Horizon Documentation

Horizons

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1 Notation

Unless explicitly stated, the following symbols will be used with the meaning defined here below:

1. N: standard cumulative normal
2. : $n(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$: standard normal probability density

2 Normal Black-Scholes Model

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Let us consider an asset having the following dynamics for the evolution from t to a certain maturity $T > t$:

$$S(T) = F + \sigma z \tag{1}$$

where z is a standard normal and F is called T-maturity asset forward. Let us compute the undiscounted price of a vanilla option on $S(T)$:

$$\Pi[\phi] = E\left[(\phi(S(T) - K))^+\right] = E\left[(\phi(\sigma z - (K - F)))^+\right] \tag{2}$$

We have

$$\Pi[\phi] = \int_{z^*}^{\phi\infty} n(z)(\sigma z - (K - F))dz \tag{3}$$

with

$$z^* = \frac{K - F}{\sigma} \tag{4}$$

which gives

$$\Pi[\phi] = \phi(F - K)N(-\phi z^*) + \sigma n(z^*) \tag{5}$$

3 Girsanov's Theorem

Proposition: (From [1]) Let W^P be a d-dimensional standard (i.e. independent components) P-Wiener process on $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ and let φ be any d-dimensional adapted column vector process. Choose a fixed $T > 0$ and define the (scalar) process L on $[0, T]$ by

$$dL_t = \varphi_t^* L_t dW_t^P; L_0 = 1$$

that is

$$L_t = \exp\left[\int_0^t \varphi_s^* dW_s^P - \frac{1}{2}\int_0^t \|\varphi_s\|^2 ds\right]$$

Assume that

$$E^P[L_T] = 1$$

and define the new probability measure Q on $\mathcal{F}_T$ by

$$L_T = \frac{dQ}{dP}$$

Then

$$dW_t^P = \varphi_t dt + dW_t^Q \quad (6)$$

$$W_t^Q = W_t^P - \int_0^t \varphi_s ds \quad (7)$$

where W^Q is a Q-Wiener process.

Proof:

TODO:

4 Changing measures between equivalent martingale measures

For every non dividend paying tradable asset H, measurable on $\mathcal{F}_\tau$, and considering two numerairs $N(\tau)$, $M(\tau)$, we know that the following holds ($t \leq \tau$):

$$H(t) = N(t)E^N\left[\frac{H(\tau)}{N(\tau)}|\mathcal{F}_t\right] = M(t)E^M\left[\frac{H(\tau)}{M(\tau)}|\mathcal{F}_t\right]$$

which, defining $G(\tau) = \frac{H(\tau)}{N(\tau)}$, yields

$$E^N\left[G(\tau)|\mathcal{F}_t\right] = E^M\left[G(\tau)\frac{M(t)}{M(\tau)}\frac{N(\tau)}{N(t)}|\mathcal{F}_t\right] = E^M\left[G(\tau)\frac{dQ^N}{dQ^M}(\tau)|\mathcal{F}_t\right]$$

which gives

$$\frac{dQ^N}{dQ^M}(\tau) = \frac{M(t)}{M(\tau)}\frac{N(\tau)}{N(t)} \quad (8)$$

5 Changing measures between T-fwd martingale measures

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Let $t_0 < t_1 < \cdots < t_N$ be a time schedule and referring to (8) we assume

$$M(t) = P(t, t_k) \tag{9}$$

$$N(t) = P(t, t_h) \tag{10}$$

with $t \leq \tau \leq \min(t_k, t_h)$ and $h, k \geq 0$. By $P(\tau, T)$ we denote the risk free zero coupon bond price as observed from τ for maturity T, delivering 1 unit of currency at T. We then get

$$\frac{dQ^N}{dQ^M}(\tau) = \frac{dQ^h}{dQ^k}(\tau) = \frac{P(t, t_k)}{P(\tau, t_k)} \frac{P(\tau, t_h)}{P(t, t_h)} = L_{t,h/k}(\tau) \tag{11}$$

where $L_{t,h/k}(t) = 1$ and

$$E^h \left[G(\tau)|\mathcal{F}_t\right] = E^k \left[G(\tau)L_{t,h/k}(\tau)|\mathcal{F}_t\right] \tag{12}$$

We now write

$$P(\tau, h/k) \equiv \frac{P(\tau, t_h)}{P(\tau, t_k)} = \left\{\Pi_{i=m_{h,k}+1}^{M_{h,k}} \left[1 + F_i(\tau)\tau_i\right]\right\}^{s(h,k)} \tag{13}$$

$$m_{h,k} = \min(h, k) \tag{14}$$

$$M_{h,k} = \max(h, k) \tag{15}$$

$$s(h, k) = 1 \text{ if } h \leq k, -1 \text{ if } h > k \tag{16}$$

and

$$F_i(\tau) \equiv \left(\frac{P(\tau, t_{i-1})}{P(\tau, t_i)} - 1\right) \frac{1}{\tau_i} \tag{17}$$

We now compute the following Ito differential in the t_k-forward measure

$$dP(\tau, h/k) = \sum_{l=m_{h,k}+1}^{M_{h,k}} dF_l(\tau) \frac{\partial}{\partial F_l(\tau)} P(\tau, h/k) = \tag{18}$$

$$= \sum_{l=m_{h,k}+1}^{M_{h,k}} \tau_l s(h, k) dF_l(\tau) \left[1 + F_l(\tau)\tau_l\right]^{s(h,k)-1} \left\{\Pi_{i=m_{h,k}+1, i\neq l}^{M_{h,k}} \left[1 + F_i(\tau)\tau_i\right]\right\}^{s(h,k)} = \tag{19}$$

$$= s(h, k) P(\tau, h/k) \sum_{l=m_{h,k}+1}^{M_{h,k}} \frac{\tau_l dF_l(\tau)}{1 + F_l(\tau)\tau_l} \tag{20}$$

From the last equation and (11) we get (always in the t_k-forward measure)

$$dL_{t,h/k}(\tau) = L_{t,h/k}(\tau) s(h, k) \sum_{l=m_{h,k}+1}^{M_{h,k}} \frac{\tau_l dF_l(\tau)}{1 + F_l(\tau)\tau_l} \tag{21}$$

We now use this last equation, together with the Girsanov's theorem (6) to get the following result. Suppose to consider a 1-dimensional Wiener process $Z^{(k)}(\tau)$ where the (k)-apex is used to stress that it is a Wiener process in the t_k-fwd measure. We can then conclude that $Z^{(h)}(\tau)$ as defined in the following is a Wiener process in the t_h-fwd measure:

$$dZ^{(h)}(\tau) = dZ^{(k)}(\tau) - s(h,k) \sum_{l=m_{h,k}+1}^{M_{h,k}} \frac{\tau_l \left\langle dF_l(\tau) \cdot dZ^{(k)}(\tau) \right\rangle}{1 + F_l(\tau)\tau_l} \tag{22}$$

5.1 Shifted log-normal fwd rate model and measure change between T-fwd martingale measures

We now assume that

$$F_l(\tau) = \lambda_l + f_l(\tau) \tag{23}$$

where λ_l is a constant and $f_l(\tau)$ is the following (martingale) log-normal process in the t_l-fwd measure:

$$\frac{df_l(\tau)}{f_l(\tau)} = -\frac{1}{2}\sigma_l^2 d\tau + \sigma_l dW_l^{(l)}(\tau) \tag{24}$$

where we used a notation that stresses that $W_l^{(l)}$ is the Wiener process driving $f_l(\tau)$ in the t_l-fwd measure and we introduced an annual volatility process σ_l (not a constant in general). We now further observe the obvious relation

$$dF_l(\tau) = df_l(\tau) \tag{25}$$

Equation (49) then becomes

$$dZ^{(h)}(\tau) = dZ^{(k)}(\tau) - s(h,k) \sum_{l=m_{h,k}+1}^{M_{h,k}} \frac{\tau_l \sigma_l f_l(\tau) \left\langle dW_l^{(l)}(\tau) \cdot dZ^{(k)}(\tau) \right\rangle}{1 + F_l(\tau)\tau_l} = \tag{26}$$

$$= dZ^{(k)}(\tau) - s(h,k) \sum_{l=m_{h,k}+1}^{M_{h,k}} \frac{\tau_l \sigma_l (F_l(\tau) - \lambda_l) \left\langle dW_l^{(l)}(\tau) \cdot dZ^{(k)}(\tau) \right\rangle}{1 + F_l(\tau)\tau_l} \tag{27}$$

We further notice that one could apply the 'freezing the drift approximation' in the last equation to get

$$dZ^{(h)}(\tau) = dZ^{(k)}(\tau) - s(h,k) \sum_{l=m_{h,k}+1}^{M_{h,k}} \frac{\tau_l \sigma_l (F_l(t) - \lambda_l) \left\langle dW_l^{(l)}(\tau) \cdot dZ^{(k)}(\tau) \right\rangle}{1 + F_l(t)\tau_l} \tag{28}$$

where t has been substituted in place of τ in some of the factors.

5.2 Changing measures between T-fwd martingale measures of different currencies

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Let us consider a tradable asset A quoted in currency α and suppose to know its market-agreed forward value as seen from time t for maturity t_k. For example, A could be a stock with t_k any future maturity or a fwd-rate ibor index with t_k equal to its 'natural pay date' (typically its calendar adjusted end accrual date). By definition of fwd contract we can write

$$F_A^\alpha(t,t_k)P^\alpha(t,t_k) = E^\alpha\left[A(f_k)D^\alpha(t,t_k)|F_t\right] = P^\alpha(t,t_k)E^{k,\alpha}\left[A(f_k)|F_t\right] \tag{29}$$

where D^α denotes the stochastic risk neutral discount factor in α currency and E^α the expectation operator in the corresponding measure. Furthermore $f_k \leq t_k$ denotes the fixing date of A associated to the forward contract delivering at t_k (typically $f_k = t_k$ for stocks or $f_k = t_{k-1}$ for ibor indexes). We can then write

$$F_A^\alpha(t,t_k) = E^{k,\alpha}\left[A(f_k)|F_t\right] \tag{30}$$

We now consider the problem of quanto-ing the forward contract in a β-currency and at the same time of changing its payment date from t_k to t_h, with $t_h \geq f_k$. We define X as the value of 1 unit of β currency expressed in α currency. The quanto fwd price will be

$$E^\beta\left[A(f_k)D^\beta(t,t_h)|F_t\right] = P^\beta(t,t_h)E^{h,\beta}\left[A(f_k)|F_t\right] = F_A^\beta(t,t_h)P^\beta(t,t_h) = \phi_A(t,t_h,\beta) \tag{31}$$

where the t_h-fwd quantoed in β-currency is then defined as

$$F_A^\beta(t,t_h) = E^{h,\beta}\left[A(f_k)|F_t\right] \tag{32}$$

By no arbitrage, it must also hold that

$$\phi_A(t,t_h,\beta) = E^\alpha\left[A(f_k)D^\alpha(t,t_h)X(t_h)|F_t\right]\frac{1}{X(t)} = \tag{33}$$

$$= E^\alpha\left[E^\alpha\left[A(f_k)D^\alpha(t,t_h)\frac{X(t_h)}{X(t)}|F_{f_k}\right]|F_t\right] = \tag{34}$$

$$= E^\alpha\left[D^\alpha(t,f_k)A(f_k)E^\alpha\left[D^\alpha(f_k,t_h)\frac{X(t_h)}{X(t)}|F_{f_k}\right]|F_t\right] \tag{35}$$

Again by no arbitrage it must hold that

$$E^\alpha\left[D^\alpha(f_k,t_h)\frac{X(t_h)}{X(f_k)}|F_{f_k}\right] = E^\beta\left[1\cdot D^\beta(f_k,t_h)|F_{f_k}\right] = P^\beta(f_k,t_h) \tag{36}$$

meaning that the contract delivering 1 unit of β currency at t_h must have the same price as seen form f_k irrespective of the measure we use to compute it. Then we get

$$\phi_A(t,t_h,\beta) = E^\alpha\left[D^\alpha(t,f_k)A(f_k)\frac{X(f_k)}{X(t)}P^\beta(f_k,t_h)|F_t\right] \tag{37}$$

Equating (31) to (37) we obtain

$$P^{\beta}(t,t_h)E^{h,\beta}\left[A(f_k)|F_t\right] = E^{\alpha}\left[D^{\alpha}(t,f_k)A(f_k)\frac{X(f_k)}{X(t)}P^{\beta}(f_k,t_h)|F_t\right] = \tag{38}$$

$$= E^{\alpha}\left[\frac{D^{\alpha}(t,t_k)}{P^{\alpha}(f_k,t_k)}A(f_k)\frac{X(f_k)}{X(t)}P^{\beta}(f_k,t_h)|F_t\right] = E^{k,\alpha}\left[\frac{P^{\alpha}(t,t_k)}{P^{\alpha}(f_k,t_k)}A(f_k)\frac{X(f_k)}{X(t)}P^{\beta}(f_k,t_h)|F_t\right] \tag{39}$$

where we also applied a payoff deferring formula and changed from the risk neutral expectation E^{α} to the t_k-fwd measure expectation of the α currency. Summarizing, we proved that for any asset A it holds that

$$E^{h,\beta}\left[A(f_k)|F_t\right] = E^{k,\alpha}\left[A(f_k)\frac{L_X(f_k;\alpha,\beta,t_h,t_k)}{L_X(t;\alpha,\beta,t_h,t_k)}|F_t\right] \tag{40}$$

where

$$L_X(\tau;\alpha,\beta,t_h,t_k) = X(\tau)\frac{P^{\beta}(\tau,t_h)}{P^{\alpha}(\tau,t_k)} = \frac{dQ^{h,\beta}}{dQ^{k,\alpha}}(\tau) \tag{41}$$

is the Radon-Nikodyn derivative. We also write L in another illuminating form

$$L_X(\tau;\alpha,\beta,t_h,t_k) = \frac{dQ^{h,\beta}}{dQ^{k,\alpha}}(\tau) = X(\tau)\frac{P^{\beta}(\tau,t_h)}{P^{\alpha}(\tau,t_h)}\frac{P^{\alpha}(\tau,t_h)}{P^{\alpha}(\tau,t_k)} \tag{42}$$

We notice that $L_X(\tau;\alpha,\beta,t_h,t_k)$ is a martingale in the t_k-fwd measure of the α currency (being a $X(\tau)P^{\beta}(\tau,t_h)$ a tradable asset). We now define the process of the forward of the X fx rate for date t_h as

$$F_X(\tau,t_h) = X(\tau)\frac{P^{\beta}(\tau,t_h)}{P^{\alpha}(\tau,t_h)} \tag{43}$$

and finally obtain (see (13))

$$L_X(\tau;\alpha,\beta,t_h,t_k) = \frac{dQ^{h,\beta}}{dQ^{k,\alpha}}(\tau) = F_X(\tau,t_h)\cdot\frac{P^{\alpha}(\tau,t_h)}{P^{\alpha}(\tau,t_k)} = F_X(\tau,t_h)\cdot P^{\alpha}(\tau,h/k) \tag{44}$$

We now write (in the t_k-fwd measure of the α currency)

$$dF_X(\tau,t_h) = \cdots dt + \sigma_{F_X(t_h)}F_X(\tau,t_h)dW_X(\tau) \tag{45}$$

We now consider the following Ito differential (see (20)):

$$\frac{dL_X(\tau;\alpha,\beta,t_h,t_k)}{L_X(\tau;\alpha,\beta,t_h,t_k)} = \cdots dt + \sigma_{F_X(t_h)}dW_X(\tau) + \frac{1}{P^{\alpha}(\tau,h/k)}dP^{\alpha}(\tau,h/k) = \tag{46}$$

$$= \cdots dt + \sigma_{F_X(t_h)}dW_X(\tau) + s(h,k)\sum_{l=m_{h,k}+1}^{M_{h,k}}\frac{\tau_l dF_l^{\alpha}(\tau)}{1+F_l^{\alpha}(\tau)\tau_l} \tag{47}$$

We now use this last equation, together with the Girsanov's theorem (6) to get the following result. Suppose to consider a 1-dimensional Wiener process $Z^{(k,\alpha)}(\tau)$ where the (k,α)-apex is used to stress that it is a Wiener process in the t_k-fwd measure of the α currency. We can then conclude that $Z^{(h,\beta)}(\tau)$ as defined in the following is a Wiener process in the t_h-fwd measure of the β currency:

$$dZ^{(h,\beta)}(\tau) = dZ^{(k,\alpha)}(\tau) - s(h,k)\sum_{l=m_{h,k}+1}^{M_{h,k}}\frac{\tau_l\left\langle dF_l^{\alpha}(\tau)\cdot dZ^{(k,\alpha)}(\tau)\right\rangle}{1+F_l^{\alpha}(\tau)\tau_l} - \tag{48}$$

$$-\sigma_{F_X(t_h)}\left\langle dW_X(\tau)\cdot dZ^{(k,\alpha)}(\tau)\right\rangle \tag{49}$$

References

[1] Bjork, 'Arbitrage Theory in Continuos Time', 2nd ed, Oxford