QUIZ 6

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(1) $\mathbb{Z}_{24}$ is not isomorphic to D_{12} since $\mathbb{Z}_{24}$ is cyclic and D_{12} is not, also $\mathbb{Z}_{24}$ is Abelian and D_{12} is not. Isomorphisms take cyclic groups to cyclic groups and Abelian groups to Abelian groups so $\mathbb{Z}_{24}$ and D_{12} cannot be isomorphic.

(2) $\langle \mathbb{Q}, + \rangle$ cannot be isomorphic to $\langle \mathbb{Z}, + \rangle$ since $\langle \mathbb{Z}, + \rangle$ is cyclic but $\langle \mathbb{Q}, + \rangle$ is not.

(3) $U(8)$ and $\mathbb{Z}_4$ are not isomorphic. Every element in $U(8)$ is its own inverse, so none of them generates the group, but $\mathbb{Z}_4$ is cyclic and generated by the element 1, so no isomorphism can exist between $\mathbb{Z}_4$ and $U(8)$.

(4) Let $\phi : G_1 \to G_2$ and $\psi : G_2 \to G_3$ be two isomorphisms. ϕ^{-1} is bijective since ϕ is bijective. Consider $\phi(x), \phi(y) \in G_2$. $\phi^{-1}(\phi(x)\phi(y)) = \phi^{-1}(\phi(xy)) = xy = \phi^{-1}\phi(x)\phi^{-1}\phi(y)$, so ϕ^{-1} is an isomorphism. Consider $\psi \circ \phi(xy) = \psi(\phi(xy))$. Since ϕ is an isomorphism this equals $\psi(\phi(x)\phi(y))$ and since ψ is an isomorphism it equals $\psi(\phi(x))\psi(\phi(y)) = \psi \circ \phi(x)\psi \circ \phi(y)$, so $\psi \circ \phi$ is an isomorphism.

Consider the set of groups and define $(G_1, G_2) \in R$ iff $G_1 \equiv G_2$. $(G_1, G_1) \in R$ for all groups since $\epsilon : G \to G$ given by $\epsilon(x) = x$ is clearly an isomorphism. Syppose $(G_1, G_2) \in R$ then there exists a function $\phi : G_1 \to G_2$ such that ϕ is an isomorphism. Since ϕ is bijective it has an inverse, which was shown above to be an isomorphism. So isomorphism is transitive, symmetric, and reflexive and forms an equivalence relation on the set of groups.

(5) $\Phi(a + bi) = (a - bi)$ is an isomorphism from $\mathbb{C}$ to $\mathbb{C}$. Suppose $\Phi(a_1 + b_1 i) = \Phi(a_2 + b_2 i)$, then $a_1 - b_1 i = a_2 - b_2 i$. These two complex numbers being equal implies $a_1 = a_2$ and $b_1 = b_2$, which implies $a_1 + b_1 i = a_2 + b_2 i$, so Φ is injective.

Let $y \in \mathbb{C}$ be arbitrary, that is $y = a + bi$ for some $a, b \in \mathbb{R}$ then $\Psi(a - bi) = a + bi = y$, so Φ is surjective.

$$\begin{aligned}
\Phi(a_1 + b_1 i + a_2 + b_2 i &= \Psi((a_1 + a_2) + (b_1 + b_2)i) \\
&= a_1 + a_2 - (b_1 + b_2)i \\
&= a_1 - b_1 i + a_2 - b_2 i \\
&= \Phi(a_1 + b_1 i) + \Phi(a_2 + b_2)i
\end{aligned}$$

So Φ is an isomorphism from $\mathbb{C}$ to $\mathbb{C}$.

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(6) Φ was shown to be bijective above.

$$\begin{aligned}\Phi((a_1+b_1i)(a_2+b_2i)) &= \Phi(a_1a_2+a_1b_2i+a_2b_1i-b_1b_2)\\ &= \Phi((a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i)\\ &= a_1a_2-b_1b_2-(a_1b_2+a_2b_1)i\\ &= a_1a_2-b_1b_2-a_1b_2i-a_2b_1i\\ &= (a_1-b_1i)(a_2-b_2i)\\ &= \Psi(a_1-b_1i)\Psi(a_2-b_2i)\end{aligned}$$

So Ψ is an isomorphism from $\mathbb{C}^*$ to $\mathbb{C}^*$

(7) Let G be a group and let $g \in G$. Define $i_g : G \to G$ by $i_g(x) = gxg^{-1}$.
Suppose $i_g(x_1) = i_g(x_2)$ then $gx_1g^{-1} = gx_2g^{-1}$. Multiply on the left by g^{-1} and the right by g to get $x_1 = x_2$ so i_g is injective.
Let $y \in G$ then $i_g(g^{-1}yg) = gg^{-1}ygg^{-1} = y$ so i_g is surjective.
$i_g(xy) = i_g(xey) = i_g(xg^{-1}gy) = gxg^{-1}gyg^{-1} = i_g(x)i_g(y)$ so i_g is an isomorphism from G to G.

(8) let ξ and ϕ be two inner automorphisms and consider $\xi\phi^{-1}(x) = \xi(\phi^{-1}(x)) = \xi(g^{-1}xg) = hg^{-1}xgh^{-1}$ for $g, h \in G$.
Suppose $hg^{-1}xgh^{-1} = hg^{-1}ygh^{-1}$ then multiply on the left by $h^{-1}g$ and the right by hg^{-1} to get $x = y$ so $\xi\phi^{-1}$ is injective.
Let $z \in G$ be arbitrary then $z = \xi\phi^{-1}(gh^{-1}zhg^{-1})$ so $\xi\phi^{-1}$ is surjective.
$\xi\phi^{-1}(xy) = \xi\phi^{-1}(xgh^{-1}hg^{-1}y) = hg^{-1}xgh^{-1}hg^{-1}ygh^{-1} = \xi\phi^{-1}(x)\xi\phi^{-1}(y)$
So $\xi\phi^{-1} \in Aut(G)$ so $Inn(G)$ is a subgroup of $Aut(G)$.

(9) let G be a group and let $g \in G$. Define $\lambda_g : G \to G$ and $\rho_g : G \to G$ such that $\lambda_g(x) = gx$ and $\rho_g(x) = xg^{-1}$. Define $i_g = \rho_g \circ \lambda_g$, then $i_g = \rho_g(\lambda_g(x)) = \rho_g(gx) = gxg^{-1}$ so $i_g \in Inn(G)$ which is a subgroup of $Aut(G)$ so $i_g \in Aut(G)$
Define $f : G \to S_G$ such that $f(g) = \rho_g$. It must be shown that f is an injective homomorphism from G to S_G. Suppose

$$\begin{aligned}f(g_1) &= f(g_2)\\ \rho_{g1} &= \rho_{g2}\\ \rho_{g1}(x) &= \rho_{g2}(x)\\ xg_1^{-1} &= xg_2^{-1}\\ g_1^{-1} &= g_2^{-1}\\ g_1 &= g_2\end{aligned}$$

So f is injective.

to see that $f(g_1 g_2) = f(g_1) \circ f(g_2)$ observe that

$$\begin{aligned}
f(g_1) \circ f(g_2) &= \rho_{g1} \circ \rho_{g2} \\
&= \rho_{g1}(\rho_{g2}(x)) \\
&= \rho_{g1}(x g_2^{-1}) \\
&= x g_2^{-1} g_1^{-1} \\
&= x(g_1 g_2)^{-1} \\
&= \rho_{g_1 g_2} \\
&= f(g_1 g_2)
\end{aligned}$$

So f is an injective homomorphism between G and S_G as required for Cayley's Theorem.

(10) Automorphisms take generators to generators. Generators of $\mathbb{Z}_{\ntriangleleft}$ are 1 and 5, so possible automorphisms are $\epsilon(1) = 1$ or $f(1) = 5$, which gives

$$\begin{aligned}
f(2) &= f(1) + f(1) = 4 \\
f(3) &= f(2) + f(1) = 3 \\
f(4) &= f(3) + f(1) = 2 \\
f(5) &= f(4) + f(1) = 1 \\
f(0) &= f(5) + f(1) = 0
\end{aligned}$$

Also $\mathbb{Z}$ has two generators, 1 and -1 so $Aut(Z)$ consists of $\epsilon(x) = x$ and $\tau(x) = -x$.