

Derivation of standard and directional dipole quantities

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1 Integrating the radiative transfer equation

We start from the radiative transfer equation [1]:

$$(\nabla \cdot \vec{\omega})L(\mathbf{x}, \vec{\omega}) = -\sigma_t L(\mathbf{x}, \vec{\omega}) + \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) L(\mathbf{x}, \vec{\omega}') d\omega' + q(\mathbf{x}, \vec{\omega}),$$

where L is radiance at the position \mathbf{x} in the direction $\vec{\omega}$. The equation describes how the directional derivative of L depends on the scattering properties of the surrounding medium, where σ_t is the extinction coefficient, σ_s is the scattering coefficient, and p is the phase function. Finally, q is the emitted radiance in the medium per unit length that we move along a ray. If we then integrate over all directions $\vec{\omega}$, we get

$$\int_{4\pi} (\nabla \cdot \vec{\omega})L(\mathbf{x}, \vec{\omega}) d\omega = \int_{4\pi} -\sigma_t L(\mathbf{x}, \vec{\omega}) d\omega + \int_{4\pi} \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) L(\mathbf{x}, \vec{\omega}') d\omega' d\omega + \int_{4\pi} q(\mathbf{x}, \vec{\omega}) d\omega.$$

Rearranging, we obtain

$$\nabla \cdot \left(\int_{4\pi} \vec{\omega} L(\mathbf{x}, \vec{\omega}) d\omega \right) = -\sigma_t \int_{4\pi} L(\mathbf{x}, \vec{\omega}) d\omega + \sigma_s \int_{4\pi} \left(\int_{4\pi} p(\vec{\omega}', \vec{\omega}) d\omega \right) L(\mathbf{x}, \vec{\omega}') d\omega' + Q_0(\mathbf{x}),$$

where we used the regularity of the operators to switch divergence and integral operations on the left-hand side, and to switch the integrals on the right-hand side. The integral of the phase function is 1, since it is normalized, so by further simplifying and applying the definitions of fluence ϕ and vector irradiance \mathbf{E} , we obtain

$$\begin{aligned} \nabla \cdot \mathbf{E}(\mathbf{x}) &= -\sigma_t \phi(\mathbf{x}) + \sigma_s \int_{4\pi} L(\mathbf{x}, \vec{\omega}') d\omega' + Q_0(\mathbf{x}) \\ \nabla \cdot \mathbf{E}(\mathbf{x}) &= -\sigma_t \phi(\mathbf{x}) + \sigma_s \phi(\mathbf{x}) + Q_0(\mathbf{x}) \\ \nabla \cdot \mathbf{E}(\mathbf{x}) &= -\sigma_a \phi(\mathbf{x}) + Q_0(\mathbf{x}), \end{aligned} \tag{1}$$

where we introduced the absorption coefficient $\sigma_a = \sigma_t - \sigma_s$. Our Equation 1 then corresponds to Equation 1 in the work of Jensen et al. [6].

2 The diffusion approximation

To get the diffusion approximation, we approximate radiance using a second order spherical harmonics expansion:

$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{n=0}^1 \sum_{m=-n}^n L_{n,m}(\mathbf{x}) Y_{n,m}(\vec{\omega}),$$

where $Y_{n,m}(\vec{\omega})$ are normalized spherical harmonics basis functions, and $L_{n,m}(\mathbf{x})$ is the projection of L against the n, m basis function:

$$L_{n,m}(\mathbf{x}) = \int_{4\pi} L(\mathbf{x}, \vec{\omega}) Y_{n,m}(\vec{\omega}) d\omega.$$

For $n = 0$, the integral is trivial. The first term of the sum becomes:

$$L_{0,0}(\mathbf{x}) Y_{0,0}(\vec{\omega}) = \int_{4\pi} \sqrt{\frac{1}{4\pi}} L(\mathbf{x}, \vec{\omega}) d\omega \sqrt{\frac{1}{4\pi}} = \frac{1}{4\pi} \phi(\mathbf{x}).$$

As for the other basis functions, we use the Cartesian form. We have

$$L_{-1,1}(\mathbf{x}) Y_{-1,1}(\vec{\omega}) = \omega_x \int_{4\pi} \sqrt{\frac{3}{4\pi}} \omega_x L(\mathbf{x}, \vec{\omega}) d\omega \sqrt{\frac{3}{4\pi}} = \frac{3}{4\pi} \omega_x E_x(\mathbf{x}),$$

where the x subscript indicates the first component. Similarly, we obtain

$$L_{0,1}(\mathbf{x}) Y_{0,1}(\vec{\omega}) = \frac{3}{4\pi} \omega_z E_z(\mathbf{x})$$

$$L_{1,1}(\mathbf{x}) Y_{1,1}(\vec{\omega}) = \frac{3}{4\pi} \omega_y E_y(\mathbf{x}).$$

By applying the approximation, we finally obtain:

$$L(\mathbf{x}, \vec{\omega}) \approx \frac{1}{4\pi} \phi(\mathbf{x}) + \frac{3}{4\pi} \omega_x E_x(\mathbf{x}) + \frac{3}{4\pi} \omega_y E_y(\mathbf{x}) + \frac{3}{4\pi} \omega_z E_z(\mathbf{x}) = \frac{\phi(\mathbf{x})}{4\pi} + \frac{3}{4\pi} \vec{\omega} \cdot \mathbf{E}(\mathbf{x}).$$

3 The diffusion equation

To find an equation for the relation between the fluence and the optical properties of the medium, we substitute the diffusion approximation into the radiative transfer equation:

$$\begin{aligned} (\nabla \cdot \vec{\omega}) \left(\frac{\phi(\mathbf{x})}{4\pi} + \frac{3}{4\pi} \vec{\omega} \cdot \mathbf{E}(\mathbf{x}) \right) &= -\sigma_t \left(\frac{\phi(\mathbf{x})}{4\pi} + \frac{3}{4\pi} \vec{\omega} \cdot \mathbf{E}(\mathbf{x}) \right) \\ &+ \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) \left(\frac{\phi(\mathbf{x})}{4\pi} + \frac{3}{4\pi} \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) \right) d\omega' + q(\mathbf{x}, \vec{\omega}). \end{aligned}$$

For simplification, we need the following three identities:

$$\begin{aligned} \int_{4\pi} \vec{\omega} d\omega &= 0 \\ \int_{4\pi} \vec{\omega} (\vec{\omega} \cdot \mathbf{A}) d\omega &= \frac{4\pi}{3} \mathbf{A} \end{aligned} \tag{2}$$

$$\begin{aligned} \int_{4\pi} \vec{\omega} [\vec{\omega} \cdot \nabla (\vec{\omega} \cdot \mathbf{A})] d\omega &= 0 \\ \int_{4\pi} (\vec{\omega} \cdot \mathbf{A})(\vec{\omega} \cdot \mathbf{B}) d\omega &= \frac{4\pi}{3} \mathbf{A} \cdot \mathbf{B}. \end{aligned} \quad (3)$$

We first multiply into parentheses in the equation above:

$$\begin{aligned} \frac{1}{4\pi} \vec{\omega} \cdot \nabla \phi(\mathbf{x}) + \frac{3}{4\pi} \vec{\omega} \cdot \nabla (\vec{\omega} \cdot \mathbf{E}(\mathbf{x})) &= -\sigma_t \frac{\phi(\mathbf{x})}{4\pi} - \sigma_t \frac{3}{4\pi} \vec{\omega} \cdot \mathbf{E}(\mathbf{x}) + \sigma_s \frac{\phi(\mathbf{x})}{4\pi} \int_{4\pi} p(\vec{\omega}', \vec{\omega}) d\omega' \\ &\quad + \frac{3}{4\pi} \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) d\omega' + q(\mathbf{x}, \vec{\omega}). \end{aligned}$$

Now, we multiply each term by $\vec{\omega}$ and integrate over the sphere. Take all the terms separately, we have

$$\begin{aligned} \int_{4\pi} \frac{1}{4\pi} \vec{\omega} \cdot \nabla \phi(\mathbf{x}) \vec{\omega} d\omega &= \frac{1}{4\pi} \frac{4\pi}{3} \nabla \phi(\mathbf{x}) = \frac{\nabla \phi(\mathbf{x})}{3} \\ \int_{4\pi} \frac{3}{4\pi} \vec{\omega} \cdot \nabla (\vec{\omega} \cdot \mathbf{E}(\mathbf{x})) \vec{\omega} d\omega &= \frac{3}{4\pi} \int_{4\pi} \vec{\omega} [\vec{\omega} \cdot \nabla (\vec{\omega} \cdot \mathbf{E}(\mathbf{x}))] d\omega = 0 \\ \int_{4\pi} -\sigma_t \frac{\phi(\mathbf{x})}{4\pi} \vec{\omega} d\omega &= -\sigma_t \frac{\phi(\mathbf{x})}{4\pi} \int_{4\pi} \vec{\omega} d\omega = 0 \\ \int_{4\pi} -\sigma_t \frac{3}{4\pi} \vec{\omega} \cdot \mathbf{E}(\mathbf{x}) \vec{\omega} d\omega &= -\sigma_t \frac{3}{4\pi} \int_{4\pi} \vec{\omega} (\vec{\omega} \cdot \mathbf{E}(\mathbf{x})) d\omega = -\sigma_t \frac{3}{4\pi} \frac{4\pi}{3} \mathbf{E}(\mathbf{x}) = -\sigma_t \mathbf{E}(\mathbf{x}) \\ \int_{4\pi} \sigma_s \frac{\phi(\mathbf{x})}{4\pi} \int_{4\pi} p(\vec{\omega}', \vec{\omega}) d\omega' \vec{\omega} d\omega &= \sigma_s \frac{\phi(\mathbf{x})}{4\pi} \int_{4\pi} \vec{\omega} d\omega = 0 \\ \int_{4\pi} \frac{3}{4\pi} \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) d\omega' \vec{\omega} d\omega &\stackrel{(*)}{\approx} g \sigma_s \mathbf{E}(\mathbf{x}) \\ \int_{4\pi} q(\mathbf{x}, \vec{\omega}) \vec{\omega} d\omega &= \mathbf{Q}_1(\mathbf{x}), \end{aligned}$$

Passage (*) is a bit more delicate as it requires a spherical harmonics expansion of the phase function, as well as the assumption that the phase function depends only on the cosine between the two vectors ($p(\vec{\omega}', \vec{\omega}) = p(\vec{\omega}' \cdot \vec{\omega})$). For the interested reader, we derive the complete solution at the end of this chapter (Section 3.1). Putting everything together:

$$\begin{aligned} \frac{\nabla \phi(\mathbf{x})}{3} + 0 &= 0 - \sigma_t \mathbf{E}(\mathbf{x}) + 0 + g \sigma_s \mathbf{E}(\mathbf{x}) + \mathbf{Q}_1(\mathbf{x}) \\ \nabla \phi(\mathbf{x}) &= -3\sigma_t' \mathbf{E}(\mathbf{x}) + 3\mathbf{Q}_1(\mathbf{x}), \end{aligned} \quad (4)$$

where we used $\sigma_t' = \sigma_s' + \sigma_a = \sigma_s(1 - g) + \sigma_a = \sigma_t - g\sigma_s$. To obtain the diffusion equation, we need to combine Equations 4 and 1. We first rearrange Equation 4:

$$\mathbf{E}(\mathbf{x}) = 3D\mathbf{Q}_1(\mathbf{x}) - D\nabla \phi(\mathbf{x}),$$

where $D = \frac{1}{3\sigma_t'}$. Inserting into Equation 1 (and assuming a homogeneous medium), we have

$$\begin{aligned} \nabla \cdot (3D\mathbf{Q}_1(\mathbf{x}) - D\nabla \phi(\mathbf{x})) &= -\sigma_a \phi(\mathbf{x}) + Q_0(\mathbf{x}) \\ 3D\nabla \cdot \mathbf{Q}_1(\mathbf{x}) - D\nabla^2 \phi(\mathbf{x}) &= -\sigma_a \phi(\mathbf{x}) + Q_0(\mathbf{x}) \\ D\nabla^2 \phi(\mathbf{x}) &= \sigma_a \phi(\mathbf{x}) - Q_0(\mathbf{x}) + 3D\nabla \cdot \mathbf{Q}_1(\mathbf{x}), \end{aligned} \quad (5)$$

which is the diffusion equation as it appears in the work of Jensen et al. [6].

3.1 Phase function approximation

We have assumed that p depends only on the cosine of the two vectors so that $p(\vec{\omega}', \vec{\omega}) = p(\vec{\omega}' \cdot \vec{\omega}) = p(\vec{\omega}' \cdot \vec{\omega})$. As before, we want to simplify our function using spherical harmonics:

$$p(\vec{\omega}' \cdot \vec{\omega}) \approx \sum_{n=0}^1 \frac{2n+1}{4\pi} p_n(\vec{\omega}' \cdot \vec{\omega}) \sum_{m=-n}^n P_{n,m}(\vec{\omega}' \cdot \vec{\omega}).$$

Where p_n is the n -th order projection. P are the Legendre polynomials in spherical coordinates. Through the addition theorem and the regularity of spherical harmonics, we can deduce the following two identities:

$$P_{0,0}(\vec{\omega}' \cdot \vec{\omega}) = 1, \quad \sum_{m=-1}^1 P_{1,m}(\vec{\omega}' \cdot \vec{\omega}) = \vec{\omega}' \cdot \vec{\omega}.$$

We can easily calculate:

$$p_0 = \int_{4\pi} p(\vec{\omega}' \cdot \vec{\omega}) P_{0,0}(\vec{\omega}' \cdot \vec{\omega}) d\omega = 1,$$

and the other projection:

$$p_1 = \sum_{m=-1}^1 \int_{4\pi} p(\vec{\omega}' \cdot \vec{\omega}) Y_{1,m}(\vec{\omega}' \cdot \vec{\omega}) d\omega = \int_{4\pi} p(\vec{\omega}' \cdot \vec{\omega}) \vec{\omega}' \cdot \vec{\omega} d\omega = g.$$

Putting it all together, we get the final approximation:

$$p(\vec{\omega}' \cdot \vec{\omega}) \approx \frac{1 + 3g(\vec{\omega}' \cdot \vec{\omega})}{4\pi}.$$

Let us plug it in passage (*):

$$\begin{aligned} & \int_{4\pi} \frac{3}{4\pi} \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) d\omega' \vec{\omega} d\omega \approx \\ & \approx \int_{4\pi} \frac{3}{4\pi} \sigma_s \int_{4\pi} \frac{1 + 3g(\vec{\omega}' \cdot \vec{\omega})}{4\pi} \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) d\omega' \vec{\omega} d\omega = \\ & = \int_{4\pi} \frac{3}{4\pi} \sigma_s \left[\int_{4\pi} \frac{1}{4\pi} \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) d\omega' + \int_{4\pi} \frac{3g(\vec{\omega}' \cdot \vec{\omega})}{4\pi} \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) d\omega' \right] \vec{\omega} d\omega. \end{aligned}$$

Now the first integral within the square parenthesis is zero, given that \mathbf{E} is independent of $\vec{\omega}'$. The second integral within the parenthesis can be solved with the fourth identity in Equation 3.

$$\begin{aligned} & \int_{4\pi} \frac{3}{4\pi} \sigma_s \left[\int_{4\pi} \frac{1}{4\pi} \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) d\omega' + \int_{4\pi} \frac{3g(\vec{\omega}' \cdot \vec{\omega})}{4\pi} \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) d\omega' \right] \vec{\omega} d\omega = \\ & = \int_{4\pi} \frac{3}{4\pi} \sigma_s \left[0 + \frac{4\pi}{3} \frac{3g(\vec{\omega} \cdot \mathbf{E}(\mathbf{x}))}{4\pi} \right] \vec{\omega} d\omega = \\ & = \frac{3g}{4\pi} \sigma_s \int_{4\pi} (\vec{\omega} \cdot \mathbf{E}(\mathbf{x})) \vec{\omega} d\omega = \\ & = g\sigma_s \mathbf{E}(\mathbf{x}), \end{aligned}$$

which is the expected result.

4 Boundary condition

In the case of a scattering medium in a finite region of space, we impose the classic boundary condition that the net inward flux on each surface point \mathbf{x}_s with (inward) normal \vec{n}_s is zero:

$$\int_{2\pi_+} L(\mathbf{x}_s, \vec{\omega})(\vec{\omega} \cdot \vec{n}_s) d\omega = 0.$$

We use the diffusion approximation:

$$\begin{aligned} \int_{2\pi_+} \left(\frac{\phi(\mathbf{x}_s)}{4\pi} + \frac{3}{4\pi} \vec{\omega} \cdot \mathbf{E}(\mathbf{x}_s) \right) (\vec{\omega} \cdot \vec{n}_s) d\omega &= 0 \\ \phi(\mathbf{x}) \int_{2\pi_+} (\vec{\omega} \cdot \vec{n}_s) d\omega + 3 \int_{2\pi_+} (\vec{\omega} \cdot \mathbf{E}(\mathbf{x}_s)) (\vec{n}_s \cdot \vec{\omega}) d\omega &= 0. \end{aligned}$$

Given the standard spherical coordinates convention, $n_s = (0, 0, 1)$ and $\vec{\omega} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$. We then obtain

$$\int_{2\pi_+} (\vec{\omega} \cdot \vec{n}_s) d\omega = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos \theta \sin \theta d\theta d\phi = \pi,$$

and

$$\begin{aligned} \int_{2\pi_+} (\vec{\omega} \cdot \mathbf{E}(\mathbf{x})) (\vec{n}_s \cdot \vec{\omega}) d\omega &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (\cos \phi \sin \theta E_x + \sin \phi \sin \theta E_y + \cos \theta E_z) \cos \theta \sin \theta d\theta d\phi \\ &= \frac{2\pi}{3} E_z = \frac{2\pi}{3} \vec{n}_s \cdot \mathbf{E}(\mathbf{x}_s). \end{aligned}$$

Using the last two results and simplifying, we get:

$$\begin{aligned} \phi(\mathbf{x})\pi + 3 \left(\frac{2\pi}{3} \vec{n}_s \cdot \mathbf{E}(\mathbf{x}_s) \right) &= 0 \\ \phi(\mathbf{x}) + 2\vec{n}_s \cdot \mathbf{E}(\mathbf{x}_s) &= 0. \end{aligned}$$

From Equation 4, assuming no emission in \mathbf{x}_s , we have $\mathbf{Q}_1(\mathbf{x}_s) = \mathbf{0}$, so

$$\mathbf{E}(\mathbf{x}_s) = -D\nabla\phi(\mathbf{x}_s).$$

Inserting this result and simplifying, we get the final boundary condition:

$$\phi(\mathbf{x}_s) - 2D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}_s) = 0.$$

5 Different media assumption

To include nonzero inward flux at boundaries, we need to change the above equations. The boundary condition then becomes:

$$I_+ = \int_{2\pi_+} L(\mathbf{x}_s, \vec{\omega})(\vec{\omega} \cdot \vec{n}_s) d\omega = \int_{2\pi_-} R(\eta, \vec{\omega})L(\mathbf{x}_s, \vec{\omega})(-\vec{\omega} \cdot \vec{n}_s) d\omega = I_- \quad (6)$$

Keeping the above conventions, we define the Fresnel reflectance R by:

$$R(\eta, \vec{\omega}) = \begin{cases} 1 & \text{for } \frac{\pi}{2} \leq \theta \leq \pi - \theta_c \\ F_r(\eta, \vec{\omega} \cdot \vec{n}_s) & \text{for } \pi - \theta_c \leq \theta \leq \pi, \end{cases}$$

where θ_c is the critical angle, and

$$F_r(\eta, \mu) = \frac{1}{2} \left[\left(\frac{\mu - \eta\mu_0}{\mu + \eta\mu_0} \right)^2 + \left(\frac{\eta\mu - \mu_0}{\eta\mu + \mu_0} \right)^2 \right] \quad \text{with} \quad \mu_0^2 = 1 - \eta^2(1 - \mu^2)$$

and $\cos^2 \theta_c = \max(1 - \eta^{-2}, 0)$. In principle, if we allow complex numbers, we would have $R(\eta, \vec{\omega}) = F_r(\eta, \vec{\omega} \cdot \vec{n}_s)$.

The left side of Equation 6 is:

$$I_+ = \frac{1}{4}(\phi(\mathbf{x}) - 2D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x})).$$

The other side is more tricky, since it requires splitting the integration along the different angles. We proceed as before, introducing the diffusion approximation:

$$I_- = \frac{\phi(\mathbf{x})}{4\pi} \int_{2\pi_-} R(\eta, \vec{\omega})(-\vec{\omega} \cdot \vec{n}_s) d\omega + \frac{3}{4\pi} \int_{2\pi_-} R(\eta, \vec{\omega})(\vec{\omega} \cdot \mathbf{E}(\mathbf{x}))(-\vec{\omega} \cdot \vec{n}_s) d\omega.$$

The cosine-weighted integration of the Fresnel reflectance is sometimes referred to as diffuse Fresnel reflectance F_{dr} . If we, outside the region of total internal reflection, approximate the R function by Fresnel reflectance for normal incidence $R_0 = F_r(\eta, 1)$, we can find an approximate analytical solution. The first part is then:

$$\begin{aligned} \int_{2\pi_-} R(\eta, \vec{\omega})(-\vec{\omega} \cdot \vec{n}_s) d\omega &= \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi - \theta_c} (-\cos \theta) \sin \theta d\theta d\phi + \int_0^{2\pi} \int_{\pi - \theta_c}^{\pi} R_0(-\cos \theta) \sin \theta d\theta d\phi \\ &= \pi((1 - R_0) \cos^2 \theta_c + R_0). \end{aligned}$$

The second part (only on the z -coordinate, since the other coordinates are zero):

$$\begin{aligned} &\int_{2\pi_-} R(\eta, \vec{\omega})(\vec{\omega} \cdot \mathbf{E}(\mathbf{x}))(-\vec{\omega} \cdot \vec{n}_s) d\omega \\ &= E_z \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi - \theta_c} (-\cos^2 \theta) \sin \theta d\theta d\phi + E_z \int_0^{2\pi} \int_{\pi - \theta_c}^{\pi} R_0(-\cos^2 \theta) \sin \theta d\theta d\phi \\ &= \frac{2\pi}{3} E_z (R_0(\cos^3 \theta_c - 1) - \cos^3 \theta_c). \end{aligned}$$

Performing all simplifications, we finally get:

$$I_- = \frac{1}{4} [((1 - R_0) \cos^2 \theta_c + R_0)\phi(x) - 2D(R_0(\cos^3 \theta_c - 1) - \cos^3 \theta_c)(\vec{n}_s \cdot \nabla)\phi(\mathbf{x})].$$

We can now impose $I_+ = I_-$:

$$\phi(\mathbf{x}) - 2D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}) = ((1 - R_0) \cos^2 \theta_c + R_0)\phi(x) - 2D(R_0(\cos^3 \theta_c - 1) - \cos^3 \theta_c)(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}),$$

which we can simplify as follows:

$$\phi(\mathbf{x}) - 2 \frac{1 + R_0 + (1 - R_0) \cos^3 \theta_c}{1 - R_0 - (1 - R_0) \cos^2 \theta_c} D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}) = 0$$

$$\begin{aligned}\phi(\mathbf{x}) - 2 \frac{\frac{1+R_0}{1-R_0} + \cos^3 \theta_c}{1 - \cos^2 \theta_c} D(\vec{n}_s \cdot \nabla) \phi(\mathbf{x}) &= 0 \\ \phi(\mathbf{x}) - 2AD(\vec{n}_s \cdot \nabla) \phi(\mathbf{x}) &= 0.\end{aligned}$$

So, to handle reflective boundaries, we need to add a correction factor A in our boundary condition. With our current approximation, we have

$$A = \frac{\frac{1+R_0}{1-R_0} + \cos^3 \theta_c}{1 - \cos^2 \theta_c} = \frac{\frac{\eta^2+1}{2\eta} + [\max(1 - \eta^{-2}, 0)]^{\frac{3}{2}}}{1 - \max(1 - \eta^{-2}, 0)}.$$

5.1 Approximating the corrective factor

Assuming separability, we can rewrite the I_- term in Equation 6 as:

$$\begin{aligned}\int_{2\pi_-} R(\eta, \vec{\omega}) L(x_s, \vec{\omega}) (-\vec{\omega} \cdot \vec{n}_s) d\omega &\approx \int_{2\pi_-} R(\eta, \vec{\omega}) (-\vec{\omega} \cdot \vec{n}_s) d\omega \int_{2\pi_-} L(x_s, \vec{\omega}) (-\vec{\omega} \cdot \vec{n}_s) d\omega \\ &= F_{dr}(\eta) \frac{1}{4} (\phi(\mathbf{x}) + 2D(\vec{n}_s \cdot \nabla) \phi(\mathbf{x}))\end{aligned}$$

An approximate fit of the $F_{dr}(\eta)$ integral is [3]

$$F_{dr}(\eta) = \begin{cases} -0.4399 + \frac{0.7099}{\eta} - \frac{0.3319}{\eta^2} + \frac{0.0636}{\eta^3} & , \quad \eta < 1 \\ -\frac{1.4399}{\eta^2} + \frac{0.7099}{\eta} + 0.6681 + 0.0636\eta & , \quad \eta > 1. \end{cases}$$

So we can express the boundary condition as

$$\begin{aligned}\phi(\mathbf{x}) - 2\pi D(\vec{n}_s \cdot \nabla) \phi(\mathbf{x}) &= F_{dr}(\eta) (\phi(\mathbf{x}) + 2D(\vec{n}_s \cdot \nabla) \phi(\mathbf{x})) \\ \phi(\mathbf{x})(1 - F_{dr}) - 2D(1 + F_{dr})(\vec{n}_s \cdot \nabla) \phi(\mathbf{x}) &= 0 \\ \phi(\mathbf{x}) - 2D \frac{1 + F_{dr}}{1 - F_{dr}} (\vec{n}_s \cdot \nabla) \phi(\mathbf{x}) &= 0 \\ \phi(\mathbf{x}) - 2AD(\vec{n}_s \cdot \nabla) \phi(\mathbf{x}) &= 0\end{aligned}$$

with $A = \frac{1+F_{dr}}{1-F_{dr}}$, which returns values fairly close to the A found in the previous section. The A in this section is the one employed by Jensen et al. [6]. With respect to F_{dr} , they only provide the more common case of $\eta > 1$.

6 Solutions for an infinite medium

From the diffusion equation (5), we have

$$\begin{aligned}(D\nabla^2 - \sigma_a)\phi(\mathbf{x}) &= -Q_0(\mathbf{x}) + 3D\nabla \cdot \mathbf{Q}_1(\mathbf{x}) \\ (\nabla^2 - \sigma_{tr}^2)\phi(\mathbf{x}) &= -\frac{Q_0(\mathbf{x})}{D} + 3\nabla \cdot \mathbf{Q}_1(\mathbf{x}),\end{aligned}$$

which is a particular case of the screened Poisson equation. This has a generic solution based on the method of Green's functions:

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|} \left(\frac{Q_0(\mathbf{r}')}{D} - 3\nabla \cdot \mathbf{Q}_1(\mathbf{r}') \right) d^3\mathbf{r}'.$$

6.1 Point source solutions

If we use a point source placed at the origin, we have

$$Q_0(\mathbf{x}) = \Phi_i \delta(\mathbf{x})$$

$$\mathbf{Q}_1(\mathbf{x}) = 0.$$

Inserting in the solution based on Green's function:

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr} \|\mathbf{x} - \mathbf{r}'\|}}{\|\mathbf{x} - \mathbf{r}'\|} \left(\frac{\Phi_i \delta(\mathbf{r}')}{D} \right) d^3 \mathbf{r}'$$

and applying the delta function, the result is

$$\phi(\mathbf{x}) = \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r},$$

where $r = \|\mathbf{x}\|$ is the distance to the point of interest. A similar result is obtained if we consider a ray source at the origin with direction along the inward surface normal (z -axis). Suppose the medium exhibits isotropic scattering, then the source of first scattering events is [10]

$$Q_0(\mathbf{x}) = \Phi_i \sigma'_s \delta(x) \delta(y) \Theta(z) e^{-\sigma'_t z}$$

$$\mathbf{Q}_1(\mathbf{x}) = 0,$$

where $\Theta(z)$ is the Heaviside step function, which is 1 for $z \geq 0$ and otherwise 0. Inserting in the solution based on Green's function with $\mathbf{r}' = (x', y', z')$:

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr} \|\mathbf{x} - \mathbf{r}'\|}}{\|\mathbf{x} - \mathbf{r}'\|} \left(\frac{\Phi_i \sigma'_s \delta(x') \delta(y') \Theta(z') e^{-\sigma'_t z'}}{D} \right) d^3 \mathbf{r}'$$

and applying the deltas:

$$\phi(\mathbf{x}) = \frac{\Phi_i}{4\pi D} \int_0^{+\infty} \frac{e^{-\sigma_{tr} \|\mathbf{x} - z' \vec{n}_s\|}}{\|\mathbf{x} - z' \vec{n}_s\|} \left(\sigma'_s e^{-\sigma'_t z'} \right) dz'.$$

Considering positions \mathbf{x} in xy -plane far from the origin and the exponential attenuation with increasing z' , we use the assumption $\|\mathbf{x} - z' \vec{n}_s\| \approx \|\mathbf{x}\| = r$ and get

$$\phi(\mathbf{x}) = \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r} \sigma'_s \int_0^{+\infty} e^{-\sigma'_t z'} dz' = \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r} \frac{\sigma'_s}{\sigma'_t} = \alpha' \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r},$$

where $\alpha' = \frac{\sigma'_s}{\sigma'_t}$ is the reduced albedo. Thus, we get the monopole solution for a ray source of normal incidence:

$$\phi(\mathbf{x}) = \frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r}, \quad (7)$$

where $\Phi = \alpha' \Phi_i$.

6.2 Ray source solution

In case of a ray source that is not along the normal direction and not necessarily in an isotropic medium, we can use the following equations for the source terms [8]:

$$Q_0(\mathbf{x}) = \Phi_i \tilde{\sigma}_s \delta(x) \delta(y) \Theta(z) e^{-\tilde{\sigma}_t z}$$

$$\mathbf{Q}_1(\mathbf{x}) = \tilde{g} Q_0(\mathbf{x}) \vec{n}_s,$$

where we have used the delta-Eddington scattering properties [7]:

$$\tilde{\sigma}_s = \sigma_s(1 - g^2) \quad , \quad \tilde{\sigma}_t = \tilde{\sigma}_s + \sigma_a \quad , \quad \tilde{g} = g/(g + 1).$$

Inserting in the diffusion equation (5), we get two integrals when using the Green's function solution. Splitting the solution accordingly: $\phi(\mathbf{x}) = \phi_1(\mathbf{x}) + \phi_2(\mathbf{x})$, we have

$$\begin{aligned} \phi_1(\mathbf{x}) &= \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr} \|\mathbf{x} - \mathbf{r}'\|}}{\|\mathbf{x} - \mathbf{r}'\|} \left(\frac{\Phi_i \tilde{\sigma}_s \delta(x') \delta(y') \Theta(z') e^{-\tilde{\sigma}_t z'}}{D} \right) d^3 \mathbf{r}' \\ &= \frac{\Phi_i \tilde{\sigma}_s}{4\pi D} \int_0^{+\infty} \frac{e^{-\sigma_{tr} \|\mathbf{x} - z' \vec{n}_s\|}}{\|\mathbf{x} - z' \vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz' \\ &= \frac{3\tilde{\sigma}_s \Phi_i}{4\pi} \tilde{\sigma}_t \int_0^{+\infty} \frac{e^{-\sigma_{tr} \|\mathbf{x} - z' \vec{n}_s\|}}{\|\mathbf{x} - z' \vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz' \end{aligned}$$

and

$$\phi_2(\mathbf{x}) = -\frac{3}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr} \|\mathbf{x} - \mathbf{r}'\|}}{\|\mathbf{x} - \mathbf{r}'\|} \nabla \cdot \left(\Phi_i \tilde{\sigma}_s \tilde{g} \delta(x') \delta(y') \Theta(z') e^{-\tilde{\sigma}_t z'} \right) \vec{n}_s d^3 \mathbf{r}'.$$

We now apply the divergence operator:

$$\phi_2(\mathbf{x}) = -\frac{3\Phi_i \tilde{\sigma}_s \tilde{g}}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr} \|\mathbf{x} - \mathbf{r}'\|}}{\|\mathbf{x} - \mathbf{r}'\|} \left[\delta(x) \delta(y) \frac{\partial}{\partial z'} (\Theta(z') e^{-\tilde{\sigma}_t z'}) \right] d^3 \mathbf{r}'.$$

Note that we have only the z -term given that we multiply by $\vec{n}_s = (0, 0, 1)$. Using that $\frac{\partial \Theta(z')}{\partial z'} = \delta(z')$, we have

$$\phi_2(\mathbf{x}) = -\frac{3\Phi_i \tilde{\sigma}_s \tilde{g}}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr} \|\mathbf{x} - \mathbf{r}'\|}}{\|\mathbf{x} - \mathbf{r}'\|} \delta(x') \delta(y') \left[\delta(z') e^{-\tilde{\sigma}_t z'} - \tilde{\sigma}_t \Theta(z') e^{-\tilde{\sigma}_t z'} \right] d^3 \mathbf{r}'.$$

Applying the deltas, we get

$$\begin{aligned} \phi_2(\mathbf{x}) &= -\frac{3\Phi_i \tilde{\sigma}_s \tilde{g}}{4\pi} \frac{e^{-\sigma_{tr} r}}{r} + \frac{3\Phi_i \tilde{\sigma}_s \tilde{g} \tilde{\sigma}_t}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr} \|\mathbf{x} - \mathbf{r}'\|}}{\|\mathbf{x} - \mathbf{r}'\|} \delta(x') \delta(y') \Theta(z') e^{-\tilde{\sigma}_t z'} d^3 \mathbf{r}' \\ &= -\frac{3\Phi_i \tilde{\sigma}_s \tilde{g}}{4\pi} \frac{e^{-\sigma_{tr} r}}{r} + \frac{3\Phi_i \tilde{\sigma}_s \tilde{g} \tilde{\sigma}_t}{4\pi} \int_0^{+\infty} \frac{e^{-\sigma_{tr} \|\mathbf{x} - z' \vec{n}_s\|}}{\|\mathbf{x} - z' \vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz'. \end{aligned}$$

Putting it together:

$$\phi(\mathbf{x}) = -\frac{3\Phi_i \tilde{\sigma}_s \tilde{g}}{4\pi} \frac{e^{-\sigma_{tr} r}}{r} + \frac{3\tilde{\sigma}_s \Phi_i}{4\pi} (\tilde{\sigma}_t + \tilde{\sigma}_s \tilde{g} + \sigma_a \tilde{g}) \int_0^{+\infty} \frac{e^{-\sigma_{tr} \|\mathbf{x} - z' \vec{n}_s\|}}{\|\mathbf{x} - z' \vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz'$$

$$\phi(\mathbf{x}) = \frac{3\Phi_i\tilde{\sigma}_s}{4\pi} \left(-\tilde{g} \frac{e^{-\sigma_{tr}r}}{r} + (\tilde{\sigma}_s + \sigma_a(1 + \tilde{g})) \int_0^{+\infty} \frac{e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x}-z'\vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz' \right). \quad (8)$$

We can interpret the second term in Eq. 8 as the fluence from an exponentially decaying line source along the z -axis. Due to this exponentially decaying factor, the integrand will only have a significant weight for small z . Hence, in the asymptotic limit, $r \gg 1/\tilde{\sigma}_s$, we can approximate the distances in the integrand

$$\begin{aligned} \|\mathbf{x} - z'\vec{n}_s\| &= \sqrt{r^2 + z'^2 - 2z'r\cos\theta} = r \left(1 - 2\frac{z'}{r}\cos\theta + \frac{z'^2}{r^2} \right)^{\frac{1}{2}} \\ &\approx r \left(1 - \frac{z'}{r}\cos\theta \right) = r - z'\cos\theta, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\|\mathbf{x} - z'\vec{n}_s\|} &= \frac{1}{\sqrt{r^2 + z'^2 - 2z'r\cos\theta}} = \frac{1}{r} \left(1 - 2\frac{z'}{r}\cos\theta + \frac{z'^2}{r^2} \right)^{-\frac{1}{2}} \\ &\approx \frac{1}{r} \left(1 + \frac{z'}{r}\cos\theta \right). \end{aligned}$$

Then

$$\begin{aligned} \int_0^\infty \frac{e^{-\tilde{\sigma}_t z'} e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x} - z'\vec{n}_s\|} dz' &\approx \frac{e^{-\sigma_{tr}r}}{r} \int_0^\infty e^{-(\tilde{\sigma}_t - \sigma_{tr}\cos\theta)z'} \left(1 + \frac{z'}{r}\cos\theta \right) dz' \\ &= \frac{e^{-\sigma_{tr}r}}{r} \left(\frac{1}{\tilde{\sigma}_t - \sigma_{tr}\cos\theta} + \frac{\cos\theta}{r} \frac{1}{(\tilde{\sigma}_t - \sigma_{tr}\cos\theta)^2} \right). \end{aligned}$$

In a highly scattering medium, $\sigma_a \ll \tilde{\sigma}_s$. For $g \neq 1$ this will imply and $\sigma_a \ll \sigma_{tr} \ll \tilde{\sigma}_s$, and, hence

$$\frac{1}{\tilde{\sigma}_t - \sigma_{tr}\cos\theta} = \frac{1}{\tilde{\sigma}_s} \left(1 + \frac{\sigma_a - \sigma_{tr}\cos\theta}{\tilde{\sigma}_s} \right)^{-1} \approx \frac{1}{\tilde{\sigma}_s} \left(1 - \frac{\sigma_a - \sigma_{tr}\cos\theta}{\tilde{\sigma}_s} \right)$$

and

$$\frac{1}{(\tilde{\sigma}_t - \sigma_{tr}\cos\theta)^2} = \frac{1}{\tilde{\sigma}_s^2} \left(1 + \frac{\sigma_a - \sigma_{tr}\cos\theta}{\tilde{\sigma}_s} \right)^{-2} \approx \frac{1}{\tilde{\sigma}_s^2} \left(1 - 2\frac{\sigma_a - \sigma_{tr}\cos\theta}{\tilde{\sigma}_s} \right).$$

The integral can now be approximated by

$$\begin{aligned} \int_0^\infty \frac{e^{-\tilde{\sigma}_t z'} e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x} - z'\vec{n}_s\|} dz' &\approx \frac{e^{-\sigma_{tr}r}}{r} \left(\frac{1}{\tilde{\sigma}_s + \sigma_a - \sigma_{tr}\cos\theta} + \frac{\cos\theta}{r} \frac{1}{(\tilde{\sigma}_s + \sigma_a - \sigma_{tr}\cos\theta)^2} \right) \\ &\approx \frac{e^{-\sigma_{tr}r}}{\tilde{\sigma}_s r} \left(1 - \frac{\sigma_a - \sigma_{tr}\cos\theta}{\tilde{\sigma}_s} + \frac{\cos\theta}{\tilde{\sigma}_s r} \left(1 - 2\frac{\sigma_a - \sigma_{tr}\cos\theta}{\tilde{\sigma}_s} \right) \right) \\ &= \frac{e^{-\sigma_{tr}r}}{\tilde{\sigma}_s r} \left(1 - \frac{\sigma_a}{\tilde{\sigma}_s} + \frac{\cos\theta}{\tilde{\sigma}_s} \left(\sigma_{tr} + \frac{1}{r} - 2\frac{\sigma_a}{\tilde{\sigma}_s r} \right) + 2\frac{\sigma_{tr}\cos^2\theta}{\tilde{\sigma}_s^2 r} \right). \end{aligned}$$

Inserting this expression in Eq. 8,

$$\begin{aligned}
\frac{\phi(\mathbf{x})}{\Phi_i} &\approx -\frac{3\tilde{g}\tilde{\sigma}_s e^{-\sigma_{tr}r}}{4\pi r} \\
&+ \frac{3(\tilde{\sigma}_s + (\tilde{g} + 1)\sigma_a) e^{-\sigma_{tr}r}}{4\pi r} \left(1 - \frac{\sigma_a}{\tilde{\sigma}_s} + \frac{\cos\theta}{\tilde{\sigma}_s} \left(\sigma_{tr} + \frac{1}{r} - 2\frac{\sigma_a}{\tilde{\sigma}_s r} \right) + 2\frac{\sigma_{tr} \cos^2\theta}{\tilde{\sigma}_s^2 r} \right) \\
&= \frac{3e^{-\sigma_{tr}r}}{4\pi r} \left((\tilde{\sigma}_s + (\tilde{g} + 1)\sigma_a) \left(1 - \frac{\sigma_a}{\tilde{\sigma}_s} \right) - \tilde{g}\tilde{\sigma}_s \right) \\
&+ \frac{3e^{-\sigma_{tr}r}}{4\pi r} \cos\theta \left(1 + (\tilde{g} + 1)\frac{\sigma_a}{\tilde{\sigma}_s} \right) \left(\sigma_{tr} + \frac{1}{r} \left(1 - 2\frac{\sigma_a}{\tilde{\sigma}_s} \right) \right) \\
&+ \frac{3(\tilde{\sigma}_s + (\tilde{g} + 1)\sigma_a) e^{-\sigma_{tr}r}}{4\pi r} 2\frac{\sigma_{tr} \cos^2\theta}{\tilde{\sigma}_s^2 r}.
\end{aligned}$$

Now we neglect terms $\sigma_a/\tilde{\sigma}_s$ compared to unity

$$\frac{\phi(\mathbf{x})}{\Phi_i} \approx \frac{3e^{-\sigma_{tr}r}}{4\pi r} \tilde{\sigma}_s (1 - \tilde{g}) + \frac{3e^{-\sigma_{tr}r}}{4\pi r} \cos\theta \left(\sigma_{tr} + \frac{1}{r} \right) + \frac{3e^{-\sigma_{tr}r}}{4\pi r} 2\frac{\sigma_{tr} \cos^2\theta}{\tilde{\sigma}_s r}.$$

Introducing the reduced scattering coefficient $\sigma'_s = \tilde{\sigma}_s(1 - \tilde{g}) = \sigma_s(1 - g)$, and neglecting terms $\sigma_{tr}/\tilde{\sigma}_s \cdot 1/(\tilde{\sigma}_s r)$ compared to unity,

$$\phi(\mathbf{x}) \approx \frac{3\Phi_i e^{-\sigma_{tr}r}}{4\pi r} \sigma'_s + \frac{3\Phi_i e^{-\sigma_{tr}r}}{4\pi r} \cos\theta \left(\sigma_{tr} + \frac{1}{r} \right) = \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r} \left(1 + 3D \frac{1 + \sigma_{tr}r}{r} \cos\theta \right),$$

where $1/D$ is used in place of $3\sigma'_s$, since the σ'_s can be approximated by σ'_t when absorption is negligible compared to scattering. This is a valid assumption in highly scattering media.

7 Fresnel integrals

To aid in our calculations, we define the Fresnel transmittance integrals of the first two orders:

$$\begin{aligned}
C_\phi(\eta) &= \frac{1}{4\pi} \int_{2\pi} T_{21}(\eta, \theta_o) \cos\theta_o d\vec{\omega}_o \\
C_E(\eta) &= \frac{3}{4\pi} \int_{2\pi} T_{21}(\eta, \theta_o) \cos^2\theta_o d\vec{\omega}_o,
\end{aligned}$$

where $\cos\theta_o = \vec{n}_o \cdot \vec{\omega}_o$, and the integral is on the whole hemisphere where $\vec{n}_o \cdot \vec{\omega}_o > 0$. These integrals are similar to F_{dr} , but based on Fresnel transmittance instead of Fresnel reflectance.

8 BSSRDF theory

The BSSRDF is defined as the ratio of an element of emergent radiance L_r to an element of incident flux Φ_i [9]:

$$S(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o) = \frac{dL_r(\mathbf{x}_o, \vec{\omega}_o)}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)}.$$

There are various approximations available for the BSSRDF. For the directional dipole [5], the BSSRDF is split into the following terms:

$$S = T_{12}(S_{\delta E} + S_d)T_{21}.$$

Let us consider only the diffusive part, S_d . The emergent radiance due to diffusion events is given by

$$L_{r,d}(\mathbf{x}_o, \vec{\omega}_o) = \eta^2 T_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21}),$$

where $\vec{\omega}_{21}$ is the refracted vector corresponding to $\vec{\omega}_o$:

$$\vec{\omega}_{21} = \frac{\vec{\omega}_o}{\eta} - \left(\frac{\vec{n}_o \cdot \vec{\omega}_o}{\eta} - \sqrt{1 - \frac{1 - (\vec{n}_o \cdot \vec{\omega}_o)^2}{\eta^2}} \right) \vec{n}_o.$$

Combining the above equations we obtain:

$$S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o) = T_{12} S_d T_{21} = \frac{dL_{r,d}(\mathbf{x}_o, \vec{\omega}_o)}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)} = \eta^2 \frac{dT_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21})}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)}$$

We now integrate over the cosine-weighted hemisphere with $\vec{n}_o \cdot \vec{\omega}_o > 0$ on both sides of the equation:

$$\int_{2\pi} T_{12} S_d T_{21} \cos \theta_o d\vec{\omega}_o = \int_{2\pi} \eta^2 \frac{dT_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21})}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)} \cos \theta_o d\vec{\omega}_o.$$

Assuming no dependency on the outgoing direction, $S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o) = S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o)$, and we have

$$\begin{aligned} T_{12} S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) \int_{2\pi} T_{21} \cos \theta_o d\vec{\omega}_o &= \eta^2 \frac{d \int_{2\pi} T_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21}) \cos \theta_o d\vec{\omega}_o}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)} \\ T_{12} S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) 4\pi \frac{C_\phi(\eta)}{\eta^2} &= \frac{dM_d(\mathbf{x}_o)}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)} \\ T_{12} S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) 4\pi C_\phi(1/\eta) &= \frac{dM_d(\mathbf{x}_o)}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)}. \end{aligned} \quad (9)$$

Let us calculate the diffuse radiant exitance first. We insert the diffusion approximation in place of L_d to find

$$\begin{aligned} M_d(\mathbf{x}_o) &= \int_{2\pi} T_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21}) \cos \theta_o d\vec{\omega}_o = \int_{2\pi} T_{21} \left(\frac{\phi(\mathbf{x}_o)}{4\pi} - \frac{3}{4\pi} D \vec{\omega}_{21} \cdot \nabla \phi(\mathbf{x}_o) \right) \cos \theta_o d\vec{\omega}_o \\ &= \underbrace{\frac{\phi(\mathbf{x}_o)}{4\pi} \int_{2\pi} T_{21} \cos \theta_o d\vec{\omega}_o}_{I_\phi} - \underbrace{\frac{3}{4\pi} D \int_{2\pi} \vec{\omega}_{21} \cdot \nabla \phi(\mathbf{x}_o) \cos \theta_o d\vec{\omega}_o}_{I_E}. \end{aligned}$$

And, in a straightforward way,

$$I_\phi = C_\phi(\eta) \phi(\mathbf{x}_o).$$

The second part is more complicated. Without loss of generality, we rotate the reference coordinate system so that $\vec{\omega}_o = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$ and $\vec{n}_o = (0, 0, 1)$. Given this reference system, the refracted vector becomes:

$$\begin{aligned}\vec{\omega}_{21} &= \frac{(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)}{\eta} - \left(\frac{\cos \theta}{\eta} - \sqrt{1 - \frac{\sin^2 \theta}{\eta^2}} \right) (0, 0, 1) \\ &= \left(\frac{\cos \phi \sin \theta}{\eta}, \frac{\sin \phi \sin \theta}{\eta}, \sqrt{1 - \frac{\sin^2 \theta}{\eta^2}} \right)\end{aligned}$$

When inserted in the dot product in the expression for $I_{\mathbf{E}}$, we get a sum of three components. The first two terms of this sum are zero, since we can first integrate over ϕ :

$$\int_0^{2\pi} \cos \phi d\phi = \int_0^{2\pi} \sin \phi d\phi = 0.$$

So we are left only with the cumbersome z term:

$$I_{\mathbf{E}} = \frac{3}{4\pi} D \frac{\partial \phi_z(\mathbf{x}_o)}{\partial z} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} T_{21}(\eta, \theta_o) \sqrt{1 - \frac{\sin^2 \theta_o}{\eta^2}} \cos \theta_o \sin \theta_o d\theta_o d\phi,$$

where we note that $\frac{\partial \phi_z}{\partial z} = \vec{n}_o \cdot \nabla \phi$. We assume $\eta > 1$, so that the argument of the square root is never negative. We now perform a substitution using the law of refraction, $\sin \theta_i = \eta \sin \theta_o$. We obtain the following identities:

$$d\theta_o = \frac{\eta \cos \theta_i}{\cos \theta_o} d\theta_i, \quad T_{21}(\eta, \theta_o) = T_{21}(1/\eta, \theta_i), \quad \sqrt{1 - \frac{\sin^2 \theta_o}{\eta^2}} = \cos \theta_i.$$

If we introduce a critical angle $\theta_c = \arcsin(1/\eta)$, we get

$$\begin{aligned}I_{\mathbf{E}} &= \frac{3}{4\pi} D \vec{n}_o \cdot \nabla \phi(\mathbf{x}_o) \int_0^{2\pi} \int_0^{\theta_c} T_{21}(1/\eta, \theta_i) \cos \theta_i \cos \theta_o \eta \sin \theta_i \frac{\eta \cos \theta_i}{\cos \theta_o} d\theta_i d\phi \\ &= D \vec{n}_o \cdot \nabla \phi(\mathbf{x}_o) \frac{3}{4\pi} \eta^2 \int_0^{2\pi} \int_0^{\theta_c} T_{21}(1/\eta, \theta_i) \cos^2 \theta_i \sin \theta_i d\theta_i d\phi \\ &= D \vec{n}_o \cdot \nabla \phi(\mathbf{x}_o) \eta^2 C_{\mathbf{E}}(1/\eta) = C_{\mathbf{E}}(\eta) D \vec{n}_o \cdot \nabla \phi(\mathbf{x}_o).\end{aligned}$$

We can then get our final expression for $M_d(\mathbf{x}_o)$

$$M_d(\mathbf{x}_o) = C_{\phi}(\eta) \phi(\mathbf{x}_o) - C_{\mathbf{E}}(\eta) D \vec{n}_o \cdot \nabla \phi(\mathbf{x}_o).$$

Inserting into the expression (9), we derive the monopole BSSRDF:

$$S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) = \frac{1}{4\pi T_{12} C_{\phi}(1/\eta)} \frac{dM_d(\mathbf{x}_o)}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)}$$

8.1 Diffuse monopole BSSRDF

Using the monopole solution (7) for a ray normally incident on an isotropic half-space:

$$\phi(\mathbf{x}) = \alpha' \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r},$$

we can derive the gradient:

$$\nabla\phi(\mathbf{x}) = -\alpha' \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r^3} (1 + \sigma_{tr}r) \mathbf{x}.$$

Thus, we can find the monopole BSSRDF for this configuration:

$$S_d(\mathbf{x}_i, \mathbf{x}_o) = \frac{1}{4C_\phi(1/\eta)} \frac{\alpha'}{4\pi^2} \frac{e^{-\sigma_{tr}r}}{r^3} \left[C_\phi(\eta) \frac{r^2}{D} + C_{\mathbf{E}}(\eta)(1 + \sigma_{tr}r)(\mathbf{x}_o - \mathbf{x}_i) \cdot \vec{n}_o \right],$$

where we used $\Phi_i = T_{12}L_i(\mathbf{x}_i, \vec{\omega}_i)$. This is the monopole version of the better dipole from Eugene d'Eon [2]. To get a monopole version of the standard dipole [4, 6], we further approximate the above expression using $C_\phi(1/\eta) \approx 1/4$, $C_\phi(\eta) \approx 0$, and $C_{\mathbf{E}}(\eta) \approx 1$:

$$S_d(\mathbf{x}_i, \mathbf{x}_o) = \frac{\alpha'}{4\pi^2} \frac{e^{-\sigma_{tr}r}}{r^3} (1 + \sigma_{tr}r) \mathbf{x} \cdot \vec{n}_o$$

This is the monopole version of the standard dipole.

8.2 Directional monopole BSSRDF

For the directional dipole, we have the following form:

$$\phi(\mathbf{x}) = \frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r} \left(1 + 3D \frac{1 + \sigma_{tr}r}{r} \cos\theta \right).$$

It is more convenient to do the gradient in spherical coordinates:

$$\nabla\phi(\mathbf{x}) = \frac{\partial}{\partial r}\phi(\mathbf{x})\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\phi(\mathbf{x})\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\phi(\mathbf{x})\vec{e}_\phi,$$

where

$$\frac{\partial}{\partial r}\phi(\mathbf{x}) = -\frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r^2} \left(3D \frac{2(1 + \sigma_{tr}r) + (\sigma_{tr}r)^2}{r} \cos\theta + (1 + \sigma_{tr}r) \right)$$

and

$$\frac{\partial}{\partial\theta}\phi(\mathbf{x}) = -\frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r^2} 3D(1 + \sigma_{tr}r) \sin\theta.$$

Finally, $\frac{\partial}{\partial\phi}\phi(\mathbf{x}) = 0$. Given our choice of basis for derivation, we can use the following identities:

$$\begin{aligned} \vec{e}_r &= \frac{\mathbf{x}}{r} \\ -\vec{e}_\theta \sin\theta &= \vec{\omega}_{12} - \vec{e}_r \cos\theta. \end{aligned}$$

Inserting the identities, we get an expression for the gradient:

$$\begin{aligned}\nabla\phi(\mathbf{x}) &= -\frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r^2} \left(3D \frac{2(1+\sigma_{tr}r) + (\sigma_{tr}r)^2}{r} \cos\theta + (1+\sigma_{tr}r) \right) \vec{e}_r \\ &\quad + \frac{1}{r} \frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r^2} 3D(1+\sigma_{tr}r)(-\vec{e}_\theta \sin\theta) \\ \nabla\phi(\mathbf{x}) &= \frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r^3} \left[\left(-3D \frac{2(1+\sigma_{tr}r) + (\sigma_{tr}r)^2}{r} \cos\theta - (1+\sigma_{tr}r) \right) r\vec{e}_r \right. \\ &\quad \left. - 3D(1+\sigma_{tr}r) \cos\theta \vec{e}_r + 3D(1+\sigma_{tr}r)\vec{\omega}_{12} \right] \\ \nabla\phi(\mathbf{x}) &= \frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r^3} \left[- \left(3D \frac{3(1+\sigma_{tr}r) + (\sigma_{tr}r)^2}{r} \cos\theta + (1+\sigma_{tr}r) \right) \mathbf{x} \right. \\ &\quad \left. + 3D(1+\sigma_{tr}r)\vec{\omega}_{12} \right].\end{aligned}$$

We can now do the same as above, obtaining the directional monopole BSSRDF with $\mathbf{x} = \mathbf{x}_o - \mathbf{x}_i$ and $r = \|\mathbf{x}\|$:

$$\begin{aligned}S'_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) &= \frac{1}{4C_\phi(1/\eta)} \frac{1}{4\pi^2} \frac{e^{-\sigma_{tr}r}}{r^3} \left[C_\phi(\eta) \left(\frac{r^2}{D} + 3(1+\sigma_{tr}r)\mathbf{x} \cdot \vec{\omega}_{12} \right) \right. \\ &\quad \left. - C_E(\eta) \left(3D(1+\sigma_{tr}r)\vec{\omega}_{12} \cdot \vec{n}_o - \left((1+\sigma_{tr}r) + 3D \frac{3(1+\sigma_{tr}r) + (\sigma_{tr}r)^2}{r^2} \mathbf{x} \cdot \vec{\omega}_{12} \right) \mathbf{x} \cdot \vec{n}_o \right) \right].\end{aligned}$$

References

- [1] Subrahmanyan Chandrasekhar. *Radiative Transfer*. Oxford, Clarendon Press, 1950. Unabridged and slightly revised version published by Dover Publications, Inc., in 1960.
- [2] Eugene d'Eon. A better dipole. Publicly available technical report. <http://www.eugenedeon.com/?project=a-better-dipole>, 2012.
- [3] W. G. Egan, T. Hilgeman, and J. Reichman. Determination of absorption and scattering coefficients for nonhomogeneous media. 2: Experiment. *Applied Optics*, 12(8):1816–1823, August 1973.
- [4] Thomas J. Farrell, Michael S. Patterson, and Brian Wilson. A diffusion theory model of spatially resolved, steady-state diffuse reflectance for the non-invasive determination of tissue optical properties *in vivo*. *Medical Physics*, 19(4):879–888, July/August 1992.
- [5] Jeppe Revall Frisvad, Toshiya Hachisuka, and Thomas Kim Kjeldsen. Directional dipole model for subsurface scattering. *ACM Transactions on Graphics*, 34(1):5:1–5:12, November 2014.
- [6] Henrik Wann Jensen, Stephen R. Marschner, Marc Levoy, and Pat Hanrahan. A practical model for subsurface light transport. In *Proceedings of SIGGRAPH 2001*, pages 511–518. ACM, August 2001.

- [7] J. H. Joseph, W. J. Wiscombe, and J. A. Weinman. The delta-Eddington approximation for radiative flux transfer. *Journal of Atmospheric Sciences*, 33:2452–2459, December 1976.
- [8] S. Menon, Q. Su, and R. Grobe. Determination of g and μ using multiply scattered light in turbid media. *Physical Review Letters*, 94:153904, April 2005.
- [9] F. E. Nicodemus, J. C. Richmond, J. J. Hsia, I. W. Ginsberg, and T. Limperis. Geometrical considerations and nomenclature for reflectance. Technical report, National Bureau of Standards (US), October 1977.
- [10] M. S. Patterson, E. Schwartz, and B. C. Wilson. Quantitative reflectance spectrophotometry for the non-invasive measurement of photosensitizer concentration in tissue during photodynamic therapy. In *Proceedings of SPIE*, volume 1065 of *Photodynamic Therapy: Mechanisms*, pages 115–122, June 1989.