Cosserat Test suite

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Tests involving Cosserat mechanics are described. The notation is defined in the Theory manual.

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1 Cosserat glide: elasticity

Forest[1] describes a "glide" test of Cosserat elasticity in his Appendix A. The test involves a 3D material, but all quantities are assumed to be functions of the $x_2 = y$ direction only. The displacement field, u, and Cosserat rotation, θ^{c} are assumed to obey

$$\begin{aligned} u_i &= (u_x(y),\ 0,\ 0)\ , && (1.1)\\ \theta^{\mathrm{c}}_i &= (0,\ 0,\ \theta^{\mathrm{c}}_z(y))\ . && (1.2) \end{aligned}$$

These mean that the strain tensor, γ, and curvature tensor, κ, are

$$\gamma = \begin{pmatrix} 0 & \theta^{\mathrm{c}}_z + \nabla_y u_x & 0 \\ -\theta^{\mathrm{c}}_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \kappa = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \nabla_y \theta^{\mathrm{c}}_z & 0 \end{pmatrix} \tag{1.3}$$

The solution below has boundary conditions $u_x(0) = 0 = \theta^{\mathrm{c}}_z(y)$.

The stress and couple-stress tensors are assumed to be zero, except for the following components:

$$\begin{aligned} \sigma_{yx} &\neq 0\ , && (1.4)\\ m_{zy} &\neq 0\ . && (1.5) \end{aligned}$$

If standard (non-Cosserat, Cauchy) elasticity were being used, the solution is the trivial $u_x = 0$. However, using Cosserat elasticity a nontrivial solution is found. Physically this setup corresponds to a 3D object with infinite extent in the x and z directions subjected to a moment m_{zy} that rotates the Cosserat grains. The 3D object may or may not be finite in the y direction. The MOOSE simulation actually uses a unit cube of material: the infiniteness in the x and z directions is irrelevant since there is no dependence of the variables on these directions.

Isotropic elasticity is assumed, with the additional assumption that the couple-stress equation only involves two moduli:

$$\begin{aligned} \sigma_{ij} &= \lambda\delta_{ij}\mathrm{Tr}\gamma + 2\mu\gamma_{(ij)} + 2\alpha\gamma_{[ij]}\ , \\ m_{ij} &= \beta\delta_{ij}\mathrm{Tr}\kappa + 2\varepsilon\kappa_{(ij)} + 2\varepsilon\kappa_{[ij]}\ . && (1.6) \end{aligned}$$

With these assumptions, the moment and force balance equations reduce to a second-order ODE

[1] S Forest "Mechanics of Cosserat media An introduction". Available from http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.154.4476&rep=rep1&type=pdf

that has solution

$$
\begin{aligned}
\theta_z^c &= B\sinh(\omega_e y) \ , & (1.7) \\
u_x &= \frac{2\alpha B}{\omega_e(\mu+\alpha)}(1-\cosh(\omega_e y)) \ , & (1.8) \\
m_{zy} &= 2B\varepsilon\omega_e\cosh(\omega_e y) \ , & (1.9) \\
\sigma_{yx} &= -\frac{4\mu\alpha}{\mu+\alpha}B\sinh(\omega_e y) & (1.10)
\end{aligned}
$$

with B being an arbitrary constant of integration, and

$$
\omega_e = \sqrt{\frac{2\mu\alpha}{\varepsilon(\mu+\alpha)}} \ . \qquad (1.11)
$$

Forest's notation is slightly different: for α he writes μ_c, and for ε he writes β.

The MOOSE simulation uses 100 elements in the y direction, with $\mu = 2$, $\alpha = 3$ and $\varepsilon = 0.6$. This gives $w_e = 2$. Preset boundary conditions at $y = 0$ and $y = 1$ are used, and the system relaxes to the equilirium solution within 1 iteration. Figure 1.1 reveals that the MOOSE simulation agrees with expectations. The displacements agree well for the 100-element simulation, but the stress components agree less well (this is not really observable to the eye in Figure 1.1) and even a non-zero σ_{xy} appears. However, as the number of elements is increased, the stresses tend to the analytical formulae given above.

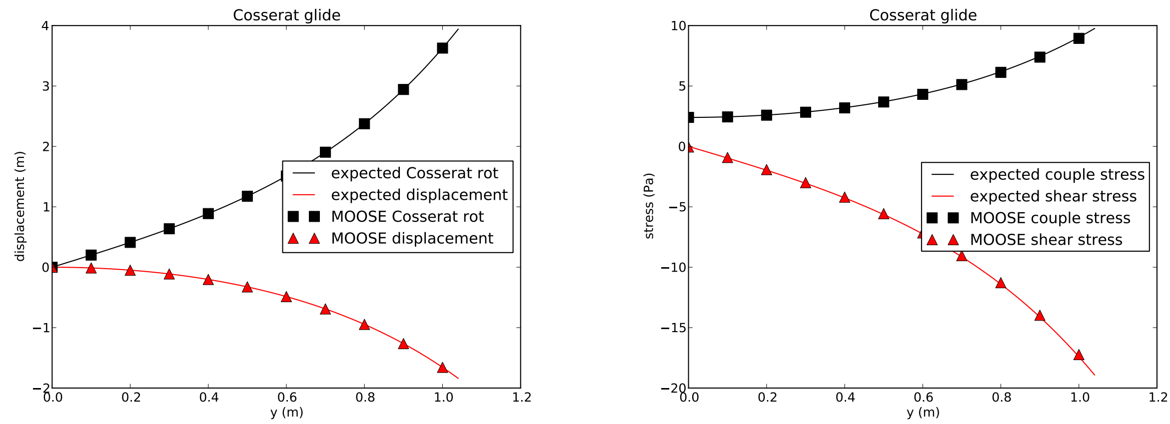

Figure 1.1: Results from the elastic Cosserat glide test. Left: displacements. Right: stresses

2 Cosserat tension

This is a simple test where a 3D sample is subjected to a normal load on its top surface. The sample is allowed to shrink in directions perpendicular to the force, via the Poisson's ratio. Specifically, all components of the stress tensor are zero except for

$$\sigma_{22} \neq 0 \ , \tag{2.1}$$

(which is constant). There are no Cosserat rotations involved:

$$m = 0 = \kappa \ . \tag{2.2}$$

A general isotropic elasticity tensor is assumed so that the constitutive relation reads

$$\sigma_{ij} = \lambda \delta_{ij} \mathrm{Tr}\gamma + 2\mu\gamma_{(ij)} + 2\alpha\gamma_{[ij]} \ . \tag{2.3}$$

The solution is identical to the standard (non-Cosserat) case, which is independent of α, and has strain components

$$\varepsilon_{22} = \frac{(\lambda+\mu)}{\mu(3\lambda+2\mu)}\sigma_{22} \quad \text{and} \quad \varepsilon_{11} = \varepsilon_{33} = -\frac{\lambda}{2(\mu+\lambda)}\varepsilon_{22} \ . \tag{2.4}$$

MOOSE generates this solution exactly.