Two Special Cases of Divergence

Allen Williams

November 27 2017

Problem. $\lim_{n\to\infty} n^2 = +\infty$

Proof. Recall that if for all $M > 0$ there exists a number N for which $n > N$ implies $s_n > M$ then $\lim_{n\to\infty} s_n = +\infty$. Consider $s_n = n^2$. Fix $M > 0$ and let $N = \sqrt{M}$ then $n > M$ means $n > \sqrt{M}$, so $n^2 > M$. So for all $M > 0$ there exists a number N for which $n > N$ implies $n^2 > M$ meaning $\lim_{n\to\infty} n^2 = +\infty$. QED

Problem. *Suppose there exists an N_0 such that $s_n \leq t_n$ for all $n > N_0$. If $\lim_{n\to\infty} s_n = +\infty$ then $\lim_{n\to\infty} t_n = +\infty$.*

Proof. Since $\lim_{n\to\infty} = +\infty$, for all $M > 0$ there exists a number N_1 such that $n > N_1$ implies $s_n > M$. further, there exists a number N_0 such that for $n > N_0$, $s_n \leq t_n$. Then for $n > max\{N_0, N_1\}$ $t_n \geq s_n > M$ implying that $t_n > M$. So since there exists an N such that $n > N$ implies $t_n > M$, $\lim_{n\to\infty} t_n = +\infty$. QED

Problem. *If $\lim_{n\to\infty} s_n = +\infty$ and $k < 0$ then $\lim_{n\to\infty} ks_n = -\infty$.*

Proof. Since $\lim_{n\to\infty} s_n = +\infty$, for all $M > 0$ there exists an N such that $n > N$ implies $s_n > -\frac{M}{k}$ (since $k < 0$ and $M > 0$, $-\frac{M}{k} > 0$) Then $ks_n < -M$. Since M was arbitrary and positive, $-M$ is arbitrary and negative. Call it M_1. So for all $M_1 < 0$ there exists an N such that $n > N$ implies $ks_n < M_1$ meaning $\lim_{n\to\infty} ks_n = -\infty$. QED

Problem. *If $\lim_{n\to\infty} s_n = +\infty$ and $\inf\{t_n \mid n \in N\} > -\infty$, then $\lim_{n\to\infty}(s_n + t_n) = +\infty$.*

Proof. Since $\inf\{t_n \mid n \in N\} > -\infty$ either $\inf\{t_n \mid n \in N\} \in R$ or $\inf\{t_n \mid n \in N\} = +\infty$. Since $\{t_n \mid n \in N\}$ contains at least one element, $\inf\{t_n \mid n \in N\} \in R$, let l denote this element. Since $\lim_{n\to\infty} s_n = +\infty$, for all $M + |l| > 0$ there exists an N_1 such that $n > N_1$ implies $s_n > M + |l|$. Then for $n > N_1$, $s_n + t_n > M + |l| + l \geq M$, so $\lim_{n\to\infty}(s_n + t_n) = +\infty$. QED