QUIZ 7

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(1) Are the following functions homomorphisms? If so give the kernel.
  (a) $\phi : \mathbb{R}^* \to GL_2(\mathbb{R})$ defined by $\phi(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ is a homomorphism and $ker(\phi) = \{1\}$.
  (b) $\phi : \mathbb{R} \to GL_2(\mathbb{R})$ defined by $\phi(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ is a homomorphism with $ker(\phi) = \{0\}$.
  (c) $\phi : GL_2(\mathbb{R}) \to \mathbb{R}$ defined by $\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$=$a + d$ is not a homomorphism.
  (d) $\phi : GL_2(\mathbb{R}) \to \mathbb{R}^*$ defined by $\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$ is a homomorphism with $ker(\phi) = SL_2(\mathbb{R})$
  (e) $\phi : \mathbb{M}_2(\mathbb{R}) \to \mathbb{R}$ defined by $\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = b$ is a homomorphism with $ker(\phi) = \left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\right\}$.

(2) Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be given by $\phi(n) = 7n$. Prove the ϕ is a group homomorphism. Find the kernel and the image of ϕ. $\phi(m + n) = 7(m + n) = 7m + 7n = \phi(m) + \phi(n)$ so ϕ is a homomorphism. $ker(\phi) = \{n \mid \phi(n) = 0\}$ or the solution set to the equation $7n = 0$, so $ker(\phi) = \{0\}$. The image of ϕ is $\phi(\mathbb{Z})$ or $\{7z \mid z \in \mathbb{Z}\}$, or $7\mathbb{Z}$.

(3) In $\mathbb{Z}_{40}$ let $H = \langle 4 \rangle$ and let $N = \langle 10 \rangle$.
  (a) List the elements in $H + N$ and $H \cap N$
  $H+N = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38\}$
  $H \cap N = \{0, 20\}$
  (b) list the cosets in $\frac{HN}{N}$
  $\frac{H+N}{N} = \{\{0, 10, 20, 30\}N = \{0, 10, 20, 30\}$,
  $\{2, 12, 22, 32\}N = \{2, 12, 22, 32\}$,
  $\{4, 14, 24, 34\}N = \{4, 14, 24, 34\}$,
  $\{6, 16, 26, 36\}N = \{6, 16, 26, 36\}$,
  $\{8, 18, 28, 38\}N = \{8, 18, 28, 38\}\}$

  (c) List the cosets in $\frac{H}{H \cap N}$ $\frac{H}{H \cap N} = \{0(H \cap N) = \{0, 20\}, 4(H \cap N) = \{4, 24\}, 8(H \cap N) = \{8, 28\}, 12(H \cap N) = \{12, 32\}, 16(H \cap N) = \{16, 36\}\}$
  (d) Give the correspondence between $\frac{H+N}{N}$ and $\frac{H}{H \cap N}$. Define $\varphi : \frac{H}{H \cap N} \to \frac{H+N}{N}$ by $\varphi(g(H \cap N)) = gN$ Suppose $\varphi(g(H \cap N)) = \varphi(h(H \cap N))$

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Observe that $gN = g'N$ when $g \equiv g' \mod (10)$, so $g \equiv h \mod (10)$. Also notice that $g(H \cap N) = g'(H \cap N)$ whenever $g \equiv g' \mod (10)$ so $g(H \cap N) = h(H \cap N)$, so φ is injective. Let gN be arbitrary, then $gN = \varphi(g(H \cap N))$ so φ is surjective. Also $\varphi(g(H \cap N)h(H \cap N)) = \varphi(gh(H \cap N)) = ghN = gNhN = \varphi(g(H \cap N))\varphi(h(H \cap N))$ so φ is an isomorphism meaning $\frac{H}{H \cap N} \cong \frac{H+N}{N}$.

(4) If $\phi : G \to H$ is a group homomorphism and G is abelian, prove that $\phi(G)$ is also abelian. Let $\phi : G \to H$ be a homomorphism and let G be an abelian group. Let $a, b \in G$ then $ab = ba$ so $\phi(ab) = \phi(a)\phi(b) = \phi(ba) = \phi(b)\phi(a)$ so since $\phi(a)\phi(b) = \phi(b)\phi(a)$, $\phi(G)$ is abelian.

If $\phi : G \to H$ is a group homomorphism and G is cyclic, prove that $\phi(G)$ is also cyclic. Let G be a cyclic group and let $\phi : G \to H$ be a homomorphism then $G = \langle g \rangle$ for some element $g \in G$, and an arbitrary element in G can be written g^k for integer k. Then $\phi(g^k) = \phi(g)^k$. The proof of which is by induction. Clearly when $k = 1$ $\phi(g^1) = \phi(g)^1$ establishing the basis of induction. Now assume $\phi(g^k) = \phi(g)^k$ for arbitrary $k \in \mathbb{Z}$. Then

$$\phi(g^k)\phi(g) = \phi(g)^k\phi(g)\phi(g^{k+1} = \phi(g)^{k+1}$$

So by induction $\phi(g^k) = \phi(g)^k$ for all $k \in \mathbb{N}$, so $\phi(G)$ is cyclic.

(5) Let G be a group with N a normal subgroup and both N and $\frac{G}{N}$ are abelian. Does this imply that G is abelian? No. Consider $G = S_3$ and $N = A_3$. A_3 is cyclic and therefore abelian since $|A_3| = 3$ and 3 is prime. Also $|\frac{G}{N}| = \frac{|G|}{|N|} = \frac{6}{3} = 2$ which is prime so $\frac{G}{N}$ is cyclic and therefore abelian. But $G = S_3$ is not abelian since $(12)(13) = (132) \neq (123) = (13)(12)$.

(6) Let G be a group with N a normal subgroup and both N and $\frac{G}{N}$ are cyclic. Does this imply that G is cyclic? Again, no. The response to question 5 holds for question 6 as well as groups with prime order are cyclic and therefore abelian.

(7) Let $\phi : G \to H$ be a homomorphism. Define a relation on G by a b if $\phi(a) = \phi(b)$. Let $a, b, c \in G$ be arbitrary. Clearly $\phi(a) = \phi(a)$ since ϕ is a function, so is reflexive. Suppose a b, that is $\phi(a) = \phi(b)$ then by the symmetric property of equality $\phi(b) = \phi(a)$, so b a, so ϕ is symmetric. Now suppose a b and b c, that is $\phi(a) = \phi(b)$ and $\phi(b) = \phi(c)$, then by the transitive property of equality $\phi(a) = \phi(c)$ so ϕ is transitive, and therefore an equivalence relation on G. The equivalence classes are sets of elements of G that map to the same element in H under ϕ.

(8) Let G be a group and define $\phi : G \to Aut(G)$ by $g \mapsto i_g$. Then $\phi(gh) = i_{gh} = ghxh^{-1}g^{-1} = i_g(hxh^{-1}) = i_g i_h(x) = \phi(g)\phi(h)$ so ϕ is a homomorphism. $i_g(x) = gxg^{-1}$ so $\phi(G) = inn(G)$. $ker(\phi) = \{g \in G \mid \phi(g) = e_{Aut(G)}\}$ and $e_{Aut(G)} = \epsilon$ where $\epsilon(x) = x$, then g must commute with an arbitrary element of G so $g \in Z(G)$, so $ker(\phi) = Z(G)$. By the first isomorphism theorem $\frac{G}{Z(G)} \cong Inn(G)$.

(9)

$\circ$	$1\cdot\langle 7\rangle$	$3\cdot\langle 7\rangle$	$9\cdot\langle 7\rangle$	$11\cdot\langle 7\rangle$
$1\cdot\langle 7\rangle$	$1\cdot\langle 7\rangle$	$3\cdot\langle 7\rangle$	$9\cdot\langle 7\rangle$	$11\cdot\langle 7\rangle$
$3\cdot\langle 7\rangle$	$3\cdot\langle 7\rangle$	$9\cdot\langle 7\rangle$	$11\cdot\langle 7\rangle$	$1\cdot\langle 7\rangle$
$9\cdot\langle 7\rangle$	$9\cdot\langle 7\rangle$	$11\cdot\langle 7\rangle$	$1\cdot\langle 7\rangle$	$3\cdot\langle 7\rangle$
$11\cdot\langle 7\rangle$	$11\cdot\langle 7\rangle$	$1\cdot\langle 7\rangle$	$3\cdot\langle 7\rangle$	$9\cdot\langle 7\rangle$

Where $U(16) = \{1, 3, 5, 7, 9, 11, 13, 15\}$,

$\langle 7\rangle = \{1, 7\}$
$1\langle 7\rangle = \{1, 7\}$
$3\langle 7\rangle = \{3, 5\}$
$9\langle 7\rangle = \{9, 15\}$
$11\langle 7\rangle = \{11, 13\}$

(10) (a) Let G be a group with H a normal subgroup and both H and $\frac{G}{H}$ are abelian and $\frac{G}{H}$ is cyclic. Does this imply that G is cyclic?
(b) Let G be a group with H a normal subgroup and both H and $\frac{G}{H}$ are abelian. Does this imply that G is abelian?

For both a and b let $G = S_3$, $H = A_3$ then as described in question 5 both H and $\frac{G}{H}$ have prime order so are cyclic and therefore abelian, although G was shown to not be abelian and therefore cannot be cyclic. So the answer to both a and b is no.