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Optimal Choice of Sample Fraction in Extreme-Value Estimation

ARNOLD L. M. DEKKERS

National Institute of Public Health and Environmental Protection, Bilthoven, The Netherlands

AND

LAURENS DE HAAN

Erasmus University, Rotterdam, The Netherlands

We study the asymptotic bias of the moment estimator $\hat{\gamma}_n$ for the extreme-value index $\gamma \in \mathscr{R}$ under quite natural and general conditions on the underlying distribution function. Furthermore the optimal choice for the sample franction in estimating γ is considered by minimizing the mean squared error of $\hat{\gamma}_n - \gamma$. The results cover all three limiting types of extreme-value theory. The connection between statistics and regular variation and Π-variation is handled in a systematic way.

1. INTRODUCTION

Suppose one is given a sequence $X_1, X_2, \ldots$ of i.i.d. observations from some unknown distribution function F. Suppose for some constants $a_n > 0$ and b_n and some $\gamma \in \mathscr{R}$

$$\lim_{n \to \infty} P\left\{\frac{\max\{X_1, X_2, \ldots, X_n\} - b_n}{a_n} \leqslant x\right\} = G_\gamma(x) \tag{1}$$

for all x, where $G_\gamma(x)$ is one of the extreme-value distributions, given by

$$G_\gamma(x) := \exp - (1 + \gamma x)^{-1/\gamma}. \tag{2}$$

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Here γ is a real parameter, x such that $1+\gamma x>0$. Interpret $(1+\gamma x)^{-1/\gamma}$ as e^{-x} for $\gamma=0$. The question is how to estimate γ, the extreme-value index, from a finite sample $X_1, X_2, ..., X_n$. If (1) holds, F is said to be in the domain of attraction of the generalized extreme-value distribution G_γ [notation $F\in\mathscr{D}(G_\gamma)$]. For the extreme-value distributions itself one has $G_\gamma\in\mathscr{D}(G_\gamma)$.

In the last decade much attention has been paid to the estimation of the tail-index of a distribution. This corresponds to estimating γ when $\gamma>0$. Most of the publications are based on the work of Pickands (1975) and Hill (1975).

Pickands proposed the following estimator for $\gamma\in\mathscr{R}$ and $1\leqslant k\leqslant[n/4]$

$$\hat{\gamma}_n^{(P)}:=(\log 2)^{-1}\log\frac{X_{(n-k,n)}-X_{(n-2k,n)}}{X_{(n-2k,n)}-X_{(n-4k,n)}},$$

where $X_{(1,n)}\leqslant X_{(2,n)}\leqslant\cdots\leqslant X_{(n,n)}$ are the ascending order statistics of $X_1, X_2, ..., X_n$. He proved weak consistency of the estimate.

Dekkers and de Haan (1989) gave quite natural and general conditions under which $\sqrt{k}\,(\hat{\gamma}_n^{(P)}-\gamma)$ is asymptotically normal. Conditions on $k=k(n)$ include $k=k(n)\to\infty$ and $k/n\to 0$ $(n\to\infty)$.

For γ positive, Hill introduced the estimator

$$M_n^{(1)}:=\frac{1}{k}\sum_{i=0}^{k-1}\log X_{(n-i,n)}-\log X_{(n-k,n)}$$

which involves all $k+1$ upper order statistics instead of only $X_{(n-k,n)}$, $X_{(n-2k,n)}$ and $X_{(n-4k,n)}$. Mason (1982) proved weak consistency of $M_n^{(1)}$ for any sequence $k=k(n)\to\infty$, $k(n)/n\to 0$ $(n\to\infty)$ and Deheuvels *et al.* (1988) proved also strong consistency for sequences $k(n)$, with $k/\log\log n\to\infty$ and $k/n\to 0$, $n\to\infty$. Under certain extra conditions $\sqrt{k}\,(M_n^{(1)}-\gamma)$ is asymptotically normal with mean zero and variance γ^2 (see Hall, 1982; Davis and Resnick, 1984; Csörgö and Mason, 1985; Häusler and Teugels, 1985; Goldie and Smith, 1987; and Dekkers *et al.* 1989).

Hall (1982) considered distribution functions F which satisfy

$$1-F(x)=Ax^{-1/\gamma}\{1+Bx^{-\beta}+o(x^{-\beta})\},\qquad x\to\infty,$$

for $\gamma>0$, $A>0$, $B\neq 0$, and $\beta>0$. He proved asymptotic normality for the Hill estimator and derived an optimal choice for k, the number of upper order statistics used in estimating γ, by minimizing the asymptotic mean squared error of $M_n^{(1)}$. Although he considered an important class of distribution functions, his approach is limited to only γ positive.

Using Pickands' well-known key idea [the conditional distribution function of $X-u$, given X exceeds threshold u, can be approximated by the generalized Pareto distribution (GPD)], Smith (1987) fits the GPD-distribution by the method of maximum likelihood. The shape-parameter of the fitted GPD-distribution is an estimator of γ. He obtains asymptotic normality for the MLE-estimators in case $\gamma > -1/2$ and under some extra conditions he obtains also the asymptotic bias of the estimators.

Dekkers *et al.* (1989) considered the problem how to estimate γ for general $\gamma \in \mathscr{R}$. They introduced the moment estimator given by

$$\gamma_n^{(M)} := M_n^{(1)} + 1 - \tfrac{1}{2}\{1 - (M_n^{(1)})^2/M_n^{(2)}\}^{-1}. \tag{3}$$

where $M_n^{(1)}$ is the Hill estimator and

$$M_n^{(2)} := \frac{1}{k}\sum_{i=0}^{k-1} \{\log X_{(n-i,n)} - \log X_{(n-k,n)}\}^2,$$

provided that $x^* = x^*(F) > 0$, which can always be achieved by a simple shift $[x^*(F) := \sup\{x \mid F(x) < 1\}]$. The moment estimator has some intuitive background (cf. Dekkers *et al.*, 1989, Sect. 6) and covers all limiting types of extreme-value theory. Under natural and general conditions the estimator has asymptotically a normal distribution.

All the mentioned estimators for γ have one common property. When the number of upper order statistics used in estimating γ is small, the variance of the estimator will be large. But on the other hand the use of a large number of upper order statistics will introduce a bias in the estimation in most cases. Balancing the variance and bias components will lead to an optimal choice for k. Therefore we want to study the bias of the moment estimator in a systematic way.

So the two main problems which return in all the work and where we like to focus on in this paper are

- how to choose the number of upper order statistics, k, involved in estimating γ,
- are the conditions in some way natural and do they cover all possibilities of tail behaviour?

In Section 2 we give more in detail some conditions and we claim that these conditions are quite natural and general (see de Haan and Stadtmüller, 1992). In Section 3 we study the moment-estimator for the cases $\gamma > 0$, $\gamma < 0$, and γ equals zero. Finally, we give some examples in Section 4.

2. Regular Variation, Π-variation, and Extreme-Value Theory

In this section we want to give some details how the tail behaviour of distribution function F can be translated into terms of the inverse function of $1/(1-F)$. Next we will formulate our "second order" conditions on F. Finally we will give a lemma which we need for minimizing the asymptotic mean squared error of $\hat{\gamma}_n$.

Define the function $U: \mathscr{R}^+ \to \mathscr{R}$ by

$$U(x) := \begin{cases} 0 & 0 \leqslant x < 1 \\ \left(\dfrac{1}{1-F}\right)^{\leftarrow}(x) & 1 \leqslant x \end{cases},$$

where the arrow indicates the inverse function, i.e., for $x \geqslant 1$ U is defined by $U(x) := \inf\{y \mid 1/(1-F(y)) \geqslant x\}$. Now the domain of attraction condition (1) can be stated in the following way in terms of U.

Lemma 2.1. *For a distribution function F holds $F \in \mathscr{D}(G_\gamma)$ if and only if there exists a positive function a_1 such that*

$$\lim_{t \to \infty} \frac{U(tx) - U(t)}{a_1(t)} = \frac{x^\gamma - 1}{\gamma}, \qquad x > 0, \tag{4}$$

where the right hand side of (4) *has to be interpreted as* $\log x$ *for* $\gamma = 0$.

Proof. Cf. de Haan (1984, Lemma 1).

Lemma 2.2. *For $\gamma > 0$,* (4) *is equivalent to*

$$\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^\gamma, \tag{5}$$

for all $x > 0$, i.e., U is regularly varying with index γ [notation $U(t) \in RV_\gamma$], and hence $a_1(t) \sim \gamma U(t)$, $t \to \infty$, i.e., $\lim_{t \to \infty} a_1(t)/(\gamma U(t)) = 1$.

For $\gamma < 0$, F has a finite right endpoint, so $U(\infty) = x^* < \infty$, and (4) *is equivalent to*

$$U(\infty) - U(t) \in RV_\gamma. \tag{6}$$

In this case $a_1(t) \sim -\gamma\{U(\infty) - U(t)\}$, $t \to \infty$.

Proof. Cf. de Haan (1984, Coro. 3).

We call (5) and (6) the first order regular variation conditions on U and for $\gamma = 0$ property (4) the first order Π-variation condition on U [notation $U \in \Pi(a_1)$].

In the following two lemmas the second order conditions are formulated and equivalent conditions are given. See also de Haan and Stadmüller (1992) for a complete theory of extended regular variation of second order.

LEMMA 2.3 (Second Order Regular Variation). *Suppose $\rho > 0$ and $c > 0$.*

1. *For $\gamma < 0$ the following conditions are equivalent [with either choice of sign]:*

(a) $$\pm\{x^{-1/\gamma}[1 - F(U(\infty) - x^{-1})] - c^{1/\gamma}\} \in RV_{-\rho}$$

(b) $$\mp\{t^{-\gamma}[U(\infty) - U(t)] - c\} \in RV_{\gamma\rho}.$$

For $U(\infty) > 0$ these conditions imply the following equivalent conditions:

(c) $$\pm\{x^{-1/\gamma}[1 - F(U(\infty)\, e^{-1/x})] - (c/U(\infty))^{1/\gamma}\} \in RV_{-\rho}$$

(d) $$\mp\{t^{-\gamma}[\log U(\infty) - \log U(t)] - c/U(\infty)\} \in RV_{\gamma\rho}.$$

2. *For $\gamma > 0$ the following conditions are equivalent [with either choice of sign]:*

(e) $$\pm\{x^{1/\gamma}(1 - F(x)) - c^{1/\gamma}\} \in RV_{-\rho}$$

(f) $$\pm\{t^{-\gamma}U(t) - c\} \in RV_{-\gamma\rho}$$

(g) $$\pm\{\log U(t) - \gamma \log t - \log c\} \in RV_{-\gamma\rho}.$$

Proof. See Appendix A.

Remark 2.4. Note that the conditions (d) and (g) are different, (g) is equivalent to (f), but (d) is not equivalent to (b). A counter example is the uniform distribution with $U(t) = 1 - 1/t$, which does not satisfy (b) although it satisfies (d) with $\gamma = 1$, $\rho = 1$, and $c = U(\infty) = 1$.

LEMMA 2.5 (Second Order Π-Variation). *Suppose the functions b_1, b_2, b_3, b_4, f, and α are positive.*

1. *For $\gamma < 0$ the following conditions are equivalent [with either choice of sign]:*

(a) $$\pm\{x^{-1/\gamma}[1 - F(U(\infty) - x^{-1})]\} \in \Pi$$

(b) $$\mp\{t^{-\gamma}[U(\infty) - U(t)]\} \in \Pi(b_1).$$

For $U(\infty) > 0$ these conditions imply the following equivalent conditions:

(c) $$\pm\{x^{-1/\gamma}[1 - F(U(\infty)\, e^{-1/x})]\} \in \Pi$$

(d) $$\mp\{t^{-\gamma}[\log U(\infty) - \log U(t)]\} \in \Pi(b_1/U(\infty)).$$

2. *For $\gamma=0$ the following conditions are equivalent with $\alpha(t)\to 0$, $t\to x^*$ and $b_2(t)\to 0$, $t\to\infty$ [with either choice of sign]:*

(e) $$\lim_{t\uparrow x^*}\left(\frac{1-F(\exp(t+xf(t)))}{1-F(\exp(t))}-e^{-x}\right)\Big/\alpha(t)=\frac{x^2}{2}e^{-x}$$

(f) $$\lim_{t\to\infty}\frac{\log U(tx)-\log U(t)-b_2(t)\log x}{b_3(t)}=-\frac{(\log x)^2}{2}$$

3. *For $\gamma>0$ the following conditions are equivalent [with either choice of sign]:*

(g) $$\pm\{x^{1/\gamma}(1-F(x))\}\in\Pi$$

(h) $$\pm t^{-\gamma}U(t)\in\Pi(b_4)$$

(i) $$\pm\{\log U(t)-\gamma\log t\}\in\Pi(b_4/(t^{-\gamma}U(t))).$$

Proof. For the proof we refer to the Appendix of Dekkers and de Haan (1989) and to Theorem 3.3 of Dekkers *et al.* (1989).

Remark 2.6. Note that all conditions imply $F\in\mathscr{D}(G_\gamma)$ for appropriate γ.

Remark 2.7. In the case of second order Π-variation with $\gamma=0$ we have in (e) only plus sign and in (f) only the minus sign, instead of both choices as for $\gamma\neq 0$. The reason is the following. Let $V(t):=\log U(t)$, then condition (f) implies for $x>1$ and $y>1$,

$$\begin{aligned}\frac{V(txy)-V(t)-b_2(t)\log xy}{b_3(t)}&=\frac{V(txy)-V(tx)-b^2(tx)\log y}{b_3(tx)}\cdot\frac{b_3(tx)}{b_3(t)}\\&\quad+\frac{V(tx)-V(t)-b_2(t)\log x}{b_3(t)}\\&\quad+\frac{b_2(tx)-b_2(t)}{b_3(t)}\log y.\end{aligned}\tag{7}$$

Now suppose that the left-hand side of (7) tends to $\pm(\log xy)^2/2$ and thus the right hand side converges also. So $(b_2(tx)-b_2(t))/b_3(t)$ converges to $\pm\log x$ and hence $\pm b_2\in\Pi(b_3)$. Note that $b_2(t)>0$ and $b_2(t)\to 0$, $t\to\infty$, which is not compatible with $b_2\in\Pi(b_3)$. This implies $-b_2\in\Pi(b_3)$ and therefore only the minus sign is possible in condition (f).

In the last part of this section we describe in a general way how to minimize the mean squared error

$$\frac{\sigma^2(\gamma)}{k}+f\left(\frac{n}{k}\right),$$

where $\sigma^2(\gamma)$ denotes the asymptotic variance of the estimator, n the sample size, k the number of used upper order statistics and f the bias squared, hence f is positive. When the bias is not equal to zero, the mean squared error can be minimized. Let k_o be the value for k for which the minimum is attained. If f is differentiable then $k_o = s^{\leftarrow}(\sigma^2(\gamma)/n)$, where s is defined as minus the first derivative of f, i.e., $-f'$.

In general $f \in RV_{-2\alpha}$ with $\alpha \geqslant 0$ and moreover for $\alpha = 0$, $f(t) \to 0$, $t \to \infty$. The following lemma about the inverse complementary function of f, shows that these conditions are already sufficient for obtaining the asymptotic value of k_o. For more information concerning the inverse complementary function of a regularly varying function, we refer to Geluk and de Haan (1987, Sect. II.1).

LEMMA 2.8. *Suppose $\alpha \geqslant 0$ and $f \in RV_{-\alpha}$. Moreover for $\alpha = 0$ suppose $f(t) \to 0$, $t \to \infty$ and f is asymptotic to a non-increasing function. There exists a positive decreasing function $s \in RV_{-(\alpha+1)}$, such that*

$$f(t) \sim \int_t^{\infty} s(u)\, du, \qquad t \to \infty. \tag{8}$$

Let f_c denote the inverse complementary function of f defined as

$$f_c(x) := \inf_{y>0} \{f(y) + xy\}, \qquad x > 0, \tag{9}$$

then $f_c(x)$ exists for sufficiently small x and

$$f_c(x) \sim \int_0^x s^{\leftarrow}(u)\, du, \qquad x \to 0,$$

where $s^{\leftarrow}$ is the generalized inverse function of s and $s^{\leftarrow} \in RV^0_{-1/(\alpha+1)}$, i.e., $\lim_{x \to 0} s^{\leftarrow}(xy)/s^{\leftarrow}(x) = y^{-1/(\alpha+1)}$ for $y > 0$.

The value $y_o(x)$ for which the infimum in (9) is attained, is determined asymptotically by $y_o(x) \sim s^{\leftarrow}(x)$, $x \to 0$.

Proof. For $\alpha = 0$ the conditions imply $-f$ is asymptotic to an element of Π (see Theorem B.1 of Appendix B, due to A. A. Balkema). For (8), see Proposition 1.7.3 [$\alpha > 0$] or Proposition 1.19.3 [$\alpha = 0$] of Geluk and de Haan (1987). Let $f_1(t) := \int_t^{\infty} s(u)\, du$, $c > 1$ and

$$0 < \varepsilon < \min\left(\sqrt{c - c^{-\alpha} + \left\{\frac{1 + c^{-\alpha}}{2}\right\}^2} - \frac{1 + c^{-\alpha}}{2}, \frac{c - c^{-\alpha}}{1 + c}\right),$$

then there exists $t_o(c)$ such that for $t > t_o(c)$

$$(1 - \varepsilon) f_1(t) \leqslant f(t) \leqslant (1 + \varepsilon) f_1(t)$$

and

$$(c^{-\alpha}-\varepsilon)\, f(t) \leqslant f(ct) \leqslant (c^{-\alpha}+\varepsilon)\, f(t),$$

hence $f(ct) \leqslant (c^{-\alpha}+\varepsilon)\, f(t) \leqslant (c^{-\alpha}+\varepsilon)(1+\varepsilon)\, f_1(t) \leqslant cf_1(t)$, since $(c^{-\alpha}+\varepsilon) \times (1+\varepsilon)-c<0$.

In a similar way, $f_1(ct) \leqslant f(ct)/(1-\varepsilon) \leqslant (c^{-\alpha}+\varepsilon)\, f(t)/(1-\varepsilon) \leqslant c(f(t))$ and hence

$$\frac{1}{c}\int_{ct}^{\infty} s(u)\, du \leqslant f(t) \leqslant c\int_{t/c}^{\infty} s(u)\, du,$$

which implies

$$\inf_{y>0}\left\{\frac{1}{c}\int_{cy}^{\infty} s(u)\, du + xy\right\} \leqslant f_c(x) \leqslant \inf_{y>0}\left\{c\int_{y/c}^{\infty} s(u)\, du + xy\right\},$$

and thus for all $c>1$

$$\frac{1}{c}\int_0^x s^{\leftarrow}(u)\, du \leqslant f_c(x) \leqslant c\int_0^x s^{\leftarrow}(u)\, du, \qquad x\to 0.$$

We have also proved $y_o(x) \sim s^{\leftarrow}(x)$, $x \to 0$, since $s^{\leftarrow}(x)/c \leqslant y_o(x) \leqslant cs^{\leftarrow}(x)$ for all $c>1$.

3. Optimal Choice of Sample Fraction for the Moment Estimator

In this section we will state our main results for the optimal choice of k and the corresponding bias for the moment estimator.

Let $X_1, X_2, \ldots, X_n$ be n i.i.d. random variables of an unknown distribution function F, with $F \in \mathscr{D}(G_\gamma)$, and let $Y_1, Y_2, \ldots, Y_n$ be n i.i.d. random variables of distribution function $1-x^{-1}$, $(x \geqslant 1)$. Note that $X_{(n-i,n)} =^d U(Y_{(n-i,n)})$ for $0 \leqslant i \leqslant n$. The next lemma gives important properties of $Y_1, Y_2, \ldots, Y_n$ in relation to the moment estimator $\hat{\gamma}_n$ as defined in (3).

Then we give the main results for distributions with a second order regularly varying tail [Theorem 3.2 for $\gamma<0$ and theorem 3.4 for $\gamma>0$]. In Theorem 3.6 we will consider distribution functions with a second order Π-varying tail.

Lemma 3.1. *Let $Y_{(1,n)} \leqslant Y_{(2,n)} \leqslant \cdots \leqslant Y_{(n,n)}$ be the order statistics of $Y_1, Y_2, \ldots Y_n$. Let $0<k(n)<n$ and $k(n)\to\infty$, $n\to\infty$, then*

1. *for* $n \to \infty$, $Y_{(n-k,n)}/(n/k) \to 1$ *in probability.*
2. *for* $n \to \infty$,

$$P_n^o := \left(\frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \log Y_{(n-i,n)} - \log Y_{(n-k(n),n)} - 1 \right)$$

and

$$Q_n^o := \left(\frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \{\log Y_{(n-i,n)} - \log Y_{(n-k(n),n)}\}^2 - 2 \right),$$

$\sqrt{k}\,(P_n^o, Q_n^o)$ *is asymptotically normal with means zero, variances* 1 *and* 20, *respectively, and covariance* 4.

3. *for* $\gamma < 0$, $n \to \infty$,

$$P_n := \left(\frac{1}{k(n)} \sum_{i=0}^{k(n)-1} 1 - \left(\frac{Y_{(n-i,n)}}{Y_{(n-k(n),n)}} \right)^{\gamma} + \frac{\gamma}{1-\gamma} \right),$$

and

$$Q_n := \left(\frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \left\{ 1 - \left(\frac{Y_{(n-i,n)}}{Y_{(n-k(n),n)}} \right)^{\gamma} \right\}^2 - \frac{2\gamma^2}{(1-\gamma)(1-2\gamma)} \right),$$

$\sqrt{k}\,(P_n, Q_n)$ *is asymptotically normal with means zero and covariance matrix*

$$\frac{\gamma^2}{(1-\gamma)^2(1-2\gamma)} \begin{pmatrix} 1 & \dfrac{-4\gamma}{1-3\gamma} \\ \dfrac{-4\gamma}{1-3\gamma} & \dfrac{4\gamma^2(5-11\gamma)}{(1-2\gamma)(1-3\gamma)(1-4\gamma)} \end{pmatrix}.$$

Proof. Cf. Lemma 3.4 Dekkers *et al.* (1989).

THEOREM 3.2. *Suppose* $\gamma < 0$, $U(\infty) > 0$, *and condition* (d) *of Lemma* 2.3 *holds for* $\rho \neq 1$. *Define for* $t > 0$,

$$b(t) := \frac{c}{U(\infty)} \frac{-\gamma}{1-\gamma} t^{\gamma} + \frac{U(\infty)}{c} \frac{\gamma(1-\gamma)(1-2\gamma)\,\rho(1+\rho)}{\{1-\gamma(1+\rho)\}\{1-\gamma(2+\rho)\}} \times \left[t^{-\gamma}\{\log U(\infty) - \log U(t)\} - \frac{c}{U(\infty)} \right]. \tag{10}$$

Determine $k_o = k_o(n)$ *such that the asymptotic second moment of* $\hat{\gamma}_n - \gamma$ *is minimal and let* $\hat{\gamma}_{n,o}$ *be the corresponding estimator, then*

$$\sqrt{k_o(n)}\,(\hat{\gamma}_{n,o} - \gamma) \xrightarrow{d} N(b, \sigma^2(\gamma)),$$

where the asymptotic bias b *and variance* $\sigma^2(\gamma)$ *are given by*

$$b = \operatorname{sign}(b(t)) \sqrt{\frac{\sigma^2(\gamma)}{-2\gamma \min(1, \rho)}},$$

for t *sufficiently large, and*

$$\sigma^2(\gamma) := (1-\gamma)^2 (1-2\gamma) \left(4 - 8\,\frac{1-2\gamma}{1-3\gamma} + \frac{(5-11\gamma)(1-2\gamma)}{(1-3\gamma)(1-4\gamma)}\right). \qquad (11)$$

Moreover $k_o(n) = n/s^{\leftarrow}(1/n)(1+o(1)) \in RV_{(2\gamma\min(1,\rho))/(2\gamma\min(1,\rho)-1)}$, $n \to \infty$, *where* $s^{\leftarrow}$ *is the inverse function of* s, *with* s *given by*

$$\frac{\{b(t)\}^2}{\sigma^2(\gamma)} = \int_t^{\infty} s(u)\,du(1+o(1)), \qquad t \to \infty.$$

The existence of such function s *is guaranteed by the fact that* $b^2(t)$ *is regularly varying with index* $2\gamma \min(1, \rho)$.

Proof. Assume $\gamma < 0$ and (d) of Lemma 2.3 holds. Define $c_1 := c/U(\infty)$ and let $a(t) := t^{-\gamma}\{\log U(\infty) - \log U(t)\} - c_1$ then, since $|a(t)| \in RV_{\gamma\rho}$, for $x > 0$

$$\begin{aligned}
&\log U(tx) - \log U(t)\\
&\quad = \log U(\infty) - \log U(t) - \{\log U(\infty) - \log U(tx)\}\\
&\quad = t^{\gamma}[t^{-\gamma}\{\log U(\infty) - \log U(t)\} - x^{\gamma}(tx)^{-\gamma}\{\log U(\infty) - \log U(tx)\}]\\
&\quad = c_1 t^{\gamma}(1 - x^{\gamma}) + t^{\gamma} a(t)\left\{1 - x^{\gamma}\frac{a(tx)}{a(t)}\right\}\\
&\quad = c_1 t^{\gamma}(1 - x^{\gamma}) + t^{\gamma} a(t)\{1 - x^{\gamma} x^{\gamma\rho}\} + o(t^{\gamma} a(t)), \qquad t \to \infty.
\end{aligned}$$

Also

$$\frac{(Y_{(n-k,n)})^{\gamma}\, a(Y_{(n-k,n)})}{(n/k)^{\gamma}\, a(n/k)} \to 1, \qquad n \to \infty,$$

in probability by Lemma 3.1 [we will use the notation $(Y_{(n-k,n)})^{\gamma} a(Y_{(n-k,n)} = (n/k)^{\gamma} a(n/k) \times (1+o_p(1))$]. Now one obtains by straightforward calculations using Lemma 3.1

$$
\begin{aligned}
M_n^{(1)} &= \frac{1}{k}\sum_{i=0}^{k-1} \log X_{(n-i,n)} - \log X_{(n-k,n)} \\
&\stackrel{d}{=} \frac{1}{k}\sum_{i=0}^{k-1} \log U\left(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}} Y_{(n-k,n)}\right) - \log U(Y_{(n-k,n)}) \\
&= (Y_{(n-k,n)})^{\gamma} \frac{1}{k}\sum_{i=0}^{k-1}\left[c_1\left\{1-\left(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\right)^{\gamma}\right\}\right. \\
&\quad \left. + a(Y_{(n-k,n)})\left\{1-\left(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\right)^{\gamma(1+\rho)}\right\}\right] + o_p\left(\left(\frac{n}{k}\right)^{\gamma} a\left(\frac{n}{k}\right)\right) \\
&\stackrel{d}{=} (Y_{(n-k,n)})^{\gamma}\left[\frac{-\gamma c_1}{1-\gamma} + c_1\frac{P_n}{\sqrt{k}} + d_1 a(Y_{(n-k,n)})\right] \\
&\quad + o_p\left(\left(\frac{n}{k}\right)^{\gamma} a\left(\frac{n}{k}\right)\right),
\end{aligned}
\tag{12}
$$

where

$$
d_1 := \int_1^{\infty} (1 - x^{\gamma(1+\rho)})\frac{dx}{x^2} = \frac{-\gamma(1+\rho)}{1-\gamma(1+\rho)}
$$

and hence

$$
\begin{aligned}
\{M_n^{(1)}\}^2 &\stackrel{d}{=} (Y_{(n-k,n)})^{2\gamma}\left[\frac{\gamma^2 c_1^2}{(1-\gamma)^2} - \frac{2\gamma c_1^2}{1-\gamma}\frac{P_n}{\sqrt{k}} - \frac{2\gamma c_1}{1-\gamma}\cdot d_1 \cdot a(Y_{(n-k,n)})\right] \\
&\quad + o_p\left(\left(\frac{n}{k}\right)^{2\gamma} a\left(\frac{n}{k}\right)\right).
\end{aligned}
\tag{13}
$$

Similarly one gets

$$
\begin{aligned}
M_n^{(2)} &\stackrel{d}{=} (Y_{(n-k,n)})^{2\gamma}\left[c_1^2\frac{2\gamma^2}{(1-\gamma)(1-2\gamma)} + c_1^2\frac{Q_n}{\sqrt{k}} + d_2\, a(Y_{(n-k,n)})\right] \\
&\quad + o_p\left(\left(\frac{n}{k}\right)^{2\gamma} a\left(\frac{n}{k}\right)\right),
\end{aligned}
\tag{14}
$$

with

$$
\begin{aligned}
d_2 &:= 2c_1\int_1^{\infty} (1 - x^{\gamma} - x^{\gamma(1+\rho)} + x^{\gamma(2+\rho)})\frac{dx}{x^2} \\
&= \frac{2c_1\gamma^2(1+\rho)(2-\gamma(2+\rho))}{(1-\gamma)\{1-\gamma(1+\rho)\}\{1-\gamma(2+\rho)\}}.
\end{aligned}
$$

Combining finally (12), (13) and (14):

$$\hat{\gamma}_n = M_n^{(1)} + \frac{1}{2}\frac{M_n^{(2)} - 2\{M_n^{(1)}\}^2}{M_n^{(2)} - \{M_n^{(1)}\}^2} \stackrel{d}{=} \gamma + \frac{R_n}{\sqrt{k}} + b\left(\frac{n}{k}\right) + o_p\left(b\left(\frac{n}{k}\right)\right),$$

with $b(t)$ as defined in (10) and

$$R_n := \frac{1}{2}\frac{(1-\gamma)^2(1-2\gamma)^2}{\gamma^2} Q_n + \frac{2(1-\gamma)^2(1-2\gamma)}{\gamma} P_n,$$

which is asymptotically normal with mean zero and variance $\sigma^2(\gamma)$ as defined in (11). Hence the asymptotic mean squared error of $\hat{\gamma}_n$ equals

$$\frac{\sigma^2(\gamma)}{k} + \left\{b\left(\frac{n}{k}\right) + o\left(b\left(\frac{n}{k}\right)\right)\right\}^2.$$

Write $r := n/k$. We are interested in the optimization problem

$$\inf_r \left\{\frac{r}{n} + \frac{\{b(r)\}^2}{\sigma^2(\gamma)} + o(\{b(r)\}^2)\right\} \sim \inf_r \left\{\frac{r}{n} + \frac{\{b(r)\}^2}{\sigma^2(\gamma)}\right\}. \tag{15}$$

The asymptotic equality in (15) follows from Lemma 2.8. Define $f(t) := \{b(t)\}^2/\sigma^2(\gamma)$ then $f \in RV_{2\gamma\rho_1}$ with $\rho_1 := \min(1, \rho)$, since $|b(t)| \in RV_{\gamma\min(1,\rho)}$, and so by Lemma 2.8 there exists a positive function $s \in RV_{2\gamma\rho_1 - 1}$ such that

$$\frac{\{b(t)\}^2}{\sigma^2(\gamma)} = \int_t^\infty s(u)\,du(1+o(1)), \qquad t \to \infty. \tag{16}$$

Let r_o denote the optimal value for r in (15), then [again by Lemma 2.8] $r_o(n) = s^{\leftarrow}(1/n)(1+o(1))$, $n \to \infty$, where $s^{\leftarrow}(1/n) \in RV_{1/(1-2\gamma\rho_1)}$ and hence $k_o(n) = n/s^{\leftarrow}(1/n) \times (1+o(1)) \in RV_{(2\gamma\rho_1)/(2\gamma\rho_1 - 1)}$. Note that $r_o \to \infty$ $(n \to \infty)$ and substitution of $t = n/k_o(n)$ in (16) gives [all the o-terms are regularly varying with index $2\gamma\rho_1$]

$$\begin{aligned}
\frac{\{b(n/k_o(n))\}^2}{\sigma^2(\gamma)} &= \int_{r_o}^\infty s(u)\,du \cdot (1+o(1)) \\
&= \frac{1}{k_o} \cdot \frac{\int_{r_o}^\infty s(u)\,du}{r_o s(r_o)} \cdot (1+o(1)) \\
&= \frac{1}{k_o} \cdot \frac{1}{-2\gamma\rho_1} \cdot (1+o(1)), \qquad n \to \infty,
\end{aligned}$$

since $s \in RV_{2\gamma\rho_1 - 1}$ (cf. Theorem 1.4 in Geluk and de Haan (1987)) and hence

$$b\left(\frac{n}{k_o}\right) = \frac{\operatorname{sign}(b(t))}{\sqrt{k_o}} \cdot \sqrt{\frac{\sigma^2(\gamma)}{-2\gamma \min(1, \rho)}} \cdot (1 + o(1)), \qquad n \to \infty.$$

This completes the proof.

Remark 3.3. The above theorem holds also for $\rho = 1$ under the extra condition $|b(t)| \in RV_\gamma$. This condition is not necessarily satisfied because in spite of the fact that both terms of $b(t)$ in (10) are regularly varying with index γ, they may not have the same sign. In this case the theorem holds also but now with bias b equal to $b = \operatorname{sign}(b(t)) \sqrt{\sigma^2(\gamma)/(-2\gamma\rho)}$, where ρ is the index of $b(t)$. The uniform distribution is an example, for which $\rho = 1$ and $b(t)$ is regularly varying but with index 2.

THEOREM 3.4. *Suppose $\gamma > 0$, condition* (g) *of Lemma* 2.3 *holds for $(1 - \gamma)\rho \neq 1$ and define for $t > 0$*

$$b(t) := \frac{\gamma\rho[(1 - \gamma)\rho - 1]}{(1 + \gamma\rho)^2} \{\log U(t) - \gamma \log t - \log c\}.$$

Determine $k_o = k_o(n)$ such that the asymptotic second moment of $\hat{\gamma}_n - \gamma$ is minimal and let $\hat{\gamma}_{n,o}$ be the corresponding estimator, then

$$\sqrt{k_o}\,(\hat{\gamma}_{n,o} - \gamma) \xrightarrow{d} N(b, 1 + \gamma^2),$$

where b denotes the bias given by

$$b = \operatorname{sign}(b(t)) \sqrt{\frac{1 + \gamma^2}{2\gamma\rho}},$$

for t sufficiently large.

Moreover $k_o(n) = n/s^{\leftarrow}(1/n)(1 + o(1))$, $n \to \infty$, where $s^{\leftarrow}$ is the inverse function of s, with s given by

$$\frac{\{b(t)\}^2}{1 + \gamma^2} = \int_t^\infty s(u)\, du \cdot (1 + o(1)), \qquad t \to \infty$$

and furthermore $k_o(n) \in RV_{(2\gamma\rho)/(2\gamma\rho + 1)}$.

Proof. Suppose $\gamma > 0$ and suppose that condition (g) of Lemma 2.3 holds. Define $a(t) := \log U(t) - \gamma \log t - \log c$. Since $|a(t)| \in RV_{-\gamma\rho}$, for $x > 0$,

$$\begin{aligned}
&\log U(tx) - \log U(t) \\
&\quad = \log U(tx) - \gamma \log tx - \log c - \{\log U(t) - \gamma \log t - \log c\} + \gamma \log x \\
&\quad = \gamma \log x + (x^{-\gamma\rho} - 1)\, a(t)(1 + o(1)), \qquad t \to \infty.
\end{aligned}$$

One obtains in a similar way as before

$$\begin{aligned}M_n^{(1)} &\stackrel{d}{=} \frac{1}{k}\sum_{i=0}^{k-1} \log U\left(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}} Y_{(n-k,n)}\right) - \log U(Y_{(n-k,n)}) \\ &= \gamma + \frac{1}{k}\sum_{i=1}^{k-1}\left[\gamma\left(\log\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}} - 1\right)\right. \\ &\qquad \left. + a(Y_{(n-k,n)})\left\{\left(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\right)^{-\gamma\rho} - 1\right\}\right] + o_p\left(a\left(\frac{n}{k}\right)\right) \\ &= \gamma + \gamma\frac{P_n^o}{\sqrt{k}} + d_1 a(Y_{(n-k,n)}) + o_p\left(a\left(\frac{n}{k}\right)\right),\end{aligned} \tag{17}$$

by Lemma 3.1, with

$$d_1 := \int_1^\infty (x^{-\gamma\rho} - 1)\frac{dx}{x^2} = -\gamma\rho/(1+\gamma\rho)$$

(cf. Proof of Lemma 3.4 in Dekkers *et al.* (1989)] and hence

$$(M_n^{(1)})^2 = \gamma^2 + 2\gamma^2\frac{P_n^o}{\sqrt{k}} + 2\gamma\, d_1 a(Y_{(n-k,n)}) + o_p\left(a\left(\frac{n}{k}\right)\right). \tag{18}$$

Furthermore

$$\{\log U(tx) - \log U(t)\}^2 = \{\gamma \log x^2\}^2 + 2\gamma(x^{-\gamma\rho} - 1)(\log x)\, a(t) + o(a(t)),$$

$t \to \infty$ and hence

$$\begin{aligned}M_n^{(2)} &\stackrel{d}{=} \frac{1}{k}\sum_{i=0}^{k-1}\left[\gamma^2\left\{\log\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\right\}^2 + 2\gamma\left\{\left(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\right)^{-\gamma\rho} - 1\right\}\right. \\ &\qquad \left.\times \log\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\, a(Y_{(n-k,n)})\right] + o_p\left(a\left(\frac{n}{k}\right)\right) \\ &\stackrel{d}{=} 2\gamma^2 + \gamma^2\frac{Q_n^o}{\sqrt{k}} + d_2\, a(Y_{(n-k,n)}) + o_p\left(a\left(\frac{n}{k}\right)\right),\end{aligned} \tag{19}$$

where

$$d_2 := 2\gamma\int_1^\infty (x^{-\gamma\rho} - 1)\log x\frac{dx}{x^2} = -2\gamma^2\rho(2+\gamma\rho)(1+\gamma\rho)^{-2}$$

and where (P_n^o, Q_n^o) are asymptotically normal distributed as in Lemma 3.1. By combining (18), (19), and (20) one obtains

$$\begin{aligned}\hat{\gamma}_n &= \gamma + \gamma \frac{P_n^o}{\sqrt{k}} + d_1\, a(Y_{(n-k,n)}) + o_p\left(a\left(\frac{n}{k}\right)\right) \\ &\quad + \{2\gamma^2 + \gamma^2(Q_n^o/\sqrt{k}) + d_2 a(Y_{(n-k,n)}) - 2\gamma^2 \\ &\quad - 4\gamma^2(P_n^o/\sqrt{k}) - 4\gamma\, d_1 a(Y_{(n-k,n)})\}\{2\gamma^2[2 + (Q_n^o/\sqrt{k}) \\ &\quad + (d_2/\gamma^2)\, a(Y_{(n-k,n)}) - 1 - 2(P_n^o/\sqrt{k}) - (2d_1/\gamma)\, a(Y_{(n-k,n)})]\}^{-1} \\ &= \gamma + \frac{Q_n^o}{2\sqrt{k}} + (\gamma - 2)\frac{P_n^o}{\sqrt{k}} + \left(\frac{d_2}{2\gamma^2} + \frac{\gamma-2}{\gamma} d_1\right) a(Y_{(n-k,n)} + o_p\left(a\left(\frac{n}{k}\right)\right) \\ &= \gamma + \frac{R_n^o}{\sqrt{k}} + b\left(\frac{n}{k}\right) + o_p\left(b\left(\frac{n}{k}\right)\right),\end{aligned}$$

with R_n^o asymptotically normal with mean zero and variance $1 + \gamma^2$, and where $|b| \in RV_{-\gamma\rho}$ for $(1-\gamma)\rho \neq 1$. The rest of the proof is omitted since it follows the same line as the previous one.

Remark 3.5. In order to calculate the asymptotic bias for $(1-\gamma)\rho = 1$, one has to impose further conditions.

In the next theorem the case of second order Π-variation is considered. The conditions and the proofs are slightly different for all the three cases $\gamma < 0$, $\gamma = 0$ and $\gamma > 0$.

THEOREM 3.6. *Suppose one of the following second order Π-variation conditions of Lemma* 2.5 *holds*: (d) $[\gamma < 0]$, (f) $[\gamma = 0]$, or (i) $[\gamma > 0]$. *Define for $t > 0$, the function b as follows*

$$b(t) := \begin{cases} b_1(t)/[t^{-\gamma}\{\log U(\infty) - \log U(t)\}], & \gamma < 0 \\ b_2(t) - b_3(t)/b_2(t), & \gamma = 0 \\ b_4(t)/\{\log U(t) - \gamma \log t\}, & \gamma > 0, \end{cases}$$

and assume that b^2 is asymptotic to a non-increasing function and b_2 and b_3/b_2 are not of the same order. Determine $k_o = k_o(n)$ such that the asymptotic second moment of $\hat{\gamma}_n - \gamma$ is minimal and let $\hat{\gamma}_{n,o}$ be the corresponding estimator. Then for $\gamma \in \mathscr{R}$

$$\sqrt{k_o}\,(\hat{\gamma}_{n,o} - \gamma) - b_n \xrightarrow{d} N(0, \sigma^2(\gamma)), \tag{20}$$

with variance

$$\sigma^2(\gamma) := \begin{cases} 1+\gamma^2, & \gamma \geqslant 0 \\ (1-\gamma)^2\,(1-2\gamma)\left(4-8\,\dfrac{1-2\gamma}{1-3\gamma}+\dfrac{(5-11\gamma)(1-2\gamma)}{(1-3\gamma)(1-4\gamma)}\right) & \gamma \leqslant 0 \end{cases} \tag{21}$$

and where b_n denotes the bias which is a slowly varying sequence and tends to infinity for $n \to \infty$. Moreover k_o is a slowly varying sequence.

Remark 3.7. Note that (21) implies $\sqrt{k_o}\,(\hat{\gamma}_{n,o}-\gamma)/b_n \to 1$, $n \to \infty$ in probability. Hence the optimal rate of convergence of $\hat{\gamma}_n \to \gamma$ is given by $b_n/\sqrt{k_o}$.

Remark 3.8. In case $b_3(t) = [b_2(t)]^2\,(1+o(1))$, $t \to \infty$, we are in a similar situation as in Theorem 3.2 with $\rho = 1$. In this case one has to consider the asymptotic expansion of $b(t)$ and the proof of the theorem to obtain an expression for the bias. An example is the exponential distribution and the Gumbel distribution [cf. Section 4].

Proof. For $\gamma < 0$ we give the proof for the plus sign in (d) of Lemma 2.5. The condition implies for

$$\begin{aligned} \log U(tx) - \log U(t) &= \{\log U(\infty) - \log U(t)\} \\ &\quad \times [1 - x^\gamma - (x^\gamma \log x)\, b(t)(1+o(1))], \qquad t \to \infty, \end{aligned}$$

where $|b| \in RV_0$ and $b(t) \to 0$, $t \to \infty$. Now one obtains

$$\begin{aligned} \hat{\gamma}_n &= \gamma + \frac{R_n}{\sqrt{k}} + \frac{-\gamma}{1-\gamma}\,\{\log U(\infty) - \log U(Y_{(n-k,n)})\} \\ &\quad + b(Y_{(n-k,n)}) + o_p\left(b\left(\frac{n}{k}\right)\right) \\ &= \gamma + \frac{R_n}{\sqrt{k}} + b\left(\frac{n}{k}\right) + o_p\left(b\left(\frac{n}{k}\right)\right), \end{aligned}$$

where R_n is asymptotically normal with mean zero and variance $\sigma^2(\gamma)$. The last approximation is valid since $\log U(\infty) - \log U(Y_{(n-k,n)})$ is of lower order than b, $|b| \in RV_0$ and $Y_{(n-k,n)}/(n/k) \to 1$ in probability. The mean squared error of $\hat{\gamma}_n - \gamma$ equals

$$\frac{\sigma^2(\gamma)}{k} + \left\{b\left(\frac{n}{k}\right)\right\}^2 (1+o(1)), \qquad n \to \infty.$$

Write $r := n/k$. We are interested in the optimization problem

$$\inf_r \left\{ \frac{r}{n} + \frac{\{b(r)\}^2}{\sigma^2(\gamma)} (1+o(1)) \right\} \sim \inf_r \left\{ \frac{r}{n} + \frac{\{b(r)\}^2}{\sigma^2(\gamma)} \right\}, \tag{22}$$

with $b^2(t) \to 0$, $t \to \infty$. Hence the asymptotic equality in (23) follows from Lemma 2.8 and by the same lemma there exists a positive function $s \in RV_{-1}$, such that

$$\frac{\{b(t)\}^2}{\sigma^2(\gamma)} = \int_t^\infty s(u)\, d(u) \cdot (1+o(1)), \qquad t \to \infty. \tag{23}$$

Let r_o denote the optimal value for r in (23), then [again by Lemma 2.8] $r_o(n) = s^{\leftarrow}(1/n)(1+o(1))$, $n \to \infty$, where $s^{\leftarrow} \in RV_{-1}$. Note that $r_o \to \infty$ $(n \to \infty)$ and $k_o(n) = n/s^{\leftarrow}(1/n)(1+o(1)) \in RV_0$. Substitution of $t = n/k_o$ in (23) gives

$$\begin{aligned} \frac{\{b(n/k_o)\}^2}{\sigma^2(\gamma)} &= \int_{n/k_o}^\infty s(u)\, du \cdot (1+o(1)) \\ &= \frac{1}{k_o} \cdot \frac{\int_{s^{\leftarrow}(1/n)}^\infty s(u)\, du}{s^{\leftarrow}(1/n)/n} \cdot (1+o(1)), \qquad n \to \infty. \end{aligned} \tag{24}$$

The fraction in (24) tends to infinity [cf. Geluk and de Haan, 1987, Rmk 1 following Coro. 1.18]. Hence the asymptotic bias of $\sqrt{k_o}\,(\hat{\gamma}_{n,o} - \gamma)$ equals

$$b_n = \operatorname{sign}(b(t)) \left(\frac{\sigma^2(\gamma) \int_{s^{\leftarrow}(1/n)}^\infty s(u)\, du}{s^{\leftarrow}(1/n)/n} \right)^{1/2} (1+o(1)), \qquad n \to \infty,$$

where $|b_n|$ is slowly varying and tends to infinity for $n \to \infty$.

For $\gamma = 0$ condition (f) of Lemma 2.5 implies for $x > 1$

$$\begin{aligned} &\log U(tx) - \log U(t) \\ &\quad = b_2(t)[\log x - \tfrac{1}{2}(\log x)^2\, [b_3(t)/b_2(t)](1+o(1))], \qquad t \to \infty \end{aligned}$$

and hence

$$\begin{aligned} \hat{\gamma}_n &= b_2(Y_{(n-k,n)}) - 2\frac{P_n^o}{\sqrt{k}} + \frac{Q_n^o}{2\sqrt{k}} \\ &\quad - b_3(Y_{(n-k,n)})/b_2(Y_{(n-k,n)}) + o_p\left(a\left(\frac{n}{k}\right)\right) \\ &= \frac{R_n}{\sqrt{k}} + b\left(\frac{n}{k}\right) + o_p\left(b\left(\frac{n}{k}\right)\right), \end{aligned}$$

where R_n is asymptotically standard normal and $b_3(t) \sim [b_2(t)]^2$, $t \to \infty$, is excluded. The rest of the proof is as before and is therefore omitted.

For $\gamma > 0$ we give the proof with a plus sign in condition (i) of Lemma 2.5 and hence

$$\log U(tx) - \log U(t) = \gamma \log x + b(t) \log x(1 + o(1)), \qquad t \to \infty.$$

Similar calculations as before give

$$\hat{\gamma}_n = \gamma + \frac{R_n}{\sqrt{k}} + b\left(\frac{n}{k}\right)(1 + o_p(1)),$$

where R_n is asymptotically normal with mean zero and variance $\sigma^2(\gamma)$ as defined in (22). The rest of the proof is omitted since it follows the same line as the part for $\gamma < 0$.

4. Examples

In this section we discuss the above results applied to some distribution functions.

4.1. *Uniform Distribution*

The uniform distribution does not satisfy condition (b) of Lemma 2.3 since $U(t) = 1 - 1/t$, $t \to \infty$. But the uniform distribution function satisfies condition (d) of Lemma 2.3 with $\gamma = -1$, $\rho = 1$, $U(\infty) = c = 1$ and hence $t^{-\gamma}\{\log U(\infty) - \log U(t)\} - c/U(\infty) = t\{-\log(1 - 1/t)\} - 1$, which leads to $b_3(t) = 1/(2t) - [1/(2t) + 1/(3t^2)(1 + o(1))] \in RV_{-2}$. So $b(t) = -1/(3t^2)(1 + o(1))$, $t \to \infty$. The asymptotic bias of $\hat{\gamma}_{n,o} - \gamma$ is equal to $-\sqrt{6/5}$ and moreover $k_o(n) = (27/10)^{1/5} \cdot n^{4/5}(1 + o(1))$, $n \to \infty$.

4.2. *Cauchy Distribution*

Define

$$F(x) := \frac{1}{2} + \frac{1}{\pi} \arctan x, \qquad x \in \mathscr{R},$$

the Cauchy distribution function. Then

$$U(t) = \tan\left(\frac{\pi}{2} - \frac{\pi}{t}\right) = \frac{t}{\pi}\left\{1 - \frac{\pi^2}{3t^2} + o(t^{-2})\right\}, \qquad t \to \infty.$$

The Cauchy distribution satisfies the condition of Theorem 3.4 with $\gamma = 1$, $c = 1/\pi$ and $\rho = 2$. The bias b of $\sqrt{k_o}\,(\hat{\gamma}_{n,o} - \gamma)$ equals $(1/2)\sqrt{2}$ and $k_o(n) \in RV_{4/5}$, or more precisely

$$b(t) = \frac{-2}{9}\log(\pi t^{-1}U(t)) = \frac{2\pi^2}{27}t^{-2} + o(t^{-2}), \qquad t \to \infty$$

and hence $s(t) = 2^3 \cdot 3^{-6} \cdot \pi^4 \cdot t^{-5} + o(t^{-5})$, $t \to \infty$. One obtains $s^{\leftarrow}(t) = 2^{3/5} \cdot 3^{-6/5} \cdot \pi^{4/5} \cdot t^{-1/5} \cdot (1 + o(1))$, $t \to \infty$ and finally $k_o(n) = 2^{-3/5} \cdot 3^{6/5} \cdot (n/\pi)^{4/5} \cdot (1 + o(1))$, $n \to \infty$.

4.3. *Exponential Distribution*

The exponential distribution satisfies condition (f) of Lemma 2.5 with $U(t) = \log t$. Note that for $x > 0$

$$\begin{aligned} &\log U(tx) - \log U(t) \\ &\quad = \frac{\log x}{\log t} - \frac{1}{2}\left(\frac{\log x}{\log t}\right)^2 + \frac{1}{3}\left(\frac{\log x}{\log t}\right)^3 (1 + o(1)), \qquad t \to \infty, \end{aligned} \tag{25}$$

and hence $b_2(t) = 1/(\log t)$ and $b_3(t) = 1/(\log t)^2$. Therefore b_2 equals b_3/b_2 and Theorem 3.6 cannot be used directly.

4.4. *Generalized Extreme-Value Distribution*

Let G_γ denote the GEV-distribution as defined in (2), then $U(t) = (1/\gamma)\{[-\log(1 - t^{-1})]^{-\gamma} - 1\}$.

For $\gamma < 0$ holds $U(\infty) = 1/-\gamma > 0$ and $t^{-\gamma}[\log U(\infty) - \log U(t)] - c/U(\infty) = (1/2)[-\gamma t^{-1} + t^{\gamma}](1 + o(1))$, $t \to \infty$, hence U satisfies the condition of Theorem 3.2 with $c = 1/(-\gamma)$ and $\rho = \min(1, 1/(-\gamma))$. The bias b of $\sqrt{k_o}\,(\hat{\gamma}_{n,o} - \gamma)$ equals $-\sqrt{\sigma^2(\gamma)/2}$ for $\gamma \leqslant -1$, and $\sqrt{\sigma^2(\gamma)/(-2\gamma)}$ for $-1 < \gamma < 0$. The optimal value $k_o(n)$ is for $n \to \infty$,

$$k_o(n) = \begin{cases} \left[\dfrac{(1-\gamma)^2\,(1-2\gamma)^2}{8(2-\gamma)^2\,\sigma^2(\gamma)}\right]^{-1/3} n^{2/3}(1 + o(1)) & \gamma < -1 \\ [2\sigma^2(-1)]^{1/3}\, n^{2/3}(1 + o(1)) & \gamma = -1 \\ \left[\dfrac{-2\gamma^5(1+\gamma)^2}{(1-\gamma)^2\,(1-3\gamma)^2\,\sigma^2(\gamma)}\right]^{-1/(1-2\gamma)} & \\ \qquad \times\, n^{-2\gamma/(1-2\gamma)}(1 + o(1)) & -1 < \gamma < 0. \end{cases}$$

For $\gamma = 0$ holds $U(t) = -\log(-\log(1 - 1/t)) = \log t - 1/(2t) + o(1/t)$, $t \to \infty$, hence $\log U(tx) - \log U(t)$ equals asymptotically the right hand side of (26). So we are in the same situation as in the example of the exponential distribution.

For $\gamma > 0$, $\log(t^{-\gamma}U(t)/c) = -\gamma t^{-\gamma}/2 - t^{\gamma} + o(t^{-2} + t^{-2\gamma})$, $t \to \infty$, which satisfies the condition of Theorem 3.4 with $c = 1/\gamma$ and $\rho = \min(1, 1/\gamma)$. The bias b of $\sqrt{k_o}\,(\hat{\gamma}_{n,o} - \gamma)$ equals $\sqrt{(1+\gamma^2)/(2\gamma)}$ for $0 < \gamma \leqslant 1$ and $\sqrt{(1+\gamma^2)/2}$ for $\gamma > 1$. Finally, one obtains for the optimal value $k_o(n)$, $n \to \infty$,

$$k_o(n) = \begin{cases} \left[\dfrac{(1+\gamma)^4\,(1+\gamma^2)}{2\gamma^5}\right]^{1/(1+2\gamma)} n^{2\gamma/(1+2\gamma)}(1+o(1)) & 0 < \gamma < 1 \\ [64/9]^{1/3}\, n^{2/3}(1+o(1)) & \gamma = 1 \\ [8(1+\gamma^2)(2\gamma-1)^{-2}]^{1/3}\, n^{2/3}(1+o(1)) & \gamma > 1. \end{cases}$$

Appendix A

In this Appendix we give the proof of Lemma 2.3 (Second Order Regular Variation).

(b) $\Rightarrow$ (a): Suppose $\gamma < 0$ and $t^{-\gamma}\{U(\infty) - U(t)\} - c =: H(t)$ for t sufficiently large, with $H \in RV_{\gamma\rho}$. Replacing now t by $\{1 - F(U(\infty) - x^{-1})\}^{-1}$ one obtains $\{1 - F(U(\infty) - x^{-1})\}^{\gamma} x^{-1} - c = H(\{1 - F(U(\infty) - x^{-1})\}^{-1})$ for x sufficiently large, and $H(\{1 - F(U(\infty) - x^{-1})\}^{-1}) \in RV_{-\rho}$ since $U(\infty) - U(t) \in RV_{\gamma}$ and $U(\infty) - U(\{1 - F(U(\infty) - x^{-1})\}^{-1} \in RV_{-1}$.

Now one obtains for t sufficiently large

$$\begin{aligned} &-\{t^{-1/\gamma}[1 - F(U(\infty) - t^{-1})] - c^{-1/\gamma}\} \\ &= -\left[c^{1/\gamma}\left\{\frac{t^{-1}[1 - F(U(\infty) - t^{-1})]^{\gamma} - c}{c} + 1\right\}^{1/\gamma} - c^{1/\gamma}\right] \\ &= \frac{c^{-1+1/\gamma}}{-\gamma} H\left(\frac{1}{1 - F(U(\infty) - t^{-1})}\right)(1 + o(1)), \qquad t \to \infty, \end{aligned}$$

where the latter term is positive and $\in RV_{-\rho}$.

(a) $\Rightarrow$ (b): This part of the proof follows the same line.

(b) $\Rightarrow$ (d): Note that (b) is equivalent with

$$\mp\{t^{-\gamma}[1 - U(t)/U(\infty)] - c/U(\infty)\} \in RV_{\gamma\rho}$$

and use $\log x = (x-1)(1+o(1))$, $x \to 1$.

(c) $\Leftrightarrow$ (d): Use the equivalence of (a) and (b).

(f) $\Rightarrow$ (e): Suppose $\gamma > 0$ and $t^{-\gamma}U(t) - c =: H(t)$, $t \to \infty$, H positive and $H \in RV_{-\gamma\rho}$. Since $U \in RV_{\gamma}$, $1/\{1 - F\} \in RV_{1/\gamma}$ and, replacing t by $1/\{1 - F(x)\}$,

$$\{1 - F(x)\}^{\gamma}\, x - c = H\left(\frac{1}{1 - F(x)}\right) \in RV_{-\rho}.$$

Since $x^{1/\gamma}\{1-F(x)\}-c^{1/\gamma}=[x\{1-F(x)\}^{\gamma}-c+c]^{1/\gamma}-c^{1/\gamma}=$

$$c^{1/\gamma}\left[1+\frac{x\{1-F(x)\}^{\gamma}-c}{\gamma c}(1+o(x^{-\gamma}))\right]-c^{1/\gamma}, \qquad x\to\infty,$$

one obtains for t sufficiently large

$$t^{1/\gamma}\{1-F(t)\}-c^{1/\gamma}=\frac{c^{-1+1/\gamma}}{\gamma}H\left(\frac{1}{1-F(t)}\right)(1+O(t^{-\rho}))$$

with $c^{-1+1/\gamma}\gamma^{-1}H(1/\{1-F(t)\})\in RV_{-\rho}$.

(e)$\Rightarrow$(f): This part of the proof is omitted since it follows the same line as the previous part.

(f)$\Rightarrow$(g): Suppose $t^{-\gamma}U(t)-c\in RV_{-\gamma\rho}$, then also $t^{-\gamma}U(t)/c-1\in RV_{-\gamma\rho}$ and hence $t^{-\gamma}U(t)/c\to 1$, $t\to\infty$. Now $\log(t^{-\gamma}U(t)/c)=(t^{-\gamma}U(t)/c-1)(1+o(1))=c^{-1}(t^{-\gamma}U(t)-c)(1+o(1))$ which is regularly varying with index $-\gamma\rho$.

(g)$\Rightarrow$(f): Follows the same line as (f)$\Rightarrow$(g).

Appendix B

The following theorem has been communicated to us by A. A. Balkema.

Theorem B.1. *Let $U>0$ vary slowly and be asymptotic to a non-decreasing function. Then U is asymptotic to an element of Π.*

Proof. Write $g(t)=U(e^t)$. Slow variation of U means that $g(t+x)/g(t)\to 1$ uniformly on bounded x-intervals for $t\to\infty$. We shall construct a function $f\sim g$ such that $\log f'$ is continuous and piecewise linear, and $(\log f')'\to 0$. This implies that $V(t):=f(\log t)$ lies in Π. We may assume that $g(t)\to\infty$ for $t\to\infty$, since else g is asymptotic to a function $f(t)=C-1/t$, $C>0$, which satisfies the condition $(\log f')'(t)=2/t\to 0$. We may also assume that g is strictly increasing and continuous.

For $t\in\mathscr{R}$ and $c>1$ define $t_c>t$ by $g(t_c)=cg(t)$. Obviously $t_c-t\to\infty$. This implies that there exists a sequence $y_n=g(x_n)$ such that $y_{n+1}\sim y_n\to\infty$ and $y_{n+1}-y_n=:v_n\sim v_{n-1}$ and such that $x_{n+1}-x_n=:u_n\to\infty$. Indeed choose x_{n+1} so that $g(x_{n+1})=c_n g(x_n)$ with $c_n>1$ and $c_n\to 1$ so slowly that $x_{n+1}-x_n\to\infty$. We may assume c_n to be weakly decreasing. In addition we may choose c_n of the form $1+1/m$ with $m=m_n$ a positive integer and $m_{n+1}-m_n\in\{0,1\}$. Increase the value of c_n if necessary. Then

$$\frac{v_{n-1}}{v_n}=\frac{(c_{n-1}-1)\,y_{n-1}}{(c_n-1)\,y_n}\sim\frac{c_{n-1}-1}{c_n-1}=\frac{m_{n-1}}{m_n}\to 1.$$

Let h be piecewise linear such that $h(x_n)=y_n$. The derivative $h'(x)=a_n=v_n/u_n$ is constant on the interval $J_n=(x_n, x_{n+1})$, and $a_n/v_n=1/u_n\to 0$. The asymptotic relation $v_{n+1}\sim v_n$ implies $a_{n+m}/v_n\to 0$ for any integer m. Hence $b_n/v_n\to 0$ where $b_n=a_{n-1}+a_n$ is the sum of the left and right derivative of h in the point x_n. Similarly $b_{n+1}/v_n\to 0$.

We now give an explicit construction of the function f.

Set $f(x_n)=y_n$ so that f agrees with g in the points x_n. Since f will be strictly increasing and $y_{n+1}\sim y_n$ this ensures that $f\sim g$. We divide the interval $J_n=(x_n, x_{n+1})$ into two parts by a point ξ_n to be determined later and define

$$f(x+u)=\begin{cases}\varphi_n(x_n+u)=y_n+b_n\displaystyle\int_0^u e^{-\lambda_n t}\,dt & x_n+u\leqslant\xi_n\\[2ex] \psi(x_{n+1}-u)=y_{n+1}-b_{n+1}\displaystyle\int_0^u e^{-\lambda_n t}\,dt & x_{n+1}-u>\xi_n.\end{cases}$$

We shall choose ξ_n and $\lambda_n>0$ so that f is C^1 on the interval J_n.

It is best to look at the derivatives. The function φ_n' is decreasing with initial value $b_n>a_n$ in the point x_n; the function ψ_n' is increasing with boundary value $b_{n+1}>a_n$ in the point x_{n+1}. For $\lambda=0$ the two derivatives are constant and as λ increases, the slopes of the two derivatives increase. Let $\xi(\lambda)$ be the point where they intersect. The function f' agrees with $\max(\varphi_n', \psi_n')$ on the interval J_n, and we have to choose $\lambda_n>0$ so that the average slope over the interval J_n is a_n, since this is the derivative of the linear function h on J_n. Hence $\xi_n=\xi(\lambda_n)\in J_n$ and $f'(\xi_n)<a_n$. Now observe that $\varphi_n>\psi_n$ on J_n if $\lambda=0$ since the slopes exceed a_n, and that $\psi_n-\varphi_n>v_n-(b_n+b_{n+1})/\lambda\geqslant 0$ for $\lambda\geqslant(b_n+b_{n+1})/v_n\to 0$. This implies $\lambda_n\to 0$, and since $|(\log f')'|=\lambda_n$ on J_n we obtain the desired limit relation $(\log f')'(x)\to 0$ for $x\to\infty$.

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