï€, G, V) a continuous representation of G in a topological vector space V . A vector v âˆˆ V is called analytic if the function Î¾ v : g â†' Ï€(g)v is a real analytic function on G with values in V . This means that there exists a neighborhood U of G in its complexification G C such that Î¾ v extends to a holomorphic function on U . In other words, for each element g âˆˆ U we can unambiguously define the vector Ï€(g)v as Î¾ v (g), i.e., we can extend the action of G to a somewhat larger set. In this paper we will show that the possibility of such an extension sometimes allows one to prove some highly nontrivial estimates. Unless otherwise stated, G = SL(2, R), so G C = SL(2, C). We consider a typical representation of G, i.e., a representation of the principal series. Namely, fix Î» âˆˆ C and consider the space D Î» of smooth homogeneous functions of degree Î» âˆ' 1 on R 2 \ 0, i.e., D Î» = {Ï† âˆˆ C âˆž (R 2 \ 0) : Ï†(ax, ay) = |a| Î»âˆ'1 Ï†(x, y)}; we denote by (Ï€ Î» , G, D Î») the natural representation of G in the space D Î» . Restriction to S 1 gives an isomorphism D Î» C âˆž even (S 1), and for basis vectors of D Î» one can take the vectors e k = exp(2ikÎ¸). If Î» = it, then (Ï€ Î» , D Î») is a unitary representation of G with the invariant norm ||Ï†|| 2 = 1 2Ï€ S 1 |Ï†| 2 dÎ¸. Consider the vector v = e 0 âˆˆ D Î» . We claim that v is an analytic vector and we want to exhibit a large set of elements g âˆˆ G C for which the expression Ï€(g)v makes sense. The vector v is represented by the function (x 2 + y 2) For any a > 0 consider the diagonal matrix g a = diag(a âˆ'1 , a). Then This last expression makes sense as a vector in D Î» for any complex a such that |arg(a)| < Ï€ 4 (since in this case Re (a 2 x 2 + a âˆ'2 y 2) > 0). Hence, we see that the function Î¾ v extends analytically to the subset I = {g a : |arg(a)| < Ï€ 4 } âŠ‚ SL(2, C). The same argument shows that the function Î¾ v extends analytically to the domain U = SL(2, R) Â· I Â· K C âŠ‚ SL(2, C) (open in the usual topology), 330 JOSEPH BERNSTEIN AND ANDRE REZNIKOV where K = SO(2, R) and K C = SO(2, C) C * ; thus, for any g âˆˆ U we unambiguously define the vector Ï€(g)v. As g approaches the boundary of U , the vector Ï€(g)v âˆˆ D Î» has very specific asymptotic behavior that we will use in order to obtain information about this vector. 0.2. Triple products. Let us describe an application of the principle of analytic continuation to a problem in the theory of automorphic functions. Namely, we will show how to apply the principle in order to settle a conjecture of Peter Sarnak on triple products. As a corollary of our result we will get a new bound on Fourier coefficients of cusp forms. Recall the setting. Let H be the upper half-plane with the hyperbolic metric of constant curvature âˆ'1. We consider the natural action of the group G = SL(2, R) on H and identify H with G/K by means of this action. Fix a lattice Î“ âŠ‚ G and consider the Riemann surface Y = Î“ \ H. In this paper we will discuss both cocompact and noncocompact lattices of finite covolume. For simplicity of exposition, in most of the paper we will only discuss the cocompact case. Then in Section 4 we will describe how to overcome the extra difficulties in case of noncocompact lattices. The Laplace-Beltrami operator âˆ† acts on the space of functions on Y. When Y is compact it has discrete spectrum; we denote by Âµ 0 < Âµ 1 â‰¤ . . . its eigenvalues on Y and by Ï† i the corresponding eigenfunctions. (We assume that Ï† i are L 2 normalized: ||Ï† i || = 1.) These functions Ï† i are usually called automorphic functions or Maass forms (see [B]). To state the problem about triple products, fix one automorphic function, Ï†, and consider the function Ï† 2 on Y . Since Ï† 2 is not an eigenfunction, it is not an automorphic function. Since Ï† 2 âˆˆ L 2 (Y), we may consider its spectral decomposition in the basis {Ï† i }: Here the coefficients are given by the triple product integrals: Later we will explain why these triple products are of interest and how they are related to the theory of Rankin-Selberg L-functions (see also [S], which was our starting point). Claim. The coefficients c i decay exponentially as exp(âˆ' Ï€ 2 âˆš Âµ i). More precisely, let us introduce new parameters (the meaning of this parametrization will become clear in subsection 0.3). The main result of the paper is the proof of the following theorem which settles a conjecture of P. Sarnak (see [S]): Theorem. There exists a constant C > 0 such that Corollary. There exists a constant C > 0 such that Remarks. 1. The bound in the theorem is essentially sharp. Namely, our method gives the following lower bound on the average: For a single triple product we cannot do better than the bound in the corollary. For congruence subgroups we can speculate about the true "size" of these triple products. It is known (see 0.6) that in certain cases the c i are equal (up to an explicit factor) to the value of the triple Garrett L-function at 1 2 . For these L-functions, the LindelÃ¶f conjecture predicts b i |Î» i | âˆ'2+Îµ . This is consistent with our bound together with the Weyl law: the number of eigenfunctions with |Î» i | â‰¤ T is proportional to T 2 . 2. We will prove similar results for nonuniform lattices (see Â§4). 3. This type of question has been considered before. The first result on exponential decay of the coefficients c i for a holomorphic cusp form Ï† was proven by A. Good ([G]) for the general (i.e., nonarithmetic) nonuniform lattices Î“ thanks to a special feature of holomorphic PoincarÃ© series. Recently, M. Jutila ([J]) extended these results to the nonholomorphic case (Maass forms), but only for the group SL(2, Z), using Kuznetsov's formula and nontrivial arithmetic information (Weil's bounds on Kloosterman's sums and deep results of Iwaniec). In particular, all these methods work only for nonuniform lattices. In [S], P. Sarnak introduced a new method to estimate the triple products based on analytic continuation of certain matrix coefficients of the function Ï†; this method works for uniform lattices as well. Being partly based on the theory of spherical harmonics, it led to a weaker bound (by a power of T). Our method, in addition to the analytic continuation, uses more sophisticated representation theory, in particular, an idea of G-invariant norms on representations and gives the optimal result (possibly, up to a power of logarithm). 4. Our method gives a more general result than Theorem 0.2. We can obtain similar logarithmic bounds for any polynomial expression in any finite number of automorphic functions Ï† k instead of Ï† 2 , as above. 332 JOSEPH BERNSTEIN AND ANDRE REZNIKOV 5. One can ask the same question about growth of triple products for polynomial expressions in automorphic functions of nonzero weight. In this case the decay is also exponential with the same exponent as in Claim 0.2, but the bound in the analogue of Theorem 0.2 is a power of T and not logarithmic as above. The main interest in triple products and their bounds stems from their relation to the theory of automorphic L-functions. We will discuss this relation in 0.6. We also show in 0.7 that Theorem 0.2 implies a new bound on the Fourier coefficients of automorphic functions in the case of nonuniform lattices. 0.3. Automorphic representations. To explain our method, we first recall the relation of automorphic functions to automorphic representations of G. For a given lattice Î“ in G we denote by X the quotient space X = Î“ \ G. The group G acts on X, hence, on the space of functions on X. We can identify This induces an isometric embedding L 2 (Y) âŠ‚ L 2 (X), the image consisting of all K-invariant functions. For any eigenfunction Ï† of the Laplace operator âˆ† on Y we consider the closed G-invariant subspace L Ï† âŠ‚ L 2 (X) generated by Ï† under the G-action. Conversely, fix an irreducible unitary representation (Ï€, L) of the group G and a K-fixed unit vector v 0 âˆˆ L. Then any G-morphism Î½ : L â†' L 2 (X) defines an eigenfunction Ï† = Î½(v 0) of âˆ† on Y ; if Î½ is an isometric embedding, then ||Ï†|| = 1. Thus, the eigenfunctions Ï† correspond to the tuples (Ï€, L, v 0 , Î½). Usually it is more convenient to work with smooth vectors. Let V = L âˆž be the subspace of smooth vectors in L. Then Î½ gives a morphism Î½ : V â†' (L 2 (X)) âˆž âŠ‚ C âˆž (X). If X is compact, then Mor G (L, L 2 (X)) Mor G (V, C âˆž (X)). Thus, the eigenfunctions correspond to the tuples (Ï€, V, v 0 , Î½ : V â†' C âˆž (X)). All irreducible unitary representations of G with K-fixed vector are classified: these are representations of the principal and complementary series and the trivial representation. For simplicity, consider representations of the principal series only. In this case the representation (Ï€, V) in the space of smooth vectors is isomorphic to the representation (Ï€ Î» , D Î») for some Î» = it (see 0.1). The eigenvalue of the corresponding automorphic function equals Âµ = 1âˆ'Î» 2 4 . 0.4. The method. We describe here the idea behind the proof of Theorem 0.2. Let L i âŠ‚ L 2 (X) be the space corresponding to the automorphic function Ï† i as above (see 0.3). Let pr i : L 2 (X) â†' L i be the orthogonal projection. ANALYTIC CONTINUATION 333 Since the function Ï† 2 is K-invariant and there is at most one K-fixed vector in each irreducible representation of

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abstract	Properties of analytic vectors in representations of SL(2,R) are used to give new bounds for the triple products recently considered by P. Sarnak. A conjecture of Sarnak about such products is proved. The results of this paper generalize results of A. Good and M. Jutila about special cases, but the techniques are entirely different. One consequence of these results is a new estimate of the magnitude of the Fourier coefficients of cusp forms for non-arithmetic sub-groups of SL(2,R).
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	SL(2, R), we have pr i (Ï† 2) = c i Ï† i . Since the G-action commutes with the multiplication of functions on X, By the principle of analytic continuation, the same identity holds for the complex points g âˆˆ U (see 0.1). Since all the spaces L i are orthogonal, we get the following basic relation for the complex points g: Here ‖ Â· ‖ = ‖ Â· ‖ L 2 denotes the L 2 -norm in L 2 (X). It is important that in (0.4.1) we deal with complex points g and for such g the operators Ï€(g) are nonunitary. As a result, relation (0.4.1) gives nontrivial information. Now, consider the behavior of the function (Ï€(g)Ï†) 2 near the boundary of U . Take Îµ > 0 and an element g Îµ âˆˆ U which is approximately at the distance Îµ from the boundary of U . For example, set Our goal is to give an upper bound on the left-hand side of (0.4.2) and a lower bound of each of the ‖Ï† i,Îµ ‖ 2 as i â†' âˆž and Îµ â†' 0. The latter problem is simpler since it is invariantly defined in terms of representation theory; thus it can be computed in any model of the representation Ï€ i (e.g., in D Î» i). A direct computation gives On the other hand, we will prove the bound Ï† 2 Îµ |ln Îµ| 3 . These two bounds easily imply Theorem 0.2 (see 2.3). The last bound follows from the bound |Ï† Îµ (x)| â‰¤ C| ln Îµ| which holds pointwise on X and which we consider to be our main achievement in this paper. Its proof is based on the use of invariant norms which we now explain. 0.5. Invariant norms. The most difficult part of the proof is that of the pointwise bound |Ï† Îµ | â‰¤ C| ln Îµ|. Note that the L 2 -norm of Ï† Îµ is of order | ln Îµ| 1 2 ; hence, the pointwise bound only differs from it by a power of logarithm. In order to obtain such a bound, we use invariant (non-Hermitian!) norms on the representation Ï€. Namely, as we have explained, any automorphic 334 JOSEPH BERNSTEIN AND ANDRE REZNIKOV function gives rise to an embedding Î½ : D Î» â†' C âˆž (X). We consider the supremum norm N sup on D Î» induced by Î½: For a discussion of L p -norms on X see Appendix A. From the Sobolev restriction theorem on X (for more details see Appendix B), it follows that N sup is bounded by some Sobolev norm S = S k on the space D Î» . Hence, the main properties of the norm N sup are: it is G-invariant and N sup â‰¤ S. We will show that there exists a maximal norm S G on the space D Î» satisfying these two conditions. This norm is defined in terms of the representation Ï€ Î» only and it is independent of the automorphic form picture. We then use the model D Î» of Ï€ Î» in order to prove the bound S G (v Îµ) â‰¤ C| ln Îµ|. The proof uses the standard method of dyadic decomposition from harmonic analysis; it is based on the observation that, in D Î» , the vector v Îµ is represented by a function which is roughly homogeneous. As a result we get a pointwise bound Remark. A new feature of our method, which seems to be absent in the classical approaches to automorphic forms, is the essential use of representation theory. First of all, in order to study the automorphic function Ï† that lives on the space Y , we pass to a bigger space, X, and work directly with the representation (Ï€, G, V) âŠ‚ C âˆž (X) which corresponds to Ï†. In some classical approaches, the space V is actually also present, albeit very implicitly. And when present, it appears only as a collection of vectors Ï€(h)Ï† created from the automorphic function Ï† by operators Ï€(h) corresponding to various functions (or distributions) h on G. Though, in principle, one can show that such functions exhaust V , in most cases it is very difficult to work with such an implicit description. In this paper we directly use the space V in order to prove Theorem 0.2. For example, the central technical result is the pointwise bound of the function u = Ï† Îµ âˆˆ V . This bound is proven in Section 5 by means of dyadic decomposition. The idea of the method is to break the function u into the sum of "pieces" u i âˆˆ V which we can move to a better position (for more detail see Â§ Â§5.2). We describe these u i using the explicit model D Î» of V . We do not know how to realize the u i 's in the form Ï€(h)Ï†. So we do not see how to prove this crucial estimate without using the space V as a whole. ANALYTIC CONTINUATION 335 0.6. Relation to L-functions. The main interest in triple products and their bounds stems from their relation to the theory of automorphic L-functions. A particular case of these triple products is the scalar product of Ï† 2 with the Eisenstein series E(s). This is the original example of Rankin and Selberg of the L-function associated to two cusp forms (see [B]). Namely, L(Ï† âŠ— Ï†, s) = g(s) Ï† 2 , E(s) , where g(s) is an explicit factor. M. Harris and S. Kudla ([HK]) discovered that such triple products are related to the special value at s = 1 2 of L(Ï†âŠ—Ï†âŠ—Ï† i , s). This gives further reason for the study of such triple products, at least when Ï† and Ï† i are holomorphic cusp forms for a congruence subgroup of a division algebra. 0.7. Bounds on Fourier coefficients of cusp forms. As we mentioned above, our result implies certain bounds for the Rankin-Selberg L-functions on the critical line. This, in turn, has implication for the classical problem of obtaining bounds of the Fourier coefficients of cusp forms. Recall the setting (see [Se], [G], [S]). Let Î“ be a nonuniform lattice in SL(2, R), which can be nonarithmetic (the standard example of a nonuniform lattice is Î“ = SL(2, Z)). Let 1 1 0 1 be a generator of its unipotent subgroup. Let Ï† be a cusp form with eigenvalue Âµ = 1âˆ'Î» 2 4 . We have then the following Fourier decomposition (see [B]): Ï†(x + iy) = n =0 a n y where K Î» 2 is the K-Bessel function. In order to study the coefficients a n , Rankin and Selberg introduced the series L(s) = n>0 |an| 2 n s , the Rankin-Selberg L-function (we assume that Ï† is real valued; hence, a n = a âˆ'n). The significance of this Dirichlet series is that it has an integral representation and as a result a spectral interpretation (as well as an analytic continuation!) which we will use. Let E(s) be the Eisenstein series associated to the cusp at âˆž. The series E(s) is unitary for Re(s) = 1/2 and L(s) = 2Ï€ s Î"(s) Î"(s/2) 2 Î"(s/2 + it)Î"(s/2 âˆ' it) Ï† 2 , E(s) ; hence, our method gives an upper bound for L(s). Namely, taking into account the asymptotic behavior of the Î"-function we obtain, for example, the following: Corollary 1.			
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