Monotonic Convergence Theorem

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Problem. *Show that the sequence $x_1 = 3$ and $x_{n+1} = \frac{1}{4-x_n}$ where $n \in N$ converges and find its limit.*
It will first be shown that $0 < x_n \leq 3$, then that x_n is monotonically decreasing proving that x_n converges to a limit $s \in R$. The limit can then be computed using limit theorems for convergent sequences.
The proof that $0 < x_n \leq 3$ is by induction. Clearly $0 < x_1 = 3 \leq 3$ establishing the basis of induction. Assume $0 < x_n \leq 3$. $x_{n+1} = \frac{1}{4-x_n}$ and by the induction hypothesis $0 < \frac{1}{4} \leq x_{n+1} \leq 1 < 3$, so by induction $0 < x_n \leq 3$.
The proof that x_n is decreasing is also by induction. The first two terms of the sequence, x_1 and x_2 are 3 and 1 respectively. Clearly $x_2 = 1 \leq x_1 = 3$, establishing the basis of induction. Assume that $x_{n+1} \leq x_n$. It must be shown that $x_{n+2} \leq x_{n+1}$. Observe that $x_{n+2} = \frac{1}{4-x_{n+1}} \leq \frac{1}{4-x_n} = x_{n+1}$ by the induction hypothesis since it was proved above that x_n is bounded by 0 and 3. So by induction x_n is decreasing.
Since x_n is bounded and monotonic it converges to a limit $s \in R$. Since it is known that $x_{n+1} = \frac{1}{4-x_n}$, $\lim_{n\to\infty} x_{n+1} = \frac{1}{4-\lim_{n\to\infty} x_n}$. Noting that $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n$, we have

$$\begin{aligned} s &= \frac{1}{4-s} \\ 4s - s^2 &= 1 \\ -s^2 + 4s - 1 &= 0 \\ (s-2)^2 &= 3 \\ s &= 2 \pm \sqrt{3} \end{aligned}$$

Since $2+\sqrt{3} > 3$ and it was proved above that $x_n \leq 3$ for all n, x_n must converge to the limit $2 - \sqrt{3}$.

Problem. *Show that the sequence $y_1 = 1$ and $y_{n+1} = 4 - \frac{1}{y_n}$ converges and find its limit.*
It will first be shown that $1 \leq y_n < 4$, then that y_n is monotonically increasing proving that y_n converges to a limit $s \in R$. The limit can then be computed using limit theorems for convergent sequences.

The proof that $1 \leq y_n < 4$ is by induction. Clearly $1 \leq y_1 = 1 < 4$ establishing the basis of induction. Assume $1 \leq y_n < 4$ then by the induction hypothesis $1 < 3 \leq y_{n+1} \leq \frac{15}{4} < 4$ so by induction $1 \leq y_n < 4$.
The proof that y_n is increasing is also by induction. Clearly $y_2 = 3 > y_1 = 1$ establishing the basis of induction. Assume $y_{k+1} \geq y_k$ for some k, it must be shown that $y_{k+2} \geq y_{k+1}$. $y_{k+2} = 4 - 1y_{k+1} \geq 4 - \frac{1}{y_k} = y_{k+1}$ by the induction hypothesis, so by induction y_n is increasing.
Since y_n is bounded and monotonic, by the monotonic convergence theorem it converges to a limit $s \in R$. Since it is known that $y_{n+1} = 4 - \frac{1}{y_n}$ it must be true that $\lim_{n\to\infty} y_{n+1} = 4 - \frac{1}{\lim_{n\to\infty} y_n}$. So we have

$$\begin{aligned} s &= 4 - \frac{1}{s} \\ 4 - s &= \frac{1}{s} \\ 4s - s^2 - 1 &= 0 \\ (s-2)^2 &= 3 \\ s &= 2 \pm \sqrt{3} \end{aligned}$$

Since $2 - \sqrt{3} < 1$ and it was proved above that $1 \leq y_n$ for all n, y_n must converge to $2 + \sqrt{3}$.

Show that the sequence $s_1 = 1$ and $s_{n+1} = \sqrt{2 + 7s_n}$ converges and find its limit.
It will first be shown that $1 \leq s_n < 8$, then that s_n is monotonically increasing proving that s_n converges to a limit $s \in R$. The limit can then be computed using limit theorems for convergent sequences.
The proof that $1 \leq s_n < 8$ is by induction. Clearly $1 \leq s_1 = 1 < 8$ establishing the basis of induction. Assume $1 \leq s_n < 8$, then $1 \leq 3 \leq s_{n+1} \leq \sqrt{58} < 8$, so by induction $1 \leq s_n < 8$.
The proof that s_n is monotonically increasing is also by induction. Clearly $s_1 = 1 < 3 = s_2$ establishing the basis of induction. Assume $s_{n+1} \geq s_n$, it must be shown that $s_{n+2} \geq s_{n+1}$. $s_{n+2} = \sqrt{2_7 s_{n+1}} \geq \sqrt{2 + 7s_n} = s_{n+1}$ by the induction hypothesis, so by induction s_n is increasing.
Since s_n is increasing and bounded, it converges to a limit $s \in R$. Since it is known that $s_{n+1} = \sqrt{2 + 7s_n}$, it must be true that $\lim_{n\to\infty} s_{n+1} = \sqrt{2 = 7 \lim_{n\to\infty} s_n}$. Then,

$$\begin{aligned} s &= \sqrt{2 + 7s} \\ s^2 - 7s - s &= 0 \\ (s - \frac{7}{2})^2 - \frac{57}{4} &= 0 \\ s &= \frac{7}{2} \pm \frac{\sqrt{57}}{2} \end{aligned}$$

Since s_n cannot converge to $\frac{7}{2} - \frac{\sqrt{57}}{2}$ it must converge to $\frac{7}{2} + \frac{\sqrt{57}}{2}$

Problem. *Show that* $\sqrt{2}$, $\sqrt{2\sqrt{2}}$, $\sqrt{2\sqrt{2\sqrt{2}}}$... *converges and find its limit.*
Note that this sequence can be defined recursively as the sequence $s_1 = \sqrt{2}$ *and* $s_{n+1} = \sqrt{2s_n}$.
It will first be shown that $\sqrt{2} \leq s_n < 2$, *then that* s_n *is monotonically increasing proving that* s_n *converges to a limit* $s \in R$. *The limit can then be computed using limit theorems for convergent sequences.*
The proof that $\sqrt{2} \leq s_n < 2$ *is by induction. Clearly* $\sqrt{2} \leq \sqrt{2} = s_1 < 2$ *establishing the basis of induction. Assume* $\sqrt{2} \leq s_k < 2$ *for some* k. *Then* $s_{k+1} = \sqrt{2s_k} = \sqrt{2}\sqrt{s_k}$ *so* $\sqrt{2} < \sqrt{2\sqrt{2}} \leq s_{k+1} < 2$ *and by induction* $\sqrt{2} \leq s_k < 1$.
The proof that s_n *is monotonically increasing is also by induction. Clearly* $\sqrt{2} \approx 1.41 \leq \sqrt{2\sqrt{2}} \approx 1.68$ *establishing the basis of induction. Assume* $s_{k+1} \geq s_k$ *for some* k. *Then* $s_{k+2} = \sqrt{2s_{k+1}} \geq \sqrt{2s_k} = s_{k+1}$ *by the induction hypothesis. So by induction* s_n *is increasing.*
Since s_n *is bounded and monotonic it converges to a limit* $s \in R$. *Since it is known that* $s_{n+1} = \sqrt{2s_n}$, *it must be true that*

$$\begin{aligned}
\lim_{n\to\infty} s_{n+1} &= \sqrt{2 \lim_{n\to\infty} s_n} \\
s &= \sqrt{2s} \\
s^2 - 2s &= 0 \\
s(s-2) &= 0 \\
s = 2 &\ \mathit{or}\ s = 0
\end{aligned}$$

s_n *cannot converge to 0 since* s_n *is increasing and begins with a number* $s_1 > 0$ *so* s_n *converges to limit 2.*