

Derivation of standard and directional dipole quantities

Alessandro Dal Corso
Technical University of Denmark

Jeppe Revall Frisvad
Technical University of Denmark

January 2018

1 Measuring BSSRDFs

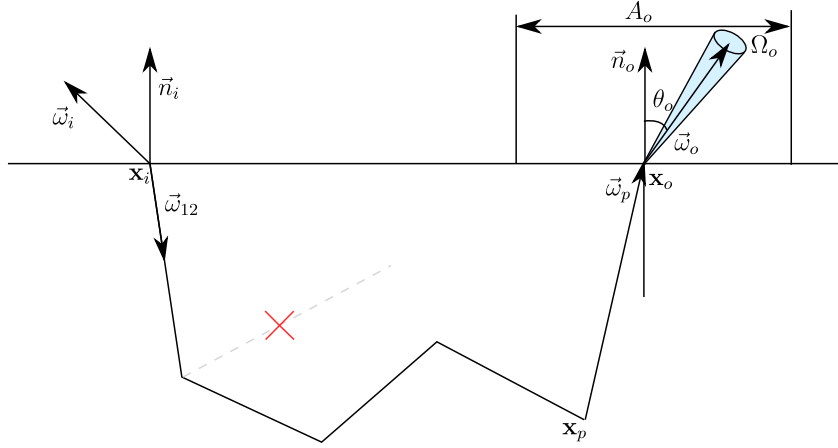


Figure 1: Configuration for random walk procedure.

In our configuration, we want to measure the BSSRDF for a specific spatial area bin with area A_o and for a specific angular bin with solid angle Ω_o . In a measurement environment [3], we have the following incoming flux:

$$\Phi_i(A_i, \Omega_i) = \int_{A_i} \int_{\Omega_i} L(\mathbf{x}'_i, \vec{\omega}'_i) (\vec{\omega}'_i \cdot \vec{n}_i) d\vec{\omega}'_i dA'_i$$

See Figure 1 for an explanation of the variables. The corresponding response weighted emergent flux is:

$$\Phi_o(A_o, \Omega_o) = \int_{A_o} \int_{\Omega_o} \int_{A_i} \int_{\Omega_i} L(\mathbf{x}'_i, \vec{\omega}'_i) S(\mathbf{x}'_i, \vec{\omega}'_i, \mathbf{x}'_o, \vec{\omega}'_o) R(\mathbf{x}'_o, \vec{\omega}'_o) (\vec{\omega}'_i \cdot \vec{n}_i) d\vec{\omega}'_i dA_i (\vec{\omega}'_o \cdot \vec{n}_o) d\vec{\omega}'_o dA'_o$$

Note that we choose the notation $\Phi_o(A_o, \Omega_o)$ to represent the emergent flux arriving in the bin. In our ideal precomputed case, we ignore the response of the sensor $R(\mathbf{x}'_o, \vec{\omega}'_o) = 1$. If we choose a unit response function and unit incident radiance $L(\mathbf{x}_i, \vec{\omega}_i) = \delta(\mathbf{x}_i)\delta(\vec{\omega}_i)$, the ratio of emergent to incident flux G can be easily calculated:

$$G(A_o, \Omega_o) = \frac{\Phi_o(A_o, \Omega_o)}{\Phi_i(A_i, \Omega_i)} = \int_{A_o} \int_{\Omega_o} S(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}'_o, \vec{\omega}'_o)(\vec{\omega}'_o \cdot \vec{n}_o) d\vec{\omega}'_o dA'_o$$

Once we have obtained G , we can invert the above equation to obtain the final BSSRDF:

$$S(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o) = \frac{dL_o(\mathbf{x}_o, \vec{\omega}_o)}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)} \approx \frac{\Phi_o(\mathbf{x}_o, \vec{\omega}_o)}{\Phi_i(\mathbf{x}_i, \vec{\omega}_i) A_o \Omega_o (\vec{\omega}_o \cdot \vec{n}_o)} = \frac{G(A_o, \Omega_o)}{A_o \Omega_o (\vec{\omega}_o \cdot \vec{n}_o)} \quad (1)$$

Which means that we can represent the transmitted light between two points, the BSSRDF, as the ratio of emergent and incoming flux, weighted by the cosine weighted area of a bin times the bin solid angle.

The BSSRDF of a medium can be defined as an operator that includes all radiance resulting from the scattering events in the medium [1]. This radiance is described locally by the radiative transfer equation [1]. Thus, we can evaluate the scattering process given by the BSSRDF using the radiative transfer equation. So, the above quantity G becomes all flux carried by a flux packet arriving in a bin after going through the random walk. This is the procedure used in [4].

The random walk procedure works as follows:

1. Create a new photon with flux $\Phi_t = 1/N$, where N is the number of photons. Start assigning $\mathbf{x}_p = \mathbf{x}_i$ and $\vec{\omega}_p = \vec{\omega}_{12}$, the refracted incoming direction.
2. Sample a new distance to the next scattering event s . We assume an homogenous medium, so we use an exponential distribution with decay σ_t , that can be easily importance sampled:

$$s = \frac{-\log(1 - \xi)}{\sigma_t}.$$

where $\xi \in [0, 1)$.

3. Check if the next scattering event $\mathbf{x}'_p = \mathbf{x}_p + s\vec{\omega}_p$ is within the medium.
4. If the next scattering event is within the medium:
 - (a) Kill the path with probability $1 - \alpha$. If killed, start from step 1.
 - (b) If not killed, sample a new direction $\vec{\omega}_p$ according to the Heyney-Greenstein phase function.
 - (c) Continue from step 2.
5. If the next scattering event is outside of the medium:
6. Calculate the reflection Fresnel coefficient R_{21} and update \mathbf{x}_p to be the intersection point on the surface of the medium.

- (a) With probability R_{21} reflect the path around $-\vec{n}_o$, and continue from point 2.
- (b) With probability $1 - R_{21}$ refract the path outside, storing Φ_t in the corresponding bin. Start from step 1.

Note that all the sampling is chosen to cancel out the terms that would change the stored flux. So, the flux within the medium should change for each scattering event as:

$$\Phi'_t = \Phi_t \sigma_s p(\vec{\omega}_p \cdot \vec{\omega}_p) e^{-\sigma_t s}$$

Tough, applying the pdfs, $p(\vec{\omega}_p \cdot \vec{\omega}_p')$ cancels out sampling $\vec{\omega}_p$ according to the correct phase function, $e^{-\sigma_t s}$ goes away with the pdf of distance sampling, leaving $\sigma_s/\sigma_t = \alpha$. This last factor is divided out by the path killing probability, leading $\Phi'_t = \Phi_t$ as expected. If the flux goes outside the medium, we do not multiply it by the corresponding Fresnel coefficients to account for reflection and refraction probabilities.

2 Variance reduction through connections

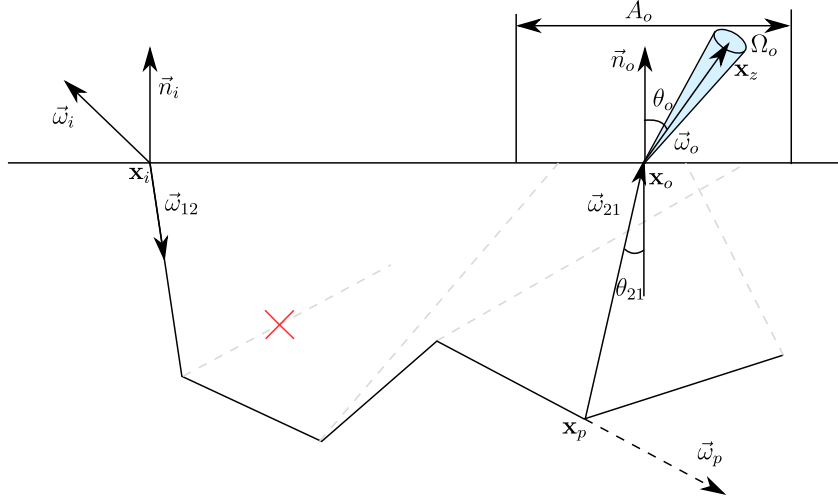


Figure 2: Configuration for the connection event described in the text.

The procedure from the previous section leaves a correct BSSRDF. However, this process requires a huge number of photons to converge to an acceptable solution. To reduce convergence times, we connect each scattering event (except the first, to measure multiple scattering only), to a point \mathbf{x}_o on the surface. We then use the three point form of the local scattering equation [2] to calculate the resulting measurement:

$$L(\mathbf{x}_o, \vec{\omega}_o) = L_e(\mathbf{x}_o, \vec{\omega}_o) + L(\mathbf{x}_p \rightarrow \mathbf{x}_o) f(\mathbf{x}_p \rightarrow \mathbf{x}_o \rightarrow \mathbf{x}_z) G(\mathbf{x}_p \leftrightarrow \mathbf{x}_o) V(\mathbf{x}_p \leftrightarrow \mathbf{x}_o)$$

We assume a non emissive body ($L_e(\mathbf{x}_o, \vec{\omega}_o) = 0$). In our case, we have a point in the volume generating a scattering event (\mathbf{x}_p), a point in a bin on the surface

(\mathbf{x}_o) , and a point specifying a direction into the bin (\mathbf{x}_z) , see Figure 2. So, we have the incoming radiance:

$$L(\mathbf{x}_p \rightarrow \mathbf{x}_o) = \alpha \Phi_t(\mathbf{x}_p) p(\vec{\omega}_p \cdot \vec{\omega}_{21}, g)$$

The three point scattering function, corresponding to the BSDF for a perfectly reflective material at the surface:

$$f(\mathbf{x}_p \rightarrow \mathbf{x}_o \rightarrow \mathbf{x}_z) = T_{21}(\eta, \vec{\omega}_o) \frac{\delta(\vec{\omega}_o)}{\vec{\omega}_o \cdot \vec{n}_o}$$

The geometry term:

$$G(\mathbf{x}_p \leftrightarrow \mathbf{x}_o) = \frac{|\vec{\omega}_{21} \cdot \vec{n}_o|}{\|\mathbf{x}_o - \mathbf{x}_p\|^2}$$

And the generic visibility term:

$$V(\mathbf{x}_p \leftrightarrow \mathbf{x}_o) = V'(\mathbf{x}_p \leftrightarrow \mathbf{x}_o) \exp \left(- \int_0^{\|\mathbf{x}_o - \mathbf{x}_p\|} \sigma_t(\mathbf{x}_p + t\vec{\omega}_{21}) dt \right)$$

In our configuration of a semi infinite plane, the binary visibility function $V'(\mathbf{x}_p \leftrightarrow \mathbf{x}_o)$ is always one (i.e., it is always possible to connect to the emergent point). Since the have an homogenous medium, we can rewrite V as the beam transmittance:

$$V(\mathbf{x}_p \leftrightarrow \mathbf{x}_o) = \exp(-\sigma_t \|\mathbf{x}_o - \mathbf{x}_p\|)$$

And the final radiance of the p -th scattering event of the k -th photon as:

$$L_{k,p}(\mathbf{x}_o, \vec{\omega}_o) = \alpha \Phi_t(\mathbf{x}_p) p(\vec{\omega}_p \cdot \vec{\omega}_{21}, g) T_{21}(\eta, \vec{\omega}_o) \frac{\delta(\vec{\omega}_o)}{\vec{\omega}_o \cdot \vec{n}_o} \frac{|\vec{\omega}_{21} \cdot \vec{n}_o|}{\|\mathbf{x}_o - \mathbf{x}_p\|^2} \exp(-\sigma_t \|\mathbf{x}_o - \mathbf{x}_p\|)$$

To get the whole contribution of a photon, we need to sum all the elements of the random walk:

$$L_k(\mathbf{x}_o, \vec{\omega}_o) = \sum_p L_{k,p}(\mathbf{x}_o, \vec{\omega}_o)$$

To get the final emerging flux, we need to solve the measurement equation for each bin. This is integration of incident flux (given by the three point form of the local scattering equation) across the cosine weighted area and the solid angle of the bin:

$$\Phi_o(A_o, \Omega_o) = \int_{A_o} \int_{\Omega_o} L(\mathbf{x}'_o, \vec{\omega}'_o) (\vec{\omega}'_o \cdot \vec{n}_o) d\vec{\omega}'_o dA'_o$$

Now, since the three point scattering function is a delta function in cosine-weighted direction, the cosine weight and the integration over bin solid angle cancel with this delta. As for the area integral, we solve it using Monte Carlo integration. The samples for the integral then become the photons, giving:

$$\Phi_o(A_o, \Omega_o) = \frac{1}{N} \sum_{k=1}^N \sum_p \alpha \Phi_t p(\vec{\omega}_p \cdot \vec{\omega}_{21}, g) T_{21}(\eta, \vec{\omega}_o) \frac{|\vec{\omega}_{21} \cdot \vec{n}_o|}{\|\mathbf{x}_o - \mathbf{x}_p\|^2} e^{-\sigma_t \|\mathbf{x}_o - \mathbf{x}_p\|} \frac{\|\mathbf{x}_o - \mathbf{x}_i\|}{\text{pdf}(\mathbf{x}_o)}$$

Where we sampled \mathbf{x}_o according to an uniform pdf in polar coordinates. If we think our spatial bin A_o as a circular sector delimited by r_{\min} and r_{\max} in radius,

and $\theta_{s,\min}$ and $\theta_{s,\max}$. We have $\text{pdf}(\mathbf{x}_o) = 1/(\Delta r \Delta \theta_s)$, where $\Delta r = r_{\max} - r_{\min}$ and $\Delta \theta_s = \theta_{s,\max} - \theta_{s,\min}$. Since we use a circular sector, we obtain:

$$A_o = \iint_A dA = \int_{r_{\min}}^{r_{\max}} \int_{\theta_{s,\min}}^{\theta_{s,\max}} r dr d\theta_s = \frac{r_{\max}^2 - r_{\min}^2}{2} \Delta \theta_s = \frac{r_{\min} + r_{\max}}{2} \Delta r \Delta \theta_s$$

Also, since we are using a uniform sampling of the hemisphere, we obtain a constant solid angle for all directional bins:

$$\Omega_o = \frac{2\pi}{N_{bins}}$$

Plugging everything in the original BSSRDF equation 1 and simplifying, we obtain the final BSSRDF:

$$\begin{aligned} S(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o) = \\ \frac{1}{N} \sum_{k=1}^N \sum_p \alpha \Phi_t p(\vec{\omega}_p \cdot \vec{\omega}_{21}, g) \frac{T_{21}(\eta, \vec{\omega}_o)}{\vec{n}_o \cdot \vec{\omega}_o} \frac{|\vec{\omega}_{21} \cdot \vec{n}_o|}{\|\mathbf{x}_o - \mathbf{x}_p\|^2} e^{-\sigma_t \|\mathbf{x}_o - \mathbf{x}_p\|} \\ \|\mathbf{x}_o - \mathbf{x}_i\| \frac{2}{r_{\min} + r_{\max}} \frac{N_{bins}}{2\pi} \end{aligned}$$

Note that we used $\Phi(\mathbf{x}_i, \vec{\omega}_i) = 1$.

References

- [1] Radiative transfer on discrete spaces. by r. w. preisendorfer. oxford, london, edinburgh, new york, paris, frankfurt. (pergamon press), (1st edn.) 1965. pp. xi, 462: Many figures, tables. £7. *Quarterly Journal of the Royal Meteorological Society*, 93(395):148–148, 1967.
- [2] Matthias Raab, Daniel Seibert, and Alexander Keller. Unbiased global illumination with participating media. In Alexander Keller, Stefan Heinrich, and Harald Niederreiter, editors, *Monte Carlo and Quasi-Monte Carlo Methods 2006*, pages 591–605, Berlin, Heidelberg, 2008. Springer Berlin Heidelberg.
- [3] William H. Venable, Jr. and Jack J. Hsia. Optical Radiation Measurements: Describing Spectrophotometric Measurements. 1974.
- [4] Lihong Wang, Steven L. Jacques, and Liqiong Zheng. Mcm1—monte carlo modeling of light transport in multi-layered tissues. *Computer Methods and Programs in Biomedicine*, 47(2):131 – 146, 1995.