

# Derivation of standard and directional dipole quantities

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## 1 Integrating the radiative transfer equation

We start from the radiative transfer equation [1]:

$$(\nabla \cdot \vec{\omega})L(\mathbf{x}, \vec{\omega}) = -\sigma_t L(\mathbf{x}, \vec{\omega}) + \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) L(\mathbf{x}, \vec{\omega}') d\omega' + q(\mathbf{x}, \vec{\omega}),$$

where  $L$  is radiance at the position  $\mathbf{x}$  in the direction  $\vec{\omega}$ . The equation describes how the directional derivative of  $L$  depends on the scattering properties of the surrounding medium, where  $\sigma_t$  is the extinction coefficient,  $\sigma_s$  is the scattering coefficient, and  $p$  is the phase function. Finally,  $q$  is the emitted radiance in the medium per unit length that we move along a ray. If we then integrate over all directions  $\vec{\omega}$ , we get

$$\int_{4\pi} (\nabla \cdot \vec{\omega})L(\mathbf{x}, \vec{\omega}) d\omega = \int_{4\pi} -\sigma_t L(\mathbf{x}, \vec{\omega}) d\omega + \int_{4\pi} \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) L(\mathbf{x}, \vec{\omega}') d\omega' d\omega + \int_{4\pi} q(\mathbf{x}, \vec{\omega}) d\omega.$$

Rearranging, we obtain

$$\nabla \cdot \left( \int_{4\pi} \vec{\omega} L(\mathbf{x}, \vec{\omega}) d\omega \right) = -\sigma_t \int_{4\pi} L(\mathbf{x}, \vec{\omega}) d\omega + \sigma_s \int_{4\pi} \left( \int_{4\pi} p(\vec{\omega}', \vec{\omega}) d\omega \right) L(\mathbf{x}, \vec{\omega}') d\omega' + Q_0(\mathbf{x}),$$

where we used the regularity of the operators to switch divergence and integral operations on the left-hand side, and to switch the integrals on the right-hand side. The integral of the phase function is 1, since it is normalized, so by further simplifying and applying the definitions of fluence  $\phi$  and vector irradiance  $\mathbf{E}$ , we obtain

$$\begin{aligned} \nabla \cdot \mathbf{E}(\mathbf{x}) &= -\sigma_t \phi(\mathbf{x}) + \sigma_s \int_{4\pi} L(\mathbf{x}, \vec{\omega}') d\omega' + Q_0(\mathbf{x}) \\ \nabla \cdot \mathbf{E}(\mathbf{x}) &= -\sigma_t \phi(\mathbf{x}) + \sigma_s \phi(\mathbf{x}) + Q_0(\mathbf{x}) \\ \nabla \cdot \mathbf{E}(\mathbf{x}) &= -\sigma_a \phi(\mathbf{x}) + Q_0(\mathbf{x}), \end{aligned} \tag{1}$$

where we introduced the absorption coefficient  $\sigma_a = \sigma_t - \sigma_s$ . Our Equation 1 then corresponds to Equation 1 in the work of Jensen et al. [6].

## 2 The diffusion approximation

To get the diffusion approximation, we approximate radiance using a second order spherical harmonics expansion:

$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{n=0}^1 \sum_{m=-n}^n L_{n,m}(\mathbf{x}) Y_{n,m}(\vec{\omega}),$$

where  $Y_{n,m}(\vec{\omega})$  are normalized real-valued spherical harmonics basis functions, and  $L_{n,m}(\mathbf{x})$  is the projection of  $L$  against the  $n, m$  basis function:

$$L_{n,m}(\mathbf{x}) = \int_{4\pi} L(\mathbf{x}, \vec{\omega}) Y_{n,m}(\vec{\omega}) d\omega.$$

For  $n = 0$ , the integral is trivial. The first term of the sum becomes:

$$L_{0,0}(\mathbf{x}) Y_{0,0}(\vec{\omega}) = \int_{4\pi} \sqrt{\frac{1}{4\pi}} L(\mathbf{x}, \vec{\omega}) d\omega \sqrt{\frac{1}{4\pi}} = \frac{1}{4\pi} \phi(\mathbf{x}).$$

As for the other basis functions, we use the Cartesian form. We have

$$L_{1,-1}(\mathbf{x}) Y_{1,-1}(\vec{\omega}) = \omega_x \int_{4\pi} \sqrt{\frac{3}{4\pi}} \omega_x L(\mathbf{x}, \vec{\omega}) d\omega \sqrt{\frac{3}{4\pi}} = \frac{3}{4\pi} \omega_x E_x(\mathbf{x}),$$

where the  $x$  subscript indicates the first component. Similarly, we obtain

$$L_{1,0}(\mathbf{x}) Y_{1,0}(\vec{\omega}) = \frac{3}{4\pi} \omega_z E_z(\mathbf{x})$$

$$L_{1,1}(\mathbf{x}) Y_{1,1}(\vec{\omega}) = \frac{3}{4\pi} \omega_y E_y(\mathbf{x}).$$

By applying the approximation, we finally obtain:

$$L(\mathbf{x}, \vec{\omega}) \approx \frac{1}{4\pi} \phi(\mathbf{x}) + \frac{3}{4\pi} \omega_x E_x(\mathbf{x}) + \frac{3}{4\pi} \omega_y E_y(\mathbf{x}) + \frac{3}{4\pi} \omega_z E_z(\mathbf{x}) = \frac{\phi(\mathbf{x})}{4\pi} + \frac{3}{4\pi} \vec{\omega} \cdot \mathbf{E}(\mathbf{x}).$$

## 3 The diffusion equation

To find an equation for the relation between the fluence and the optical properties of the medium, we substitute the diffusion approximation into the radiative transfer equation:

$$\begin{aligned} (\nabla \cdot \vec{\omega}) \left( \frac{\phi(\mathbf{x})}{4\pi} + \frac{3}{4\pi} \vec{\omega} \cdot \mathbf{E}(\mathbf{x}) \right) &= -\sigma_t \left( \frac{\phi(\mathbf{x})}{4\pi} + \frac{3}{4\pi} \vec{\omega} \cdot \mathbf{E}(\mathbf{x}) \right) \\ &+ \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) \left( \frac{\phi(\mathbf{x})}{4\pi} + \frac{3}{4\pi} \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) \right) d\omega' + q(\mathbf{x}, \vec{\omega}). \end{aligned}$$

For simplification, we need the following three identities:

$$\begin{aligned} \int_{4\pi} \vec{\omega} d\omega &= 0 \\ \int_{4\pi} \vec{\omega} (\vec{\omega} \cdot \mathbf{A}) d\omega &= \frac{4\pi}{3} \mathbf{A} \end{aligned} \tag{2}$$

$$\int_{4\pi} \vec{\omega} [\vec{\omega} \cdot \nabla (\vec{\omega} \cdot \mathbf{A})] d\omega = 0$$

We first multiply into parentheses in the equation above:

$$\begin{aligned} \frac{1}{4\pi} \vec{\omega} \cdot \nabla \phi(\mathbf{x}) + \frac{3}{4\pi} \vec{\omega} \cdot \nabla (\vec{\omega} \cdot \mathbf{E}(\mathbf{x})) &= -\sigma_t \frac{\phi(\mathbf{x})}{4\pi} - \sigma_t \frac{3}{4\pi} \vec{\omega} \cdot \mathbf{E}(\mathbf{x}) + \sigma_s \frac{\phi(\mathbf{x})}{4\pi} \int_{4\pi} p(\vec{\omega}', \vec{\omega}) d\omega' \\ &\quad + \frac{3}{4\pi} \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) d\omega' + q(\mathbf{x}, \vec{\omega}). \end{aligned}$$

Now, we multiply each term by  $\vec{\omega}$  and integrate over the sphere. Taking all the terms separately, we have

$$\begin{aligned} \int_{4\pi} \frac{1}{4\pi} \vec{\omega} \cdot \nabla \phi(\mathbf{x}) \vec{\omega} d\omega &= \frac{1}{4\pi} \frac{4\pi}{3} \nabla \phi(\mathbf{x}) = \frac{\nabla \phi(\mathbf{x})}{3} \\ \int_{4\pi} \frac{3}{4\pi} \vec{\omega} \cdot \nabla (\vec{\omega} \cdot \mathbf{E}(\mathbf{x})) \vec{\omega} d\omega &= \frac{3}{4\pi} \int_{4\pi} \vec{\omega} [\vec{\omega} \cdot \nabla (\vec{\omega} \cdot \mathbf{E}(\mathbf{x}))] d\omega = 0 \\ \int_{4\pi} -\sigma_t \frac{\phi(\mathbf{x})}{4\pi} \vec{\omega} d\omega &= -\sigma_t \frac{\phi(\mathbf{x})}{4\pi} \int_{4\pi} \vec{\omega} d\omega = 0 \\ \int_{4\pi} -\sigma_t \frac{3}{4\pi} \vec{\omega} \cdot \mathbf{E}(\mathbf{x}) \vec{\omega} d\omega &= -\sigma_t \frac{3}{4\pi} \int_{4\pi} \vec{\omega} (\vec{\omega} \cdot \mathbf{E}(\mathbf{x})) d\omega = -\sigma_t \frac{3}{4\pi} \frac{4\pi}{3} \mathbf{E}(\mathbf{x}) = -\sigma_t \mathbf{E}(\mathbf{x}) \\ \int_{4\pi} \sigma_s \frac{\phi(\mathbf{x})}{4\pi} \int_{4\pi} p(\vec{\omega}', \vec{\omega}) d\omega' \vec{\omega} d\omega &= \sigma_s \frac{\phi(\mathbf{x})}{4\pi} \int_{4\pi} \vec{\omega} d\omega = 0 \\ \int_{4\pi} \frac{3}{4\pi} \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) d\omega' \vec{\omega} d\omega &\stackrel{(*)}{=} g \sigma_s \mathbf{E}(\mathbf{x}) \\ \int_{4\pi} q(\mathbf{x}, \vec{\omega}) \vec{\omega} d\omega &= \mathbf{Q}_1(\mathbf{x}), \end{aligned}$$

Passage (\*) is a bit more delicate as it requires the assumption that the phase function is rotationally symmetric, which means that it depends only on the cosine between the two direction vector arguments ( $p(\vec{\omega}', \vec{\omega}) = p(\vec{\omega}' \cdot \vec{\omega})$ ). For the interested reader, we further discuss this result at the end of this section (Section 3.1). Putting everything together:

$$\begin{aligned} \frac{\nabla \phi(\mathbf{x})}{3} + 0 &= 0 - \sigma_t \mathbf{E}(\mathbf{x}) + 0 + g \sigma_s \mathbf{E}(\mathbf{x}) + \mathbf{Q}_1(\mathbf{x}) \\ \nabla \phi(\mathbf{x}) &= -3\sigma_t' \mathbf{E}(\mathbf{x}) + 3\mathbf{Q}_1(\mathbf{x}), \end{aligned} \tag{3}$$

where we used  $\sigma_t' = \sigma_s' + \sigma_a = \sigma_s(1 - g) + \sigma_a = \sigma_t - g\sigma_s$ . To obtain the diffusion equation, we need to combine Equations 3 and 1. We first rearrange Equation 3:

$$\mathbf{E}(\mathbf{x}) = 3D\mathbf{Q}_1(\mathbf{x}) - D\nabla \phi(\mathbf{x}),$$

where  $D = \frac{1}{3\sigma_t'}$ . Inserting into Equation 1 (and assuming a homogeneous medium), we have

$$\begin{aligned} \nabla \cdot (3D\mathbf{Q}_1(\mathbf{x}) - D\nabla \phi(\mathbf{x})) &= -\sigma_a \phi(\mathbf{x}) + Q_0(\mathbf{x}) \\ 3D\nabla \cdot \mathbf{Q}_1(\mathbf{x}) - D\nabla^2 \phi(\mathbf{x}) &= -\sigma_a \phi(\mathbf{x}) + Q_0(\mathbf{x}) \\ D\nabla^2 \phi(\mathbf{x}) &= \sigma_a \phi(\mathbf{x}) - Q_0(\mathbf{x}) + 3D\nabla \cdot \mathbf{Q}_1(\mathbf{x}), \end{aligned} \tag{4}$$

which is the diffusion equation as it appears in the work of Jensen et al. [6].

### 3.1 Rotationally symmetric phase function

Assuming that  $p$  is rotationally symmetric so that  $p(\vec{\omega}', \vec{\omega}) = p(\vec{\omega}' \cdot \vec{\omega})$ , we can expand it in Legendre polynomials:

$$p(\vec{\omega}' \cdot \vec{\omega}) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} p_n P_n(\vec{\omega}' \cdot \vec{\omega}),$$

where  $p_n$  are the expansion coefficients and  $P_n$  are the Legendre polynomials. Since

$$P_0(\vec{\omega}' \cdot \vec{\omega}) = 1, \quad P_1(\vec{\omega}' \cdot \vec{\omega}) = \vec{\omega}' \cdot \vec{\omega},$$

we have

$$p_0 = \int_{4\pi} p(\vec{\omega}' \cdot \vec{\omega}) P_0(\vec{\omega}' \cdot \vec{\omega}) d\omega = 1,$$

$$p_1 = \int_{4\pi} p(\vec{\omega}' \cdot \vec{\omega}) P_1(\vec{\omega}' \cdot \vec{\omega}) d\omega = \int_{4\pi} p(\vec{\omega}' \cdot \vec{\omega}) (\vec{\omega}' \cdot \vec{\omega}) d\omega = g.$$

In addition, the orthogonality relations for Legendre polynomials are

$$\int_{4\pi} P_n(\vec{\omega}' \cdot \vec{\omega}) P_m(\vec{\omega}' \cdot \vec{\omega}) d\omega' = \begin{cases} \frac{4\pi}{2n+1} & , \text{ for } n = m \\ 0 & , \text{ otherwise.} \end{cases}$$

Noting that in spherical harmonics with  $\vec{\omega}$  as the local  $z$ -axis:

$$\begin{aligned} \omega'_x &= -\sqrt{\frac{4\pi}{3}} Y_{1,1}(\vec{\omega}') \\ \omega'_y &= -\sqrt{\frac{4\pi}{3}} Y_{1,-1}(\vec{\omega}') \\ \omega'_z &= \sqrt{\frac{4\pi}{3}} Y_{1,0}(\vec{\omega}') = P_1(\vec{\omega}' \cdot \vec{\omega}), \end{aligned}$$

and that integration over  $\omega'_x$  and  $\omega'_y$  are zero, the expansion of the phase function combined with the orthogonality of the Legendre polynomials leave us with

$$\begin{aligned} & \int_{4\pi} \frac{3}{4\pi} \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) d\omega' \vec{\omega} d\omega \\ &= \int_{4\pi} \frac{3}{4\pi} \sigma_s \frac{3}{4\pi} g \left( \frac{4\pi}{3} \vec{\omega} \cdot \mathbf{E}(\mathbf{x}) \right) \vec{\omega} d\omega = g \sigma_s \mathbf{E}(\mathbf{x}), \end{aligned}$$

which is the expected result.

## 4 Boundary condition

In the case of a scattering medium in a half-space, we impose the classic boundary condition that the net inward flux on each surface point  $\mathbf{x}_s$  with (inward) normal  $\vec{n}_s$  is zero:

$$\int_{2\pi_+} L(\mathbf{x}_s, \vec{\omega}) (\vec{\omega} \cdot \vec{n}_s) d\omega = 0.$$

We use the diffusion approximation:

$$\int_{2\pi_+} \left( \frac{\phi(\mathbf{x}_s)}{4\pi} + \frac{3}{4\pi} \vec{\omega} \cdot \mathbf{E}(\mathbf{x}_s) \right) (\vec{\omega} \cdot \vec{n}_s) d\omega = 0$$

$$\phi(\mathbf{x}) \int_{2\pi_+} (\vec{\omega} \cdot \vec{n}_s) d\omega + 3 \int_{2\pi_+} (\vec{\omega} \cdot \mathbf{E}(\mathbf{x}_s)) (\vec{n}_s \cdot \vec{\omega}) d\omega = 0.$$

Given the standard spherical coordinates convention,  $n_s = (0, 0, 1)$  and  $\vec{\omega} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$ . We then obtain

$$\int_{2\pi_+} (\vec{\omega} \cdot \vec{n}_s) d\omega = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos \theta \sin \theta d\theta d\phi = \pi,$$

and

$$\int_{2\pi_+} (\vec{\omega} \cdot \mathbf{E}(\mathbf{x})) (\vec{n}_s \cdot \vec{\omega}) d\omega = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (\cos \phi \sin \theta E_x + \sin \phi \sin \theta E_y + \cos \theta E_z) \cos \theta \sin \theta d\theta d\phi$$

$$= \frac{2\pi}{3} E_z = \frac{2\pi}{3} \vec{n}_s \cdot \mathbf{E}(\mathbf{x}_s).$$

Using the last two results and simplifying, we get:

$$\phi(\mathbf{x})\pi + 3 \left( \frac{2\pi}{3} \vec{n}_s \cdot \mathbf{E}(\mathbf{x}_s) \right) = 0$$

$$\phi(\mathbf{x}) + 2\vec{n}_s \cdot \mathbf{E}(\mathbf{x}_s) = 0.$$

From Equation 3, assuming no emission in  $\mathbf{x}_s$ , we have  $\mathbf{Q}_1(\mathbf{x}_s) = \mathbf{0}$ , so

$$\mathbf{E}(\mathbf{x}_s) = -D\nabla\phi(\mathbf{x}_s).$$

Inserting this result and simplifying, we get the final boundary condition:

$$\phi(\mathbf{x}_s) - 2D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}_s) = 0.$$

## 5 Different media assumption

To include nonzero inward flux at boundaries, we need to change the above equations. The boundary condition then becomes:

$$I_+ = \int_{2\pi_+} L(\mathbf{x}_s, \vec{\omega}) (\vec{\omega} \cdot \vec{n}_s) d\omega = \int_{2\pi_-} R(\eta, \vec{\omega}) L(\mathbf{x}_s, \vec{\omega}) (-\vec{\omega} \cdot \vec{n}_s) d\omega = I_- . \quad (5)$$

Keeping the above conventions, we define the Fresnel reflectance  $R$  by:

$$R(\eta, \vec{\omega}) = \begin{cases} 1 & \text{for } \frac{\pi}{2} \leq \theta \leq \pi - \theta_c \\ F_r(\eta, \vec{\omega} \cdot \vec{n}_s) & \text{for } \pi - \theta_c \leq \theta \leq \pi, \end{cases}$$

where  $\theta_c$  is the critical angle, and

$$F_r(\eta, \mu) = \frac{1}{2} \left[ \left( \frac{\mu - \eta\mu_0}{\mu + \eta\mu_0} \right)^2 + \left( \frac{\eta\mu - \mu_0}{\eta\mu + \mu_0} \right)^2 \right] \quad \text{with} \quad \mu_0^2 = 1 - \eta^2(1 - \mu^2)$$

and  $\cos^2 \theta_c = \max(1 - \eta^{-2}, 0)$ . In principle, if we allow complex numbers, we would have  $R(\eta, \vec{\omega}) = F_r(\eta, \vec{\omega} \cdot \vec{n}_s)$ .

The left side of Equation 5 is:

$$I_+ = \frac{1}{4}(\phi(\mathbf{x}) - 2D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x})).$$

The other side is more tricky, since it requires splitting the integration along the different angles. We proceed as before, introducing the diffusion approximation:

$$I_- = \frac{\phi(\mathbf{x})}{4\pi} \int_{2\pi_-} R(\eta, \vec{\omega})(-\vec{\omega} \cdot \vec{n}_s) d\omega + \frac{3}{4\pi} \int_{2\pi_-} R(\eta, \vec{\omega})(\vec{\omega} \cdot \mathbf{E}(\mathbf{x}))(-\vec{\omega} \cdot \vec{n}_s) d\omega.$$

The cosine-weighted integration of the Fresnel reflectance is sometimes referred to as diffuse Fresnel reflectance  $F_{dr}$ . If we, outside the region of total internal reflection, approximate the  $R$  function by Fresnel reflectance for normal incidence  $R_0 = F_r(\eta, 1)$ , we can find an approximate analytical solution. The first part is then:

$$\begin{aligned} \int_{2\pi_-} R(\eta, \vec{\omega})(-\vec{\omega} \cdot \vec{n}_s) d\omega &= \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi-\theta_c} (-\cos \theta) \sin \theta d\theta d\phi + \int_0^{2\pi} \int_{\pi-\theta_c}^{\pi} R_0(-\cos \theta) \sin \theta d\theta d\phi \\ &= \pi((1 - R_0) \cos^2 \theta_c + R_0). \end{aligned}$$

The second part (only on the  $z$ -coordinate, since the other coordinates are zero):

$$\begin{aligned} &\int_{2\pi_-} R(\eta, \vec{\omega})(\vec{\omega} \cdot \mathbf{E}(\mathbf{x}))(-\vec{\omega} \cdot \vec{n}_s) d\omega \\ &= E_z \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi-\theta_c} (-\cos^2 \theta) \sin \theta d\theta d\phi + E_z \int_0^{2\pi} \int_{\pi-\theta_c}^{\pi} R_0(-\cos^2 \theta) \sin \theta d\theta d\phi \\ &= \frac{2\pi}{3} E_z (R_0(\cos^3 \theta_c - 1) - \cos^3 \theta_c). \end{aligned}$$

Performing all simplifications, we finally get:

$$I_- = \frac{1}{4}[(1 - R_0) \cos^2 \theta_c + R_0]\phi(x) - 2D(R_0(\cos^3 \theta_c - 1) - \cos^3 \theta_c)(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}).$$

We can now impose  $I_+ = I_-$ :

$$\phi(\mathbf{x}) - 2D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}) = ((1 - R_0) \cos^2 \theta_c + R_0)\phi(x) - 2D(R_0(\cos^3 \theta_c - 1) - \cos^3 \theta_c)(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}),$$

which we can simplify as follows:

$$\begin{aligned} \phi(\mathbf{x}) - 2 \frac{1 + R_0 + (1 - R_0) \cos^3 \theta_c}{1 - R_0 - (1 - R_0) \cos^2 \theta_c} D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}) &= 0 \\ \phi(\mathbf{x}) - 2 \frac{\frac{1+R_0}{1-R_0} + \cos^3 \theta_c}{1 - \cos^2 \theta_c} D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}) &= 0 \\ \phi(\mathbf{x}) - 2AD(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}) &= 0. \end{aligned}$$

So, to handle reflective boundaries, we need to add a correction factor  $A$  in our boundary condition. With our current approximation, we have

$$A = \frac{\frac{1+R_0}{1-R_0} + \cos^3 \theta_c}{1 - \cos^2 \theta_c} = \frac{\frac{\eta^2+1}{2\eta} + [\max(1 - \eta^{-2}, 0)]^{\frac{3}{2}}}{1 - \max(1 - \eta^{-2}, 0)}.$$

## 5.1 Approximating the corrective factor

Assuming separability, we can rewrite the  $I_-$  term in Equation 5 as:

$$\begin{aligned} \int_{2\pi_-} R(\eta, \vec{\omega}) L(x_s, \vec{\omega}) (-\vec{\omega} \cdot \vec{n}_s) d\omega &\approx \int_{2\pi_-} R(\eta, \vec{\omega}) (-\vec{\omega} \cdot \vec{n}_s) d\omega \int_{2\pi_-} L(x_s, \vec{\omega}) (-\vec{\omega} \cdot \vec{n}_s) d\omega \\ &= F_{dr}(\eta) \frac{1}{4} (\phi(\mathbf{x}) + 2D(\vec{n}_s \cdot \nabla) \phi(\mathbf{x})) \end{aligned}$$

An approximate fit of the  $F_{dr}(\eta)$  integral is [3]

$$F_{dr}(\eta) = \begin{cases} -0.4399 + \frac{0.7099}{\eta} - \frac{0.3319}{\eta^2} + \frac{0.0636}{\eta^3} & , \quad \eta < 1 \\ -\frac{1.4399}{\eta^2} + \frac{0.7099}{\eta} + 0.6681 + 0.0636\eta & , \quad \eta > 1. \end{cases}$$

So we can express the boundary condition as

$$\begin{aligned} \phi(\mathbf{x}) - 2\pi D(\vec{n}_s \cdot \nabla) \phi(\mathbf{x}) &= F_{dr}(\eta) (\phi(\mathbf{x}) + 2D(\vec{n}_s \cdot \nabla) \phi(\mathbf{x})) \\ \phi(\mathbf{x})(1 - F_{dr}) - 2D(1 + F_{dr})(\vec{n}_s \cdot \nabla) \phi(\mathbf{x}) &= 0 \\ \phi(\mathbf{x}) - 2D \frac{1 + F_{dr}}{1 - F_{dr}} (\vec{n}_s \cdot \nabla) \phi(\mathbf{x}) &= 0 \\ \phi(\mathbf{x}) - 2AD(\vec{n}_s \cdot \nabla) \phi(\mathbf{x}) &= 0 \end{aligned}$$

with  $A = \frac{1+F_{dr}}{1-F_{dr}}$ , which returns values fairly close to the  $A$  found in the previous section. The  $A$  in this section is the one employed by Jensen et al. [6]. With respect to  $F_{dr}$ , they only provide the more common case of  $\eta > 1$ .

## 6 Solutions for an infinite medium

From the diffusion equation (4), we have

$$\begin{aligned} (D\nabla^2 - \sigma_a)\phi(\mathbf{x}) &= -Q_0(\mathbf{x}) + 3D\nabla \cdot \mathbf{Q}_1(\mathbf{x}) \\ (\nabla^2 - \sigma_{tr}^2)\phi(\mathbf{x}) &= -\frac{Q_0(\mathbf{x})}{D} + 3\nabla \cdot \mathbf{Q}_1(\mathbf{x}), \end{aligned}$$

which is a particular case of the screened Poisson equation. This has a generic solution based on the method of Green's functions:

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|} \left( \frac{Q_0(\mathbf{r}')}{D} - 3\nabla \cdot \mathbf{Q}_1(\mathbf{r}') \right) d^3\mathbf{r}'.$$

### 6.1 Point source solutions

If we use a point source placed at the origin, we have

$$\begin{aligned} Q_0(\mathbf{x}) &= \Phi_i \delta(\mathbf{x}) \\ \mathbf{Q}_1(\mathbf{x}) &= 0. \end{aligned}$$

Inserting in the solution based on Green's function:

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|} \left( \frac{\Phi_i \delta(\mathbf{r}')}{D} \right) d^3\mathbf{r}'$$

and applying the delta function, the result is

$$\phi(\mathbf{x}) = \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r},$$

where  $r = \|\mathbf{x}\|$  is the distance to the point of interest. A similar result is obtained if we consider a ray source at the origin with direction along the inward surface normal ( $z$ -axis). Suppose the medium exhibits isotropic scattering, then the source of first scattering events is [10]

$$Q_0(\mathbf{x}) = \Phi_i \sigma'_s \delta(x) \delta(y) \Theta(z) e^{-\sigma'_t z}$$

$$\mathbf{Q}_1(\mathbf{x}) = 0,$$

where  $\Theta(z)$  is the Heaviside step function, which is 1 for  $z \geq 0$  and 0 otherwise. Inserting in the solution based on Green's function with  $\mathbf{r}' = (x', y', z')$ :

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|} \left( \frac{\Phi_i \sigma'_s \delta(x') \delta(y') \Theta(z') e^{-\sigma'_t z'}}{D} \right) d^3\mathbf{r}'$$

and applying the deltas:

$$\phi(\mathbf{x}) = \frac{\Phi_i}{4\pi D} \int_0^{+\infty} \frac{e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x}-z'\vec{n}_s\|} \left( \sigma'_s e^{-\sigma'_t z'} \right) dz'.$$

Considering positions  $\mathbf{x}$  in the  $xy$ -plane far from the origin and the exponential attenuation with increasing  $z'$ , we use the assumption  $\|\mathbf{x} - z'\vec{n}_s\| \approx \|\mathbf{x}\| = r$  and get

$$\phi(\mathbf{x}) = \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r} \sigma'_s \int_0^{+\infty} e^{-\sigma'_t z'} dz' = \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r} \frac{\sigma'_s}{\sigma'_t} = \alpha' \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r},$$

where  $\alpha' = \frac{\sigma'_s}{\sigma'_t}$  is the reduced scattering albedo. Thus, we get the monopole solution for a ray source of normal incidence:

$$\phi(\mathbf{x}) = \frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r} \quad \text{with } \Phi = \alpha' \Phi_i. \quad (6)$$

## 6.2 Ray source solution

In case of a ray source that is not along the normal direction and not necessarily in an isotropic medium, we can use the following equations for the source terms [8]:

$$Q_0(\mathbf{x}) = \Phi_i \tilde{\sigma}_s \delta(x) \delta(y) \Theta(z) e^{-\tilde{\sigma}_t z}$$

$$\mathbf{Q}_1(\mathbf{x}) = \tilde{g} Q_0(\mathbf{x}) \vec{n}_s,$$

where we have used the delta-Eddington scattering properties [7]:

$$\tilde{\sigma}_s = \sigma_s(1 - g^2) \quad , \quad \tilde{\sigma}_t = \tilde{\sigma}_s + \sigma_a \quad , \quad \tilde{g} = g/(g + 1).$$



Inserting in the diffusion equation (4), we get two integrals when using the Green's function solution. Splitting the solution accordingly:  $\phi(\mathbf{x}) = \phi_1(\mathbf{x}) + \phi_2(\mathbf{x})$ , we have

$$\begin{aligned}\phi_1(\mathbf{x}) &= \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|} \left( \frac{\Phi_i \tilde{\sigma}_s \delta(x') \delta(y') \Theta(z') e^{-\tilde{\sigma}_t z'}}{D} \right) d^3 \mathbf{r}' \\ &= \frac{\Phi_i \tilde{\sigma}_s}{4\pi D} \int_0^{+\infty} \frac{e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x}-z'\vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz' \\ &= \frac{3\tilde{\sigma}_s \Phi_i}{4\pi} \tilde{\sigma}_t \int_0^{+\infty} \frac{e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x}-z'\vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz'\end{aligned}$$

and

$$\phi_2(\mathbf{x}) = -\frac{3}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|} \nabla \cdot \left( \Phi_i \tilde{\sigma}_s \tilde{g} \delta(x') \delta(y') \Theta(z') e^{-\tilde{\sigma}_t z'} \right) \vec{n}_s d^3 \mathbf{r}'.$$

We now apply the divergence operator:

$$\phi_2(\mathbf{x}) = -\frac{3\Phi_i \tilde{\sigma}_s \tilde{g}}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|} \left[ \delta(x) \delta(y) \frac{\partial}{\partial z'} (\Theta(z) e^{-\tilde{\sigma}_t z'}) \right] d^3 \mathbf{r}'.$$

Note that we have only the  $z$ -term given that we multiply by  $\vec{n}_s = (0, 0, 1)$ .

Using that  $\frac{\partial \Theta(z')}{\partial z'} = \delta(z')$ , we have

$$\phi_2(\mathbf{x}) = -\frac{3\Phi_i \tilde{\sigma}_s \tilde{g}}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|} \delta(x') \delta(y') \left[ \delta(z') e^{-\tilde{\sigma}_t z'} - \tilde{\sigma}_t \Theta(z') e^{-\tilde{\sigma}_t z'} \right] d^3 \mathbf{r}'.$$

Applying the deltas, we get

$$\begin{aligned}\phi_2(\mathbf{x}) &= -\frac{3\Phi_i \tilde{\sigma}_s \tilde{g}}{4\pi} \frac{e^{-\sigma_{tr}r}}{r} + \frac{3\Phi_i \tilde{\sigma}_s \tilde{g} \tilde{\sigma}_t}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|} \delta(x') \delta(y') \Theta(z') e^{-\tilde{\sigma}_t z'} d^3 \mathbf{r}' \\ &= -\frac{3\Phi_i \tilde{\sigma}_s \tilde{g}}{4\pi} \frac{e^{-\sigma_{tr}r}}{r} + \frac{3\Phi_i \tilde{\sigma}_s \tilde{g} \tilde{\sigma}_t}{4\pi} \int_0^{+\infty} \frac{e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x}-z'\vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz' .\end{aligned}$$

Putting it together:

$$\begin{aligned}\phi(\mathbf{x}) &= -\frac{3\Phi_i \tilde{\sigma}_s \tilde{g}}{4\pi} \frac{e^{-\sigma_{tr}r}}{r} + \frac{3\tilde{\sigma}_s \Phi_i}{4\pi} (\tilde{\sigma}_t + \tilde{\sigma}_s \tilde{g} + \sigma_a \tilde{g}) \int_0^{+\infty} \frac{e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x}-z'\vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz' \\ \phi(\mathbf{x}) &= \frac{3\Phi_i \tilde{\sigma}_s}{4\pi} \left( -\tilde{g} \frac{e^{-\sigma_{tr}r}}{r} + (\tilde{\sigma}_s + \sigma_a(1 + \tilde{g})) \int_0^{+\infty} \frac{e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x}-z'\vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz' \right).\end{aligned}\tag{7}$$

We can interpret the second term in Eq. 7 as the fluence from an exponentially decaying line source along the  $z$ -axis. Due to this exponentially decaying factor, the integrand will only have a significant weight for small  $z$ . Hence, in the asymptotic limit,  $r \gg 1/\tilde{\sigma}_s$ , we can approximate the distances in the integrand

$$\begin{aligned}\|\mathbf{x} - z'\vec{n}_s\| &= \sqrt{r^2 + z'^2 - 2z'r \cos \theta} = r \left( 1 - 2\frac{z'}{r} \cos \theta + \frac{z'^2}{r^2} \right)^{\frac{1}{2}} \\ &\approx r \left( 1 - \frac{z'}{r} \cos \theta \right) = r - z' \cos \theta ,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\|\mathbf{x} - z'\vec{n}_s\|} &= \frac{1}{\sqrt{r^2 + z'^2 - 2z'r\cos\theta}} = \frac{1}{r} \left(1 - 2\frac{z'}{r}\cos\theta + \frac{z'^2}{r^2}\right)^{-\frac{1}{2}} \\ &\approx \frac{1}{r} \left(1 + \frac{z'}{r}\cos\theta\right).\end{aligned}$$

Then

$$\begin{aligned}\int_0^\infty \frac{e^{-\tilde{\sigma}_t z'} e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x} - z'\vec{n}_s\|} dz' &\approx \frac{e^{-\sigma_{tr}r}}{r} \int_0^\infty e^{-(\tilde{\sigma}_t - \sigma_{tr}\cos\theta)z'} \left(1 + \frac{z'}{r}\cos\theta\right) dz' \\ &= \frac{e^{-\sigma_{tr}r}}{r} \left( \frac{1}{\tilde{\sigma}_t - \sigma_{tr}\cos\theta} + \frac{\cos\theta}{r} \frac{1}{(\tilde{\sigma}_t - \sigma_{tr}\cos\theta)^2} \right).\end{aligned}$$

In a highly scattering medium,  $\sigma_a \ll \tilde{\sigma}_s$ . For  $g \neq 1$  and  $\sigma_a \ll \sigma_{tr} \ll \tilde{\sigma}_s$ , this will imply

$$\frac{1}{\tilde{\sigma}_t - \sigma_{tr}\cos\theta} = \frac{1}{\tilde{\sigma}_s} \left(1 + \frac{\sigma_a - \sigma_{tr}\cos\theta}{\tilde{\sigma}_s}\right)^{-1} \approx \frac{1}{\tilde{\sigma}_s} \left(1 - \frac{\sigma_a - \sigma_{tr}\cos\theta}{\tilde{\sigma}_s}\right)$$

and

$$\frac{1}{(\tilde{\sigma}_t - \sigma_{tr}\cos\theta)^2} = \frac{1}{\tilde{\sigma}_s^2} \left(1 + \frac{\sigma_a - \sigma_{tr}\cos\theta}{\tilde{\sigma}_s}\right)^{-2} \approx \frac{1}{\tilde{\sigma}_s^2} \left(1 - 2\frac{\sigma_a - \sigma_{tr}\cos\theta}{\tilde{\sigma}_s}\right).$$

The integral can now be approximated by

$$\begin{aligned}\int_0^\infty \frac{e^{-\tilde{\sigma}_t z'} e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x} - z'\vec{n}_s\|} dz' &\approx \frac{e^{-\sigma_{tr}r}}{r} \left( \frac{1}{\tilde{\sigma}_s + \sigma_a - \sigma_{tr}\cos\theta} + \frac{\cos\theta}{r} \frac{1}{(\tilde{\sigma}_s + \sigma_a - \sigma_{tr}\cos\theta)^2} \right) \\ &\approx \frac{e^{-\sigma_{tr}r}}{\tilde{\sigma}_s r} \left( 1 - \frac{\sigma_a - \sigma_{tr}\cos\theta}{\tilde{\sigma}_s} + \frac{\cos\theta}{\tilde{\sigma}_s r} \left( 1 - 2\frac{\sigma_a - \sigma_{tr}\cos\theta}{\tilde{\sigma}_s} \right) \right) \\ &= \frac{e^{-\sigma_{tr}r}}{\tilde{\sigma}_s r} \left( 1 - \frac{\sigma_a}{\tilde{\sigma}_s} + \frac{\cos\theta}{\tilde{\sigma}_s} \left( \sigma_{tr} + \frac{1}{r} - 2\frac{\sigma_a}{\tilde{\sigma}_s r} \right) + 2\frac{\sigma_{tr}\cos^2\theta}{\tilde{\sigma}_s^2 r} \right).\end{aligned}$$

Inserting this expression in Eq. 7,

$$\begin{aligned}\frac{\phi(\mathbf{x})}{\Phi_i} &\approx -\frac{3\tilde{g}\tilde{\sigma}_s e^{-\sigma_{tr}r}}{4\pi r} \\ &\quad + \frac{3(\tilde{\sigma}_s + (\tilde{g}+1)\sigma_a) e^{-\sigma_{tr}r}}{4\pi r} \left( 1 - \frac{\sigma_a}{\tilde{\sigma}_s} + \frac{\cos\theta}{\tilde{\sigma}_s} \left( \sigma_{tr} + \frac{1}{r} - 2\frac{\sigma_a}{\tilde{\sigma}_s r} \right) + 2\frac{\sigma_{tr}\cos^2\theta}{\tilde{\sigma}_s^2 r} \right) \\ &= \frac{3e^{-\sigma_{tr}r}}{4\pi r} \left( (\tilde{\sigma}_s + (\tilde{g}+1)\sigma_a) \left( 1 - \frac{\sigma_a}{\tilde{\sigma}_s} \right) - \tilde{g}\tilde{\sigma}_s \right) \\ &\quad + \frac{3e^{-\sigma_{tr}r}}{4\pi r} \cos\theta \left( 1 + (\tilde{g}+1)\frac{\sigma_a}{\tilde{\sigma}_s} \right) \left( \sigma_{tr} + \frac{1}{r} \left( 1 - 2\frac{\sigma_a}{\tilde{\sigma}_s} \right) \right) \\ &\quad + \frac{3(\tilde{\sigma}_s + (\tilde{g}+1)\sigma_a) e^{-\sigma_{tr}r}}{4\pi r} 2\frac{\sigma_{tr}\cos^2\theta}{\tilde{\sigma}_s^2 r}.\end{aligned}$$

Now we neglect terms  $\sigma_a/\tilde{\sigma}_s$  compared to unity

$$\frac{\phi(\mathbf{x})}{\Phi_i} \approx \frac{3e^{-\sigma_{tr}r}}{4\pi r} \tilde{\sigma}_s (1 - \tilde{g}) + \frac{3e^{-\sigma_{tr}r}}{4\pi r} \cos\theta \left( \sigma_{tr} + \frac{1}{r} \right) + \frac{3e^{-\sigma_{tr}r}}{4\pi r} 2\frac{\sigma_{tr}\cos^2\theta}{\tilde{\sigma}_s r}.$$

Introducing the reduced scattering coefficient  $\sigma'_s = \tilde{\sigma}_s(1 - \tilde{g}) = \sigma_s(1 - g)$ , and neglecting terms  $\sigma_{tr}/\tilde{\sigma}_s \cdot 1/(\tilde{\sigma}_s r)$  compared to unity,

$$\phi(\mathbf{x}) \approx \frac{3\Phi_i e^{-\sigma_{tr}r}}{4\pi r} \sigma'_s + \frac{3\Phi_i e^{-\sigma_{tr}r}}{4\pi r} \cos \theta \left( \sigma_{tr} + \frac{1}{r} \right) = \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r} \left( 1 + 3D \frac{1 + \sigma_{tr}r}{r} \cos \theta \right),$$

where  $1/D$  is used in place of  $3\sigma'_s$ , since the  $\sigma'_s$  can be approximated by  $\sigma'_t$  when absorption is negligible compared to scattering. This is a valid assumption in highly scattering media.

## 7 Fresnel integrals

To aid in our calculations, we define the Fresnel transmittance integrals of the first two orders:

$$C_\phi(\eta) = \frac{1}{4\pi} \int_{2\pi} T_{21}(\eta, \theta_o) \cos \theta_o d\vec{\omega}_o$$

$$C_E(\eta) = \frac{3}{4\pi} \int_{2\pi} T_{21}(\eta, \theta_o) \cos^2 \theta_o d\vec{\omega}_o,$$

where  $\cos \theta_o = \vec{n}_o \cdot \vec{\omega}_o$ , and the integral is on the whole hemisphere where  $\vec{n}_o \cdot \vec{\omega}_o > 0$ . These integrals are similar to  $F_{dr}$ , but based on Fresnel transmittance instead of Fresnel reflectance.

## 8 BSSRDF theory

The BSSRDF is defined as the ratio of an element of emergent radiance  $L_r$  to an element of incident flux  $\Phi_i$  [9]:

$$S(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o) = \frac{dL_r(\mathbf{x}_o, \vec{\omega}_o)}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)}.$$

There are various approximations available for the BSSRDF. For the directional dipole [5], the BSSRDF is split into the following terms:

$$S = T_{12}(S_{\delta E} + S_d)T_{21}.$$

Let us consider only the diffusive part,  $S_d$ . The emergent radiance due to diffusion events is given by

$$L_{r,d}(\mathbf{x}_o, \vec{\omega}_o) = \eta^2 T_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21}),$$

where  $\vec{\omega}_{21}$  is the refracted vector corresponding to  $\vec{\omega}_o$ :

$$\vec{\omega}_{21} = \frac{\vec{\omega}_o}{\eta} - \left( \frac{\vec{n}_o \cdot \vec{\omega}_o}{\eta} - \sqrt{1 - \frac{(\vec{n}_o \cdot \vec{\omega}_o)^2}{\eta^2}} \right) \vec{n}_o.$$

Combining the above equations we obtain:

$$S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o) = T_{12} S_d T_{21} = \frac{dL_{r,d}(\mathbf{x}_o, \vec{\omega}_o)}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)} = \eta^2 \frac{dT_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21})}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)}$$

We now integrate over the cosine-weighted hemisphere with  $\vec{n}_o \cdot \vec{\omega}_o > 0$  on both sides of the equation:

$$\int_{2\pi} T_{12} S_d T_{21} \cos \theta_o d\vec{\omega}_o = \int_{2\pi} \eta^2 \frac{dT_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21})}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)} \cos \theta_o d\vec{\omega}_o.$$

Assuming no dependency on the outgoing direction,  $S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o) = S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o)$ , and we have

$$\begin{aligned} T_{12} S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) \int_{2\pi} T_{21} \cos \theta_o d\vec{\omega}_o &= \eta^2 \frac{\int_{2\pi} T_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21}) \cos \theta_o d\vec{\omega}_o}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)} \\ T_{12} S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) 4\pi \frac{C_\phi(\eta)}{\eta^2} &= \frac{dM_d(\mathbf{x}_o)}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)} \\ T_{12} S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) 4\pi C_\phi(1/\eta) &= \frac{dM_d(\mathbf{x}_o)}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)}. \end{aligned} \quad (8)$$

Let us calculate the diffuse radiant exitance first. We insert the diffusion approximation in place of  $L_d$  to find

$$\begin{aligned} M_d(\mathbf{x}_o) &= \int_{2\pi} T_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21}) \cos \theta_o d\vec{\omega}_o = \int_{2\pi} T_{21} \left( \frac{\phi(\mathbf{x}_o)}{4\pi} - \frac{3}{4\pi} D \vec{\omega}_{21} \cdot \nabla \phi(\mathbf{x}_o) \right) \cos \theta_o d\vec{\omega}_o \\ &= \underbrace{\frac{\phi(\mathbf{x}_o)}{4\pi} \int_{2\pi} T_{21} \cos \theta_o d\vec{\omega}_o}_{I_\phi} - \underbrace{\frac{3}{4\pi} D \int_{2\pi} \vec{\omega}_{21} \cdot \nabla \phi(\mathbf{x}_o) \cos \theta_o d\vec{\omega}_o}_{I_E}. \end{aligned}$$

And, in a straightforward way,

$$I_\phi = C_\phi(\eta) \phi(\mathbf{x}_o).$$

The second part is more complicated. Without loss of generality, we rotate the reference coordinate system so that  $\vec{\omega}_o = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$  and  $\vec{n}_o = (0, 0, 1)$ . Given this reference system, the refracted vector becomes:

$$\begin{aligned} \vec{\omega}_{21} &= \frac{(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)}{\eta} - \left( \frac{\cos \theta}{\eta} - \sqrt{1 - \frac{\sin^2 \theta}{\eta^2}} \right) (0, 0, 1) \\ &= \left( \frac{\cos \phi \sin \theta}{\eta}, \frac{\sin \phi \sin \theta}{\eta}, \sqrt{1 - \frac{\sin^2 \theta}{\eta^2}} \right) \end{aligned}$$

When inserted in the dot product in the expression for  $I_E$ , we get a sum of three components. The first two terms of this sum are zero, since we can first integrate over  $\phi$ :

$$\int_0^{2\pi} \cos \phi d\phi = \int_0^{2\pi} \sin \phi d\phi = 0.$$

So we are left only with the cumbersome  $z$  term:

$$I_E = \frac{3}{4\pi} D \frac{\partial \phi_z(\mathbf{x}_o)}{\partial z} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} T_{21}(\eta, \theta_o) \sqrt{1 - \frac{\sin^2 \theta_o}{\eta^2}} \cos \theta_o \sin \theta_o d\theta_o d\phi,$$

where we note that  $\frac{\partial \phi_z}{\partial z} = \vec{n}_o \cdot \nabla \phi$ . We assume  $\eta > 1$ , so that the argument of the square root is never negative. We now perform a substitution using the law of refraction,  $\sin \theta_i = \eta \sin \theta_o$ . We obtain the following identities:

$$d\theta_o = \frac{\eta \cos \theta_i}{\cos \theta_o} d\theta_i, \quad T_{21}(\eta, \theta_o) = T_{21}(1/\eta, \theta_i), \quad \sqrt{1 - \frac{\sin^2 \theta_o}{\eta^2}} = \cos \theta_i.$$

If we introduce a critical angle  $\theta_c = \arcsin(1/\eta)$ , we get

$$\begin{aligned} I_{\mathbf{E}} &= \frac{3}{4\pi} D \vec{n}_o \cdot \nabla \phi(\mathbf{x}_o) \int_0^{2\pi} \int_0^{\theta_c} T_{21}(1/\eta, \theta_i) \cos \theta_i \cos \theta_o \eta \sin \theta_i \frac{\eta \cos \theta_i}{\cos \theta_o} d\theta_i d\phi \\ &= D \vec{n}_o \cdot \nabla \phi(\mathbf{x}_o) \frac{3}{4\pi} \eta^2 \int_0^{2\pi} \int_0^{\theta_c} T_{21}(1/\eta, \theta_i) \cos^2 \theta_i \sin \theta_i d\theta_i d\phi \\ &= D \vec{n}_o \cdot \nabla \phi(\mathbf{x}_o) \eta^2 C_{\mathbf{E}}(1/\eta) = C_{\mathbf{E}}(\eta) D \vec{n}_o \cdot \nabla \phi(\mathbf{x}_o). \end{aligned}$$

We can then get our final expression for  $M_d(\mathbf{x}_o)$

$$M_d(\mathbf{x}_o) = C_{\phi}(\eta) \phi(\mathbf{x}_o) - C_{\mathbf{E}}(\eta) D \vec{n}_o \cdot \nabla \phi(\mathbf{x}_o).$$

Inserting into the expression (8), we derive the monopole BSSRDF:

$$S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) = \frac{1}{4\pi T_{12} C_{\phi}(1/\eta)} \frac{dM_d(\mathbf{x}_o)}{d\Phi_i(\mathbf{x}_i, \vec{\omega}_i)}$$

## 8.1 Diffuse monopole BSSRDF

Using the monopole solution (6) for a ray normally incident on an isotropic half-space:

$$\phi(\mathbf{x}) = \alpha' \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r},$$

we can derive the gradient:

$$\nabla \phi(\mathbf{x}) = -\alpha' \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r^3} (1 + \sigma_{tr} r) \mathbf{x}.$$

Thus, we can find the monopole BSSRDF for this configuration:

$$S_d(\mathbf{x}_i, \mathbf{x}_o) = \frac{1}{4C_{\phi}(1/\eta)} \frac{\alpha'}{4\pi^2} \frac{e^{-\sigma_{tr} r}}{r^3} \left[ C_{\phi}(\eta) \frac{r^2}{D} + C_{\mathbf{E}}(\eta) (1 + \sigma_{tr} r) (\mathbf{x}_o - \mathbf{x}_i) \cdot \vec{n}_o \right],$$

where we used  $\Phi_i = T_{12} L_i(\mathbf{x}_i, \vec{\omega}_i)$ . This is the monopole version of the better dipole from Eugene d'Eon [2]. To get a monopole version of the standard dipole [4, 6], we further approximate the above expression using  $C_{\phi}(1/\eta) \approx 1/4$ ,  $C_{\phi}(\eta) \approx 0$ , and  $C_{\mathbf{E}}(\eta) \approx 1$ :

$$S_d(\mathbf{x}_i, \mathbf{x}_o) = \frac{\alpha'}{4\pi^2} \frac{e^{-\sigma_{tr} r}}{r^3} (1 + \sigma_{tr} r) \mathbf{x} \cdot \vec{n}_o$$

This is the monopole version of the standard dipole.

## 8.2 Directional monopole BSSRDF

For the directional dipole, we have the following form:

$$\phi(\mathbf{x}) = \frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r} \left( 1 + 3D \frac{1 + \sigma_{tr}r}{r} \cos \theta \right).$$

It is more convenient to do the gradient in spherical coordinates:

$$\nabla \phi(\mathbf{x}) = \frac{\partial}{\partial r} \phi(\mathbf{x}) \vec{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \phi(\mathbf{x}) \vec{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \phi(\mathbf{x}) \vec{e}_\phi,$$

where

$$\frac{\partial}{\partial r} \phi(\mathbf{x}) = -\frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r^2} \left( 3D \frac{2(1 + \sigma_{tr}r) + (\sigma_{tr}r)^2}{r} \cos \theta + (1 + \sigma_{tr}r) \right)$$

and

$$\frac{\partial}{\partial \theta} \phi(\mathbf{x}) = -\frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r^2} 3D(1 + \sigma_{tr}r) \sin \theta.$$

Finally,  $\frac{\partial}{\partial \phi} \phi(\mathbf{x}) = 0$ . Given our choice of basis for derivation, we can use the following identities:

$$\begin{aligned} \vec{e}_r &= \frac{\mathbf{x}}{r} \\ -\vec{e}_\theta \sin \theta &= \vec{\omega}_{12} - \vec{e}_r \cos \theta. \end{aligned}$$

Inserting the identities, we get an expression for the gradient:

$$\begin{aligned} \nabla \phi(\mathbf{x}) &= -\frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r^2} \left( 3D \frac{2(1 + \sigma_{tr}r) + (\sigma_{tr}r)^2}{r} \cos \theta + (1 + \sigma_{tr}r) \right) \vec{e}_r \\ &\quad + \frac{1}{r} \frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r^2} 3D(1 + \sigma_{tr}r) (-\vec{e}_\theta \sin \theta) \end{aligned}$$

$$\begin{aligned} \nabla \phi(\mathbf{x}) &= \frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r^3} \left[ \left( -3D \frac{2(1 + \sigma_{tr}r) + (\sigma_{tr}r)^2}{r} \cos \theta - (1 + \sigma_{tr}r) \right) r \vec{e}_r \right. \\ &\quad \left. - 3D(1 + \sigma_{tr}r) \cos \theta \vec{e}_r + 3D(1 + \sigma_{tr}r) \vec{\omega}_{12} \right] \end{aligned}$$

$$\begin{aligned} \nabla \phi(\mathbf{x}) &= \frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr}r}}{r^3} \left[ - \left( 3D \frac{3(1 + \sigma_{tr}r) + (\sigma_{tr}r)^2}{r} \cos \theta + (1 + \sigma_{tr}r) \right) \mathbf{x} \right. \\ &\quad \left. + 3D(1 + \sigma_{tr}r) \vec{\omega}_{12} \right]. \end{aligned}$$

We can now do the same as above, obtaining the directional monopole BSSRDF with  $\mathbf{x} = \mathbf{x}_o - \mathbf{x}_i$  and  $r = \|\mathbf{x}\|$ :

$$\begin{aligned} S'_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) &= \frac{1}{4C_\phi(1/\eta)} \frac{1}{4\pi^2} \frac{e^{-\sigma_{tr}r}}{r^3} \left[ C_\phi(\eta) \left( \frac{r^2}{D} + 3(1 + \sigma_{tr}r) \mathbf{x} \cdot \vec{\omega}_{12} \right) \right. \\ &\quad \left. - C_E(\eta) \left( 3D(1 + \sigma_{tr}r) \vec{\omega}_{12} \cdot \vec{n}_o - \left( (1 + \sigma_{tr}r) + 3D \frac{3(1 + \sigma_{tr}r) + (\sigma_{tr}r)^2}{r^2} \mathbf{x} \cdot \vec{\omega}_{12} \right) \mathbf{x} \cdot \vec{n}_o \right) \right]. \end{aligned}$$

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