Completeness

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Problem. *1.* $\{\frac{1}{n} \mid n \in N\}$

> *Give 3 upper bounds for this set: 1, 2, 3*
> *Give 3 lower bounds for this set: 0, -1, -2*
> *Maximum: 1*
> *Minimum: No Minimum*
> *Supremum: 1*
> *Infimum: 0*

Problem. *2. If $A, B \subset R$ and we define $C = A+B = \{a+b \mid a \in A, b \in B\}$ then if A and B have suprema, then C has a supremum and $\sup C = \sup A + \sup B$*

Proof. let a_0 and b_0 denote $\sup A$ and $\sup B$ Then for all $a \in A$, $a \leq a_0$ and for all $b \in B$, $b \leq b_0$. An arbitrary element in C can be written $a+b$ for some $a \in A$ and $b \in B$. Since $a \leq a_0$ for all $a \in A$ and $b \leq b_0$ for all $b \in B$, $a+b \leq a_0+b_0$ for all $a \in A$ and $b \in B$ and a_0+b_0 is an upper bound for the set C.
To show that a_0+b_0 is the least upper bound for C, recall that if a_0+b_0 is the least upper bound for C then for all $\epsilon > 0$ there exists an element $c \in C$ such that $a_0+b_0-\epsilon < c$, that is for some $a \in A$ and $b \in B$, $a_0+b_0 < a+b+\epsilon$, for all $\epsilon > 0$. Fix $\epsilon > 0$ and rewrite as $a_0+b_0 < (a+\frac{\epsilon}{2})+(b+\frac{\epsilon}{2})$. It must be shown that there exist $a \in A$ and $b \in B$ for which this is true. Since $a_0 = \sup A$ then for all $\epsilon_1 > 0$ there exists an $a \in A$ such that $a+\epsilon_1 > a_0$. Take $\epsilon_1 = \frac{\epsilon}{2}$, which is justified since if $\epsilon > 0$ then $\frac{\epsilon}{2} > 0$. Then there exists an $a \in A$ such that $a+\frac{\epsilon}{2} > a_0$. Also since $b_0 = \sup B$, for all $\epsilon_2 > 0$ there exists a $b \in B$ such that $b+\epsilon_2 > b_0$. Take $\epsilon_2 = \frac{\epsilon}{2}$, then there exists a $b \in B$ such that $b+\frac{\epsilon}{2} > b_0$. Taking a and b to be these values it is clear that there exist an $a \in A$ and $b \in B$ such that $a_0+b_0 < (a+\frac{\epsilon}{2})+(b+\frac{\epsilon}{2})$. Rewriting the right side and recalling that $a+b$ was an element $c \in C$, we see that $a_0+b_0 < c+\epsilon$ for some $c \in C$. Since ϵ was arbitrary the result holds for all $\epsilon > 0$ proving that $a_0+b_0 = \sup C$, that is $\sup A + \sup B = \sup C$. QED

Problem. *3. Let $m \in R$ be a lower bound for a set $S \subset R$. Then m is the greatest lower bound if and only if for all $\epsilon > 0$ there exists an $s \in S$ such that $m+\epsilon > s$.*

Proof. First to show that if m is a lower bound for S and for all $\epsilon > 0$ there exists an $s \in S$ such that $m + \epsilon > s$, then m is the greatest lower bound for S, let m be a lower bound for S with that property. Now let n be another lower bound for S and assume for contradiction that $n > m$. Chose $\epsilon = n - m$. Since $n > m$, $n - m > 0$ and there exists an $s \in S$ such that $n > s$. Then n cannot be a lower bound for S which contradicts our assumption that it was. Then if n is another lower bound for S, n must be less than or equal to m, that is, m is the greatest lower bound for S.

To prove the other direction it must be shown that if m is the greatest lower bound for S then for all $\epsilon > 0$ there exists an $s \in S$ such that $m + \epsilon > s$. Let m be the greatest lower bound for S and fix $\epsilon > 0$. Since $\epsilon > 0$ and m is the greatest lower bound for S, $m + \epsilon$ cannot be a lower bound for S. Then it is false that for all $s \in S$, $s \geq m + \epsilon$, so there exists an $s \in S$ such that $m + \epsilon > s$. Since ϵ was arbitrary that result holds for all $\epsilon > 0$. QED

Problem. *4. The statements "For all $r \in R$ there exists an $n \in N$ such that $n > r$" and "For all $a > 0$ in R and $b \in R$, there exists an $n \in N$ such that $na > b$" are equivalent.*

Proof. To show that the first statement implies the second observe that any real number r can be written as $r \cdot a \cdot \frac{1}{a}$ for any $a > 0$. Then for all $r \in R$ and $a > 0$ there exists an $n \in N$ such that $na > ra$. Since the product of two real numbers is a real number and every real number can be expressed as the product of two real numbers, in other words ra is a real number whenever r is real and a is real and greater than 0, and there is no real number which cannot be expressed in this form. Then it is justifiable to let $b = ra$ and say for all $b \in R$ and $a > 0$ in R there exists an $n \in N$ such that $na > b$ as required.

To prove the other direction fix $a > 0$ and $b \in R$. Since there exists an $n \in N$ such that $na > b$, let n be that particular n. Then $n > \frac{b}{a}$. The sets $\{\frac{b}{a} \mid b \in R, a > 0\}$ and $\{r \mid r \in R\}$ are exactly equal so every $\frac{b}{a}$ corresponds to exactly one r and there are no $r \in R$ that cannot be written $\frac{b}{a}$ for some $b \in R$ and $a > 0$. So for all $r \in R$ there exists an $n \in N$ such that $n > r$. QED