Bayesian Statitics Exercise 1

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1 Solutions Exercise 1

This document is the solution for the exercise 1 of Homework 1. All equations use $p = \frac{1}{2}$.

1.1 Question 1

1.1.1 Part A

The objective is to find the pmf of X_2 conditional to a given value of X_1. Therefore, we compute first $P(X_1)$

$$P(X_1 = n_1) = \binom{n}{n_1} p^{n_1}(1-p)^{n-n_1} \tag{1}$$

Then we compute the conditional probability

$$P(X_2 = n_2|X_1 = n_1) = \binom{n-n_1}{n_2} p^{n_2}(1-p)^{n-n_1-n_2} \tag{2}$$

1.1.2 Part B

The obective is to find the pmf joint $(X_1; X_2)$ and $(X_1; X_2; R_2)$ First we compute the joint $(X_1; X_2)$ using Bayes formula

$$P(X_2 = n_2; X_1 = n_1) = P(X_2 = n_2|X_1 = n_1)P(X_1 = n_1) \tag{3}$$

$$= \binom{n-n_1}{n_2} p^{n_2}(1-p)^{n-n_1-n_2} \binom{n}{n_1} p^{n_1}(1-p)^{n-n_1} \tag{4}$$

$$P(X_2 = n_2; X_1 = n_1) = \binom{n-n_1}{n_2}\binom{n}{n_1} p^{n_2+n_1}(1-p)^{2n-2n_1-n_2} \tag{5}$$

We define R_2 as $R_2 = n - X_1 - X_2 = R_1 - X_2$, that is, the number of coins that yield tails in the second tossing. We can clearly see how R_2 and X_2 are completely dependent, since X_2 is the number of coins that yield heads in the second tossing. So given one, the other is fully determined. Therefore:

$$P(R_2 = r_2 | X_2 = R_2) = P(X_2 = n_2 | R_2 = r_2) = 1 \tag{6}$$

with $r_2 = n - n_1 - n_2$. Thus, using it on equation (5) we obtain:

$$P(R_2 = r_2; X_1 = n_1) = P(X_2 = n - n_1 - r_2; X_1 = n_1) \tag{7}$$

$$P(R_2 = r_2; X_1 = n_1) = \binom{n - n_1}{n - n_1 - r_2}\binom{n}{n_1} p^{n-r_2}(1-p)^{n-n_1-r_2} \tag{8}$$

$$P(R_2 = r_2; X_1 = n_1) = \binom{n - n_1}{r_2}\binom{n}{n_1} p^{n-r_2}(1-p)^{n-n_1-r_2} \tag{9}$$

we observe how the joint distribution of X_1, R_2 is the same as X_1, X_2. And similarly,

$$P(R_2 = r_2; X_1 = n_1; X_2 = n_2) = \binom{n - n_1}{n - n_1 - r_2}\binom{n}{n_1} p^{n-r_2}(1-p)^{n-n_1-r_2} \tag{10}$$

or

$$P(R_2 = r_2; X_1 = n_1; X_2 = n_2) = \binom{n - n_1}{n_2}\binom{n}{n_1} p^{n_2+n_1}(1-p)^{2n-2n_1-n_2} \tag{11}$$

1.1.3 Part C

The objective is to find the marginal of X_2. We start with the Bayes definition for marginal and then we get the marginal expresion.

$$P(X_2 = n_2) = \sum_{n_1=0}^{n} P(X_2 = n_2; X_1 = n_1) \tag{12}$$

$$= \sum_{n_1=0}^{n} P(X_2 = n_2 | X_1 = n_1) P(X_1 = n_1) \tag{13}$$

$$= \sum_{n_1=0}^{n} \binom{n}{n_1}\binom{n - n_1}{n_2} p^{n_2+n_1}(1-p)^{2n-2n_1-n_2} \tag{14}$$

The marginal $P(R_2)$ is:

$$P(R_2 = r_2) = \sum_{n_1=0}^{n} P(R_2 = r_2; X_1 = n_1) \tag{15}$$

$$= \sum_{n_1=0}^{n} P(R_2 = r_2 | X_1 = n_1) P(X_1 = n_1) \tag{16}$$

$$= \sum_{n_1=0}^{n} \binom{n}{n_1}\binom{n - n_1}{r_2} p^{r_2+n_1}(1-p)^{2n-2n_1-r_2} \tag{17}$$

1.2 Question 2

Question 2 ask for generalising the before formulas with more than two group coin tosses. We define the group coin tosses with letter k. Then we observe that the joint probability of throwing up to k depends on the joint probability of $k-1$ due to the chain rule:

$$P(X_k; X_{k-1}; ...X_2; X_1) = P(X_k|X_{k-1}; ...X_2; X_1)P(X_{k-1}; ...X_2; X_1) \tag{18}$$

$$= P(X_k|X_{k-1}; ...X_2; X_1) = P(X_{k-1}|...X_2; X_1)...P(X_2|X_1)P(X_1) \tag{19}$$

If we generalize and use variable $Y = X_{k-1}; ..; X_1$ we have

$$P(X_k = n_k|Y = y) = \binom{n-y}{n_k} p^{n_k}(1-p)^{n-y-n_k} \tag{20}$$

Then, if we recursively apply this equation we obtain:

$$P(X_k = n_k; X_{k-1}; ...X_2; X_1) \tag{21}$$

$$= \binom{n}{n_1}\binom{n-n_1}{n_2}...\binom{n-n_1-...-n_{k-1}}{n_k} p^{n_1+...+n_k}(1-p)^{n-n_1-...-n_k} \tag{22}$$

1.3 Question 3

In this section we find the cdf of variable Y, total number of tosses. First we define the pmf of Z a random variable that is the number of tosses of a single coin that are necessary for obtaining head (p is probability of head)

$$P(Z = z) = p(1-p)^{z-1} \tag{23}$$

And its cdf is:

$$P(Z = z \leq n) = \sum_{i=1}^{n} p(1-p)^{i-1} \tag{24}$$

Then we can conclude that $Y = \max(Z_1, Z_2, ..., Z_n)$ as the maximum number of tosses will will be equal to the coin that takes more tosses to get head. Therefore, the cdf will be:

$$F(y) = P(Y \leq y) = P(\max(Z_1, Z_2, ..., Z_n) \leq y) \tag{25}$$

As each Z is independient:

$$F(y) = P(Y \leq y) = P(Z_1 \leq y)P(Z_2 \leq y)P(Z_n \leq y) \tag{26}$$

$$F(y) = (\sum_{i=1}^{y} p(1-p)^{i-1})^n \tag{27}$$