Derivation of standard and directional dipole quantities

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1 Integrating the radiative transfer equation

We start from the radiative transfer equation [1]:

$$(\nabla \cdot \vec{\omega})L(\mathbf{x}, \vec{\omega}) = -\sigma_t L(\mathbf{x}, \vec{\omega}) + \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) L(x, \vec{\omega}')\ d\omega' + q(\mathbf{x}, \vec{\omega})\,,$$

where L is radiance at the position $\mathbf{x}$ in the direction $\vec{\omega}$. The equation describes how the directional derivative of L depends on the scattering properties of the surrounding medium, where σ_t is the extinction coefficient, σ_s is the scattering coefficient, and p is the phase function. Finally, q is the emitted radiance in the medium per unit length that we move along a ray. If we then integrate over all directions $\vec{\omega}$, we get

$$\int_{4\pi} (\nabla \cdot \vec{\omega})L(\mathbf{x}, \vec{\omega})\ d\omega = \int_{4\pi} -\sigma_t L(\mathbf{x}, \vec{\omega})\ d\omega + \int_{4\pi} \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) L(x, \vec{\omega}')\ d\omega'\ d\omega + \int_{4\pi} q(\mathbf{x}, \vec{\omega})\ d\omega\,.$$

Rearranging, we obtain

$$\nabla \cdot \left(\int_{4\pi} \vec{\omega} L(\mathbf{x}, \vec{\omega})\ d\omega \right) = -\sigma_t \int_{4\pi} L(\mathbf{x}, \vec{\omega})\ d\omega + \sigma_s \int_{4\pi} \left(\int_{4\pi} p(\vec{\omega}', \vec{\omega})\ d\omega \right)\ L(x, \vec{\omega}')\ d\omega' + Q_0(\mathbf{x})\,,$$

where we used the regularity of the operators to switch divergence and integral operations on the left-hand side, and to switch the integrals on the right-hand side. The integral of the phase function is 1, since it is normalized, so by further simplifying and applying the definitions of fluence ϕ and vector irradiance $\mathbf{E}$, we obtain

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{x}) &= -\sigma_t \phi(\mathbf{x}) + \sigma_s \int_{4\pi} L(x, \vec{\omega}')\ d\omega' + Q_0(\mathbf{x}) \\
\nabla \cdot \mathbf{E}(\mathbf{x}) &= -\sigma_t \phi(\mathbf{x}) + \sigma_s \phi(\mathbf{x}) + Q_0(\mathbf{x}) \\
\nabla \cdot \mathbf{E}(\mathbf{x}) &= -\sigma_a \phi(\mathbf{x}) + Q_0(\mathbf{x})\,, \qquad (1)
\end{aligned}$$

where we introduced the absorption coefficient $\sigma_a = \sigma_t - \sigma_s$. Our Equation 1 then corresponds to Equation 1 in the work of Jensen et al. [6].

2 The diffusion approximation

To get the diffusion approximation, we approximate radiance using a second order spherical harmonics expansion:

$$L(\mathbf{x},\vec{\omega}) \approx \sum_{n=0}^{1}\sum_{m=-n}^{n} L_{n,m}(\mathbf{x})Y_{n,m}(\vec{\omega})\,,$$

where $Y_{n,m}(\vec{\omega})$ are normalized spherical harmonics basis functions, and $L_{n,m}(\mathbf{x})$ is the projection of L against the n,m basis function:

$$L_{n,m}(\mathbf{x}) = \int_{4\pi} L(\mathbf{x},\vec{\omega})Y_{n,m}(\vec{\omega})\;d\omega\,.$$

For $n = 0$, the integral is trivial. The first term of the sum becomes:

$$L_{0,0}(\mathbf{x})Y_{0,0}(\vec{\omega}) = \int_{4\pi}\sqrt{\frac{1}{4\pi}}L(\mathbf{x},\vec{\omega})\;d\omega\,\sqrt{\frac{1}{4\pi}} = \frac{1}{4\pi}\phi(\mathbf{x})\,.$$

As for the other basis functions, we use the Cartesian form. We have

$$L_{-1,1}(\mathbf{x})Y_{-1,1}(\vec{\omega}) = \omega_x\int_{4\pi}\sqrt{\frac{3}{4\pi}}\omega_x L(\mathbf{x},\vec{\omega})\;d\omega\,\sqrt{\frac{3}{4\pi}} = \frac{3}{4\pi}\omega_x E_x(\mathbf{x})\,,$$

where the x subscript indicates the first component. Similarly, we obtain

$$L_{0,1}(\mathbf{x})Y_{0,1}(\vec{\omega}) = \frac{3}{4\pi}\omega_z E_z(\mathbf{x})$$

$$L_{1,1}(\mathbf{x})Y_{1,1}(\vec{\omega}) = \frac{3}{4\pi}\omega_y E_y(\mathbf{x})\,.$$

By applying the approximation, we finally obtain:

$$L(\mathbf{x},\vec{\omega}) \approx \frac{1}{4\pi}\phi(\mathbf{x})+\frac{3}{4\pi}\omega_x E_x(\mathbf{x})+\frac{3}{4\pi}\omega_y E_y(\mathbf{x})+\frac{3}{4\pi}\omega_z E_z(\mathbf{x}) = \frac{\phi(\mathbf{x})}{4\pi}+\frac{3}{4\pi}\vec{\omega}\cdot\mathbf{E}(\mathbf{x})\,.$$

3 The diffusion equation

To find an equation for the relation between the fluence and the optical properties of the medium, we substitute the diffusion approximation into the radiative transfer equation:

$$\begin{aligned}(\nabla\cdot\vec{\omega})\left(\frac{\phi(\mathbf{x})}{4\pi}+\frac{3}{4\pi}\vec{\omega}\cdot\mathbf{E}(\mathbf{x})\right) &= -\sigma_t\left(\frac{\phi(\mathbf{x})}{4\pi}+\frac{3}{4\pi}\vec{\omega}\cdot\mathbf{E}(\mathbf{x})\right)\\ &\quad+\sigma_s\int_{4\pi}p(\vec{\omega}',\vec{\omega})\left(\frac{\phi(\mathbf{x})}{4\pi}+\frac{3}{4\pi}\vec{\omega}'\cdot\mathbf{E}(\mathbf{x})\right)d\omega' + q(\mathbf{x},\vec{\omega})\,.\end{aligned}$$

For simplification, we need the following three identities:

$$\int_{4\pi}\vec{\omega}\;d\omega = 0$$

$$\int_{4\pi}\vec{\omega}(\vec{\omega}\cdot\mathbf{A})\;d\omega = \frac{4\pi}{3}\mathbf{A} \tag{2}$$

$$\int_{4\pi} \vec{\omega}[\vec{\omega} \cdot \nabla(\vec{\omega} \cdot \mathbf{A})] \, d\omega = 0$$

We first multiply into parentheses in the equation above:

$$\frac{1}{4\pi}\vec{\omega}\cdot\nabla\phi(\mathbf{x})+\frac{3}{4\pi}\vec{\omega}\cdot\nabla(\vec{\omega}\cdot\mathbf{E}(\mathbf{x})) = -\sigma_t\frac{\phi(\mathbf{x})}{4\pi}-\sigma_t\frac{3}{4\pi}\vec{\omega}\cdot\mathbf{E}(\mathbf{x})+\sigma_s\frac{\phi(\mathbf{x})}{4\pi}\int_{4\pi} p(\vec{\omega}',\vec{\omega}) \, d\omega' + \frac{3}{4\pi}\sigma_s\int_{4\pi} p(\vec{\omega}',\vec{\omega})\vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) \, d\omega' + q(\mathbf{x},\vec{\omega}) \, .$$

Now, we multiply each term by $\vec{\omega}$ and integrate over the sphere. Taking all the terms separately, we have

$$\int_{4\pi} \frac{1}{4\pi}\vec{\omega} \cdot \nabla\phi(\mathbf{x})\vec{\omega} \, d\omega = \frac{1}{4\pi}\frac{4\pi}{3}\nabla\phi(\mathbf{x}) = \frac{\nabla\phi(\mathbf{x})}{3}$$

$$\int_{4\pi} \frac{3}{4\pi}\vec{\omega} \cdot \nabla(\vec{\omega} \cdot \mathbf{E}(\mathbf{x}))\vec{\omega} \, d\omega = \frac{3}{4\pi}\int_{4\pi} \vec{\omega}[\vec{\omega} \cdot \nabla(\vec{\omega} \cdot \mathbf{E}(\mathbf{x}))] \, d\omega = 0$$

$$\int_{4\pi} -\sigma_t\frac{\phi(\mathbf{x})}{4\pi}\vec{\omega} \, d\omega = -\sigma_t\frac{\phi(\mathbf{x})}{4\pi}\int_{4\pi} \vec{\omega} \, d\omega = 0$$

$$\int_{4\pi} -\sigma_t\frac{3}{4\pi}\vec{\omega} \cdot \mathbf{E}(\mathbf{x})\vec{\omega} \, d\omega = -\sigma_t\frac{3}{4\pi}\int_{4\pi} \vec{\omega}(\vec{\omega} \cdot \mathbf{E}(\mathbf{x})) \, d\omega = -\sigma_t\frac{3}{4\pi}\frac{4\pi}{3}\mathbf{E}(\mathbf{x}) = -\sigma_t\mathbf{E}(\mathbf{x})$$

$$\int_{4\pi} \sigma_s\frac{\phi(\mathbf{x})}{4\pi}\int_{4\pi} p(\vec{\omega}',\vec{\omega}) \, d\omega'\vec{\omega} \, d\omega = \sigma_s\frac{\phi(\mathbf{x})}{4\pi}\int_{4\pi} \vec{\omega} \, d\omega = 0$$

$$\int_{4\pi} \frac{3}{4\pi}\sigma_s\int_{4\pi} p(\vec{\omega}',\vec{\omega})\vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) \, d\omega'\vec{\omega} \, d\omega \overset{(*)}{\approx} g\sigma_s\mathbf{E}(\mathbf{x})$$

$$\int_{4\pi} q(\mathbf{x},\vec{\omega})\vec{\omega} \, d\omega = \mathbf{Q}_1(\mathbf{x}) \, ,$$

Passage $(*)$ is a bit more delicate as it requires the assumption that the phase function is rotationally symmetric, which means that it depends only on the cosine between the two direction vector arguments ($p(\vec{\omega}',\vec{\omega}) = p(\vec{\omega}' \cdot \vec{\omega})$). For the interested reader, we further discuss this result at the end of this section (Section 3.1). Putting everything together:

$$\frac{\nabla\phi(\mathbf{x})}{3} + 0 = 0 - \sigma_t\mathbf{E}(\mathbf{x}) + 0 + g\sigma_s\mathbf{E}(\mathbf{x}) + \mathbf{Q}_1(\mathbf{x})$$

$$\nabla\phi(\mathbf{x}) = -3\sigma_t'\mathbf{E}(\mathbf{x}) + 3\mathbf{Q}_1(\mathbf{x}) \, , \tag{3}$$

where we used $\sigma_t' = \sigma_s' + \sigma_a = \sigma_s(1-g) + \sigma_a = \sigma_t - g\sigma_s$. To obtain the diffusion equation, we need to combine Equations 3 and 1. We first rearrange Equation 3:

$$\mathbf{E}(\mathbf{x}) = 3D\mathbf{Q}_1(\mathbf{x}) - D\nabla\phi(\mathbf{x}) \, ,$$

where $D = \frac{1}{3\sigma_t'}$. Inserting into Equation 1 (and assuming a homogeneous medium), we have

$$\nabla \cdot (3D\mathbf{Q}_1(\mathbf{x}) - D\nabla\phi(\mathbf{x})) = -\sigma_a\phi(\mathbf{x}) + Q_0(\mathbf{x})$$

$$3D\nabla \cdot \mathbf{Q}_1(\mathbf{x}) - D\nabla^2\phi(\mathbf{x}) = -\sigma_a\phi(\mathbf{x}) + Q_0(\mathbf{x})$$

$$D\nabla^2\phi(\mathbf{x}) = \sigma_a\phi(\mathbf{x}) - Q_0(\mathbf{x}) + 3D\nabla \cdot \mathbf{Q}_1(\mathbf{x}) \, , \tag{4}$$

which is the diffusion equation as it appears in the work of Jensen et al. [6].

3.1 Phase function approximation

Assuming that p is rotationally symmetric so that $p(\vec{\omega}', \vec{\omega}) = p(\vec{\omega}' \cdot \vec{\omega}) = p(\vec{\omega}' \cdot \vec{\omega})$, we can it in Legendre polynomials:

$$p(\vec{\omega}' \cdot \vec{\omega}) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} p_n P_n(\vec{\omega}' \cdot \vec{\omega}) \,,$$

where p_n are the expansion coefficients and P_n are the Legendre polynomials. Since

$$P_0(\vec{\omega}' \cdot \vec{\omega}) = 1, \quad P_1(\vec{\omega}' \cdot \vec{\omega}) = \vec{\omega}' \cdot \vec{\omega} \,,$$

we have

$$p_0 = \int_{4\pi} p(\vec{\omega}' \cdot \vec{\omega}) P_{0,0}(\vec{\omega}' \cdot \vec{\omega}) \; d\omega = 1,$$

$$p_1 = \int_{4\pi} p(\vec{\omega}' \cdot \vec{\omega}) P_1(\vec{\omega}' \cdot \vec{\omega}) \; d\omega = \int_{4\pi} p(\vec{\omega}' \cdot \vec{\omega})(\vec{\omega}' \cdot \vec{\omega}) \; d\omega = g \,.$$

Inserting and truncating to second order (P_1), we get the approximation:

$$p(\vec{\omega}' \cdot \vec{\omega}) \approx \frac{1 + 3g(\vec{\omega}' \cdot \vec{\omega})}{4\pi}.$$

Let us plug it into passage $(*)$:

$$\begin{aligned}
&\int_{4\pi} \frac{3}{4\pi} \sigma_s \int_{4\pi} p(\vec{\omega}', \vec{\omega}) \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) \; d\omega' \vec{\omega} \; d\omega \\
&\approx \int_{4\pi} \frac{3}{4\pi} \sigma_s \int_{4\pi} \frac{1 + 3g(\vec{\omega}' \cdot \vec{\omega})}{4\pi} \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) \; d\omega' \vec{\omega} \; d\omega \\
&= \int_{4\pi} \frac{3}{4\pi} \sigma_s \left[\int_{4\pi} \frac{1}{4\pi} \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) \; d\omega' + \int_{4\pi} \frac{3g(\vec{\omega}' \cdot \vec{\omega})}{4\pi} \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) \; d\omega' \right] \vec{\omega} \; d\omega.
\end{aligned}$$

Now, the first integral within the square parenthesis is zero, given that $\mathbf{E}$ is independent of $\vec{\omega}'$. The second integral within the parenthesis can be solved using the following identity:

$$\int_{4\pi} (\vec{\omega} \cdot \mathbf{A})(\vec{\omega} \cdot \mathbf{B}) \; d\omega = \frac{4\pi}{3} \mathbf{A} \cdot \mathbf{B} \,.$$

With this, we have

$$\begin{aligned}
&\int_{4\pi} \frac{3}{4\pi} \sigma_s \left[\int_{4\pi} \frac{1}{4\pi} \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) \; d\omega' + \int_{4\pi} \frac{3g(\vec{\omega}' \cdot \vec{\omega})}{4\pi} \vec{\omega}' \cdot \mathbf{E}(\mathbf{x}) \; d\omega' \right] \vec{\omega} \; d\omega \\
&= \int_{4\pi} \frac{3}{4\pi} \sigma_s \left[0 + \frac{4\pi}{3} \frac{3g(\vec{\omega} \cdot \mathbf{E}(\mathbf{x}))}{4\pi} \right] \vec{\omega} \; d\omega \\
&= \frac{3g}{4\pi} \sigma_s \int_{4\pi} (\vec{\omega} \cdot \mathbf{E}(\mathbf{x})) \vec{\omega} \; d\omega = g \sigma_s \mathbf{E}(\mathbf{x}),
\end{aligned}$$

which is the expected result.

4 Boundary condition

In the case of a scattering medium in a finite region of space, we impose the classic boundary condition that the net inward flux on each surface point $\mathbf{x}_s$ with (inward) normal $\vec{n}_s$ is zero:

$$\int_{2\pi_+} L(\mathbf{x}_s, \vec{\omega})(\vec{\omega} \cdot \vec{n}_s)\ d\omega = 0\,.$$

We use the diffusion approximation:

$$\int_{2\pi_+} \left(\frac{\phi(\mathbf{x_s})}{4\pi} + \frac{3}{4\pi} \vec{\omega} \cdot \mathbf{E}(\mathbf{x_s}) \right) (\vec{\omega} \cdot \vec{n}_s)\ d\omega = 0$$

$$\phi(\mathbf{x}) \int_{2\pi_+} (\vec{\omega} \cdot \vec{n}_s)\ d\omega + 3 \int_{2\pi_+} (\vec{\omega} \cdot \mathbf{E}(\mathbf{x_s}))(\vec{n}_s \cdot \vec{\omega})\ d\omega = 0\,.$$

Given the standard spherical coordinates convention, $n_s = (0,0,1)$ and $\vec{\omega} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$. We then obtain

$$\int_{2\pi_+} (\vec{\omega} \cdot \vec{n}_s)\ d\omega = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta d\theta d\phi = \pi\,,$$

and

$$\int_{2\pi_+} (\vec{\omega} \cdot \mathbf{E}(\mathbf{x}))(\vec{n}_s \cdot \vec{\omega})\ d\omega = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (\cos\phi \sin\theta E_x + \sin\phi \sin\theta E_y + \cos\theta E_z) \cos\theta \sin\theta d\theta d\phi$$

$$= \frac{2\pi}{3} E_z = \frac{2\pi}{3} \vec{n}_s \cdot \mathbf{E}(\mathbf{x_s})\,.$$

Using the last two results and simplifying, we get:

$$\phi(\mathbf{x})\pi + 3 \left(\frac{2\pi}{3} \vec{n}_s \cdot \mathbf{E}(\mathbf{x_s}) \right) = 0$$

$$\phi(\mathbf{x}) + 2\vec{n}_s \cdot \mathbf{E}(\mathbf{x_s}) = 0\,.$$

From Equation 3, assuming no emission in $\mathbf{x}_s$, we have $\mathbf{Q}_1(\mathbf{x_s}) = \mathbf{0}$, so

$$\mathbf{E}(\mathbf{x_s}) = -D\nabla\phi(\mathbf{x_s})\,.$$

Inserting this result and simplifying, we get the final boundary condition:

$$\phi(\mathbf{x_s}) - 2D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x_s}) = 0\,.$$

5 Different media assumption

To include nonzero inward flux at boundaries, we need to change the above equations. The boundary condition then becomes:

$$I_+ = \int_{2\pi_+} L(\mathbf{x}_s, \vec{\omega})(\vec{\omega} \cdot \vec{n}_s)\ d\omega = \int_{2\pi_-} R(\eta, \vec{\omega}) L(\mathbf{x}_s, \vec{\omega})(-\vec{\omega} \cdot \vec{n}_s)\ d\omega = I_-\,. \quad (5)$$

Keeping the above conventions, we define the Fresnel reflectance R by:

$$R(\eta, \vec{\omega}) = \begin{cases} 1 & \text{for } \frac{\pi}{2} \leq \theta \leq \pi - \theta_c \\ F_r(\eta, \vec{\omega} \cdot \vec{n}_s) & \text{for } \pi - \theta_c \leq \theta \leq \pi\,, \end{cases}$$

where θ_c is the critical angle, and

$$F_r(\eta, \mu) = \frac{1}{2}\left[\left(\frac{\mu - \eta\mu_0}{\mu + \eta\mu_0}\right)^2 + \left(\frac{\eta\mu - \mu_0}{\eta\mu + \mu_0}\right)^2\right] \quad \text{with} \quad \mu_0^2 = 1 - \eta^2(1 - \mu^2)$$

and $\cos^2\theta_c = \max(1 - \eta^{-2}, 0)$. In principle, if we allow complex numbers, we would have $R(\eta, \vec{\omega}) = F_r(\eta, \vec{\omega} \cdot \vec{n}_s)$.

The left side of Equation 5 is:

$$I_+ = \frac{1}{4}(\phi(\mathbf{x}) - 2D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}))\,.$$

The other side is more tricky, since it requires splitting the integration along the different angles. We proceed as before, introducing the diffusion approximation:

$$I_- = \frac{\phi(\mathbf{x})}{4\pi}\int_{2\pi_-} R(\eta, \vec{\omega})(-\vec{\omega} \cdot \vec{n}_s)\; d\omega + \frac{3}{4\pi}\int_{2\pi_-} R(\eta, \vec{\omega})(\vec{\omega} \cdot \mathbf{E}(\mathbf{x}))(-\vec{\omega} \cdot \vec{n}_s)\; d\omega\,.$$

The cosine-weighted integration of the Fresnel reflectance is sometimes referred to as diffuse Fresnel reflectance F_{dr}. If we, outside the region of total internal reflection, approximate the R function by Fresnel reflectance for normal incidence $R_0 = F_r(\eta, 1)$, we can find an approximate analytical solution. The first part is then:

$$\begin{aligned}\int_{2\pi_-} R(\eta, \vec{\omega})(-\vec{\omega} \cdot \vec{n}_s)\; d\omega &= \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi-\theta_c}(-\cos\theta)\sin\theta d\theta d\phi + \int_0^{2\pi}\int_{\pi-\theta_c}^{\pi} R_0(-\cos\theta)\sin\theta d\theta d\phi \\ &= \pi((1 - R_0)\cos^2\theta_c + R_0)\,.\end{aligned}$$

The second part (only on the z-coordinate, since the other coordinates are zero):

$$\begin{aligned}&\int_{2\pi_-} R(\eta, \vec{\omega})(\vec{\omega} \cdot \mathbf{E}(\mathbf{x}))(-\vec{\omega} \cdot \vec{n}_s)\; d\omega \\ &= E_z\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi-\theta_c}(-\cos^2\theta)\sin\theta d\theta d\phi + E_z\int_0^{2\pi}\int_{\pi-\theta_c}^{\pi} R_0(-\cos^2\theta)\sin\theta d\theta d\phi \\ &= \frac{2\pi}{3}E_z(R_0(\cos^3\theta_c - 1) - \cos^3\theta_c)\,.\end{aligned}$$

Performing all simplifications, we finally get:

$$I_- = \frac{1}{4}[((1 - R_0)\cos^2\theta_c + R_0)\phi(x) - 2D(R_0(\cos^3\theta_c - 1) - \cos^3\theta_c)(\vec{n}_s \cdot \nabla)\phi(\mathbf{x})]\,.$$

We can now impose $I_+ = I_-$:

$$\phi(\mathbf{x}) - 2D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}) = ((1 - R_0)\cos^2\theta_c + R_0)\phi(x) - 2D(R_0(\cos^3\theta_c - 1) - \cos^3\theta_c)(\vec{n}_s \cdot \nabla)\phi(\mathbf{x})\,,$$

which we can simplify as follows:

$$\phi(\mathbf{x}) - 2\frac{1 + R_0 + (1 - R_0)\cos^3\theta_c}{1 - R_0 - (1 - R_0)\cos^2\theta_c}D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}) = 0$$

$$\phi(\mathbf{x}) - 2\frac{\frac{1+R_0}{1-R_0} + \cos^3\theta_c}{1-\cos^2\theta_c} D(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}) = 0$$
$$\phi(\mathbf{x}) - 2AD(\vec{n}_s \cdot \nabla)\phi(\mathbf{x}) = 0\,.$$

So, to handle reflective boundaries, we need to add a correction factor A in our boundary condition. With our current approximation, we have

$$A = \frac{\frac{1+R_0}{1-R_0} + \cos^3\theta_c}{1-\cos^2\theta_c} = \frac{\frac{\eta^2+1}{2\eta} + [\max(1-\eta^{-2}, 0)]^{\frac{3}{2}}}{1-\max(1-\eta^{-2},0)}\,.$$

5.1 Approximating the corrective factor

Assuming separability, we can rewrite the I_- term in Equation 5 as:

$$\int_{2\pi_-} R(\eta,\vec{\omega})L(x_s,\vec{\omega})(-\vec{\omega}\cdot\vec{n}_s)\,d\omega \approx \int_{2\pi_-} R(\eta,\vec{\omega})(-\vec{\omega}\cdot\vec{n}_s)\,d\omega \int_{2\pi_-} L(x_s,\vec{\omega})(-\vec{\omega}\cdot\vec{n}_s)\,d\omega$$
$$= F_{dr}(\eta)\;\frac{1}{4}(\phi(\mathbf{x}) + 2D(\vec{n}_s\cdot\nabla)\phi(\mathbf{x}))$$

An approximate fit of the $F_{dr}(\eta)$ integral is [3]

$$F_{dr}(\eta) = \begin{cases} -0.4399 + \dfrac{0.7099}{\eta} - \dfrac{0.3319}{\eta^2} + \dfrac{0.0636}{\eta^3} & , \quad \eta < 1 \\ -\dfrac{1.4399}{\eta^2} + \dfrac{0.7099}{\eta} + 0.6681 + 0.0636\eta & , \quad \eta > 1\,. \end{cases}$$

So we can express the boundary condition as

$$\phi(\mathbf{x}) - 2\pi D(\vec{n}_s\cdot\nabla)\phi(\mathbf{x}) = F_{dr}(\eta)\;(\phi(\mathbf{x}) + 2D(\vec{n}_s\cdot\nabla)\phi(\mathbf{x}))$$
$$\phi(\mathbf{x})(1-F_{dr}) - 2D(1+F_{dr})(\vec{n}_s\cdot\nabla)\phi(\mathbf{x}) = 0$$
$$\phi(\mathbf{x}) - 2D\frac{1+F_{dr}}{1-F_{dr}}(\vec{n}_s\cdot\nabla)\phi(\mathbf{x}) = 0$$
$$\phi(\mathbf{x}) - 2AD(\vec{n}_s\cdot\nabla)\phi(\mathbf{x}) = 0$$

with $A = \frac{1+F_{dr}}{1-F_{dr}}$, which returns values fairly close to the A found in the previous section. The A in this section is the one employed by Jensen et al. [6]. With respect to F_{dr}, they only provide the more common case of $\eta > 1$.

6 Solutions for an infinite medium

From the diffusion equation (4), we have

$$(D\nabla^2 - \sigma_a)\phi(\mathbf{x}) = -Q_0(\mathbf{x}) + 3D\nabla\cdot\mathbf{Q}_1(\mathbf{x})$$
$$(\nabla^2 - \sigma_{tr}^2)\phi(\mathbf{x}) = -\frac{Q_0(\mathbf{x})}{D} + 3\nabla\cdot\mathbf{Q}_1(\mathbf{x})\,,$$

which is a particular case of the screened Poisson equation. This has a generic solution based on the method of Green's functions:

$$\phi(\mathbf{x}) = \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|}\left(\frac{Q_0(\mathbf{r}')}{D} - 3\nabla\cdot\mathbf{Q}_1(\mathbf{r}')\right)d^3\mathbf{r}'\,.$$

6.1 Point source solutions

If we use a point source placed at the origin, we have

$$Q_0(\mathbf{x}) = \Phi_i \delta(\mathbf{x})$$

$$\mathbf{Q}_1(\mathbf{x}) = 0\,.$$

Inserting in the solution based on Green's function:

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|} \left(\frac{\Phi_i \delta(\mathbf{r}')}{D} \right) d^3\mathbf{r}'$$

and applying the delta function, the result is

$$\phi(\mathbf{x}) = \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r}\,,$$

where $r = \|\mathbf{x}\|$ is the distance to the point of interest. A similar result is obtained if we consider a ray source at the origin with direction along the inward surface normal (z-axis). Suppose the medium exhibits isotropic scattering, then the source of first scattering events is [10]

$$Q_0(\mathbf{x}) = \Phi_i \sigma'_s \delta(x)\delta(y)\Theta(z) e^{-\sigma'_t z}$$

$$\mathbf{Q}_1(\mathbf{x}) = 0\,,$$

where $\Theta(z)$ is the Heaviside step function, which is 1 for $z \geq 0$ and 0 otherwise. Inserting in the solution based on Green's function with $\mathbf{r}' = (x', y', z')$:

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|} \left(\frac{\Phi_i \sigma'_s \delta(x')\delta(y')\Theta(z') e^{-\sigma'_t z'}}{D} \right) d^3\mathbf{r}'$$

and applying the deltas:

$$\phi(\mathbf{x}) = \frac{\Phi_i}{4\pi D} \int_0^{+\infty} \frac{e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x}-z'\vec{n}_s\|} \left(\sigma'_s e^{-\sigma'_t z'} \right) dz'\,.$$

Considering positions $\mathbf{x}$ in the xy-plane far from the origin and the exponential attenuation with increasing z', we use the assumption $\|\mathbf{x} - z'\vec{n}_s\| \approx \|\mathbf{x}\| = r$ and get

$$\phi(\mathbf{x}) = \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r} \sigma'_s \int_0^{+\infty} e^{-\sigma'_t z'} dz' = \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r} \frac{\sigma'_s}{\sigma'_t} = \alpha' \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r}\,,$$

where $\alpha' = \frac{\sigma'_s}{\sigma'_t}$ is the reduced scattering albedo. Thus, we get the monopole solution for a ray source of normal incidence:

$$\phi(\mathbf{x}) = \frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r} \qquad \text{with } \Phi = \alpha' \Phi_i. \tag{6}$$

6.2 Ray source solution

In case of a ray source that is not along the normal direction and not necessarily in an isotropic medium, we can use the following equations for the source terms [8]:

$$Q_0(\mathbf{x}) = \Phi_i \tilde{\sigma}_s \delta(x)\delta(y)\Theta(z)e^{-\tilde{\sigma}_t z}$$
$$\mathbf{Q}_1(\mathbf{x}) = \tilde{g} Q_0(\mathbf{x})\vec{n}_s \, ,$$

where we have used the delta-Eddington scattering properties [7]:

$$\tilde{\sigma}_s = \sigma_s(1-g^2) \quad , \quad \tilde{\sigma}_t = \tilde{\sigma}_s + \sigma_a \quad , \quad \tilde{g} = g/(g+1)\, .$$

Inserting in the diffusion equation (4), we get two integrals when using the Green's function solution. Splitting the solution accordingly: $\phi(\mathbf{x}) = \phi_1(\mathbf{x}) + \phi_2(\mathbf{x})$, we have

$$\begin{aligned}\phi_1(\mathbf{x}) &= \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|}\left(\frac{\Phi_i\tilde{\sigma}_s\delta(x')\delta(y')\Theta(z')e^{-\tilde{\sigma}_t z'}}{D}\right) d^3\mathbf{r}' \\ &= \frac{\Phi_i\tilde{\sigma}_s}{4\pi D}\int_0^{+\infty} \frac{e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x}-z'\vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz' \\ &= \frac{3\tilde{\sigma}_s\Phi_i}{4\pi}\tilde{\sigma}_t\int_0^{+\infty} \frac{e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x}-z'\vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz'\end{aligned}$$

and

$$\phi_2(\mathbf{x}) = -\frac{3}{4\pi}\iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|}\nabla\cdot\left(\Phi_i\tilde{\sigma}_s\tilde{g}\delta(x')\delta(y')\Theta(z')e^{-\tilde{\sigma}_t z'}\right)\vec{n}_s\, d^3\mathbf{r}'\, .$$

We now apply the divergence operator:

$$\phi_2(\mathbf{x}) = -\frac{3\Phi_i\tilde{\sigma}_s\tilde{g}}{4\pi}\iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|}\left[\delta(x)\delta(y)\frac{\partial}{\partial z'}(\Theta(z)e^{-\tilde{\sigma}_t z'})\right] d^3\mathbf{r}'.$$

Note that we have only the z-term given that we multiply by $\vec{n}_s = (0,0,1)$. Using that $\frac{\partial\Theta(z')}{\partial z'} = \delta(z')$, we have

$$\phi_2(\mathbf{x}) = -\frac{3\Phi_i\tilde{\sigma}_s\tilde{g}}{4\pi}\iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|}\delta(x')\delta(y')\left[\delta(z')e^{-\tilde{\sigma}_t z'} - \tilde{\sigma}_t\Theta(z')e^{-\tilde{\sigma}_t z'}\right] d^3\mathbf{r}'.$$

Applying the deltas, we get

$$\begin{aligned}\phi_2(\mathbf{x}) &= -\frac{3\Phi_i\tilde{\sigma}_s\tilde{g}}{4\pi}\frac{e^{-\sigma_{tr}r}}{r} + \frac{3\Phi_i\tilde{\sigma}_s\tilde{g}\tilde{\sigma}_t}{4\pi}\iiint_{\mathbb{R}^3} \frac{e^{-\sigma_{tr}\|\mathbf{x}-\mathbf{r}'\|}}{\|\mathbf{x}-\mathbf{r}'\|}\delta(x')\delta(y')\Theta(z')e^{-\tilde{\sigma}_t z'} d^3\mathbf{r}' \\ &= -\frac{3\Phi_i\tilde{\sigma}_s\tilde{g}}{4\pi}\frac{e^{-\sigma_{tr}r}}{r} + \frac{3\Phi_i\tilde{\sigma}_s\tilde{g}\tilde{\sigma}_t}{4\pi}\int_0^{+\infty} \frac{e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x}-z'\vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz'\, .\end{aligned}$$

Putting it together:

$$\phi(\mathbf{x}) = -\frac{3\Phi_i\tilde{\sigma}_s\tilde{g}}{4\pi}\frac{e^{-\sigma_{tr}r}}{r} + \frac{3\tilde{\sigma}_s\Phi_i}{4\pi}(\tilde{\sigma}_t + \tilde{\sigma}_s\tilde{g} + \sigma_a\tilde{g})\int_0^{+\infty} \frac{e^{-\sigma_{tr}\|\mathbf{x}-z'\vec{n}_s\|}}{\|\mathbf{x}-z'\vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz'$$

$$\phi(\mathbf{x}) = \frac{3\Phi_i \tilde{\sigma}_s}{4\pi} \left(-\tilde{g} \frac{e^{-\sigma_{tr} r}}{r} + (\tilde{\sigma}_s + \sigma_a (1+\tilde{g})) \int_0^{+\infty} \frac{e^{-\sigma_{tr} \|\mathbf{x} - z' \vec{n}_s\|}}{\|\mathbf{x} - z' \vec{n}_s\|} e^{-\tilde{\sigma}_t z'} dz' \right). \tag{7}$$

We can interpret the second term in Eq. 7 as the fluence from an exponentially decaying line source along the z-axis. Due to this exponentially decaying factor, the integrand will only have a significant weight for small z. Hence, in the asymptotic limit, $r \gg 1/\tilde{\sigma}_s$, we can approximate the distances in the integrand

$$\begin{aligned} \|\mathbf{x} - z' \vec{n}_s\| &= \sqrt{r^2 + z'^2 - 2z' r \cos\theta} = r \left(1 - 2\frac{z'}{r} \cos\theta + \frac{z'^2}{r^2} \right)^{\frac{1}{2}} \\ &\approx r \left(1 - \frac{z'}{r} \cos\theta \right) = r - z' \cos\theta \,, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\|\mathbf{x} - z' \vec{n}_s\|} &= \frac{1}{\sqrt{r^2 + z'^2 - 2z' r \cos\theta}} = \frac{1}{r} \left(1 - 2\frac{z'}{r} \cos\theta + \frac{z'^2}{r^2} \right)^{-\frac{1}{2}} \\ &\approx \frac{1}{r} \left(1 + \frac{z'}{r} \cos\theta \right). \end{aligned}$$

Then

$$\begin{aligned} \int_0^{\infty} \frac{e^{-\tilde{\sigma}_t z'} e^{-\sigma_{tr} \|\mathbf{x} - z' \vec{n}_s\|}}{\|\mathbf{x} - z' \vec{n}_s\|} dz' &\approx \frac{e^{-\sigma_{tr} r}}{r} \int_0^{\infty} e^{-(\tilde{\sigma}_t - \sigma_{tr} \cos\theta) z'} \left(1 + \frac{z'}{r} \cos\theta \right) dz' \\ &= \frac{e^{-\sigma_{tr} r}}{r} \left(\frac{1}{\tilde{\sigma}_t - \sigma_{tr} \cos\theta} + \frac{\cos\theta}{r} \frac{1}{(\tilde{\sigma}_t - \sigma_{tr} \cos\theta)^2} \right). \end{aligned}$$

In a highly scattering medium, $\sigma_a \ll \tilde{\sigma}_s$. For $g \neq 1$ and $\sigma_a \ll \sigma_{tr} \ll \tilde{\sigma}_s$, this will imply

$$\frac{1}{\tilde{\sigma}_t - \sigma_{tr} \cos\theta} = \frac{1}{\tilde{\sigma}_s} \left(1 + \frac{\sigma_a - \sigma_{tr} \cos\theta}{\tilde{\sigma}_s} \right)^{-1} \approx \frac{1}{\tilde{\sigma}_s} \left(1 - \frac{\sigma_a - \sigma_{tr} \cos\theta}{\tilde{\sigma}_s} \right)$$

and

$$\frac{1}{(\tilde{\sigma}_t - \sigma_{tr} \cos\theta)^2} = \frac{1}{\tilde{\sigma}_s^2} \left(1 + \frac{\sigma_a - \sigma_{tr} \cos\theta}{\tilde{\sigma}_s} \right)^{-2} \approx \frac{1}{\tilde{\sigma}_s^2} \left(1 - 2\frac{\sigma_a - \sigma_{tr} \cos\theta}{\tilde{\sigma}_s} \right).$$

The integral can now be approximated by

$$\begin{aligned} \int_0^{\infty} \frac{e^{-\tilde{\sigma}_t z'} e^{-\sigma_{tr} \|\mathbf{x} - z' \vec{n}_s\|}}{\|\mathbf{x} - z' \vec{n}_s\|} dz' &\approx \frac{e^{-\sigma_{tr} r}}{r} \left(\frac{1}{\tilde{\sigma}_s + \sigma_a - \sigma_{tr} \cos\theta} + \frac{\cos\theta}{r} \frac{1}{(\tilde{\sigma}_s + \sigma_a - \sigma_{tr} \cos\theta)^2} \right) \\ &\approx \frac{e^{-\sigma_{tr} r}}{\tilde{\sigma}_s r} \left(1 - \frac{\sigma_a - \sigma_{tr} \cos\theta}{\tilde{\sigma}_s} + \frac{\cos\theta}{\tilde{\sigma}_s r} \left(1 - 2\frac{\sigma_a - \sigma_{tr} \cos\theta}{\tilde{\sigma}_s} \right) \right) \\ &= \frac{e^{-\sigma_{tr} r}}{\tilde{\sigma}_s r} \left(1 - \frac{\sigma_a}{\tilde{\sigma}_s} + \frac{\cos\theta}{\tilde{\sigma}_s} \left(\sigma_{tr} + \frac{1}{r} - 2\frac{\sigma_a}{\tilde{\sigma}_s r} \right) + 2\frac{\sigma_{tr} \cos^2\theta}{\tilde{\sigma}_s^2 r} \right). \end{aligned}$$

Inserting this expression in Eq. 7,

$$
\begin{aligned}
\frac{\phi(\mathbf{x})}{\Phi_i} \approx\ & -\frac{3\tilde{g}\tilde{\sigma}_s e^{-\sigma_{tr} r}}{4\pi r} \\
& + \frac{3\left(\tilde{\sigma}_s + (\tilde{g}+1)\sigma_a\right) e^{-\sigma_{tr} r}}{4\pi r}\left(1 - \frac{\sigma_a}{\tilde{\sigma}_s} + \frac{\cos\theta}{\tilde{\sigma}_s}\left(\sigma_{tr} + \frac{1}{r} - 2\frac{\sigma_a}{\tilde{\sigma}_s r}\right) + 2\frac{\sigma_{tr}\cos^2\theta}{\tilde{\sigma}_s^2 r}\right) \\
=\ & \frac{3e^{-\sigma_{tr} r}}{4\pi r}\left(\left(\tilde{\sigma}_s + (\tilde{g}+1)\sigma_a\right)\left(1 - \frac{\sigma_a}{\tilde{\sigma}_s}\right) - \tilde{g}\tilde{\sigma}_s\right) \\
& + \frac{3e^{-\sigma_{tr} r}}{4\pi r}\cos\theta\left(1 + (\tilde{g}+1)\frac{\sigma_a}{\tilde{\sigma}_s}\right)\left(\sigma_{tr} + \frac{1}{r}\left(1 - 2\frac{\sigma_a}{\tilde{\sigma}_s}\right)\right) \\
& + \frac{3\left(\tilde{\sigma}_s + (\tilde{g}+1)\sigma_a\right) e^{-\sigma_{tr} r}}{4\pi r} 2\frac{\sigma_{tr}\cos^2\theta}{\tilde{\sigma}_s^2 r}.
\end{aligned}
$$

Now we neglect terms $\sigma_a/\tilde{\sigma}_s$ compared to unity

$$
\frac{\phi(\mathbf{x})}{\Phi_i} \approx \frac{3e^{-\sigma_{tr} r}}{4\pi r}\tilde{\sigma}_s\left(1 - \tilde{g}\right) + \frac{3e^{-\sigma_{tr} r}}{4\pi r}\cos\theta\left(\sigma_{tr} + \frac{1}{r}\right) + \frac{3e^{-\sigma_{tr} r}}{4\pi r} 2\frac{\sigma_{tr}\cos^2\theta}{\tilde{\sigma}_s r}.
$$

Introducing the reduced scattering coefficient $\sigma_s' = \tilde{\sigma}_s(1-\tilde{g}) = \sigma_s(1-g)$, and neglecting terms $\sigma_{tr}/\tilde{\sigma}_s \cdot 1/(\tilde{\sigma}_s r)$ compared to unity,

$$
\phi(\mathbf{x}) \approx \frac{3\Phi_i e^{-\sigma_{tr} r}}{4\pi r}\sigma_s' + \frac{3\Phi_i e^{-\sigma_{tr} r}}{4\pi r}\cos\theta\left(\sigma_{tr} + \frac{1}{r}\right) = \frac{\Phi_i}{4\pi D}\frac{e^{-\sigma_{tr} r}}{r}\left(1 + 3D\frac{1 + \sigma_{tr} r}{r}\cos\theta\right),
$$

where $1/D$ is used in place of $3\sigma_s'$, since the σ_s' can be approximated by σ_t' when absorption is negligible compared to scattering. This is a valid assumption in highly scattering media.

7 Fresnel integrals

To aid in our calculations, we define the Fresnel transmittance integrals of the first two orders:

$$
\begin{aligned}
C_\phi(\eta) &= \frac{1}{4\pi}\int_{2\pi} T_{21}(\eta, \theta_o)\cos\theta_o \mathrm{d}\vec{\omega}_o \\
C_{\mathbf{E}}(\eta) &= \frac{3}{4\pi}\int_{2\pi} T_{21}(\eta, \theta_o)\cos^2\theta_o \mathrm{d}\vec{\omega}_o\,,
\end{aligned}
$$

where $\cos\theta_o = \vec{n}_o \cdot \vec{\omega}_o$, and the integral is on the whole hemisphere where $\vec{n}_o \cdot \vec{\omega}_o > 0$. These integrals are similar to F_{dr}, but based on Fresnel transmittance instead of Fresnel reflectance.

8 BSSRDF theory

The BSSRDF is defined as the ratio of an element of emergent radiance L_r to an element of incident flux Φ_i [9]:

$$
S(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o) = \frac{\mathrm{d}L_r(\mathbf{x}_o, \vec{\omega}_o)}{\mathrm{d}\Phi_i(\mathbf{x}_i, \vec{\omega}_i)}\,.
$$

There are various approximations available for the BSSRDF. For the directional dipole [5], the BSSRDF is split into the following terms:

$$S = T_{12}(S_{\delta E} + S_d)T_{21}\,.$$

Let us consider only the diffusive part, S_d. The emergent radiance due to diffusion events is given by

$$L_{r,d}(\mathbf{x}_o, \vec{\omega}_o) = \eta^2 T_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21})\,,$$

where $\vec{\omega}_{21}$ is the refracted vector corresponding to $\vec{\omega}_o$:

$$\vec{\omega}_{21} = \frac{\vec{\omega}_o}{\eta} - \left(\frac{\vec{n}_o \cdot \vec{\omega}_o}{\eta} - \sqrt{1 - \frac{1 - (\vec{n}_o \cdot \vec{\omega}_o)^2}{\eta^2}}\right)\vec{n}_o\,.$$

Combining the above equations we obtain:

$$S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o) = T_{12} S_d T_{21} = \frac{\mathrm{d}L_{r,d}(\mathbf{x}_o, \vec{\omega}_o)}{\mathrm{d}\Phi_i(\mathbf{x}_i, \vec{\omega}_i)} = \eta^2 \frac{\mathrm{d}T_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21})}{\mathrm{d}\Phi_i(\mathbf{x}_i, \vec{\omega}_i)}$$

We now integrate over the cosine-weighted hemisphere with $\vec{n}_o \cdot \vec{\omega}_o > 0$ on both sides of the equation:

$$\int_{2\pi} T_{12} S_d T_{21} \cos\theta_o \mathrm{d}\vec{\omega}_o = \int_{2\pi} \eta^2 \frac{\mathrm{d}T_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21})}{\mathrm{d}\Phi_i(\mathbf{x}_i, \vec{\omega}_i)} \cos\theta_o \mathrm{d}\vec{\omega}_o\,.$$

Assuming no dependency on the outgoing direction, $S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o, \vec{\omega}_o) = S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o)$, and we have

$$\begin{aligned}
T_{12} S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) \int_{2\pi} T_{21} \cos\theta_o \mathrm{d}\vec{\omega}_o &= \eta^2 \frac{\mathrm{d}\int_{2\pi} T_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21}) \cos\theta_o \mathrm{d}\vec{\omega}_o}{\mathrm{d}\Phi_i(\mathbf{x}_i, \vec{\omega}_i)} \\
T_{12} S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) 4\pi \frac{C_\phi(\eta)}{\eta^2} &= \frac{\mathrm{d}M_d(\mathbf{x}_o)}{\mathrm{d}\Phi_i(\mathbf{x}_i, \vec{\omega}_i)} \\
T_{12} S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) 4\pi C_\phi(1/\eta) &= \frac{\mathrm{d}M_d(\mathbf{x}_o)}{\mathrm{d}\Phi_i(\mathbf{x}_i, \vec{\omega}_i)}\,. \qquad (8)
\end{aligned}$$

Let us calculate the diffuse radiant exitance first. We insert the diffusion approximation in place of L_d to find

$$\begin{aligned}
M_d(\mathbf{x}_o) &= \int_{2\pi} T_{21} L_d(\mathbf{x}_o, \vec{\omega}_{21}) \cos\theta_o \mathrm{d}\vec{\omega}_o = \int_{2\pi} T_{21} \left(\frac{\phi(\mathbf{x}_o)}{4\pi} - \frac{3}{4\pi} D\, \vec{\omega}_{21} \cdot \nabla\phi(\mathbf{x}_o)\right) \cos\theta_o \mathrm{d}\vec{\omega}_o \\
&= \underbrace{\frac{\phi(\mathbf{x}_o)}{4\pi} \int_{2\pi} T_{21} \cos\theta_o \mathrm{d}\vec{\omega}_o}_{I_\phi} - \underbrace{\frac{3}{4\pi} D \int_{2\pi} \vec{\omega}_{21} \cdot \nabla\phi(\mathbf{x}_o) \cos\theta_o \mathrm{d}\vec{\omega}_o}_{I_{\mathbf{E}}}\,.
\end{aligned}$$

And, in a straightforward way,

$$I_\phi = C_\phi(\eta)\phi(\mathbf{x}_o)\,.$$

The second part is more complicated. Without loss of generality, we rotate the reference coordinate system so that $\vec{\omega}_o = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ and $\vec{n}_o = (0, 0, 1)$. Given this reference system, the refracted vector becomes:

$$\begin{aligned}\vec{\omega}_{21} &= \frac{(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)}{\eta} - \left(\frac{\cos\theta}{\eta} - \sqrt{1 - \frac{\sin^2\theta}{\eta^2}}\right)(0, 0, 1) \\ &= \left(\frac{\cos\phi\sin\theta}{\eta}, \frac{\sin\phi\sin\theta}{\eta}, \sqrt{1 - \frac{\sin^2\theta}{\eta^2}}\right)\end{aligned}$$

When inserted in the dot product in the expression for $I_{\mathbf{E}}$, we get a sum of three components. The first two terms of this sum are zero, since we can first integrate over ϕ:

$$\int_0^{2\pi} \cos\phi \mathrm{d}\phi = \int_0^{2\pi} \sin\phi \mathrm{d}\phi = 0\,.$$

So we are left only with the cumbersome z term:

$$I_{\mathbf{E}} = \frac{3}{4\pi} D \frac{\partial\phi_z(\mathbf{x}_o)}{\partial z} \int_0^{2\pi}\int_0^{\frac{\pi}{2}} T_{21}(\eta, \theta_o)\sqrt{1 - \frac{\sin^2\theta_o}{\eta^2}} \cos\theta_o \sin\theta_o \mathrm{d}\theta_o \mathrm{d}\phi\,,$$

where we note that $\frac{\partial\phi_z}{\partial z} = \vec{n}_o \cdot \nabla\phi$. We assume $\eta > 1$, so that the argument of the square root is never negative. We now perform a substitution using the law of refraction, $\sin\theta_i = \eta\sin\theta_o$. We obtain the following identities:

$$\mathrm{d}\theta_o = \frac{\eta\cos\theta_i}{\cos\theta_o}\mathrm{d}\theta_i, \quad T_{21}(\eta, \theta_o) = T_{21}(1/\eta, \theta_i), \quad \sqrt{1 - \frac{\sin^2\theta_o}{\eta^2}} = \cos\theta_i.$$

If we introduce a critical angle $\theta_c = \arcsin(1/\eta)$, we get

$$\begin{aligned}I_{\mathbf{E}} &= \frac{3}{4\pi} D\vec{n}_o \cdot \nabla\phi(\mathbf{x}_o) \int_0^{2\pi}\int_0^{\theta_c} T_{21}(1/\eta, \theta_i)\cos\theta_i\cos\theta_o\eta\sin\theta_i\frac{\eta\cos\theta_i}{\cos\theta_o}\mathrm{d}\theta_i\mathrm{d}\phi \\ &= D\vec{n}_o \cdot \nabla\phi(\mathbf{x}_o)\frac{3}{4\pi}\eta^2 \int_0^{2\pi}\int_0^{\theta_c} T_{21}(1/\eta, \theta_i)\cos^2\theta_i\sin\theta_i\mathrm{d}\theta_i\mathrm{d}\phi \\ &= D\vec{n}_o \cdot \nabla\phi(\mathbf{x}_o)\eta^2 C_{\mathbf{E}}(1/\eta) = C_{\mathbf{E}}(\eta)D\vec{n}_o \cdot \nabla\phi(\mathbf{x}_o)\,.\end{aligned}$$

We can then get our final expression for $M_d(\mathbf{x}_o)$

$$M_d(\mathbf{x}_o) = C_\phi(\eta)\phi(\mathbf{x}_o) - C_{\mathbf{E}}(\eta)D\vec{n}_o \cdot \nabla\phi(\mathbf{x}_o)\,.$$

Inserting into the expression (8), we derive the monopole BSSRDF:

$$S_d(\mathbf{x}_i, \vec{\omega}_i, \mathbf{x}_o) = \frac{1}{4\pi T_{12} C_\phi(1/\eta)} \frac{\mathrm{d}M_d(\mathbf{x}_o)}{\mathrm{d}\Phi_i(\mathbf{x}_i, \vec{\omega}_i)}$$

8.1 Diffuse monopole BSSRDF

Using the monopole solution (6) for a ray normally incident on an isotropic half-space:

$$\phi(\mathbf{x}) = \alpha' \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r} \,,$$

we can derive the gradient:

$$\nabla\phi(\mathbf{x}) = -\alpha' \frac{\Phi_i}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r^3} (1 + \sigma_{tr} r)\mathbf{x} \,.$$

Thus, we can find the monopole BSSRDF for this configuration:

$$S_d(\mathbf{x}_i, \mathbf{x}_o) = \frac{1}{4C_\phi(1/\eta)} \frac{\alpha'}{4\pi^2} \frac{e^{-\sigma_{tr} r}}{r^3} \left[C_\phi(\eta) \frac{r^2}{D} + C_{\mathbf{E}}(\eta)(1 + \sigma_{tr} r)(\mathbf{x}_o - \mathbf{x}_i) \cdot \vec{n}_o \right] \,,$$

where we used $\Phi_i = T_{12} L_i(\mathbf{x}_i, \vec{\omega}_i)$. This is the monopole version of the better dipole from Eugene d'Eon [2]. To get a monopole version of the standard dipole [4, 6], we further approximate the above expression using $C_\phi(1/\eta) \approx 1/4$, $C_\phi(\eta) \approx 0$, and $C_{\mathbf{E}}(\eta) \approx 1$:

$$S_d(\mathbf{x}_i, \mathbf{x}_o) = \frac{\alpha'}{4\pi^2} \frac{e^{-\sigma_{tr} r}}{r^3} (1 + \sigma_{tr} r)\mathbf{x} \cdot \vec{n}_o$$

This is the monopole version of the standard dipole.

8.2 Directional monopole BSSRDF

For the directional dipole, we have the following form:

$$\phi(\mathbf{x}) = \frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r} \left(1 + 3D \frac{1 + \sigma_{tr} r}{r} \cos\theta \right) .$$

It is more convenient to do the gradient in spherical coordinates:

$$\nabla\phi(\mathbf{x}) = \frac{\partial}{\partial r}\phi(\mathbf{x})\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\phi(\mathbf{x})\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\phi(\mathbf{x})\vec{e}_\phi \,,$$

where

$$\frac{\partial}{\partial r}\phi(\mathbf{x}) = -\frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r^2} \left(3D \frac{2(1 + \sigma_{tr} r) + (\sigma_{tr} r)^2}{r} \cos\theta + (1 + \sigma_{tr} r) \right)$$

and

$$\frac{\partial}{\partial\theta}\phi(\mathbf{x}) = -\frac{\Phi}{4\pi D} \frac{e^{-\sigma_{tr} r}}{r^2} 3D(1 + \sigma_{tr} r)\sin\theta \,.$$

Finally, $\frac{\partial}{\partial\phi}\phi(\mathbf{x}) = 0$. Given our choice of basis for derivation, we can use the following identities:

$$\vec{e}_r = \frac{\mathbf{x}}{r}$$

$$-\vec{e}_\theta \sin\theta = \vec{\omega}_{12} - \vec{e}_r \cos\theta \,.$$

Inserting the identities, we get an expression for the gradient:

$$\nabla\phi(\mathbf{x}) = -\frac{\Phi}{4\pi D}\frac{e^{-\sigma_{tr}r}}{r^2}\left(3D\frac{2(1+\sigma_{tr}r)+(\sigma_{tr}r)^2}{r}\cos\theta + (1+\sigma_{tr}r)\right)\vec{e}_r + \frac{1}{r}\frac{\Phi}{4\pi D}\frac{e^{-\sigma_{tr}r}}{r^2}3D(1+\sigma_{tr}r)(-\vec{e}_\theta\sin\theta)$$

$$\nabla\phi(\mathbf{x}) = \frac{\Phi}{4\pi D}\frac{e^{-\sigma_{tr}r}}{r^3}\Bigg[\left(-3D\frac{2(1+\sigma_{tr}r)+(\sigma_{tr}r)^2}{r}\cos\theta - (1+\sigma_{tr}r)\right)r\vec{e}_r - 3D(1+\sigma_{tr}r)\cos\theta\vec{e}_r + 3D(1+\sigma_{tr}r)\vec{\omega}_{12}\Bigg]$$

$$\nabla\phi(\mathbf{x}) = \frac{\Phi}{4\pi D}\frac{e^{-\sigma_{tr}r}}{r^3}\Bigg[-\left(3D\frac{3(1+\sigma_{tr}r)+(\sigma_{tr}r)^2}{r}\cos\theta + (1+\sigma_{tr}r)\right)\mathbf{x} + 3D(1+\sigma_{tr}r)\vec{\omega}_{12}\Bigg].$$

We can now do the same as above, obtaining the directional monopole BSSRDF with $\mathbf{x} = \mathbf{x}_o - \mathbf{x}_i$ and $r = \|\mathbf{x}\|$:

$$S'_d(\mathbf{x}_i,\vec{\omega}_i,\mathbf{x}_o) = \frac{1}{4C_\phi(1/\eta)}\frac{1}{4\pi^2}\frac{e^{-\sigma_{tr}r}}{r^3}\Bigg[C_\phi(\eta)\left(\frac{r^2}{D} + 3(1+\sigma_{tr}r)\mathbf{x}\cdot\vec{\omega}_{12}\right) - C_{\mathbf{E}}(\eta)\left(3D(1+\sigma_{tr}r)\vec{\omega}_{12}\cdot\vec{n}_o - \left((1+\sigma_{tr}r) + 3D\frac{3(1+\sigma_{tr}r)+(\sigma_{tr}r)^2}{r^2}\mathbf{x}\cdot\vec{\omega}_{12}\right)\mathbf{x}\cdot\vec{n}_o\right)\Bigg].$$

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