1 Planar surfaces

We start from the classical BSSRDF form of the rendering equation:

$$L_o(\mathbf{x}_o, \vec{\omega}_o) = \int_A \int_{\Omega(\mathbf{x}_i)} L_i(\mathbf{x}_i, \vec{\omega}_i) S(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o)(\vec{\omega}_i \cdot \vec{n}_i) d\omega_i dA$$

Where $\Omega(\mathbf{x}_i)$ is an hemisphere placed in $\mathbf{x}_i$ with normal $\vec{n}_i$. Our first step is to convert the integration into polar coordinates. We assume for now that the integration happens on a plane. In polar coordinates, we have $dA = r dr d\theta$. The integral becomes:

$$L_o(\mathbf{x}_o, \vec{\omega}_o) = \int_0^{2\pi} \int_0^{+\infty} \int_{\Omega(\mathbf{x}_i)} L_i(\mathbf{x}_i, \vec{\omega}_i) S(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o)(\vec{\omega}_i \cdot \vec{n}_i) d\omega_i r dr d\theta$$

Where $r = \|\mathbf{x}_o - \mathbf{x}_i\|$. Also, $\mathbf{x}_i = \mathbf{x}_o + r\cos\theta\ \vec{t}_o + r\sin\theta\ \vec{b}_o$, where $\vec{t}_o$ and $\vec{b}_o$ form an orthonormal basis with $\vec{n}_o$.

We can now perform Monte Carlo integration, that gives the estimator:

$$L_o^{N,M}(\mathbf{x}_o, \vec{\omega}_o) = \frac{1}{NM} \sum_{p=1}^{M} \sum_{q=1}^{N} \frac{L_i(\mathbf{x}_{i,p}, \vec{\omega}_{i,q}) S(\mathbf{x}_{i,p}, \vec{\omega}_{i,q}, \mathbf{x}_o, \vec{\omega}_o)(\vec{\omega}_{i,q} \cdot \vec{n}_{i,p}) r}{\text{pdf}(r, \theta)\text{pdf}(\omega_{i,q})}$$

Now, we can importance sample our disc coordinates (r, θ) using the joint pdf:

$$\text{pdf}(r, \theta) = \frac{1}{2\pi} \sigma_{tr} e^{-\sigma_{tr} r}$$

And we can sample the incoming light direction using a cosine weighted hemisphere:

$$\text{pdf}(\omega_{i,q}) = \frac{\vec{\omega}_{i,q} \cdot \vec{n}_{i,p}}{\pi}$$

So the integral can be finally approximated as:

$$L_o^{N,M}(\mathbf{x}_o, \vec{\omega}_o) = \frac{2\pi^2}{NM\sigma_{tr}} \sum_{p=1}^{M} \sum_{q=1}^{N} L_i(\mathbf{x}_{i,p}, \vec{\omega}_{i,q}) S(\mathbf{x}_{i,p}, \vec{\omega}_{i,q}, \mathbf{x}_o, \vec{\omega}_o) r e^{\sigma_{tr} r}$$

Note that the $re^{\sigma_{tr} r}$ cancels out some terms inside the BSSRDF.

2 Non planar surfaces

We now assume that the surface is no longer planar. So we want to make a change of variables so that our integral becomes tractable, i.e. we are able to perform it in the tangent plane. Let us call dA_{tan} an element of area in the tangent plane. The change of variables simply becomes:

$$L_o(\mathbf{x}_o, \vec{\omega}_o) = \int_{A_{tan}} \int_{\Omega(\mathbf{x}_i)} L_i(\mathbf{x}_i, \vec{\omega}_i) S(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o)(\vec{\omega}_i \cdot \vec{n}_i) d\omega_i \left| \frac{dA}{dA_{tan}} \right| dA_{tan}$$

Where dA is the corresponding area element on the real surface, continuing a camera ray. We call the camera ray $\vec{d} = \frac{\mathbf{x}_i - \mathbf{e}}{\|\mathbf{x}_i - \mathbf{e}\|}$, where $\mathbf{e}$ is the camera position.

We only need to estimate the Jacobian of the change of variables $\left|\frac{dA}{dA_{tan}}\right|$. We observe that, since we are using a pinhole camera, the solid angles subtended by the projected area elements must be equal. We calculate them and put them equal, that gives:

$$\frac{dA(\vec{n}_i \cdot -\vec{d})}{d^2} = \frac{dA_{tan}(\vec{n}_o \cdot -\vec{d})}{d_{tan}^2}$$

Where $d = \|\mathbf{x}_i - \mathbf{e}\|$ and $d_{tan} = \|\mathbf{x}_{i,tan} - \mathbf{e}\|$ are the distances with the real point and the point in tangent space, respectively. The Jacobian then becomes:

$$\left|\frac{dA}{dA_{tan}}\right| = \frac{(\vec{n}_o \cdot -\vec{d})}{(\vec{n}_i \cdot -\vec{d})} \frac{d^2}{d_{tan}^2}$$

Note that for a plane, $\vec{n}_i = \vec{n}_o$ and $d = d_{tan}$, giving a unitary Jacobian as expected.

After the change of variables, we can now proceed as in the previous section, and obtaining the final estimator:

$$L_o^{N,M}(\mathbf{x}_o, \vec{\omega}_o) = \frac{2\pi^2}{NM\sigma_{tr}} \sum_{p=1}^{M} \sum_{q=1}^{N} L_i(\mathbf{x}_{i,p}, \vec{\omega}_{i,q}) S(\mathbf{x}_{i,p}, \vec{\omega}_{i,q}, \mathbf{x}_o, \vec{\omega}_o) \frac{(\vec{n}_o \cdot -\vec{d})}{(\vec{n}_i \cdot -\vec{d})} \frac{d^2}{d_{tan}^2} r e^{\sigma_{tr} r}$$