Univariate Distribution Relationships

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Probability distributions are traditionally treated separately in introductory mathematical statistics textbooks. A figure is presented here that shows properties that individual distributions possess and many of the relationships between these distributions.

KEY WORDS: Asymptotic relationships; Distribution properties; Limiting distributions; Stochastic parameters; Transformations.

1. INTRODUCTION

Introductory probability and statistics textbooks typically introduce common univariate distributions individually, and seldom report all of the relationships between these distributions. This article contains an update of a figure presented by Leemis (1986) that shows the properties of and relationships between several common univariate distributions. More detail concerning these distributions is given by Johnson, Kotz, and Balakrishnan (1994, 1995) and Johnson, Kemp, and Kotz (2005). More concise treatments are given by Balakrishnan and Nevzorov (2003), Evans, Hastings, and Peacock (2000), Ord (1972), Patel, Kapadia, and Owen (1976), Patil, Boswell, Joshi, and Ratnaparkhi (1985), Patil, Boswell, and Ratnaparkhi (1985), and Shapiro and Gross (1981). Figures similar to the one presented here have appeared in Casella and Berger (2002), Marshall and Olkin (1985), Nakagawa and Yoda (1977), Song (2005), and Taha (1982).

Figure 1 contains 76 univariate probability distributions. There are 19 discrete and 57 continuous models. Discrete distributions are displayed in rectangular boxes; continuous distributions are displayed in rounded boxes. The discrete distributions are at the top of the figure, with the exception of the Benford distribution. A distribution is described by two lines of text in each box. The first line gives the name of the distribution and its parameters. The second line contains the properties (described in the next section) that the distribution assumes.

The parameterizations for the distributions are given in the Appendix. If the distribution is known by several names (e.g., the normal distribution is often called the Gaussian distribution), this is indicated in the Appendix following the name of the distribution. The parameters typically satisfy the following conditions:

- n, with or without subscripts, is a positive integer;
- p is a parameter satisfying $0 < p < 1$;
- α and σ, with or without subscripts, are positive scale parameters;
- β, γ, and κ are positive shape parameters;
- μ, a, and b are location parameters;
- λ and δ are positive parameters.

Exceptions to these rules, such as the rectangular parameter n, are given in the Appendix after any aliases for the distribution. Additionally, any parameters not described above are explicitly listed in the Appendix. Many of the distributions have several mathematical forms, only one of which is presented here (e.g., the extreme value and discrete Weibull distributions) for the sake of brevity.

There are numerous distributions that have not been included in the chart due to space limitations or that the distribution is not related to one of the distributions currently on the chart. These include Bézier curves (Flanigan–Wagner and Wilson 1993); the Burr distribution (Crowder et al. 1991, p. 33 and Johnson, Kotz, and Balakrishnan 1994, pp. 15–63); the generalized beta distribution (McDonald 1984); the generalized exponential distribution (Gupta and Kundu 2007); the generalized F distribution (Prentice 1975); Johnson curves (Johnson, Kotz, and Balakrishnan 1994, pp. 15–63); the kappa distribution (Hosking 1994); the Kolmogorov–Smirnov one-sample distribution (parameters estimated from data), the Kolmogorov–Smirnov two-sample distribution (Boomsma and Molenaar 1994); the generalized lambda distribution (Ramberg and Schmeiser 1974); the Maxwell distribution (Balakrishnan and Nevzorov 2003, p. 232); Pearson systems (Johnson, Kotz, and Balakrishnan 1994, pp. 15–63); the generalized Waring distribution (Hogg, McKean, and Craig 2005, p. 195). Likewise,

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Devroye (2006) refers to Dickman's, Kolmogorov–Smirnov, Kummer's, Linnik–Laha, theta, and de la Vallée–Poussin distributions in his chapter on variate generation.

2. DISTRIBUTION PROPERTIES

There are several properties that apply to individual distributions listed in Figure 1.

- The *linear combination* property (L) indicates that linear combinations of independent random variables having this particular distribution come from the same distribution family.

  Example: If $X_i \sim N\left(\mu_i, \sigma_i^2\right)$ for $i = 1, 2, \ldots, n$; $a_1, a_2, \ldots, a_n$ are real constants, and $X_1, X_2, \ldots, X_n$ are independent, then

$$\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right).$$

- The *convolution* property (C) indicates that sums of independent random variables having this particular distribution come from the same distribution family.

  Example: If $X_i \sim \chi^2(n_i)$ for $i = 1, 2, \ldots, n$, and $X_1, X_2, \ldots, X_n$ are independent, then

$$\sum_{i=1}^{n} X_i \sim \chi^2\left(\sum_{i=1}^{n} n_i\right).$$

- The *scaling* property (S) implies that any positive real constant times a random variable having this distribution comes from the same distribution family.

  Example: If $X \sim \text{Weibull}(\alpha, \beta)$ and k is a positive, real constant, then

$$kX \sim \text{Weibull}(\alpha k^{\beta}, \beta).$$

- The *product* property (P) indicates that products of independent random variables having this particular distribution come from the same distribution family.

  Example: If $X_i \sim \text{lognormal}(\mu_i, \sigma_i^2)$ for $i = 1, 2, \ldots, n$, and $X_1, X_2, \ldots, X_n$ are independent, then

$$\prod_{i=1}^{n} X_i \sim \text{lognormal}\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right).$$

- The *inverse* property (I) indicates that the reciprocal of a random variable of this type comes from the same distribution family.

  Example: If $X \sim F(n_1, n_2)$, then

$$\frac{1}{X} \sim F(n_2, n_1).$$

- The *minimum* property (M) indicates that the smallest of independent and identically distributed random variables from a distribution comes from the same distribution family.

  Example: If $X_i \sim \text{exponential}(\alpha_i)$ for $i = 1, 2, \ldots, n$, and $X_1, X_2, \ldots, X_n$ are independent, then

$$\min\{X_1, X_2, \ldots, X_n\} \sim \text{exponential}\left(1 \Big/ \sum_{i=1}^{n} (1/\alpha_i)\right).$$

- The *maximum* property (X) indicates that the largest of independent and identically distributed random variables from a distribution comes from the same distribution family.

  Example: If $X_i \sim \text{standard power}\ (\beta_i)$ for $i = 1, 2, \ldots, n$, and $X_1, X_2, \ldots, X_n$ are independent, then

$$\max\{X_1, X_2, \ldots, X_n\} \sim \text{standard power}\left(\sum_{i=1}^{n} \beta_i\right).$$

- The *forgetfulness* property (F), more commonly known as the memoryless property, indicates that the conditional distribution of a random variable is identical to the unconditional distribution. The geometric and exponential distributions are the only two distributions with this property. This property is a special case of the residual property.

- The *residual* property (R) indicates that the conditional distribution of a random variable left-truncated at a value in its support belongs to the same distribution family as the unconditional distribution.

  Example: If $X \sim \text{Uniform}(a, b)$, and k is a real constant satisfying $a < k < b$, then the conditional distribution of X given $X > k$ belongs to the uniform family.

- The *variate generation* property (V) indicates that the inverse cumulative distribution function of a continuous random variable can be expressed in closed form. For a discrete random variable, this property indicates that a variate can be generated in an $O(1)$ algorithm that does not cycle through the support values or rely on a special property.

  Example: If $X \sim \text{exponential}(\alpha)$, then

$$F^{-1}(u) = -\alpha \log(1 - u), \qquad 0 < u < 1.$$

Since property L implies properties C and S, the C and S properties are not listed on a distribution having the L property. Similarly, property $F \Rightarrow$ property R.

Some of the properties apply only in restricted cases. The minimum property applies to the Weibull distribution, for example, only when the shape parameter is fixed. The Weibull distribution has M_{β} on the second line in Figure 1 to indicate that the property is valid only in this restricted case.

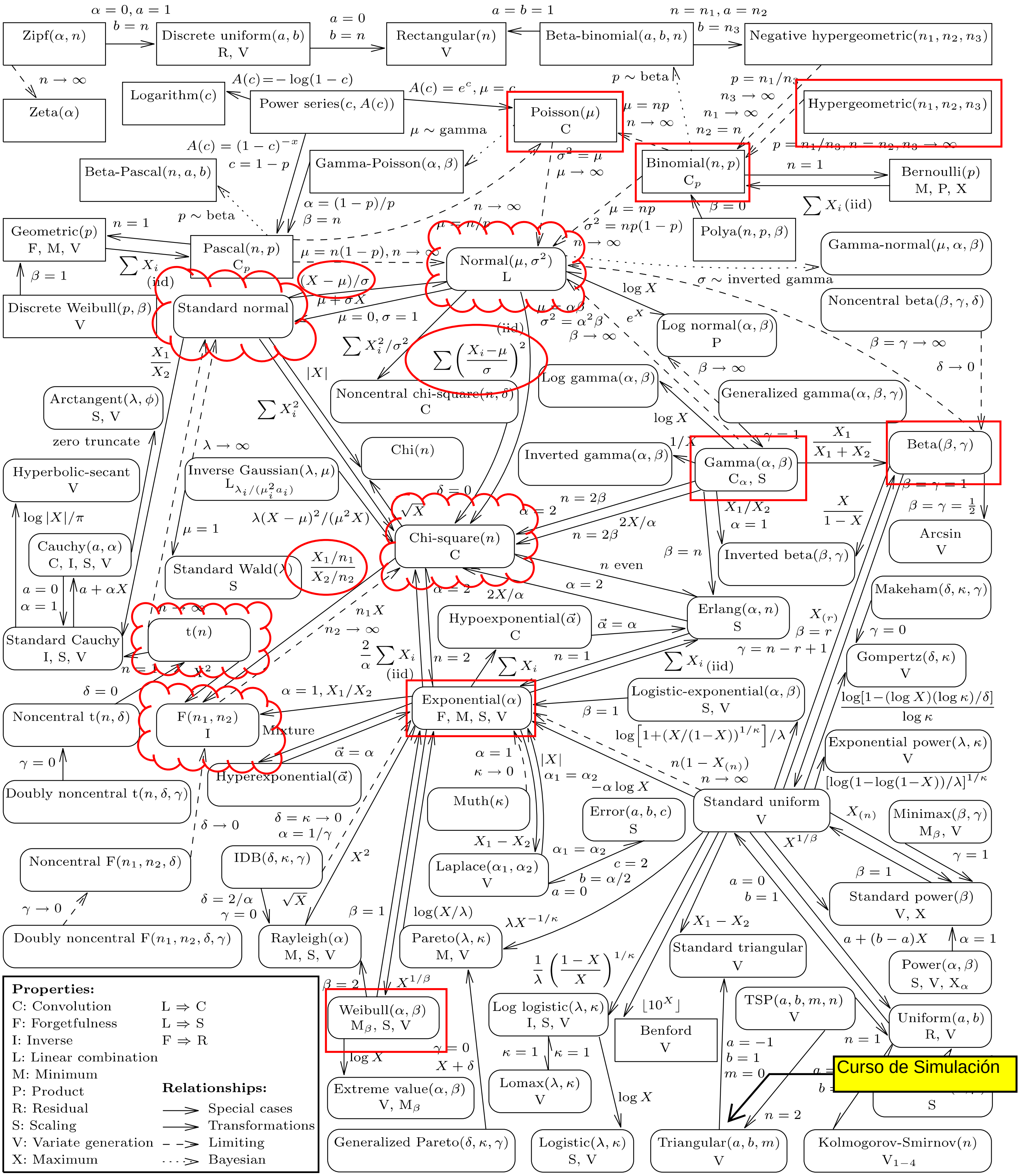

Figure 1. Univariate distribution relationships.

3. RELATIONSHIPS AMONG THE DISTRIBUTIONS

There are three types of lines used to connect the distributions to one another. The solid line is used for *special cases* and *transformations*. Transformations typically have an X on their label to distinguish them from special cases. The term "transformation" is used rather loosely here, to include the distribution of an order statistic, truncating a random variable, or taking a mixture of random variables. The dashed line is used for *asymptotic relationships*, which are typically in the limit as one or more parameters approach the boundary of the parameter space. The dotted line is used for *Bayesian relationships* (e.g., Beta–binomial, Beta–Pascal, Gamma–normal, and Gamma–Poisson). The binomial, chi-square, exponential, gamma, normal, and $U(0, 1)$ distributions emerge as hubs, highlighting their centrality in applied statistics. Summation limits run from $i = 1$ to n. The notation $X_{(r)}$ denotes the rth order statistic drawn from a random sample of size n.

There are certain special cases where distributions overlap for just a single setting of their parameters. Examples include (a) the exponential distribution with a mean of two and the chi-square distribution with two degrees of freedom, (b) the chi-square distribution with an even number of degrees of freedom and the Erlang distribution with scale parameter two, and (c) the Kolmogorov–Smirnov distribution (all parameters known case) for a sample of size $n = 1$ and the $U(1/2, 1)$ distribution. Each of these cases is indicated by a double-headed arrow.

The probability integral transformation allows a line to be drawn, in theory, between the standard uniform and all others since $F(X) \sim U(0, 1)$. Similarly, a line could be drawn between the unit exponential distribution and all others since $H(X) \sim$ exponential(1), where $H(x) = \int_{-\infty}^{x} f(t)/(1-F(t))dt$ is the cumulative hazard function.

All random variables that can be expressed as sums (e.g., the Erlang as the sum of independent and identically distributed exponential random variables) converge asymptotically in a parameter to the normal distribution by the central limit theorem. These distributions include the binomial, chi-square, Erlang, gamma, hypoexponential, and Pascal distributions. Furthermore, all distributions have an asymptotic relationship with the normal distribution (by the central limit theorem if sums of random variables are considered).

Many of the transformations can be inverted, and this is indicated on the chart by a double-headed arrow between two distributions. Consider the relationship between the normal distribution and the standard normal distribution. If $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0, 1)$ as indicated on the chart. Conversely, if $X \sim N(0, 1)$, then $\mu + \sigma X \sim N(\mu, \sigma^2)$. The first direction of the transformation is useful for standardizing random variables to be used for table lookup, while the second direction is useful for variate generation. In most cases, though, an inverse transformation is implicit and is not listed on the chart for brevity (e.g., extreme value random variable as the logarithm of a Weibull random variable and Weibull random variable as the exponential of an extreme value random variable).

Several of these relationships hint at further distributions that have not yet been developed. First, the extreme value and log gamma distributions indicate that the logarithm of any survival distribution results in a distribution with support over the entire real axis. Second, the inverted gamma distribution indicates that the reciprocal of any survival distribution results in another survival distribution. Third, switching the roles of $F(x)$ and $F^{-1}(u)$ for a random variable with support on $(0, 1)$ results in a complementary distribution (e.g., Jones 2002).

Additionally, the transformations in Figure 1 can be used to give intuition to some random variate generation routines. The Box–Muller algorithm, for example, converts a $U(0, 1)$ to an exponential to a chi-square to a standard normal to a normal random variable.

Redundant arrows have typically not been drawn. An arrow between the minimax distribution and the standard uniform distribution has not been drawn because of the two arrows connecting the minimax distribution to the standard power distribution and the standard power distribution to the standard uniform distribution. Likewise, although the exponential distribution is a special case of the gamma distribution when the shape parameter equals 1, this is not explicitly indicated because of the special case involving the Erlang distribution.

In order to preserve a planar graph, several relationships are not included, such as those that would not fit on the chart or involved distributions that were too far apart. Examples include:

- A geometric random variable is the floor of an exponential random variable.
- A rectangular random variable is the floor of a uniform random variable.
- An exponential random variable is a special case of a Makeham random variable with $\delta = 0$.
- A standard power random variable is a special case of a beta random variable with $\delta = 1$.
- If X has the F distribution with parameters n_1 and n_2, then $\frac{1}{1+(n_1/n_2)X}$ has the beta distribution (Hogg, McKean, and Craig 2005, p. 189).
- The doubly noncentral F distribution with n_1, n_2 degrees of freedom and noncentrality parameters δ, γ is defined as the distribution of

$$\left(\frac{X_1(\delta)}{n_1}\right)\left(\frac{X_2(\gamma)}{n_2}\right)^{-1},$$

  where $X_1(\delta), X_2(\gamma)$ are noncentral chi-square random variables with n_1, n_2 degrees of freedom, respectively, (Johnson, Kotz, and Balakrishnan 1995, p. 480).
- A normal and uniform random variable are special and limiting cases of an error random variable (Evans, Hastings, and Peacock 2000, p. 76).
- A binomial random variable is a special case of a power series random variable (Evans, Hastings, and Peacock 2000, p. 166).
- The limit of a von Mises random variable is a normal random variable as $\kappa \to \infty$ (Evans, Hastings, and Peacock 2000, p. 191).

- The half-normal, Rayleigh, and Maxwell–Boltzmann distributions are special cases of the chi distribution with $n = 1, 2$, and 3 degrees of freedom (Johnson, Balakrishnan, and Kotz 1994, p. 417).

- A function of the ratio of two independent generalized gamma random variables has the beta distribution (Stacy 1962).

Additionally, there are transformations where two distributions are combined to obtain a third, which were also omitted to maintain a planar graph. Two such examples are:

- The t distribution with n degrees of freedom is defined as the distribution of
$$\frac{Z}{\sqrt{\chi^2(n)/n}},$$
where Z is a standard normal random variable and $\chi^2(n)$ is a chi-square random variable with n degrees of freedom, independent of Z (Evans, Hastings, and Peacock 2000, p. 180).

- The noncentral beta distribution with noncentrality parameter δ is defined as the distribution of
$$\frac{X}{X+Y},$$
where X is a noncentral chi-square random variable with parameters (β, δ) and Y is a central chi-square random variable with γ degrees of freedom (Evans, Hastings, and Peacock 2000, p. 42).

References for distributions not typically covered in introductory probability and statistics textbooks include:

- arctan distribution: Glen and Leemis (1997)
- Benford distribution: Benford (1938)
- exponential power distribution: Smith and Bain (1975)
- extreme value distribution: de Haan and Ferreira (2006)
- generalized gamma distribution: Stacy (1962)
- generalized Pareto distribution: Davis and Feldstein (1979)
- Gompertz distribution: Jordan (1967)
- hyperexponential and hypoexponential distributions: Ross (2007)
- IDB distribution: Hjorth (1980)
- inverse Gaussian distribution: Chhikara and Folks (1989), Seshadri (1993)
- inverted gamma distribution: Casella and Berger (2002)
- logarithm distribution: Johnson, Kemp, and Kotz (2005)
- logistic–exponential distribution: Lan and Leemis (2007)
- Makeham distribution: Jordan (1967)
- Muth's distribution: Muth (1977)
- negative hypergeometric distribution: Balakrishnan and Nevzorov (2003), Miller and Fridell (2007)
- power distribution: Balakrishnan and Nevzorov (2003)
- TSP distribution: Kotz and van Dorp (2004)
- Zipf distribution: Ross (2006).

A. APPENDIX: PARAMETERIZATIONS

A.1 Discrete Distributions

Benford:
$$f(x) = \log_{10}\left(1 + \frac{1}{x}\right), \qquad x = 1, 2, \ldots, 9$$

Bernoulli:
$$f(x) = p^x(1-p)^{1-x}, \qquad x = 0, 1$$

Beta–binomial:
$$\begin{aligned} f(x) &= \frac{\Gamma(x+a)\Gamma(n-x+b)\Gamma(a+b)\Gamma(n+2)}{(n+1)\Gamma(a+b+n)\Gamma(a)\Gamma(b)\Gamma(x+1)\Gamma(n-x+1)}, \\ & x = 0, 1, \ldots, n \end{aligned}$$

Beta–Pascal (factorial):
$$f(x) = \binom{n-1+x}{x}\frac{B(n+a, b+x)}{B(a, b)}, \qquad x = 0, 1, \ldots$$

Binomial:
$$f(x) = \binom{n}{x} p^x(1-p)^{n-x}, \qquad x = 0, 1, \ldots, n$$

Discrete uniform:
$$f(x) = \frac{1}{b-a+1}, \qquad x = a, a+1, \ldots, b$$

Discrete Weibull:
$$f(x) = (1-p)^{x^\beta} - (1-p)^{(x+1)^\beta}, \qquad x = 0, 1, \ldots$$

Gamma–Poisson:
$$f(x) = \frac{\Gamma(x+\beta)\alpha^x}{\Gamma(\beta)(1+\alpha)^{\beta+x}x!}, \qquad x = 0, 1, \ldots$$

Geometric:
$$f(x) = p(1-p)^x, \qquad x = 0, 1, \ldots$$

Hypergeometric:
$$\begin{aligned} f(x) &= \binom{n_1}{x}\binom{n_3-n_1}{n_2-x} \Big/ \binom{n_3}{n_2}, \\ x &= \max(0, n_1+n_2-n_3), \ldots, \min(n_1, n_2) \end{aligned}$$

Logarithm (logarithmic series, $0 < c < 1$):

$$f(x) = \frac{-(1-c)^x}{x \log c}, \qquad x = 1, 2, \ldots$$

Negative hypergeometric:

$$\begin{aligned} f(x) &= \binom{n_1 + x - 1}{x}\binom{n_3 - n_1 + n_2 - x - 1}{n_2 - x} \\ &\quad \Big/ \binom{n_3 + n_2 - 1}{n_2}, \\ x &= \max(0, n_1 + n_2 - n_3), \ldots, n_2 \end{aligned}$$

Pascal (negative binomial):

$$f(x) = \binom{n - 1 + x}{x} p^n (1-p)^x, \qquad x = 0, 1, \ldots$$

Poisson ($\mu > 0$):

$$f(x) = \frac{\mu^x e^{-\mu}}{x!}, \qquad x = 0, 1, \ldots$$

Polya:

$$f(x) = \binom{n}{x} \prod_{j=0}^{x-1}(p + j\beta) \prod_{k=0}^{n-x-1}(1 - p + k\beta) \Big/ \prod_{i=0}^{n-1}(1 + i\beta),$$
$$x = 0, 1, \ldots, n$$

Power series ($c > 0$; $A(c) = \sum_x a_x c^x$):

$$f(x) = \frac{a_x c^x}{A(c)}, \quad x = 0, 1, \ldots$$

Rectangular (discrete uniform, $n = 0, 1, \ldots$):

$$f(x) = \frac{1}{n+1}, \quad x = 0, 1, \ldots, n$$

Zeta:

$$f(x) = \frac{1}{x^\alpha \sum_{i=1}^{\infty}(1/i)^\alpha}, \quad x = 1, 2, \ldots$$

Zipf ($\alpha \geq 0$):

$$f(x) = \frac{1}{x^\alpha \sum_{i=1}^{n}(1/i)^\alpha}, \quad x = 1, 2, \ldots, n$$

A.2 Continuous Distributions

Arcsin:

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \quad 0 < x < 1$$

Arctangent ($-\infty < \phi < \infty$):

$$f(x) = \frac{\lambda}{\left[\arctan(\lambda\phi) + \frac{\pi}{2}\right]\left[1 + \lambda^2(x - \phi)^2\right]}, \quad x \geq 0$$

Beta:

$$f(x) = \left[\frac{\Gamma(\beta + \gamma)}{\Gamma(\beta)\Gamma(\gamma)}\right] x^{\beta-1}(1-x)^{\gamma-1}, \quad 0 < x < 1$$

Cauchy (Lorentz, Breit–Wigner, $-\infty < a < \infty$):

$$f(x) = \frac{1}{\alpha\pi[1 + ((x-a)/\alpha)^2]}, \quad -\infty < x < \infty$$

Chi:

$$f(x) = \frac{1}{2^{n/2-1}\Gamma(n/2)} x^{n-1} e^{-x^2/2}, \quad x > 0$$

Chi-square:

$$f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}, \quad x > 0$$

Doubly noncentral F:

$$\begin{aligned} f(x) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left[\frac{e^{-\delta/2}\left(\frac{1}{2}\delta\right)^j}{j!}\right]\left[\frac{e^{-\gamma/2}\left(\frac{1}{2}\gamma\right)^k}{k!}\right] \\ &\quad \times n_1^{(n_1/2)+j} n_2^{(n_2/2)+k} x^{(n_1/2)+j-1} \\ &\quad \times (n_2 + n_1 x)^{-\frac{1}{2}(n_1+n_2)-j-k} \\ &\quad \times \left[B\left(\frac{1}{2}n_1 + j, \frac{1}{2}n_2 + k\right)\right]^{-1}, \qquad x > 0 \end{aligned}$$

Doubly noncentral t:

See Johnson, Kotz, and Balakrishnan (1995, p. 533)

Erlang:

$$f(x) = \frac{1}{\alpha^n (n-1)!} x^{n-1} e^{-x/\alpha}, \qquad x > 0$$

Error (exponential power, general error; $-\infty < a < \infty, b > 0, c > 0$):

$$f(x) = \frac{\exp\left[-(|x-a|/b)^{2/c}/2\right]}{b(2^{c/2+1})\Gamma(1 + c/2)}, \qquad -\infty < x < \infty$$

Exponential (negative exponential):

$$f(x) = (1/\alpha)e^{-x/\alpha}, \qquad x > 0$$

Exponential power:

$$f(x) = (e^{1-e^{\lambda x^\kappa}})e^{\lambda x^\kappa}\lambda\kappa x^{\kappa-1}, \qquad x > 0$$

Extreme value (Gumbel):

$$f(x) = (\beta/\alpha)e^{x\beta - e^{x\beta}/\alpha}, \qquad -\infty < x < \infty$$

F (variance ratio, Fisher–Snedecor):

$$f(x) = \frac{\Gamma((n_1 + n_2)/2)(n_1/n_2)^{n_1/2} x^{n_1/2-1}}{\Gamma(n_1/2)\Gamma(n_2/2)[(n_1/n_2)x + 1]^{((n_1+n_2)/2)}}, \quad x > 0$$

Gamma:

$$f(x) = \frac{1}{\alpha^\beta \Gamma(\beta)} x^{\beta-1} e^{-x/\alpha}, \qquad x > 0$$

Gamma–normal:

See Evans, Hastings, and Peacock (2000, p. 103)

Generalized gamma:

$$f(x)=\frac{\gamma}{\alpha^{\gamma\beta}\Gamma(\beta)}x^{\gamma\beta-1}e^{-(x/\alpha)^{\gamma}},\qquad x>0$$

Generalized Pareto:

$$f(x)=\left(\gamma+\frac{\kappa}{x+\delta}\right)(1+x/\delta)^{-\kappa}e^{-\gamma x},\quad x>0$$

Gompertz ($\kappa>1$):

$$f(x)=\delta\kappa^{x}e^{[-\delta(\kappa^{x}-1)/\log\kappa]},\quad x>0$$

Hyperbolic–secant:

$$f(x)=\operatorname{sech}(\pi x),\qquad -\infty<x<\infty$$

Hyperexponential ($p_i>0$, $\sum_{i=1}^{n}p_i=1$):

$$f(x)=\sum_{i=1}^{n}\frac{p_i}{\alpha_i}e^{-x/\alpha_i},\qquad x>0$$

Hypoexponential ($\alpha_i\neq\alpha_j$ for $i\neq j$):

$$f(x)=\sum_{i=1}^{n}(1/\alpha_i)e^{-x/\alpha_i}\left(\prod_{j=1,j\neq i}^{n}\frac{\alpha_i}{\alpha_i-\alpha_j}\right),\quad x>0$$

IDB ($\gamma\geq 0$):

$$f(x)=\frac{(1+\kappa x)\delta x+\gamma}{(1+\kappa x)^{\gamma/\kappa+1}}e^{-\delta x^{2}/2},\qquad x>0$$

Inverse Gaussian (Wald; $\mu>0$):

$$f(x)=\sqrt{\frac{\lambda}{2\pi x^{3}}}e^{-\frac{\lambda}{2\mu^{2}x}(x-\mu)^{2}},\qquad x>0$$

Inverted beta ($\beta>1,\gamma>1$):

$$f(x)=\frac{x^{\beta-1}(1+x)^{-\beta-\gamma}}{B(\beta,\gamma)},\qquad x>0$$

Inverted gamma:

$$f(x)=[1/\Gamma(\alpha)\beta^{\alpha}]x^{-\alpha-1}e^{-1/\beta x},\qquad x>0$$

Kolmogorov–Smirnov:

See Drew, Glen, and Leemis (2000)

Laplace (double exponential):

$$f(x)=\begin{cases}\big(1/(\alpha_1+\alpha_2)\big)e^{x/\alpha_1}, & x<0\\ \big(1/(\alpha_1+\alpha_2)\big)e^{-x/\alpha_2}, & x>0\end{cases}$$

Log gamma:

$$f(x)=[1/\alpha^{\beta}\Gamma(\beta)]e^{\beta x}e^{-e^{x}/\alpha},\quad -\infty<x<\infty$$

Log logistic:

$$f(x)=\frac{\lambda\kappa(\lambda x)^{\kappa-1}}{[1+(\lambda x)^{\kappa}]^{2}},\quad x>0$$

Log normal ($-\infty<\alpha<\infty$):

$$f(x)=\frac{1}{\sqrt{2\pi}\beta x}\exp\left[-\frac{1}{2}(\log(x/\alpha)/\beta)^{2}\right],\quad x>0$$

Logistic:

$$f(x)=\frac{\lambda^{\kappa}\kappa e^{\kappa x}}{[1+(\lambda e^{x})^{\kappa}]^{2}},\qquad -\infty<x<\infty$$

Logistic–exponential:

$$f(x)=\frac{\alpha\beta(e^{\alpha x}-1)^{\beta-1}e^{\alpha x}}{(1+(e^{\alpha x}-1)^{\beta})^{2}},\qquad x>0$$

Lomax:

$$f(x)=\frac{\lambda\kappa}{(1+\lambda x)^{\kappa+1}},\qquad x>0$$

Makeham ($\kappa>1$):

$$f(x)=(\gamma+\delta\kappa^{x})e^{-\gamma x-\frac{\delta(\kappa^{x}-1)}{\log\kappa}},\qquad x>0$$

Minimax:

$$f(x)=\beta\gamma x^{\beta-1}(1-x^{\beta})^{\gamma-1},\qquad 0<x<1$$

Muth:

$$f(x)=(e^{\kappa x}-\kappa)e^{[-\frac{1}{\kappa}e^{\kappa x}+\kappa x+\frac{1}{\kappa}]},\qquad x>0$$

Noncentral beta:

$$f(x)=\sum_{i=0}^{\infty}\frac{\Gamma(i+\beta+\gamma)}{\Gamma(\gamma)\Gamma(i+\beta)}\left(\frac{e^{-\delta/2}}{i!}\right)\left(\frac{\delta}{2}\right)^{i}$$
$$\times x^{i+\beta-1}(1-x)^{\gamma-1},\quad 0<x<1$$

Noncentral chi-square:

$$f(x)=\sum_{k=0}^{\infty}\frac{\exp(-\frac{\delta}{2})(\frac{\delta}{2})^{k}}{k!}\cdot\frac{\exp(-\frac{x}{2})x^{\frac{n+2k}{2}-1}}{2^{\frac{n+2k}{2}}\Gamma(\frac{n+2k}{2})},\qquad x>0$$

Noncentral F:

$$f(x)=\sum_{i=0}^{\infty}\frac{\Gamma\left(\frac{2i+n_1+n_2}{2}\right)\left(\frac{n_1}{n_2}\right)^{(2i+n_1)/2}x^{(2i+n_1-2)/2}e^{-\delta/2}\left(\frac{\delta}{2}\right)^{i}}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{2i+n_1}{2}\right)i!\left(1+\frac{n_1}{n_2}x\right)^{(2i+n_1+n_2)/2}},$$
$$x>0$$

Noncentral t ($-\infty<\delta<\infty$):

$$f(x)=\frac{n^{n/2}\exp(-\delta^{2}/2)}{\sqrt{\pi}\Gamma(n/2)(n+x^{2})^{(n+1)/2}}$$
$$\times\sum_{i=0}^{\infty}\frac{\Gamma[(n+i+1)/2]}{i!}\left(\frac{x\delta\sqrt{2}}{\sqrt{n+x^{2}}}\right)^{i},$$
$$-\infty<x<\infty$$

Normal (Gaussian):

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}((x-\mu)/\sigma)^2\right], \quad -\infty < x < \infty$$

Pareto:

$$f(x) = \frac{\kappa\lambda^\kappa}{x^{\kappa+1}}, \quad x > \lambda$$

Power:

$$f(x) = \frac{\beta x^{\beta-1}}{\alpha^\beta}, \quad 0 < x < \alpha$$

Rayleigh:

$$f(x) = (2x/\alpha)e^{-x^2/\alpha}, \quad x > 0$$

Standard Cauchy:

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

Standard normal:

$$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \quad -\infty < x < \infty$$

Standard power:

$$f(x) = \beta x^{\beta-1}, \quad 0 < x < 1$$

Standard triangular:

$$f(x) = \begin{cases} x+1, & -1 < x < 0 \\ 1-x, & 0 \le x < 1 \end{cases}$$

Standard uniform:

$$f(x) = 1, \quad 0 < x < 1$$

t (Student's t):

$$f(x) = \frac{\Gamma((n+1)/2)}{(n\pi)^{1/2}\Gamma(n/2)[x^2/n+1]^{(n+1)/2}}, \quad -\infty < x < \infty$$

Triangular ($a < m < b$):

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(m-a)}, & a < x < m \\ \dfrac{2(b-x)}{(b-a)(b-m)}, & m \le x < b \end{cases}$$

TSP (two-sided power):

$$f(x) = \begin{cases} \dfrac{n}{b-a}\left(\dfrac{x-a}{m-a}\right)^{n-1}, & a < x \le m \\ \dfrac{n}{b-a}\left(\dfrac{b-x}{b-m}\right)^{n-1}, & m \le x < b \end{cases}$$

Uniform (continuous rectangular; $-\infty < a < b < \infty$):

$$f(x) = 1/(b-a), \quad a < x < b$$

von Mises ($0 < \mu < 2\pi$):

$$f(x) = \frac{e^{\kappa\cos(x-\mu)}}{2\pi I_0(\kappa)}, \quad 0 < x < 2\pi$$

Wald (Standard Wald):

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}}\, e^{-\frac{\lambda}{2x}(x-1)^2}, \quad x > 0$$

Weibull:

$$f(x) = (\beta/\alpha)x^{\beta-1}\exp\left[-(1/\alpha)x^\beta\right], \quad x > 0$$

A.3 Functions

Gamma function:

$$\Gamma(c) = \int_0^\infty e^{-x}x^{c-1}\,dx$$

Beta function:

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

Modified Bessel function of the first kind of order 0:

$$I_0(\kappa) = \sum_{i=0}^\infty \frac{\kappa^{2i}}{2^{2i}(i!)^2}$$

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