Limit Theorems

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Problem. *53. $s_n = (-1)^n$ is an example of a bounded sequence which does not converge*

Problem. *Let $\lim_{n\to\infty} a_n = a$. If $a_n > 0$ for all $n \in N$ then $a \geq 0$.*

Proof. Since $\lim_{n\to\infty} a_n = a$, then for all $\epsilon > 0$ there exists an $n \in N$ such that $|a_n - a| < \epsilon$. Assume for contradiction that $a < 0$, then $|a_n - a| > a_n \geq 0$, so $a_n - a < \epsilon$ for all $\epsilon > 0$. Since $a_n \geq 0$ consider $\epsilon = a_n + \epsilon_1$ where $\epsilon_1 > 0$. Then there exists an $n \in N$ such that $a_n - a < a_n + \epsilon_1$ or $a > -\epsilon_1$. Since $a > -\epsilon_1$ it must be true that $a \geq 0$. But $a < 0$ by assumption which is contradictory and it must be true that $a \geq 0$. QED

Problem. *If there exists a $c \in R$ for which $c \leq b_n$ for all $n \in N$ then $c \leq b$. Similarly if $a_n \leq c$ for all $n \in N$ then $a \leq c$.*

Proof. Since $\lim_{n\to\infty} b_n = b$ then for all $\epsilon > 0$ there exists an $N \in N$ such that for $n > N$ $|b_n - b| < \epsilon$. Then $-\epsilon < b_n - b < \epsilon$, meaning $c \leq b_n < \epsilon + b$, so we have $c < b + \epsilon$ for all $\epsilon > 0$. Now assume for contradiction $c > b$ and consider $\epsilon = c - b$. Then $c < c$ which is absurd so it must be true that $c \leq b$. QED

Proof. Since $\lim_{n\to\infty} a_n = a$ then for all $\epsilon > 0$ there exists an $N \in N$ such that for $n > N$ $|a_n - a| < \epsilon$. Then $-\epsilon < b_n - b < \epsilon$, meaning $-\epsilon + a < a_n \leq c$. So we have $-\epsilon + a < c$ for all $\epsilon > 0$. Now assume $c < a$ for contradiction and consider $\epsilon = a - c$. Then we have $c < c$ which is absurd so it must be true that $a \leq c$ QED

Problem. *53 a) If $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$ then both x_n and y_n diverge but $x_n + y_n$ converges.*

Problem. *53 b) If $x_n = (-1)^n$ and $y_n = \frac{1}{n}$ then x_n diverges and y_n converges but $x_n + y_n$ converges.*

Problem. *53 c) If $b_n = \frac{1}{n}$ then $b_n \neq 0$ for all n and $\frac{1}{b_n} = n$ which diverges to infinity.*

Problem. *53 d) There are no sequences* a_n *and* b_n *such that* a_n *is unbounded,* b_n *is convergent, and* $a_n - b_n$ *is bounded. Since* b_n *is convergent it is also bounded. let* B_1 *be an upper bound for* b_n *and* B_2 *be an upper bound for* $a_n - b_n$ *then* $B - 1 + B_2$ *is an upper bound for* $a_n - b_n + b_n = a_n$*. Similarly let* c_1 *be a lower bound for* b_n *and* c_2 *be a lower bound for* $a_n - b_n$ *then* $c_2 + c_1$ *is a lower bound for* a_n *implying* a_n *is bounded.*

Problem. *If* $a_n = \frac{1}{n^2}$ *and* $b_n = n$ *then* $a_n b_n = \frac{1}{n}$ *which converges,* a_n *also converges, but* b_n *does not converge.*