Lab 1: Determining Hubble's Constant

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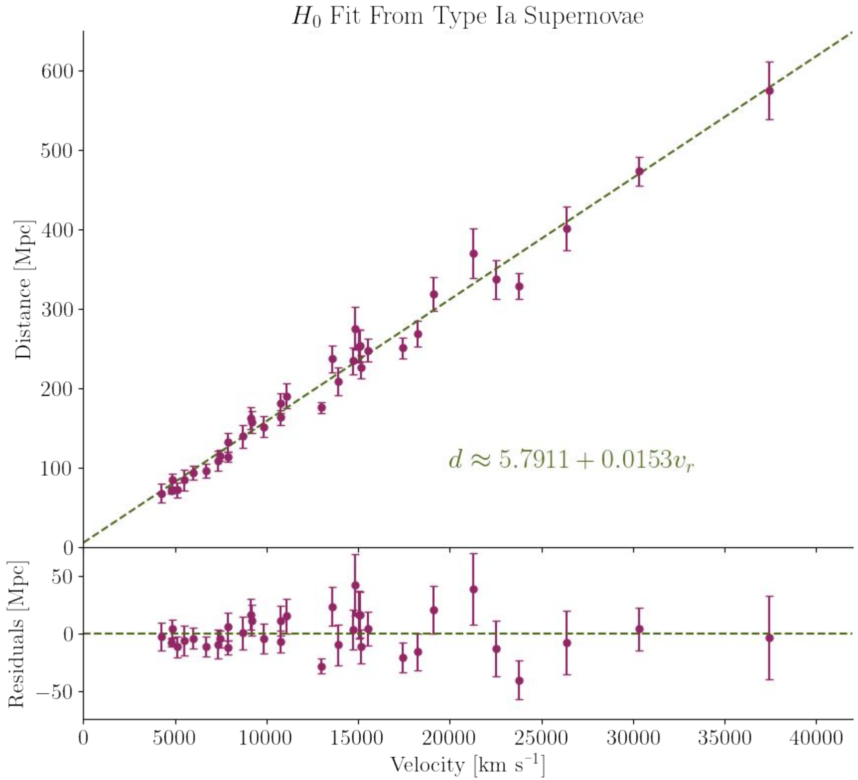
Type Ia supernovae are used as standard candles because their luminosity (and subsequently distance) can be calculated from observables. We can accurately measure radial velocity from the Doppler shift of spectral lines. We can estimate Hubble's Constant, the expansion rate of the universe, by performing a linear regression to the data.

Unweighted fit: obtain the fit coefficients β using `np.polyfit` or manually solving the normal equation

$$\mathrm{X} = \begin{bmatrix} 1 & v_1 \\ 1 & v_2 \\ \vdots & \vdots \\ 1 & v_n \end{bmatrix} \quad y = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_3 \end{bmatrix}$$

$$(\mathrm{X}^\mathsf{T}\mathrm{X})\beta = \mathrm{X}^\mathsf{T} y$$



The gradient or slope of the graph, with units of time, approximates the age of the universe, to a factor dependent on the dark energy and mass density of the universe.

$$\frac{1}{H_0} \approx t_0$$

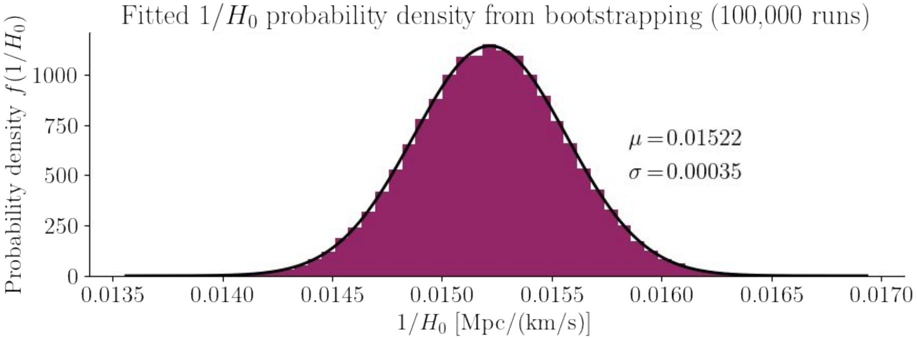
Weighted fit: we account for uncertainty in the distance measurements by passing the weights (reciprocal of individual measurement uncertainty) to `polyfit` or including the matrix W in the normal equation.

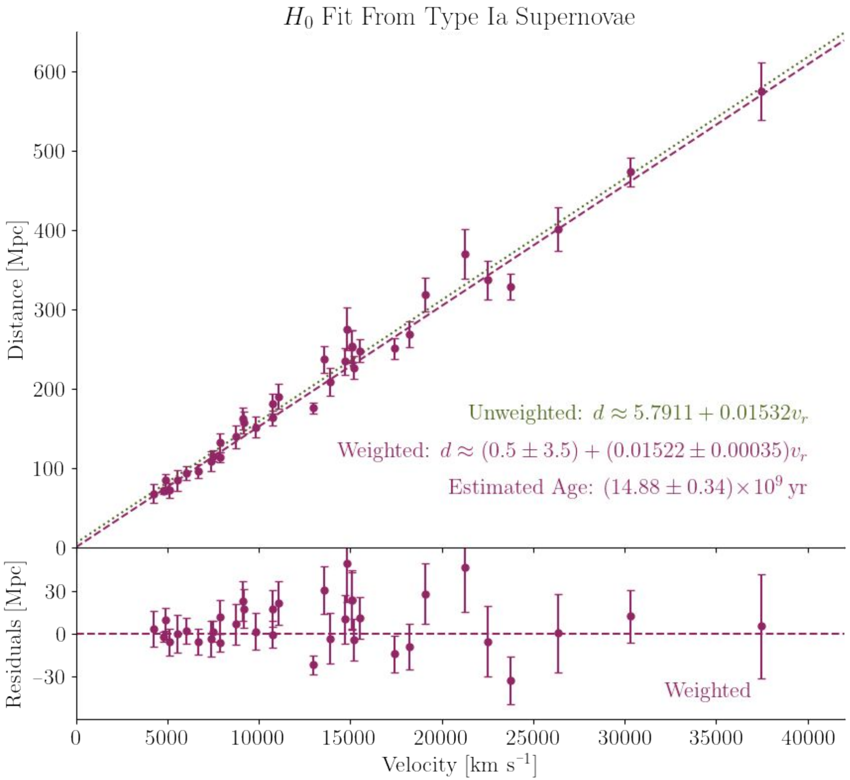
$$\mathrm{W} = \begin{bmatrix} 1/e_1^2 & & & \\ & 1/e_2^2 & & \\ & & \ddots & \\ & & & 1/e_n^2 \end{bmatrix} \qquad \begin{aligned} \mathrm{S} &= (\mathrm{X}^\mathsf{T}\mathrm{W}\mathrm{X}) \\ &= \begin{bmatrix} {\sigma_x}^2 & \sigma_x\sigma_y \\ \sigma_x\sigma_y & {\sigma_y}^2 \end{bmatrix} \end{aligned}$$

$$(\mathrm{X}^\mathsf{T}\mathrm{W}\mathrm{X})\beta = \mathrm{X}^\mathsf{T}\mathrm{W}y$$

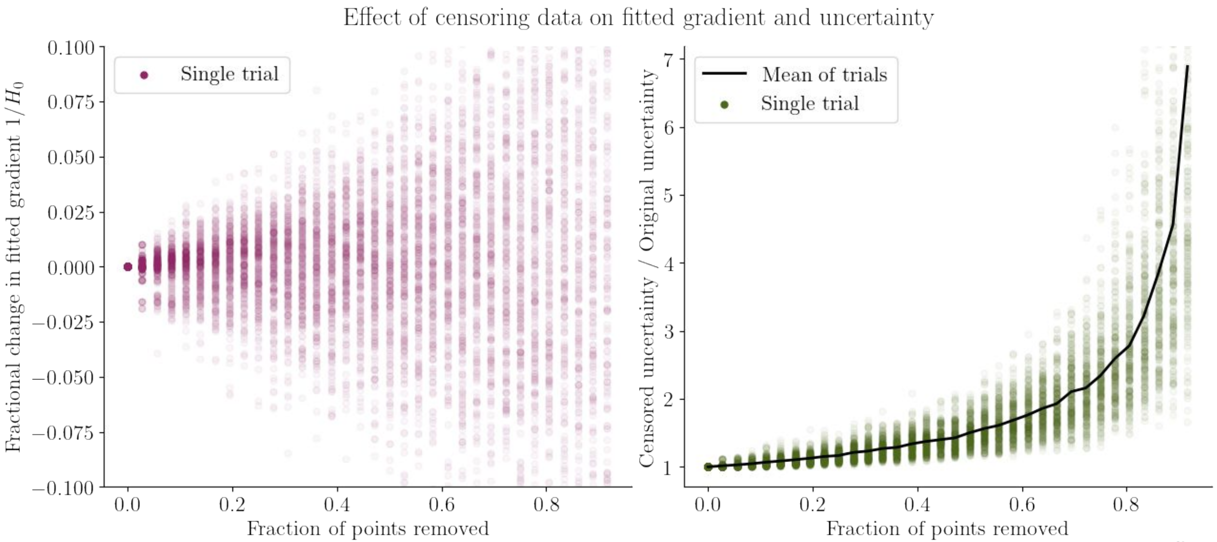
The covariance matrix S allows us to express uncertainties for the fitted gradient:

$$\begin{aligned} \frac{1}{H_0} &\approx (0.01522 \pm 0.00035)\ \mathrm{Mpc/(km/s)} \\ &\approx (14.88 \pm 0.34)\ \mathrm{Gyr} \end{aligned}$$





Bootstrapping: we can obtain an alternative uncertainty for the fit by resampling the data points from normal distributions many times and performing the fit for each sample. This gives us an approximately normal distribution for the gradient, agreeing with the covariance matrix uncertainty to two significant figures of the uncertainty.



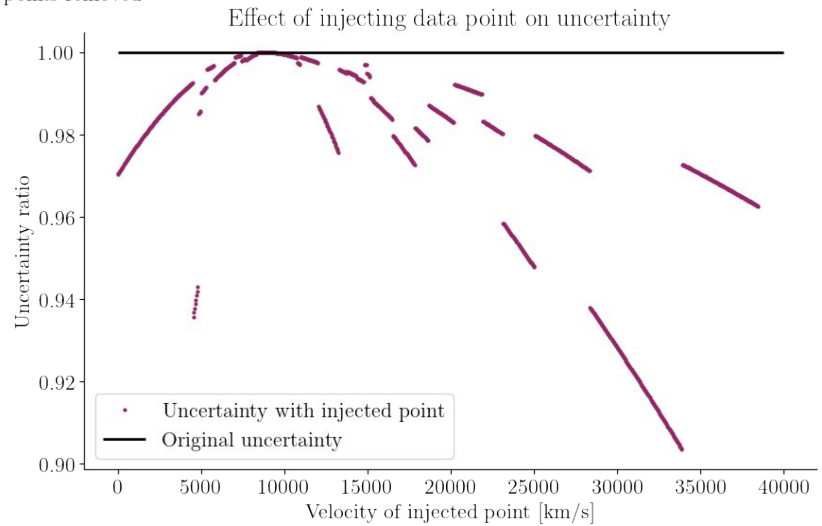
Censoring: we see how the fitted gradient and uncertainty of the fit change as an increasing fraction of the data is removed.

For each number of missing points, 256 random samples and the resulting fits were calculated.

The fit is consistent even when 25% or less of the data points are removed.

Injecting points: we see how much our uncertainty changes when a single data point is added at different velocities, with the same uncertainty as the nearest actual data point

- The covariance matrix S depends only on the velocities and the uncertainties on the distances, not the distances themselves
- The uncertainty can only decrease as more points are added, even if they are heavy outliers from the rest of the data set
- Points added far out from the median affect the uncertainty much more than points added near it.



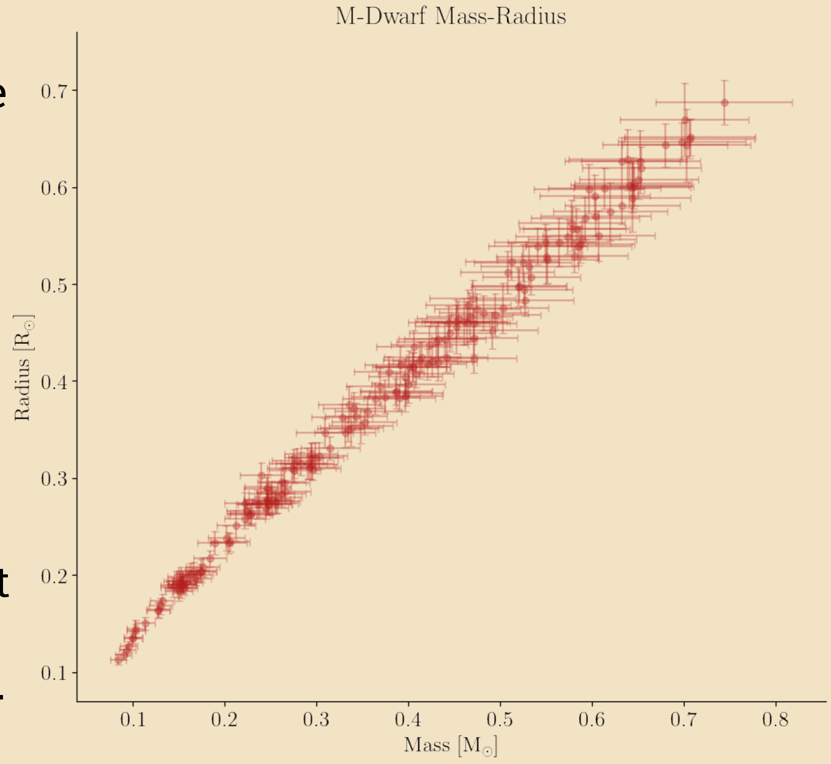
M-Dwarf Mass-Radius Relation

M-Dwarfs (cool red dwarfs) are so low mass and long lived that they cannot have evolved off of the main sequence in the ~14 Gyr history of the universe so far.

This suggests it's possible to find tight fitting models for a large population. We find a relation between the mass and radius.

Mass and radius data from Mann et al. 2015.

- Mass from mass-luminosity relation calibrated with binary systems by Delfosse et al. 2000.
- Radius from Teff and Stefan-Boltzmann Law.



Because there are uncertainties on both the dependent and independent variables, we define a likelihood function[1] of fit coefficients and maximize it (using `scipy.optimize.minimize`).

$$\Delta_i = y_i - mx_i - b$$

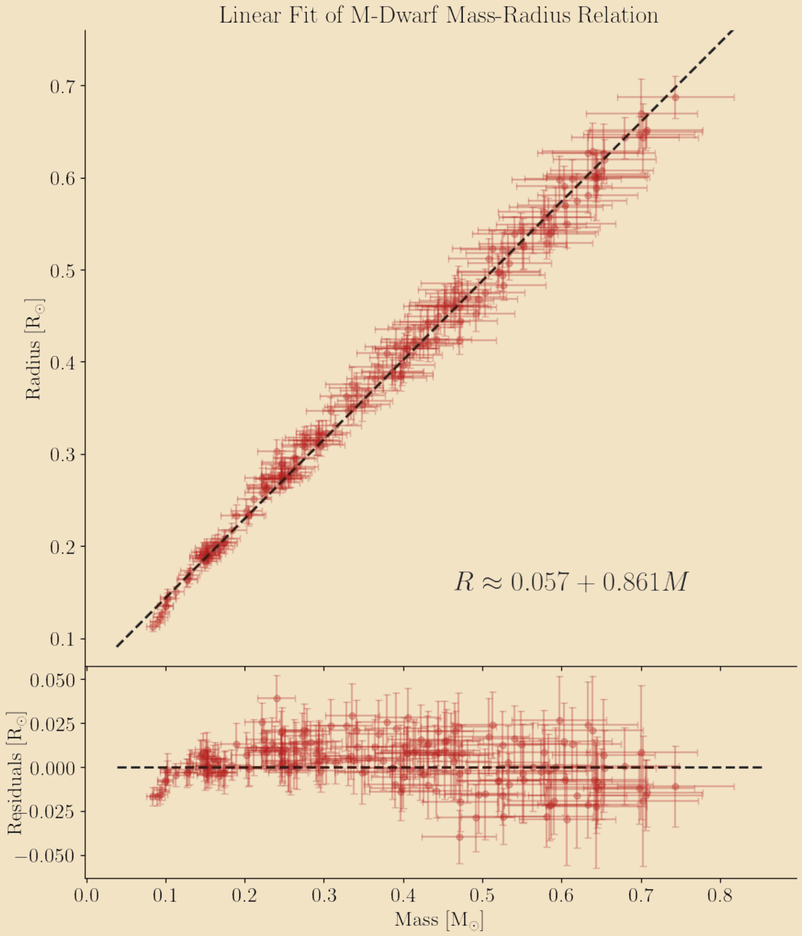
$${\Sigma_i}^2 = {\sigma_y}^2 + m^2{\sigma_x}^2 - 2m\sigma_{xy}$$

$$\ln \mathcal{L} = K - \sum_{i=1}^{n} \frac{1}{2}\left[\ln\left({\Sigma_i}^2\right) + \frac{{\Delta_i}^2}{{\Sigma_i}^2}\right]$$

The uncertainty for each data point is described by a variance in each direction ${\sigma_x}^2$, ${\sigma_y}^2$ and covariance σ_{xy}, but for our data we assume $\sigma_{xy} = 0$.

Linear regression does not fit the relation well.

[1]Correction to Eq. 32 in Hogg by D. Foreman-Mackey and J.Bovy (does not account for intrinsic scatter)
https://dfm.io/posts/fitting-a-plane/#marginalizing-over-the-true-coordinates
https://github.com/davidwhogg/DataAnalysisRecipes/issues/18#issuecomment-305301479



We can better fit the relation with a power-law relation

First we compute the logarithm of both variables, linearly propagating the uncertainty[2],

$$f = \ln A$$

$$\sigma_f = \frac{\sigma_A}{A}$$

then computing the maximum likelihood linear fit to the log-log data as before.

This gives a better fit, although Mann et al. 2015 use a quadratic model.