# Quiz 1

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**Problem 1.** *i) Let* $x, y \in \mathbb{R}$*,* $(x, y) \in R$ *iff* $x \leq y$ *is not an equivalence relation on* $\mathbb{R}$ *since it fails to be symmetric. Consider* $(x, y) = (0, 3)$*.* $(0, 3) \in R$ *since* $0 \leq 3$ *but* $(3, 0) \notin R$ *since* $3 > 0$*.*

*ii) Let* $x, y \in \mathbb{R}$*,* $(x, y) \in R$ *iff* $|x| = |y|$ *is an equivalence relation on* $\mathbb{R}$*. For all* $x \in \mathbb{R}$*,* $(x, x) \in R$ *since* $|x| = |x|$ *so* $R$ *is reflexive. Suppose* $(x, y) \in R$ *then* $|x| = |y|$ *and since "=" is symmetric* $|y| = |x|$ *meaning* $(y, x) \in R$ *so* $R$ *is symmetric. Now suppose* $(x, y) \in R$ *and* $(y, z) \in R$ *so* $|x| = |y|$ *and* $|y| = |z|$*, by transitivity of equality* $|x| = |z|$ *so* $(x, z) \in R$ *so* $R$ *is transitive. Since* $R$ *is reflexive, symmetric, and transitive* $R$ *is an equivalence relation on* $\mathbb{R}$*. The partition of* $\mathbb{R}$ *given by* $R$ *is* $\{x, -x \mid x \in \mathbb{R}\}$*.*

*iii) Let* $x, y \in \mathbb{Z}$*.* $(x, y) \in R$ *iff* $x \equiv y \pmod 5$ *is an equivalence relation on* $\mathbb{Z}$*. By definition if* $x \equiv y \pmod 5$ *then* $x - y = 5k$ *for some* $k \in \mathbb{Z}$*. Let* $x, y, z \in \mathbb{Z}$ *be arbitrary.* $x - x = 0 = 5 \cdot 0$ *so* $x \equiv x \pmod 5$ *so* $(x, x) \in R$ *and* $R$ *is reflexive. Suppose* $(x, y) \in \mathbb{R}$ *then* $x \equiv y \pmod 5$ *so* $x - y = 5k$ *for some* $k \in \mathbb{Z}$*, then* $y - x = 5(-k)$ *and since* $k \in \mathbb{Z}$*,* $-k \in \mathbb{Z}$ *so* $y \equiv x \pmod 5$ *so* $(y, x) \in R$ *and* $R$ *is symmetric. Now suppose* $(x, y) \in R$ *and* $(y, z) \in R$*, so* $x \equiv y \pmod 5$ *and* $y \equiv z \pmod 5$*. Then* $x - y = 5k_1$ *for some* $k_1 \in \mathbb{Z}$ *and* $y - z = 5k_2$ *for some* $k_2 \in \mathbb{Z}$*. Then by adding these equations,* $x - y + y - z = 5k_1 + 5k_2$*, or* $x - z = 5(k_1 + k_2)$*. Since* $k_1 \in \mathbb{Z}$ *and* $k_2 \in \mathbb{Z}$*,* $k_1 + k_2 \in \mathbb{Z}$ *so* $x \equiv z \pmod 5$*,* $(x, z) \in R$*, and* $R$ *is transitive. Since* $R$ *is reflexive, symmetric, and transitive,* $R$ *is an equivalence relation on* $\mathbb{Z}$*. The partition of* $\mathbb{Z}$ *given by* $R$ *is* $\{\{5k \mid k \in \mathbb{Z}\}, \{5k{+}1 \mid k \in \mathbb{Z}\}, \{5k{+}2 \mid k \in \mathbb{Z}\}, \{5k{+}3 \mid k \in \mathbb{Z}\}, \{5k{+}4 \mid k \in \mathbb{Z}\}\}$

**Problem 2.** *Let* $(a_n)_{n=1}^{\infty}$*,* $(b_n)_{n=1}^{\infty} \in S$ *where* $S$ *is the set of all real valued sequences. "*$((a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}) \in R$ *iff* $\lim_{n\to\infty}(a_n - b_n) = 0$*" is an equivalence relation on* $S$*. Let* $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}$ *be arbitrary. To see that* $R$ *is reflexive, let* $\varepsilon > 0$*, clearly* $|a_n - a_n - 0| = 0 < \varepsilon$ *for any value of* $n$*, so* $\lim_{n\to\infty}(a_n - a_n) = 0$ *meaning* $((a_n)_{n=1}^{\infty}, (a_n)_{n=1}^{\infty}) \in R$ *so* $R$ *is reflexive. To see that* $R$ *is symmetric, suppose* $\lim_{n\to\infty}(a_n - b_n) = 0$ *and let* $\varepsilon > 0$*. Then there exists an* $N \in \mathbb{N}$ *such that* $n > N$ *implies* $|a_n - b_n - 0| < \epsilon$*, or* $|a_n - b_n| < \varepsilon$*. Then for* $n > N$*,* $|-(a_n - b_n)| = |b_n - a_n| < \varepsilon$*, so* $\lim_{n\to\infty}(b_n - a_n) = 0$*, so* $((b_n)_{n=1}^{\infty}, (a_n)_{n=1}^{\infty}) \in R$ *so* $R$ *is symmetric. To see that* $R$ *is transitive, suppose* $\lim_{n\to\infty}(a_n - b_n) = 0$ *and* $\lim_{n\to\infty}(b_n - c_n) = 0$ *and let* $\varepsilon > 0$*. Then there* *exists an* $N_1 \in \mathbb{N}$ *such that for* $n > N_1$, $|a_n - b_n| < \frac{\varepsilon}{2}$. *Also there exists an* $N_2 \in \mathbb{N}$ *such that for* $n > N_2$, $|b_n - c_n| < \frac{\varepsilon}{2}$. *Then for* $n > \max\{N_1, N_2\}$, $|a_n - b_n + b_n - c_n| \leq |a_n - b_n| + |b_n - c_n| < \varepsilon$, *so* $|a_n - c_n| < \varepsilon$ *meaning* $\lim_{n\to\infty}(a_n - c_n) = 0$ *so* $((a_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}) \in R$ *so* $R$ *is transitive. Since* $R$ *is reflexive, symmetric, and transitive,* $R$ *is an equivalence relation on* $S$.

**Problem 3.** *Let* $A, B \in \mathcal{P}(X)$ *where* $X = \{1, 2, 3...100\}$. *"*$(A, B) \in R$ *iff* $A \subseteq B$*" is not an equivalence relation on* $\mathcal{P}(X)$ *since it fails to be symmetric. Consider* $(A, B) = (\{\}, \{1, 2, 3\})$. $(A, B) \in R$ *since* $\{\} \subseteq \{1, 2, 3\}$ *but* $(B, A) \notin R$ *since* $\{1, 2, 3\}$ *is not a subset of* $\{\}$.

**Problem 4.** *1.* *i is a partial order relation. Let* $x, y, z \in \mathbb{R}$ *be arbitrary. Since* $x = x$, *it is also true that* $x \leq x$, *so* $R$ *is reflexive. Suppose* $(x, y) \in R$ *and* $(y, x) \in R$ *then* $x \leq y$ *and* $y \leq x$ *so* $x = y$, *meaning* $R$ *is anti-symmetric. Now suppose* $(x, y) \in R$ *and* $(y, z) \in R$, *then* $x \leq y$ *and* $y \leq z$ *so by transitivity of* $\leq$, $x \leq z$ *so* $R$ *is transitive. Since* $R$ *is reflexive, anti-symmetric, and transitive* $R$ *is a partial order relation on* $\mathbb{R}$.

*ii is not a partial order relation on* $\mathbb{R}$ *since it fails to be anti-symmetric. Consider* $(x, y) = (-2, 2)$. *Then* $(x, y) \in R$ *since* $|-2| = |2|$ *and* $(y, x) \in R$ *by the symmetry of* $R$ *but* $-2 \neq 2$.

*iii is not a partial order relation on* $\mathbb{Z}$ *since it fails to be anti-symmetric. Consider* $(x, y) = (1, 6)$. *Then* $(x, y) \in R$ *since* $1 \equiv 6 \pmod 5$ *and* $(y, x) \in R$ *by the symmetry of* $R$, *but* $1 \neq 6$.

*2. is not a partial order on the set of all real valued sequences since it fails to be anti-symmetric. Let* $(a_n)_{n=1}^{\infty}$ *be the sequence given by* $a_n = \frac{1}{n}$ *and let* $(b_n)_{n=1}^{\infty}$ *be the sequence given by* $b_n = \frac{1}{n^2}$. *To see that both* $((a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty})$ *and* $((b_n)_{n=1}^{\infty}, (a_n)_{n=1}^{\infty})$ *are elements of* $R$ *recall that* $R$ *is symmetric so it is enough to show that* $((a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}) \in R$. *Fix* $\varepsilon > 0$ *and let* $N = \frac{1}{\varepsilon}$, *then* $n > N$ *means* $n > \frac{1}{\varepsilon}$, *so* $\frac{1}{n} = \frac{n}{n^2} < \varepsilon$. *Further,* $\frac{n-1}{n^2} < \frac{n}{n^2} < \varepsilon$. *Since* $n \in \mathbb{N}$ *this means* $|\frac{n-1}{n^2}| < \varepsilon$, *or or* $|(\frac{1}{n} - \frac{1}{n^2}) - 0| < \varepsilon$ *so* $\lim_{n\to\infty}(\frac{1}{n} - \frac{1}{n^2}) = 0$ *meaning* $((a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}) \in R$. *Two sequences* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *are equal if for all* $n \in \mathbb{N}$ $a_n = b_n$, *but for the sequences defined above,* $a_4 = \frac{1}{4}$ *and* $b_4 = \frac{1}{16}$ *so* $(a_n)_{n=1}^{\infty} \neq (b_n)_{n=1}^{\infty}$, *so* $R$ *is not anti-symmetric so it cannot be a partial order relation on the set of all real valued sequences.*

*3. is a partial order relation on* $\mathcal{P}(X)$. *Let* $A, B, C \in \mathcal{P}(X)$ *be arbitrary.* $(A, A) \in R$ *means* $A \subseteq A$ *and since every set is a subset of itself, clearly* $R$ *is reflexive. Suppose* $(A, B) \in R$ *and* $(B, A) \in R$ *then* $A \subseteq B$ *and* $B \subseteq A$ *so* $A = B$, *so* $R$ *is anti-symmetric. To see that* $R$ *is transitive suppose* $(A, B) \in R$ *and* $(B, C) \in R$ *then* $A \subseteq B$ *and* $B \subseteq C$. *Let* $a \in A$ *be arbitrary. Since* $A \subseteq B$ *then* $a \in B$, *and since* $a \in B$ *and* $B \subseteq C$, $a \in C$ *so* $A \subseteq C$ *demonstrating that* $R$ *is transitive. Since* $R$ *is reflexive, anti-symmetric, and transitive* $R$ *is a partial order relation on* $\mathcal{P}(X)$.