Quiz 2

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Problem 1. *Prove that the set of matrices of the form* $M = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$ *is a group under matrix multiplication where* $x, y, z \in \mathbb{Z}$. *First notice that matrices in ths set have determinant 1 since they are upper triangular with 1s on the main diagonal, so this set is a subset of the group* $SL_3(\mathbb{R})$. *By letting* $x = y = z = 0$ *we get* I_3 *the identity in* $SL_3(\mathbb{R})$. *Let* $M_1 = \begin{bmatrix} 1 & x_1 & y_1 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $M_2 = \begin{bmatrix} 1 & x_2 & y_2 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix}$. *Then* $M_1M_2 = \left[M_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, M_1 \begin{bmatrix} x_2 \\ 1 \\ 0 \end{bmatrix}, M_1 \begin{bmatrix} y_2 \\ z_2 \\ 1 \end{bmatrix} \right]$ $= \begin{bmatrix} 1 & x_2 + x_1 & y_2 + x_1z_2 + y_1 \\ 0 & 1 & z_2 + z_1 \\ 0 & 0 & 1 \end{bmatrix}$. *Since all entries of* M_1 *and* M_2 *are integers, the entries in* M_1M_2 *are also integers, so this set of matrices is closed under multiplication. Let* M_1 *be an element of this set of matrices, then* $M_1M_1' = \begin{bmatrix} 1 & x + x' & y + xz' + y' \\ 0 & 1 & z + z' \\ 0 & 0 & 1 \end{bmatrix}$. *If* $x + x' = 0$, $y + xz' + y' = 0$, *and* $z + z' = 0$ *then* $M' = M^{-1}$, *so by solving these equations simultaneously, a formula for* M^{-1} *is obtained in terms of the entries of* M. *From equations 1 and 3,* $x' = -x$ *and* $z' = -z$, *and by substituting these values into equation 2,* $y' = -y + xz$ *Then* $M^{-1} = \begin{bmatrix} 1 & -x & -y + xz \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix}$ *and since* $x, y, z \in \mathbb{Z}$, *the entries of* M^{-1} *are integers, so this set of matrices is closed under inverses. So this group of matrices is a subgroup of* $SL_3(\mathbb{R})$ *and is therefore itself a group.*

Problem 2. *Give a multiplication table for the group $U(10)$.*

$\cdot$	[1]	[3]	[7]	[9]
[1]	[1]	[3]	[7]	[9]
[3]	[3]	[9]	[1]	[7]
[7]	[7]	[1]	[9]	[3]
[9]	[9]	[7]	[3]	[1]

Problem 3. *Show that if $a^2 = e$ for all elements n a group G, then G must be abelian.*
Let G be a group and let $a^2 = e$ for all $a \in G$. Let $a, b \in G$ be arbitrary. Then $ab = c$ for some $c \in G$.

$$\begin{aligned} ab &= c \\ a(ab) &= ac \\ (aa)b &= ac \\ eb &= ac \\ b &= ac \\ bc &= (ac)c \\ bc &= a(cc) \\ bc &= ae \\ bc &= a \\ b(bc &= ba \\ (bb)c &= ba \\ ec &= ba \\ c &= ba \end{aligned}$$

So $ab = c = ba$ and since a and b were arbitrary G is an abelian group.

Problem 4. *Let H and K be two subgroups of a group $< G, \circ >$. Prove that $H \cap K$ is a subgroup of G. Let H and K be subgroups of the group $< G, \circ >$. Clearly $H \cap K$ is a subset of G since H and K are subgroups of G. Let $a, b \in H \cap K$ be arbitrary. Since $b \in H$ and $< H, \circ >$ is a group, $b^{-1} \in H$. Similarly $b^{-1} \in K$, so $b^{-1} \in H \cap K$. Consider $a \circ b$. Since $a \in H$ and $b \in H$, $a \circ b \in H$. Similarly $a \circ b \in K$. Since $a \circ b \in H$ and $a \circ b \in K$, $a \circ b \in H \cap K$. Since a and b were arbitrary, $H \cap K$ is a subgroup of G.*

Problem 5. *Let $< \mathbb{Z}, + >$ be the additive group of integers. Consider $H =< 4\mathbb{Z}, + >$, $K =< 9\mathbb{Z}, + >$, and $L =< 6\mathbb{Z}, + >$, three subgroups of $< \mathbb{Z}, + >$ where $4\mathbb{Z} = \{4n \mid n \in \mathbb{Z}\}$, $9\mathbb{Z} = \{9n \mid n \in \mathbb{Z}\}$, $6\mathbb{Z} = \{6n \mid n \in \mathbb{Z}\}$. None of $H \cup K$, $K \cup L$, or $L \cup H$ are subgroups of $\mathbb{Z}$. To see that $H \cup K$ is not a group consider $4 \in H$ and $9 \in K$. $9 + 4 = 13$ which is not an element of H or K so $H \cup K$ is not closed under addition. Similarly $K \cup L$ and $L \cup H$ are not closed under addition and are not subgroups of $\mathbb{Z}$. If $H = 2\mathbb{Z}$ and $K = 4\mathbb{Z}$*

then $H \cup K$ is a subgroup of $\mathbb{Z}$. $0 = 2(0) = 4(0)$ *so both H and K contain the identity. Consider $a \in H$ and $b \in K$ then $a = 2k$ for some $k \in \mathbb{Z}$ and $b = 4j$ for some $j \in \mathbb{Z}$. Then $a + b = 2k + 4j = 2(k + 2j)$ so $a + b \in H$ and therefore $a + b \in H \cup K$. Also if $a \in H$ then $a = 2k$ for some $k \in \mathbb{Z}$. Since $a + a^{-1} = 0$, $a^{-1} = -2k = 2(-k)$ so $a^{-1} \in H$ and therefore $a^{-1} \in H \cup K$.*

Problem 6. *Show that $G = \{1, -1, i, -i\}$ form a group with respect to multiplication. Write the binary composition table. Find a nontrivial proper subgroup. Let $G = \{1, -1, i, -i\}$, then G forms a group with respect to multiplication since its multiplication table is as follows,*

*	1	-1	i	-i
1	1	-1	i	-1
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

and $G \subseteq \mathbb{C}$ so G inherits associativity from $< \mathbb{C}, * >$ As seen on the multiplication table 1 is the identity element for G and since $1 = 1^{-1}, -1 = (-1)^{-1}, -i = i^{-1}$, and $i = (-i)^{-1}$, every element has an inverse in G. The set is also closed under multiplication so it forms a group.
Consider $H = \{1, -1\}$. Clearly $H \subset G$ and its multiplication table is as follows,

*	1	-1
1	1	-1
-1	-1	1

H contains only two elements 1 and -1, so H is non-empty. Both 1 and -1 are self inverses in H so $g^{-1} \in H$ whenever $g \in H$, and H is closed under multiplication, so H forms a proper, nontrivial subgroup of G.