Limits

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Problem. *47 a)* $\lim_{n\to\infty} \frac{n-1}{n+1} = 1$

Proof. Fix $\epsilon > 0$ and let $N = \frac{2}{\epsilon}$. Then $n > N$ means $n > \frac{2}{\epsilon}$, which means $\epsilon > \frac{2}{n} > \frac{2}{n+1} = |\frac{n-1-(n+1)}{n+1}|$ which implies $\epsilon > |\frac{n-1}{n+1} - 1|$ when $n > N$, and $\lim_{n\to\infty} \frac{n-1}{n+1} = 1$. QED

Problem. *47 b)* $\lim_{n\to\infty} \frac{1}{2n^2+1} = 0$

Proof. Fix $\epsilon > 0$ and let $N = \frac{1}{\sqrt{\epsilon}}$. Then $n > N$ means $n > \frac{1}{\sqrt{\epsilon}}$, which means $\epsilon > \frac{1}{n^2} > \frac{1}{2n^2+1} = |\frac{1}{2n^2+1}|$. So $n > N$ implies $\epsilon > |\frac{1}{2n^2+1}$ so $\lim_{n\to\infty} \frac{1}{2n^2+1} = 0$. QED

Problem. *47 c)* $\lim_{n\to\infty} \frac{4n^3+2n}{2n^3+1} = 2$

Proof. Fix $\epsilon > 0$ and let $N = \frac{1}{\sqrt{\epsilon}}$. Then $n > N$ means $n > \frac{1}{\sqrt{\epsilon}}$, which means $\epsilon > \frac{1}{n^2}$, or equivalently $\epsilon > \frac{2n}{2n^3} > |\frac{2n-2}{2n^3+1}|$. Then $\epsilon > |\frac{4n^3+2n-2(2n^3+1)}{2n^3+1}|$ or $\epsilon > |\frac{4n^3+2n}{2n^3+1} - 2|$. So $n > N$ implies $\epsilon > |\frac{4n^3+2n}{2n^3+1} - 2|$ proving that $\lim_{n\to\infty} \frac{4n^3+2n}{2n^3+1} = 2$. QED

Problem. *47 d)* $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

Proof. Fix $\epsilon > 0$ and let $N = \frac{1}{\epsilon}$, then $n > N$ means $n > \frac{1}{\epsilon}$ or $\epsilon > \frac{1}{n}$, or equivalently, $|\frac{(-1)^n}{n}| < \epsilon$. So $n > N$ implies $|\frac{(-1)^n}{n}| < \epsilon$, meaning $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$. QED

Problem. *47 e)* $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

Proof. Fix $\epsilon > 0$ and let $N = \frac{1}{\epsilon}$, then $n > N$ means $n > \frac{1}{\epsilon}$ or $\epsilon > \frac{1}{n}$, which also means $\epsilon > |\frac{1}{n}|$, since $|\frac{1}{n}| = \frac{1}{n}$. $|\frac{1}{n}| \geq |\frac{\sin(n)}{n}|$ for all n so $\epsilon > |\frac{\sin(n)}{n}|$ by transitivity. Then $n > \frac{1}{\epsilon}$ implies $|\frac{\sin(n)}{n}| < \epsilon$ so $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$. QED

Problem. *47 f)* $\lim_{n\to\infty} \sqrt{4n^2+n} - 2n = \frac{1}{4}$

Proof. Fix $\epsilon > 0$ and let $N = \frac{4\epsilon+1}{2}$, then $n > N$ means $n > \frac{4\epsilon}{2}$, or $\epsilon > \frac{2n-1}{4}$, which is always positive so $\epsilon > |\frac{2n-1}{4}|$, or equivalently, $\epsilon > |\frac{n^2}{2n} - \frac{1}{4}| > |\frac{n}{\sqrt{4n^2+n}+2n} - \frac{1}{4}|$ which is equivalent to $|\sqrt{4n^2+2n} - 2n - \frac{1}{4}|$. So $n > N$ implies $|\sqrt{4n^2+n} - 2n - \frac{1}{4}| < \epsilon$ proving $\lim_{n\to\infty} \sqrt{4n^2+n} - 2n = \frac{1}{4}$ QED

Problem. *48 a)* $s_n = \sin(\frac{\pi n}{4})$ *diverges.*

Proof. Assume for contradiction thats_n converges to a limit $s \in R$ and let $\epsilon = 1$. Then by lemma there exists an $N \in R$ such that for $n > N$, $|\sin(\frac{\pi n}{4}) - s| < 1$. Consider cases $n = 2+8k$ for $k \in Z$ and $n = 6+8k$ for $k \in Z$. Observe $|1-s| < 1$ and $|-1-s| < 1$ and note that $|-(s+1)| = |s+1|$. Then $|1-s| + |-(s+1)| < 2$ and by the triangle inequality $|1-s+s+1| < 2$ or $|2| < 2$, but $2 = 2$, so a contradiction has been derived and the assumption that s_n converges must be wrong, thus s_n diverges. QED

Problem. *48 b)* $s_n = (-1)^n \cdot n$ *diverges.*

Proof. Assume for contradiction thats_n converges to a limit $s \in R$ and let $\epsilon = 1$. Then by lemma there exists an $N \in R$ such that for $n > N$, $|(-1)^n \cdot n - s| < 1$. Consider cases $n = 2k$ for $k \in Z$ and $n = 2k+1$ for $k \in Z$, both of which can be found above any large N. $|n-s| < 1$ and $|-n-s| < 1$, then by adding the two inequalities and invoking the triangle inequality, $|2n| < |n-s| + |-n-s| < 1$. This cannot happen for any $n \in N$ so the assumption that s_n converges is false. QED

Problem. *49 Let t_n be a bounded sequence, i.e. there exists an M such that $|t_n| \leq M$ for all n, and let s_n be a sequnce such that* $\lim_{n\to\infty} s_n = 0$. *Then* $\lim_{n\to\infty}(s_n t_n) = 0$.

Proof. It must be shown that for all $\epsilon > 0$ there exists an $n \in R$ such that $|s_n t_n| < \epsilon$. $|s_n t_n| = |s_n| \cdot |t_n| \leq |s_n| M$, so it is sufficient to show that $|s_n| M < \epsilon$ for sufficiently large n. Since $\lim_{n\to\infty} s_n = 0$ then for all $\frac{\epsilon}{|M|}$ there exists an $N \in R$ such that $|s_n| < \frac{\epsilon}{|M|}$, so $|s_n| M \leq |s_n| \cdot |M| < \epsilon$. So $|t_n| \leq M$ for all n and $s_n \to 0$ implies that $\lim_{n\to\infty}(s_n t_n) = 0$. QED

Problem. *50 3 sequences $a_n \leq s_n \leq b_n$ for all n and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$, *then* $\lim_{n\to\infty} s_n = s$.

Proof. We know that for all $\epsilon > 0$ there exists an $N_a \in N$ such that $|a_n - s| < \epsilon$ and there exists an $N_b \in N$ such that $|b_n - s| < \epsilon$, that is $-\epsilon + s < a_n \leq s_n \leq b_n < \epsilon + s$ for $n > \max\{N_a, N_b\}$ then $-\epsilon < a_n - s \leq s_n - s \leq b_n - s < \epsilon$ which means that $|s_n - s| < \epsilon$ for $n > \max\{N_a, N_b\}$ proving $lim_{n\to\infty} s_n = s$. QED