Vectors and Matrices

1 Vector

1.1 Column Vectors and Row Vectors

Column vectors is, in the simplest way, components written in a column.

Row vectors is similar as column vector, that is, components in row.

$$\textbf{Column Vector} \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \text{also written as} \quad \boldsymbol{v} = \begin{bmatrix} v_1, v_2, \dots, v_n \end{bmatrix}.$$

$$\textbf{Row Vector} \quad \boldsymbol{w} = \begin{bmatrix} w_1 & w_2 & \dots & w_v \end{bmatrix}, \text{or} \quad \boldsymbol{w} = \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix}.$$

Notice the only difference between row and column vector is that there is no comma among components in the latter.

1.2 Linear Combination

There are two basic operations in vectors:

$$\textbf{Vector Addition} \quad \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

$$\textbf{Scalar Multiplication} \quad t \times \boldsymbol{v} = t \times \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} tv_1 \\ tv_2 \\ \vdots \\ tv_n \end{bmatrix}$$

Linear Combination, one of the most important concepts of linear algebra, is base on these operations.

Linear Combination of $\boldsymbol{v}$ and $\boldsymbol{w}$ is $c\boldsymbol{v} + d\boldsymbol{w}$.

The definition of linear combination can be extended to more than 2 vectors. For instance, the linear combination of three vector $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ is $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$.

2 Matrix

Matrix comes from linear equations. Suppose we have linear equation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{aligned} \tag{$*$}$$

This equation can be rewritten using vector:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

even as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a matrix, we can consider it as the combination of three column vectors.

So the equation $(*)$ can be abbreviated to

$$A\boldsymbol{x} = \boldsymbol{b}$$

This is one of the most important equation in linear algebra. A major topic in linear algebra is whether the equation has a solution and how to slove it.

2.1 Matrix Operation

Matrices can be add and multipled by scalar as vectors. It can also be multipled by vector above. But the most important operation is being multipled by another matrix.

Let's say equations:

$$A \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} \quad A \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix}$$

as above we can write it like

$$A \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

that is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

Since the abbreviated formula must have the same meaning as origional one, we can conclude the princple of matrix multiplication. There are 4 ways to consider it:

1. the (i, j) entry of C is (ith row of A) $\cdot$ (jth column of B)

   Since the origional formula is

$$\begin{gathered}a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = c_{11} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = c_{21} \\ \cdots\end{gathered}$$

   we got

$$\begin{aligned}c_{ij} &= \sum_{k=1}^{n} a_{ik}b_{kj} \\ &= (i\text{th row of } A) \cdot (j\text{th column of } B)\end{aligned}$$

2. the ith row of C is (ith row of A) $\cdot B$

3. the jth column of C is $A \cdot$ (jth column of B)

4. C is the sum of all (ith column of A) $\cdot$ (jth row of B)

2.2 The Law of Matrix Operations

$$\begin{aligned}A + B &= B + A && \text{(commutative law)} \\ c(A + B) &= cA + cB && \text{(distributive law)} \\ A + (B + C) &= (A + B) + C && \text{(associative law)}\end{aligned}$$

$$\begin{aligned}AB &\neq BA && \text{(no commutative law)} \\ A(B + C) &= AB + AC && \text{(distributive law from the left)} \\ (A + B)C &= AC + BC && \text{(distributive law from the right)} \\ A(BC) &= (AB)C && \text{(associative law)}\end{aligned}$$

$$\begin{aligned}(A^p)(A^q) &= A^{p+q} \\ (A^p)^q &= A^{pq}\end{aligned}$$

2.3 Block Matrix and Block Multiplication

The entry of a matrix can be other matrices, for exmple, let's say

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \quad C = \begin{bmatrix} c \end{bmatrix}$$

then we can say

$$S = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \\ b_1 & b_2 & b_3 & c \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$$

If the block's size agree, block matrix can multiple

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}$$